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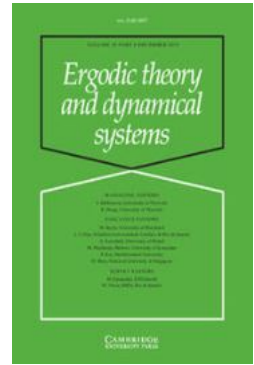
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Structure of transition classes for factor codes on shifts of finite type

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Abstract. Given a factor code π from a shift of finite type X onto a sofic shift Y, the class degree of π is defined to be the minimal number of transition classes over the points of Y. In this paper, we investigate the structure of transition classes and present several dynamical properties analogous to the properties of fibers of finite-to-one factor codes. As a corollary, we show that for an irreducible factor triple, there cannot be a transition between two distinct transition classes over a right transitive point, answering a question raised by Quas.

1. Introduction

Given a finite-to-one factor code π from a shift of finite type X onto an irreducible sofic shift Y, the *degree* of π is defined to be the minimal number of preimages of the points in Y. The notion of degree was first introduced in [**12**] for endomorphisms of full shifts, and was then extended to those of irreducible shifts of finite type and sofic shifts [**10**]. The concept is widely studied and is useful in the study of finite-to-one factor codes [**2**, **5**, **14**, **19**]. If d is the degree of a one-block factor code π, there are well-known fundamental properties of fibers of points in Y, summarized as follows.

(1) Every doubly transitive point in Y has exactly d preimages.
(2) $\pi(x)$ is doubly transitive if and only if x is.
(3) Any two distinct preimages of a doubly transitive point in Y are *mutually separated*, i.e. they do not share a common symbol at the same time.

In this work, we show that a natural generalization of the degree for general factor codes is the *class degree* introduced in [**1**]. The principal motivation for defining the class degree was to find a conjugacy-invariant upper bound on the number of ergodic measures of relative maximal entropy. Measures of relative maximal entropy have appeared in

many different topics in symbolic dynamics, owing to their connections with, for example, functions of Markov chains [**3**, **8**] and their use in computing the Hausdorff dimension of certain sets [**11**]. The class degree is, in fact, a conjugacy-invariant upper bound on the number of such measures over a fully supported ergodic measure [**1**, **16**].

The class degree is defined by using a certain equivalence relation on the fiber of each point in Y. Roughly speaking, two preimages x and $\bar{x}$ of a point y in Y are equivalent if we can find a preimage z of y which is equal to x up to an arbitrarily large given positive coordinate and is right asymptotic to $\bar{x}$, and vice versa (see Definition 2.3). The *class degree* is defined to be the minimal number of equivalence classes (called *transition classes*) over the points in Y. It is shown in [**1**] that the class degree is equal to the degree when π is finite-to-one, and, moreover, if d is the class degree of π, then every right transitive point in Y has exactly d transition classes (analogous to (1) above). This suggests that the class degree is a candidate for a generalization of the degree.

The idea of considering transitions between preimages of a point also came up independently in [**21**]. To find a condition which is invariant under conjugacy and weaker than the condition that appeared in [**9**], Yoo defined the notion of *fiber-mixing* and showed that a fiber-mixing code between two mixing shifts of finite type sends every fully supported Markov measure on X to a Gibbs measure on Y. Fiber-mixing codes were investigated in further studies, e.g. [**13**, **17**]. In our terminology, a fiber-mixing code from a shift of finite type X onto a sofic shift Y is just a code in which every point in Y has only one transition class; thus it is natural to ask what kind of properties a code can have when the code is not fiber-mixing—for example, when the code has class degree one but there exist some points with more than one transition class. Moreover, since the definition of transition classes is motivated by communicating classes in Markov chains, Quas asked whether there could be a transition between two distinct transition classes over a right transitive point.

To answer such questions, one needs to have a structural theory on transition classes. In fact, unlike the finite-to-one case, where the fibers have been well-studied, previous research on infinite-to-one codes usually concentrated on their thermodynamic formalism [**6**, **20**] or on the construction of codes with nice properties [**4**, **7**, **18**]. In this paper, we investigate the fibers and transition classes of such factor codes and provide several structural results. By these results, it is natural to consider the class degree as a natural generalization of the degree.

In particular, we provide dynamical properties analogous to (2) and (3) above (see Theorem 3.4, Corollary 3.6 and Theorem 4.4).

(2′) A point y is right (respectively, doubly) transitive if and only if each transition class over y contains a right (respectively, doubly) transitive point.

(3′) Any two points from two distinct transition classes over a right transitive point are mutually separated.

This analogy shows us that, as for a finite-to-one code, fibers over almost all images for infinite-to-one factors are well-behaved, in the sense that among the fibers over a typical point of Y a typical point of X always exists, and that any points chosen from distinct classes over a typical point of Y have orbits which neither meet nor approach each other asymptotically. As a corollary, we also show that there cannot be any transition among

distinct classes (see Corollary 4.7). Property $(3')$ above is one type of separation property between distinct transition classes. In §5, we present another type of separation property, which is stronger than the former; briefly, we make a partition on the set of preimages of a magic block of π, so that one can determine whether or not two preimages of a doubly transitive point are in the same transition class only by reading the symbols occurring in their coordinates over the magic block (see Theorem 5.4). Other structural properties on transition classes are also provided which will hopefully open new doors on further investigation of infinite-to-one factor codes.

2. *Background*

In this section, we introduce some terminology and basic results on symbolic dynamics.

If X is a subshift (or shift space) with the shift map σ, then denote by $\mathcal{B}_n(X)$ the set of all n-blocks occurring in the points of X and let $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. The alphabet of a shift space X is denoted by $\mathcal{A}(X) = \mathcal{B}_1(X)$.

A *code* $\pi : X \to Y$ is a continuous σ-commuting map between shift spaces. It is called a *factor code* if it is surjective. Every code can be recoded to become a one-block code, i.e. a code for which x_0 determines $\pi(x)_0$. Given a one-block code $\pi : X \to Y$, it naturally induces a map on $\mathcal{B}(X)$, which we also denote by π for brevity. We say that π is *finite-to-one* if $\pi^{-1}(y)$ is a finite set for all $y \in Y$.

A triple (X, Y, π) is called a *factor triple* if $\pi : X \to Y$ is a factor code from a shift of finite type X onto a (sofic) subshift Y. A factor triple is said to be irreducible when X is irreducible. It is said to be finite-to-one if π is finite-to-one.

A point x in a shift space X is said to be *right transitive* if every block in X occurs infinitely many times in $x_{[0,\infty)}$ or, equivalently, if the forward orbit of x is dense. Two points x and $\bar{x}$ in X are said to be *right asymptotic* if $x_{[N,\infty)} = \bar{x}_{[N,\infty)}$ for some $N \in \mathbb{Z}$. Left asymptotic points and left transitivity are defined similarly. A point is *doubly transitive* if it is both left and right transitive.

If π is a finite-to-one factor code from a shift of finite type X onto a sofic shift Y, there is a uniform upper bound on the number of preimages of points in Y [**15**]. The minimal number of π-preimages of the points in Y is called the *degree* of the factor code π and is denoted by d_π.

THEOREM 2.1. [**15**, §9] *Let (X, Y, π) be a finite-to-one factor triple with Y irreducible. Then every doubly transitive point of Y has exactly d_π preimages.*

Two points x and $\bar{x}$ in a shift space are *mutually separated* if x_i and $\bar{x}_i$ are different for each integer i. It is well known that if (X, Y, π) is a finite-to-one factor triple with X one-step, π one-block and Y irreducible, then each $y \in Y$ has d_π mutually separated preimages. In particular, if y is doubly transitive, then any two distinct preimages of y are mutually separated.

We say that two factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$ are *conjugate* if X is conjugate to $\tilde{X}$ under a conjugacy ϕ, Y is conjugate to $\tilde{Y}$ under a conjugacy ψ, and $\tilde{\pi} \circ \phi = \psi \circ \pi$. An immediate corollary of Theorem 2.1 is that d_π is invariant under conjugacy.

Definition 2.2. [**15**] Let (X, Y, π) be a factor triple with X one-step and π one-block. Given a block $w \in \mathcal{B}(Y)$, define

$$d(w) = \min_{1 \le k < |w|} |\{a \in \mathcal{A}(X) : \exists \, u \in \pi^{-1}(w) \text{ with } u_k = a\}|.$$

If a block w satisfies $d(w) = \min_{v \in \mathcal{B}(Y)} d(v)$, then it is called a *magic block*. In this case, a coordinate k where the minimum occurs is called a *magic coordinate*. If $|w| = 1$, then w is called a *magic symbol*.

Let w be a magic block with a magic coordinate k. Given a point y in Y and $i \in \mathbb{Z}$ with $y_{[i, i+|w|)} = w$, some block in $\pi^{-1}(w)$ may not be extendable to a point in $\pi^{-1}(y)$. However, due to the minimality of $d(w)$, the two sets $\{x_{i+k} \mid x \in \pi^{-1}(y)\}$ and $\{u_k \mid u \in \pi^{-1}(w)\}$ are the same. It is well known that for a one-block finite-to-one factor code π from a one-step shift of finite type X onto an irreducible sofic shift Y, we have $d_\pi = d(w)$ for any magic block w of π.

The class degree defined below is a quantity analogous to the degree when the factor code π is not limited to being finite-to-one.

Definition 2.3. Let (X, Y, π) be a factor triple and let $x, \bar{x} \in X$. We say that there is a *transition* from x to $\bar{x}$, and denote it by $x \to \bar{x}$, if for each integer n there exists a point z in X such that:

(1) $\pi(z) = \pi(x) = \pi(\bar{x})$; and
(2) $z_{(-\infty, n]} = x_{(-\infty, n]}$ and $z_{[i, \infty)} = \bar{x}_{[i, \infty)}$ for some $i \ge n$.

We write $x \sim \bar{x}$ and say that x and $\bar{x}$ are in the same *transition class* if $x \to \bar{x}$ and $\bar{x} \to x$. Then the relation $\sim$ is an equivalence relation. Denote the set of transition classes in X over $y \in Y$ by $\mathscr{C}(y)$. We say that there is a transition from a class $[x]$ to another class $[\bar{x}]$, and denote it by $[x] \to [\bar{x}]$, if $x \to \bar{x}$. Note that if $[x] \to [\bar{x}]$, then for each $z \sim x$ and $\bar{z} \sim \bar{x}$ we have $z \to \bar{z}$.

FACT 2.4. [**1**] *Let $\pi : X \to Y$ be a one-block factor code from a one-step shift of finite type X onto an irreducible sofic shift Y. Then the following hold.*

(1) $|\mathscr{C}(y)| < \infty$ *for each y in Y.*
(2)   *Let $x, x' \in \pi^{-1}(y)$ for some $y \in Y$. Given $x_{a_i} = x'_{a_i}$ where $(a_i)_{i \in \mathbb{N}}$ is a strictly increasing sequence in $\mathbb{Z}$, we have $x \sim x'$.*

Definition 2.5. Let (X, Y, π) be a factor triple. The minimal number of transition classes over points of Y is called the *class degree* of π and is denoted by c_π.

It is clear that c_π is invariant under conjugacy. It was shown in [**1**] that for a finite-to-one factor triple (X, Y, π) with Y irreducible, we have $c_\pi = d_\pi$.

THEOREM 2.6. [**1**] *Let (X, Y, π) be a factor triple with Y irreducible. Then every right transitive point of Y has exactly c_π transition classes.*

Theorem 2.9 below states the class degree of a factor code in terms of another quantity, which is defined concretely in terms of blocks.

Definition 2.7. Let (X, Y, π) be a factor triple with X one-step and π one-block. Let $w = w_{[0,p]} \in \mathcal{B}_{p+1}(Y)$. Also, let n be an integer in $(0, p)$ and M a subset of $\pi^{-1}(w_n)$. We say that a block $u \in \pi^{-1}(w)$ is *routable through* $a \in M$ *at time* n if there is a block $\bar{u} \in \pi^{-1}(w)$ with $\bar{u}_0 = u_0$, $\bar{u}_p = u_p$ and $\bar{u}_n = a$. A triple (w, n, M) is called a *transition block* of π if every block in $\pi^{-1}(w)$ is routable through a symbol of M at time n. The cardinality of the set M is called the *depth* of the transition block (w, n, M). When there is no confusion, for example when $y \in Y$ and $w = y_{[i,i+p]}$ are fixed, we say that the points $x, \bar{x} \in \pi^{-1}(y)$ are routable through $a \in M$ at time $i + n$ if $x_{[i,i+p]}$ and $\bar{x}_{[i,i+p]}$ are routable through a at time $i + n$.

Definition 2.8. Let

$$c_\pi^* = \min\{|M| : (w, n, M) \text{ is a transition block of } \pi\}.$$

A *minimal transition block* of π is a transition block of depth c_π^*.

THEOREM 2.9. [1] *Let (X, Y, π) be a factor triple with X one-step, π one-block and Y irreducible. Then $c_\pi = c_\pi^*$.*

For more details on symbolic dynamics, see [15]. For a perspective on the class degree and its relation to the degree, see [1].

3. *Each transition class over a right transitive point contains a right transitive point*
In this section, we prove that given an irreducible factor triple (X, Y, π), each transition class over a right transitive point contains a right transitive point. This result can be seen as an analogue of the well-known fact that for a finite-to-one irreducible factor triple (X, Y, π), every preimage of a right (respectively, doubly) transitive point is a right (respectively, doubly) transitive point. We begin with the following definition.

Definition 3.1. Let (X, Y, π) be a factor triple and let $\bar{X}$ be a proper subshift of X with $\pi(\bar{X}) = Y$. Let $\bar{v}$ be in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$. We say that two blocks u and v in $\mathcal{B}(X)$ form an $(\bar{X}, \bar{v})$-*diamond* if the following hold:
(1) $\pi(u) = \pi(v)$;
(2) $\bar{v}$ is a subblock of v;
(3) u occurs in $\bar{X}$; and
(4) u and v share the same initial symbol and the same terminal symbol.

The following lemma is a slightly stronger version of [22, Proposition 3.1]. We include a different proof here, which is also more direct than the original one. Later, in §6, a further strengthened version of Lemma 3.2 will be provided.

LEMMA 3.2. *Let (X, Y, π) be an irreducible factor triple with X one-step and π one-block. Let $\bar{X}$ be a proper subshift of X with $\pi(\bar{X}) = Y$. Then, for each block $\bar{v}$ in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$, there is an $(\bar{X}, \bar{v})$-diamond.*

Proof. For w in $\mathcal{B}(Y)$, define $n(w)$ to be the maximal number of mutually separated preimages of w in $\mathcal{B}(\bar{X})$, and let n be the infimum of $n(w)$ where w runs over all the

non-empty words in $\mathcal{B}(Y)$. Clearly, n is a positive integer. Let w be a block in $\mathcal{B}(Y)$ with $n(w) = n$ and let u be a preimage of w in $\mathcal{B}(\bar{X})$.

Let $\bar{v}$ be in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$. Since X is irreducible, there is a cycle α in $\mathcal{B}(X)$ such that $\alpha = u\gamma\bar{v}\eta$ for some blocks γ and η. Denote the length of α by l, and consider α^{n+1}. There are at least n mutually separated blocks $\beta^{(1)}, \ldots, \beta^{(n)}$ in $\mathcal{B}(\bar{X})$ all projecting to $\pi(\alpha^{n+1})$. Note that for all $0 \le j < n + 1$ and $1 \le m \le n$, we have $\pi(\alpha^{n+1}_{[jl,jl+|w|)}) = \pi(\beta^{(m)}_{[jl,jl+|w|)}) = w$. Moreover, for all $1 \le m, m' \le n$ where $m \ne m'$, the blocks $\beta^{(m)}_{[jl,jl+|w|)}$ and $\beta^{(m')}_{[jl,jl+|w|)}$ are mutually separated.

Since $n(w) = n$, for each $0 \le j < n + 1$ there is $1 \le m_j \le n$ such that $\beta^{(m_j)}_{[jl,jl+|w|)}$ meets u, i.e. there is $0 < i < |w|$ such that $\beta^{(m_j)}_{jl+i} = u_i$. Thus, by the pigeonhole principle, there is $1 \le m \le n$ such that $\beta^{(m)}$ meets u twice, say at positions $jl + i$ and $j'l + i'$ for some $0 \le j < j' < n + 1$ and $0 \le i, i' < |w|$. It is clear that the blocks $\beta^{(m)}_{[jl+i,j'l+i')}$ and $\alpha^{n+1}_{[jl+i,j'l+i')}$ form an $(\bar{X}, \bar{v})$-diamond. $\qquad\square$

Remark 3.3. Note that Lemma 3.2 is not necessarily true when X is reducible. For example, let X be the orbit closure of the point $a^{\infty}.b^{\infty}$, let $Y = \{0^{\infty}\}$, and consider the trivial map $\pi : X \to Y$. Let $\bar{X} = \{a^{\infty}\}$ and $\bar{v} = b$.

THEOREM 3.4. *Let (X, Y, π) be an irreducible factor triple and let y in Y be right transitive. Then each transition class over y contains a right transitive point.*

Proof. We may assume that X is one-step and π is one-block. Let C be a transition class over y and let x be in C. If x is right transitive, we are done. So suppose that x is not right transitive. Let $\bar{X}$ be the ω-limit set of x, i.e.

$$\bar{X} = \omega(x) = \{z \in X : \exists \, n_i \nearrow \infty \text{ with } \sigma^{n_i}(x) \to z\}.$$

Then we have $\bar{X} \subsetneq X$. Since $\pi(x) = y$ and y is right transitive, it follows that $\pi(\bar{X}) = Y$. Now consider an enumeration $\bar{v}_1, \bar{v}_2, \ldots$ of $\mathcal{B}(X)$. For each $i \in \mathbb{N}$ with $\bar{v}_i \in \mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$, by Lemma 3.2 there is an $(\bar{X}, \bar{v}_i)$-diamond (u_i, v_i). Note that for this i, $\bar{v}_i$ is a subblock of v_i, $v_i \in \mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$ and $u_i \in \mathcal{B}(\bar{X})$.

For each $i \in \mathbb{N}$, define a block $w_i \in \mathcal{B}(\bar{X})$ by

$$w_i = \begin{cases} \bar{v}_i & \text{if } \bar{v}_i \in \mathcal{B}(\bar{X}), \\ u_i & \text{otherwise.} \end{cases}$$

Then, since each w_i is in $\mathcal{B}(\bar{X})$, we can find an increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $x_{[n_i,n_i+|w_i|)} = w_i$ and $n_{i+1} > n_i + |w_i|$ for all $i \in \mathbb{N}$. Finally, define a new point $z \in X$, obtained from x by replacing each occurrence of u_i at the coordinates $x_{[n_i,n_i+|w_i|)}$ for all $i \in \mathbb{N}$ with $v_i \in \mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$. Since each (u_i, v_i) forms a diamond, z is indeed a point in X and we have $\pi(z) = y$.

Since there are infinitely many positive coordinates j for which $x_j = z_j$, we have $z \sim x$ and therefore $z \in C$. Also, since each block in X occurs infinitely many times as a subblock in the enumeration $\bar{v}_1, \bar{v}_2, \ldots$, it follows that z contains all the $\bar{v}_i$ and is therefore a right transitive point, as desired. $\qquad\square$

Remark 3.5. Let (X, Y, π) be an irreducible factor triple and let y in Y be left transitive. Then each transition class over y contains a left transitive point.

The proof of this remark is similar to the proof of Theorem 3.4 but much simpler. Let $\bar{X}$ be the α-limit set of x, i.e. $\bar{X} = \{z \in X : \exists\, n_i \searrow -\infty \text{ with } \sigma^{n_i}(x) \to z\}$. If x is not left transitive, then we may construct a point $z \in X$ similarly to the proof of Theorem 3.4: first, find a decreasing subsequence $\{n_i\}$ such that $x_{[n_i, n_i + |w_i|)} = w_i$ (the same w_i as defined in Theorem 3.4) and $n_{i+1} < n_i - |w_{i+1}|$ for all $i \in \mathbb{N}$; then, define z by replacing each occurrence of u_i at the coordinates $x_{[n_i, n_i + |w_i|)}$ with v_i for all $i \in \mathbb{Z}$. Then z is left transitive. Since z and x are right asymptotic, we have $z \sim x$.

The following corollary is an immediate result of Theorem 3.4 and Remark 3.5.

COROLLARY 3.6. *Let (X, Y, π) be an irreducible factor triple and let y in Y be doubly transitive. Then each transition class over y contains a doubly transitive point.*

With the following corollary, we see that for an irreducible factor triple, the cardinalities of the transition classes over right transitive points fall into two categories: they are either all finite (if a factor code is finite-to-one) or all uncountable (if it is infinite-to-one).

COROLLARY 3.7. *Let (X, Y, π) be an irreducible factor triple and let y in Y be right transitive. If π is infinite-to-one, then the cardinality of each transition class over y is uncountable.*

Proof. We may assume that X is one-step and π is one-block. Recall that π is infinite-to-one if and only if it has a diamond, say (u, v) [**15**, Theorem 8.1.16]. If C is a transition class over y, there is a right transitive point x in C by Theorem 3.4. Then u occurs infinitely many times to the right in x. Any point made by replacing some occurrences of u with v is equivalent to x, which implies that C is uncountable. $\square$

4. *Mutual separatedness for transition classes*

If (X, Y, π) is an irreducible finite-to-one factor triple of degree d, where X is one-step and π is one-block, then for each doubly transitive point $y \in Y$, the set of the preimages of y consists of d mutually separated points in X. This result is one of the important properties of fibers of finite-to-one factor codes, since it is used to prove that the degree indeed equals the number combinatorially defined using a magic block [**10**, **12**, **15**, §9].

In this section, we present a similar mutual separatedness property for transition classes: any two points from two distinct transition classes over a right transitive point are mutually separated (Theorem 4.4). As an application, we show that there is no transition between distinct transition classes over a right transitive point, answering a question raised by Quas.

LEMMA 4.1. *Let (X, Y, π) be an irreducible factor triple with X one-step and π one-block. Given a minimal transition block (w, n, M), any preimage of w is routable through a unique symbol of M.*

Proof. Let u be in $\pi^{-1}(w)$ and let $d = c_\pi$ be the class degree of π. If $d = 1$, then the result is trivial, so suppose $d \geq 2$. Assume that u is routable through two different members $a^{(1)}$

and $a^{(2)}$ of $M = \{a^{(1)}, a^{(2)}, \ldots, a^{(d)}\}$. Let x be a point of X such that u occurs infinitely many times to the right, say at positions $\{[i_j, i_j + |w|)\}_{j \in \mathbb{N}}$ where $i_{j+1} > i_j + |w|$.

From each transition class in $\mathscr{C}(\pi(x)) \setminus \{[x]\}$ choose one point, and denote these points by $x^{(1)}, \ldots, x^{(d-1)}$. Each of these points is routable through at least one member of M at time $i_j + n$ for each $j \in \mathbb{N}$. If there is a point in $\{x^{(1)}, \ldots, x^{(d-1)}\}$ which is routable through $a^{(1)}$ or through $a^{(2)}$ at $i_j + n$ for infinitely many j, then such a point is equivalent to x, which gives a contradiction and so we are done.

Suppose that there is no such point, i.e. each of the points $x^{(1)}, \ldots, x^{(d-1)}$ is routable through a symbol in $\{a^{(3)}, \ldots, a^{(d)}\}$ at $i_j + n$ for all but finitely many j. It follows, by the pigeonhole principle, that there are at least two points in $\{x^{(1)}, \ldots, x^{(d-1)}\}$ which are routable through the same symbol in $\{a^{(3)}, \ldots, a^{(d)}\}$ at $i_j + n$ for infinitely many j. This forces these two points to be equivalent, which is again a contradiction. $\qquad \square$

Remark 4.2. Note that Lemma 4.1 is not necessarily true when X is reducible. For example, let X be the orbit closure of the point $a^\infty.b^\infty$, let $Y = \{0^\infty\}$, and consider the trivial map $\pi : X \to Y$. The triple $(000, 1, \{a, b\})$ is a minimal transition block with a preimage abb which is routable through both a and b.

LEMMA 4.3. *Let (X, Y, π) be an irreducible factor triple with X one-step and π one-block. Suppose there are two points x and $\bar{x}$ such that:*
(1) *x and $\bar{x}$ are in two distinct transition classes over a right transitive point $y \in Y$; and*
(2) *x and $\bar{x}$ are not mutually separated.*
Then, given a right transitive point z in X, there is a transition from $[z]$ to a transition class over $\pi(z)$ other than $[z]$.

Proof. Let (w, n, M) be a minimal transition block. Since y is right transitive, there is a sequence $\{i_j\}_{j \in \mathbb{N}}$ such that $y_{[i_j, i_j + |w|)} = w$ and $i_{j+1} > i_j + |w|$. By assumption (2), for some integer i we have $x_i = \bar{x}_i$. Since x and $\bar{x}$ are in distinct transition classes, there exist $i_j > i$ and two distinct symbols $a, b \in M$ such that x and $\bar{x}$ are routable through a and b, respectively, at time $i_j + n$. By taking equivalent points in the transition classes of x and $\bar{x}$, we may assume that $x_{i_j+n} = a$ and $\bar{x}_{i_j+n} = b$.

Consider the block $y_{[i, i_j + |w|)}$ which is an extension of w. Denote this extension by $\bar{w}$ and the coordinate $i_j - i + n$ by $\bar{n}$. Then $(\bar{w}, \bar{n}, M)$ is also a minimal transition block. Note that, by the above, block $\bar{w}$ has two preimages $x_{[i, i + |\bar{w}|)}$ and $\bar{x}_{[i, i + |\bar{w}|)}$ which share the same initial symbol; moreover, $x_{i+\bar{n}} = a$ and $\bar{x}_{i+\bar{n}} = b$.

Now let z be a right transitive point in X. Note that for infinitely many $k \in \mathbb{Z}$ we have $z_{[k, k+|\bar{w}|)} = x_{[i, i+|\bar{w}|)}$. Denote the class degree by d, and let $z^{(1)} = z$. From each transition class of $\mathscr{C}(\pi(z)) \setminus \{[z]\}$ choose a point, and denote these points by $z^{(2)}, z^{(3)}, \ldots, z^{(d)}$. Since $(\bar{w}, \bar{n}, M)$ is a minimal transition block, by Lemma 4.1 we may assume that $z^{(j)}_{k+\bar{n}} \in M$ for each $1 < j \le d$. Since the $z^{(j)}$ are from distinct transition classes, the set $\{z^{(j)}_{k+\bar{n}} : 1 \le j \le d\}$ consists of d symbols for all large such k. Thus there is a point $\bar{z}$ among $z^{(2)}, z^{(3)}, \ldots, z^{(d)}$ such that among those k, at infinitely many l we have $\bar{z}_{l+\bar{n}} = b$. Then, for any such l, the points $u^{(l)}$ defined by

$$u^{(l)}_{(-\infty, l)} = z_{[-\infty, l)}, \quad u^{(l)}_{[l, l+\bar{n})} = \bar{x}_{[i, i+\bar{n})} \quad \text{and} \quad u^{(l)}_{[l+\bar{n}, \infty)} = \bar{z}_{[l+\bar{n}, \infty)}$$

give a transition $[z] \to [\bar{z}]$ over the right transitive point $\pi(z)$. $\qquad \square$

THEOREM 4.4. *Let (X, Y, π) be an irreducible factor triple with X one-step and π one-block. Let $y \in Y$ be right transitive. Then any two points from two distinct transition classes over y are mutually separated.*

Proof. Suppose not, i.e. that there are two points x and $\bar{x}$ in distinct transition classes over y such that x and $\bar{x}$ are not mutually separated. Recall that any transition class over y contains a right transitive point. Then, by Lemma 4.3, given any class $C \in \mathscr{C}(y)$ there is a transition from C to some other transition class over y. However, since there are only finitely many classes in $\mathscr{C}(y)$, there must be a transition class over y with no transition to any other class, which is a contradiction. □

One can easily check the following corollary, which is a conjugacy-invariant version of Theorem 4.4.

COROLLARY 4.5. *Let (X, Y, π) be an irreducible factor triple. Then there is $c > 0$ such that whenever $y \in Y$ is right transitive and x and $\bar{x}$ are points from two distinct transition classes over y, we have $d(x, \bar{x}) > c$.*

COROLLARY 4.6. *Let (X, Y, π) be an irreducible factor triple. Then each transition class over a right transitive point is a closed set (with respect to the usual topology on X).*

Proof. We may assume that X is one-step and π is one-block. Let $\{x^{(i)}\}_{i \in \mathbb{N}}$ be a convergent sequence in a transition class C over a right transitive point y. Denote the limit of this sequence by x. Then $\pi(x) = y$. Since $x^{(i)} \to x$, for large i we have $x_0^{(i)} = x_0$. Then x is not mutually separated from this $x^{(i)}$, and so, by Theorem 4.4, that x must belong to C. □

COROLLARY 4.7. *Let (X, Y, π) be an irreducible factor triple and let y be a right transitive point in Y. There is no transition between any two distinct transition classes over y.*

Proof. We may assume that X is one-step and π is one-block. Suppose, on the contrary, that there is a transition $[x] \to [\bar{x}]$ between two distinct transition classes $[x]$ and $[\bar{x}]$ over y. Then there is a point $z \in \pi^{-1}(y)$ such that $z_{(\infty,0]} = x_{(\infty,0]}$ and $z_{[i,\infty)} = \bar{x}_{[i,\infty)}$ for some $i > 0$. Since $z \in [\bar{x}]$ and z and x are not mutually separated, by Theorem 4.4 we have a contradiction. □

The following examples show that there may be a transition if the domain is not irreducible, or if the point in Y is not right transitive.

Example 4.8. (1) Let X be the orbit closure of the point $a^\infty.b^\infty$, let $Y = \{0^\infty\}$, and consider the trivial map $\pi : X \to Y$. Above the point 0^∞ there are two transition classes: one class, say C_1, consists of only one point a^∞, and the other class, C_2, consists of all the points of the form $a^\infty b^\infty$ and b^∞. Even though 0^∞ is a (right) transitive point, there is a transition from C_1 to C_2. Note that X is not irreducible.

(2) Let X be an irreducible edge shift given by the following diagram, and let $Y = \{0, 1\}^{\mathbb{Z}}$. Let $\pi : X \to Y$ be a one-block factor code given by $\pi(a) = \pi(b) = \pi(d) = 0$ and $\pi(c) = \pi(e) = 1$. Note that $c_\pi = 1$.
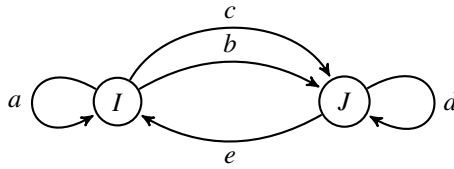
Let y in Y be any point with $y_{[0,\infty)} = 0^\infty$. Since each vertex has incoming edges labeled by 0 and 1, there are two left infinite paths, say α and β, mapping to $y_{(-\infty,0]}$ and terminating at I and J, respectively. Hence, as in the previous example, y has two transition classes: one containing $\alpha.a^\infty$ and the other containing $\alpha.a^n bd^\infty$ for all $n \in \mathbb{N}$ and $\beta.d^\infty$. It follows that there is a transition from one transition class over y to the other.

In the definition of a transition $x \to \bar{x}$ between two points x and $\bar{x}$ in X, we require an infinite number of points which are left asymptotic to x and right asymptotic to $\bar{x}$, since for each $n \in \mathbb{N}$ we need a point which equals x until time n. However, if the point $y = \pi(x) = \pi(\bar{x})$ is right transitive, to satisfy this definition, a single asymptotic point to x and $\bar{x}$ suffices, as statement (3) in the following corollary shows.

COROLLARY 4.9. *Let (X, Y, π) be an irreducible factor triple and let $y \in Y$ be right transitive. Then, for $x, \bar{x} \in \pi^{-1}(y)$, the following are equivalent.*
(1) $x \sim \bar{x}$.
(2) $x \to \bar{x}$.
(3)   *There is a point in $\pi^{-1}(y)$ which is left asymptotic to x and right asymptotic to $\bar{x}$.*

Proof. Since the conditions (1), (2) and (3) are all conjugacy-invariant, we may assume that X is one-step and π is one-block. The equivalence of (1) and (2) follows from Corollary 4.7. Statement (2) clearly implies (3). Suppose that x and $\bar{x}$ are not equivalent. If z is a point satisfying condition (3), we have $z \in [\bar{x}]$. Since z and x are left asymptotic, they are not mutually separated, which contradicts Theorem 4.4.   □

The following corollary is immediate.

COROLLARY 4.10. *Let (X, Y, π) be an irreducible finite-to-one factor triple. Let y in Y be right transitive. Then, for any x and $\bar{x}$ in $\pi^{-1}(y)$, they are either mutually separated or right asymptotic.*

Proof. First, note that since π is finite-to-one, x and $\bar{x}$ must be right transitive. Suppose that x and $\bar{x}$ are neither mutually separated nor right asymptotic. Then, by Corollary 4.9, they are equivalent and therefore produce a diamond, which implies that π is not finite-to-one (see [**15**, §8.1]).   □

5. *Block partition properties for a factor triple*

In this section, we provide further separation properties for transition classes over a right transitive point. Throughout this section, we assume that X is one-step and π is one-block

for a factor triple (X, Y, π). Also, in what follows, for a transition class C and $A \subset \mathbb{Z}$, denote by $C|_A$ the set $\{x_A : x \in C\}$. If $A = \{i\}$ for some integer i, then $C|_A$ is written simply as $C|_i$ for convenience. With this notation, Theorem 4.4 says that if $y \in Y$ is right transitive, then $C|_i \cap \bar{C}|_i = \emptyset$ for $C \neq \bar{C} \in \mathscr{C}(y)$ and all $i \in \mathbb{Z}$.

We will first see that if (w, n, M) is a minimal transition block and y is right transitive, then $|C|_{i+n} \cap M| = 1$ for any C in $\mathscr{C}(y)$ and $i \in \mathbb{Z}$ with $y_{[i,i+|w|)} = w$.

LEMMA 5.1. *Let (X, Y, π) be an irreducible factor triple. Let $y \in Y$ be right transitive with $y_{[i,i+|w|)} = w$ for some minimal transition block (w, n, M). Given $C \in \mathscr{C}(y)$, there is a unique symbol b in M such that every point in C is routable through b at time $i + n$.*

Proof. Let $M_C = M \cap C|_{i+n}$ for any C in $\mathscr{C}(y)$. Then $\bigcup_{C \in \mathscr{C}(y)} M_C = M$. As each M_C is contained in $C|_{i+n}$, by Theorem 4.4 the sets M_C are mutually disjoint. So $\sum_{C \in \mathscr{C}(y)} |M_C| = |M| = c_\pi$. Since $|\mathscr{C}(y)| = c_\pi$ and each M_C is non-empty, we have that $|M_C| = 1$ for each $C \in \mathscr{C}(y)$. $\square$

The following proposition states, intuitively, that given a right transitive point y in Y, if two preimages of the symbol y_0 appear in two distinct transition classes over y at time 0, then these two symbols cannot ever appear at the same time in a single class over any right transitive point.

PROPOSITION 5.2. *Let (X, Y, π) be an irreducible factor triple, and let $y \in Y$ be a right transitive point with a minimal transition block in $y_{(-\infty,0)}$. Then, for any right transitive point $z \in Y$ with $z_i = y_0$ for $i \in \mathbb{Z}$ and any given $D \in \mathscr{C}(z)$, we have $D|_i \cap C|_0 \neq \emptyset$ for at most one $C \in \mathscr{C}(y)$.*

Proof. Let $y_0 = a$. Suppose, on the contrary, that for some right transitive point $z \in Y$ with $z_i = a$ for $i \in \mathbb{Z}$ and some $D \in \mathscr{C}(z)$, there are two symbols $b \in C|_0$ and $c \in \bar{C}|_0$, where $C \neq \bar{C} \in \mathscr{C}(y)$, occurring at time i in D, i.e. $b, c \in D|_i$. Then there are two points $x^{(1)}$ and $x^{(2)}$ in D with $x_i^{(1)} = b$ and $x_i^{(2)} = c$. As $b \in C|_0$ and $c \in \bar{C}|_0$, we have two points $\bar{x}^{(1)} \in C$ and $\bar{x}^{(2)} \in \bar{C}$ such that $\bar{x}_0^{(1)} = b$ and $\bar{x}_0^{(2)} = c$.

Define new points $\hat{x}^{(l)}$ for $l = 1, 2$ by

$$\hat{x}^{(l)} = \bar{x}_{(-\infty,0)}^{(l)} \cdot x_{[i,\infty)}^{(l)},$$

and consider $\hat{z} = \pi(\hat{x}^{(1)}) = \pi(\hat{x}^{(2)})$. Then $\hat{z}$ is right transitive since it is right asymptotic to $\sigma^i(z)$, and the points $\hat{x}^{(l)}$ are equivalent since the $x^{(l)}$ are equivalent.

Recall that, by assumption, a minimal transition block (w, n, M) occurs in $y_{[j,j+|w|)} = \hat{z}_{[j,j+|w|)}$ for some $j \leq -|w|$. Since $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ are not equivalent, by Theorem 4.4 they are routable through different members of M at time $j + n$. It follows that $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ are routable through different members of M at time $j + n$, which contradicts Lemma 5.1, since $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ are equivalent. $\square$

A finite-to-one factor code possesses a kind of permutation property, as discussed in [15, §9], for preimages of a magic symbol. Analogous to this property, the following proposition exhibits a permutation property for preimages of a minimal transition block.

PROPOSITION 5.3. *Let (X, Y, π) be an irreducible factor triple of class degree d, and let (w, n, M) be a minimal transition block. For each block $u = wvw \in \mathcal{B}(Y)$ for some v, there is a permutation $\tau_u : M \to M$ such that given any right transitive point y in Y with $y_{[0,|u|)} = u$ and $C \in \mathcal{C}(y)$, we have $\tau_u(C|_n \cap M) = C|_{|wv|+n} \cap M$.*

Proof. Let y be a right transitive point in Y with $y_{[0,|u|)} = u$. Given C in $\mathcal{C}(y)$, by Lemma 5.1 we have $C|_n \cap M = \{a\}$ and $C|_{|wv|+n} \cap M = \{b\}$ for some symbols a and b. Define a permutation $\tau_{u,y}$ of M by $\tau_{u,y}(a) = b$, and let $x \in C$ be a point with $x_n = a$ and $x_{|wv|+n} = b$.

We show that $\tau_{u,y}$ does not depend on y. Let $\bar{y}$ be a right transitive point in Y with $\bar{y}_{[0,|u|)} = u$, and let $\bar{x}$ be a point in $\pi^{-1}(\bar{y})$ with $\bar{x}_n = a$. Consider a point $x' \in \pi^{-1}(\bar{y})$ with $x'_{|wv|+n} = b$. Then the point $\bar{x}_{(-\infty,n]} x_{(n,|wv|+n)} x'_{[|wv|+n,\infty)} \in \pi^{-1}(\bar{y})$ is left asymptotic to $\bar{x}$ and right asymptotic to x', which implies that $\bar{x}$ and x' are equivalent. It follows that $\tau_{u,\bar{y}}(a) = b$, and thus $\tau_{u,y} = \tau_{u,\bar{y}}$ as C was chosen arbitrarily. $\square$

Theorem 5.4 informally states that given any irreducible factor triple (X, Y, π), in order to determine whether two points with the same image are equivalent or not, one only needs to locally compare the preimages of a magic block which occur within these points. More precisely, there is a partition on the set of preimages of a magic block w of π such that given all right transitive points y in Y and any $i \in \mathbb{Z}$ with $y_{[i,i+|w|)} = w$, two preimages x and x' of y are equivalent if and only if $x_{[i,i+|w|)}$ and $x'_{[i,i+|w|)}$ belong to the same class of the partition on $\pi^{-1}(w)$.

THEOREM 5.4. *Let (X, Y, π) be an irreducible factor triple of class degree d. Let w be a magic block of π. There is a partition of $\pi^{-1}(w)$ into d subsets $\mathcal{B}_1, \ldots, \mathcal{B}_d$ such that for any doubly transitive z in Y and any $i \in \mathbb{Z}$ with $z_{[i,i+|w|)} = w$, we have a bijection $\rho_{z,i} : \mathcal{C}(z) \to \{1, \ldots, d\}$ with $D|_{[i,i+|w|)} \subseteq \mathcal{B}_{\rho_{z,i}(D)}$ for each $D \in \mathcal{C}(z)$.*

Proof. Let k be a magic coordinate of w and let $\mathcal{A}_{w,k} = \{u_k \mid u \in \pi^{-1}(w)\}$. Let y be a doubly transitive point in Y with $y_{[0,|w|)} = w$. For $C \in \mathcal{C}(y)$, let $\mathcal{B}_C = \{u \in \pi^{-1}(w) : u_k \in C|_k\}$. List $\mathcal{C}(y) = \{C^{(1)}, \ldots, C^{(d)}\}$ and write $\mathcal{B}_j = \mathcal{B}_{C^{(j)}}$. We claim that $\{\mathcal{B}_j : 1 \le j \le d\}$ is a desired partition.

Note that since k is a magic coordinate of w, we have $\bigcup_{C \in \mathcal{C}(y)} C|_k = \mathcal{A}_{w,k}$ and hence $\bigcup_{C \in \mathcal{C}(y)} \mathcal{B}_C = \pi^{-1}(w)$. Moreover, the $\mathcal{B}_C$ with $C \in \mathcal{C}(y)$ are mutually disjoint, since by Theorem 4.4 the sets $C|_k$ with $C \in \mathcal{C}(y)$ are mutually disjoint.

Now we show that such a partition does not depend on y. Let z be a doubly transitive point in Y with $z_{[i,i+|w|)} = w$ for some $i \in \mathbb{Z}$. By considering the point $\sigma^i(z)$, we may assume that $i = 0$. For each $D \in \mathcal{C}(z)$, let $\mathcal{B}_D^{(z)} = \{u \in \pi^{-1}(w) : u_k \in D|_k\}$. Again, since k is a magic coordinate of w, we have $\bigcup_{D \in \mathcal{C}(z)} D|_k = \mathcal{A}_{w,k}$ and thus $\bigcup_{D \in \mathcal{C}(z)} \mathcal{B}_D^{(z)} = \pi^{-1}(w)$. It follows that for each $D \in \mathcal{C}(z)$, there is a transition class $C \in \mathcal{C}(y)$ such that $C|_k \cap D|_k \ne \emptyset$, which implies that $\mathcal{B}_C \cap \mathcal{B}_D^{(z)} \ne \emptyset$. We show that $\mathcal{B}_C = \mathcal{B}_D^{(z)}$. Suppose not. First assume that there is a block u in $\mathcal{B}_C \setminus \mathcal{B}_D^{(z)}$. There must be another class $D' \in \mathcal{C}(z)$ such that $u_k \in D'|_k$. This means that $C|_k$ intersects both $D|_k$ and $D'|_k$, which contradicts Proposition 5.2. So $\mathcal{B}_C \subseteq \mathcal{B}_D^{(z)}$. The case of $u \in \mathcal{B}_D^{(z)} \setminus \mathcal{B}_C$ is also impossible by the symmetric argument, and hence we have $\mathcal{B}_C = \mathcal{B}_D^{(z)}$.
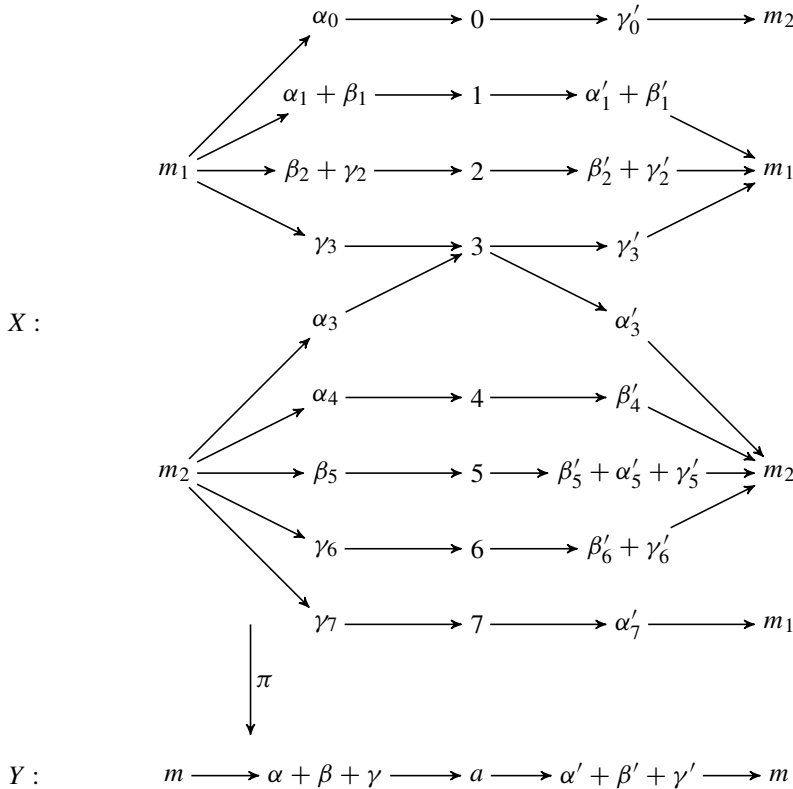
FIGURE 1. Graph for Example 5.6

Define $\rho_{z,i} : \mathscr{C}(z) \to \{1, \ldots, d\}$ to send each $D \in \mathscr{C}(z)$ to the unique $1 \le \rho_{z,i}(D) \le d$ with $\mathcal{B}_D^{(\sigma^i(z))} = \mathcal{B}_{\rho_{z,i}(D)}$. By definition, $D|_{[i, i+|w|)}$ is contained in $\mathcal{B}_{\rho_{z,i}(D)}$. $\square$

Note that, in general, the partition in Theorem 5.4 need not be unique. However, in some special cases, for example when π has a magic symbol, we obtain the uniqueness of the partition. The following remark, which follows directly from Proposition 5.2, explains such cases.

Remark 5.5. Let (X, Y, π) be an irreducible factor triple of class degree d, and let a be in $\mathcal{A}(Y)$. If for some doubly transitive point y in Y we have $y_0 = a$ and $\pi^{-1}(y)|_0 = \pi^{-1}(a)$, then there is a unique partition of $\pi^{-1}(a)$ into d subsets $\mathcal{B}_1, \ldots, \mathcal{B}_d$ such that for any right transitive $z \in Y$ with $z_i = a$, we have a bijection $\rho_{z,i} : \mathscr{C}(z) \to \{1, \ldots, d\}$ with $C|_i \subseteq \mathcal{B}_{\rho_{z,i}(C)}$ for each class C in $\mathscr{C}(z)$.

The following example shows that the partition property stated in Remark 5.5 need not hold for every symbol.

Example 5.6. Consider the irreducible vertex shifts X and Y displayed in Figure 1. In the graphs of X and Y, each pair of leftmost and rightmost vertices with the same symbol

(m_1, m_2 or m) is identified. Let π be the map that erases the subscripts and sends all numbered vertices of X to the symbol a.

A point y in Y is of the form $(m(\alpha + \beta + \gamma)a(\alpha' + \beta' + \gamma'))^\infty$, where the notation $\alpha + \beta + \gamma$ implies that one can choose freely any of α, β or γ as a symbol to appear in the given position (this is a standard notion in the theory of formal languages). The choices may differ up to positions, so that a word like $m\alpha a\beta'm\beta a\gamma'$ is allowed. If $u = m\delta a\delta'm$ with $\delta \in \{\alpha, \beta, \gamma\}$ and $\delta' \in \{\alpha', \beta', \gamma'\}$, this u defines a *unique permutation* τ_u of $\{m_1, m_2\}$: if m_1 and u are given, for any preimage v of u with $v_0 = m_1$, the last symbol of v is uniquely determined. Similarly for m_2. Note that this property is inductively true for general blocks of the form $u = m\delta^{(1)}a\delta'^{(1)}m \cdots m\delta^{(k)}a\delta'^{(k)}m$. This means that for each y in Y and x in $\pi^{-1}(y)$, there are only two choices of putting the m_i in x. Hence, there are exactly two transition classes over y. It follows that every point in Y has two transition classes and the class degree of π is 2. Note that m is a magic symbol; also, it is a minimal transition block.

Now consider the symbol $a \in \mathcal{A}(Y)$ and the sets $C|_0$ with $C \in \mathscr{C}(y)$, where y is right or doubly transitive.

(1)   If $y = \cdots m\alpha.a\alpha'm \cdots$, then we have $\{1\}$ and $\{3\}$ as the sets $C|_0$ in Proposition 5.2.
(2)   If $y = \cdots m\beta.a\beta'm \cdots$, then we have $\{1, 2\}$ and $\{5\}$.
(3)   If $y = \cdots m\gamma.a\gamma'm \cdots$, then we have $\{2, 3\}$ and $\{6\}$.
(4)   If $y = \cdots m\delta.a\delta'm \cdots$, where (δ, δ') is not a pair in the above cases, then each $C|_0$ is a singleton.

Symbols 1 and 2 appear together in one class, as do symbols 2 and 3; however, symbols 1 and 3 appear in distinct classes. So, there is no partition of $\pi^{-1}(a)$ into two sets that satisfies the property stated in Remark 5.5.

6.   *Applications and examples*

The definitions of transition, transition classes and class degree are asymmetric. So it is natural to consider reversed transitions as follows: $x \to_r \bar{x}$ if for each integer n there is a point which is left asymptotic to x and equal to $\bar{x}$ in $[n, \infty)$. We say that $x \sim_r \bar{x}$ if and only if $x \to_r \bar{x}$ and $\bar{x} \to_r x$. The *reversed transition classes* of a point in Y are the equivalence classes generated by this new relation $\sim_r$. Denote by $[x]_r$ the reversed transition class containing x. Let x^T be the point such that $(x^\mathsf{T})_i = x_{-i}$ for every integer i, and define $X^\mathsf{T} = \{x^\mathsf{T} : x \in X\}$. Note that the reversed transition classes of a point y can be obtained from the transition classes of a point y^T under the *transposed code* $\pi^\mathsf{T} : X^\mathsf{T} \to Y^\mathsf{T}$. All the results in the previous sections hold for the reversed transition classes if we replace left transitive with right transitive.

Note that the set of transition classes and the set of the reversed transition classes of a point $y \in Y$ need not coincide; nor do they need to have the same cardinality. However, they do coincide for almost all points (Proposition 6.2).

Example 6.1. We show that given any finite-to-one irreducible factor triple (X, Y, π) which is not bi-closing, there are uncountably many points in Y for which the number of transition classes differs from the number of reversed transition classes.

Suppose, without loss of generality, that π is not *left closing*, i.e. there are two distinct points $x \neq \bar{x} \in X$ which are right asymptotic and such that $y = \pi(x) = \pi(\bar{x})$. Since the

subshift X is of finite type and $x_{[i,\infty)} = \bar{x}_{[i,\infty)}$ for some i, by changing the common right tail we may assume that x and $\bar{x}$ are right transitive. Then y is also right transitive and hence y has exactly c_π transition classes. Since π does not have any diamond, x and $\bar{x}$ are not equivalent with respect to the reversed transition relation. Moreover, since any given two points z and $\bar{z}$ from distinct transition classes over y are mutually separated, they are also not equivalent with respect to the reversed transition relation. It follows that the number of reversed transition classes of y is at least $c_\pi + 1$. Since the right tail of x can be changed in uncountably many ways to produce a right transitive point, we have uncountably many points in Y with a different number of transition classes than the number of its reversed transition classes.

PROPOSITION 6.2. *Let (X, Y, π) be an irreducible factor triple. If y is doubly transitive and $x \in \pi^{-1}(y)$, then the transition class of x equals the reversed transition class of x.*

Proof. Suppose, on the contrary, that $[x] \neq [x]_r$. We may assume that there is a point $\bar{x} \in [x]_r \setminus [x]$. Since $\bar{x} \sim_r x$, there is a point $z \in \pi^{-1}(y)$ which is left asymptotic to $\bar{x}$ and right asymptotic to x. Then, by Corollary 4.9, $[x] = [z] = [\bar{x}]$ and therefore $\bar{x} \in [x]$, which is a contradiction. $\qquad\square$

Recall Definition 3.1. Let (X, Y, π) be a factor triple and $\bar{X}$ a proper subshift of X with $\pi(\bar{X}) = Y$. Let $\bar{v}$ be in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$. A block u in $\mathcal{B}(\bar{X})$ and a block v in $\mathcal{B}(X)$ form an $(\bar{X}, \bar{v})$-*diamond* if $\pi(u) = \pi(v)$, $\bar{v}$ is a subblock of v, and u and v share the same initial symbol and the same terminal symbol. Lemma 3.2 states that if X is irreducible and one-step and π is one-block, then for each block $\bar{v}$ in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$ there is an $(\bar{X}, \bar{v})$-diamond.

As mentioned before, we strengthen Lemma 3.2 in this section. Proposition 6.3 gives an upper bound on the length of the $(\bar{X}, \bar{v})$-diamond of a given word $\bar{v}$ in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$. Note that the proof of Proposition 6.3 employs Theorem 3.4, which was shown using Lemma 3.2.

PROPOSITION 6.3. *Let (X, Y, π) be an irreducible factor triple with X one-step and π one-block. Let $\bar{X}$ be a proper subshift of X with $\pi(\bar{X}) = Y$. Then there is a positive integer N such that for each block $\bar{v}$ in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$, we have an $(\bar{X}, \bar{v})$-diamond of length less than $|\bar{v}| + N$.*

Proof. Let k be a positive integer such that for any blocks u and v in $\mathcal{B}(X)$, there is a block w in $\mathcal{B}(X)$ with $|w| \leq k$ and uwv in $\mathcal{B}(X)$. Such k exists, since X is irreducible and of finite type. Let (w, n, M) be a minimal transition block of π with $|w| = l$ and u a preimage of w in $\mathcal{B}(\bar{X})$.

Consider a block $\bar{v}$ in $\mathcal{B}(X) \setminus \mathcal{B}(\bar{X})$, and let $\gamma = u\alpha\bar{v}\beta u$ for some α and β with $|\alpha|, |\beta| \leq k$. Write $|\gamma| = L$. Let y be a right transitive point of Y and $\bar{x}$ a preimage of y in $\bar{X}$. By Theorem 3.4, there is a right transitive preimage x of y in X which is equivalent to $\bar{x}$. Note that $x \neq \bar{x}$. For convenience, let $x_{[0,L)} = \gamma$. Note that x and $\bar{x}$ are both routable through the same symbol of M, say a, at time n and at time $L - l + n$. Let δ be a block of length l in $\mathcal{B}(X)$ such that $\delta_0 = u_0$, $\delta_{l-1} = u_{l-1}$ and $\delta_n = a$. Also, let $\bar{\delta}$ and $\bar{\delta}'$ be blocks of length l in $\mathcal{B}(X)$ such that $\bar{\delta}_0 = \bar{x}_0$, $\bar{\delta}_{l-1} = \bar{x}_{l-1}$, $\bar{\delta}_n = a$ and $\bar{\delta}'_0 = \bar{x}_{L-l}$, $\bar{\delta}'_{l-1} = \bar{x}_{L-1}$, $\bar{\delta}'_n = a$. Then the two blocks $\bar{\delta}_{[0,n)}\delta_{[n,l)}x_{[l,L-l)}\delta_{[0,n)}\bar{\delta}'_{[n,l)}$ and

$\bar{x}_{[0,L]}$ form an $(\bar{X}, \bar{v})$-diamond of length smaller than or equal to $|\bar{v}| + 2l + 2k$. Letting $N = 2l + 2k$ completes the proof. □

In the case of finite-to-one factor codes $\pi = \pi_2 \circ \pi_1$, we have $c_\pi = d_\pi = d_{\pi_1} \cdot d_{\pi_2} = c_{\pi_1} \cdot c_{\pi_2}$. Since class degree is a conjugacy-invariant generalization of degree, it is natural to consider whether this equality holds for the infinite-to-one case. The following example shows that it actually does not; however, we are still able to get an inequality, as stated in Proposition 6.5 below.

Example 6.4. Let X be the full 2-shift, let $Y = \{0^\infty\}$, and consider the trivial map $\pi : X \to Y$. By letting $\pi_2 = \pi$ and $\pi_1 : X \to X$ by $\pi_1(x)_i = x_i + x_{i+1}$ mod 2, we have $\pi = \pi_2 \circ \pi_1$. However, $1 = c_\pi < c_{\pi_1} \cdot c_{\pi_2} = 2$.

PROPOSITION 6.5. *Let (X, Y, π_1) and (Y, Z, π_2) be irreducible factor triples. If $\pi = \pi_2 \circ \pi_1$, then $c_\pi \leq c_{\pi_1} \cdot c_{\pi_2}$.*

Proof. Since class degree is invariant under conjugacy, we may assume that X and Y are one-step and that π_1 and π_2 (and hence π) are one-block. For convenience, rename $c_1 = c_{\pi_1}$ and $c_2 = c_{\pi_2}$. Fix a doubly transitive point z in Z, and let C be a transition class over z with respect to π_2. By Corollary 3.6, C contains a doubly transitive point y. Moreover, by the same corollary, there are c_1 doubly transitive points $x^{(1)}, \dots, x^{(c_1)}$ in $\pi_1^{-1}(y)$ which are not equivalent to each other with respect to π_1.

We claim that any doubly transitive point x' in $\pi_1^{-1}(C)$ is equivalent to some $x^{(i)}$, $1 \leq i \leq c_1$. To show this claim, observe that $y' = \pi_1(x')$ lies in C and is equivalent to y with respect to π_2. It follows that there is a point $\vec{y}$ in Y such that $\pi_2(\vec{y}) = z$, $\vec{y}_{(-\infty,0]} = y_{(-\infty,0]}$ and $\vec{y}_{[j,\infty)} = y'_{[j,\infty)}$ for some $j > 0$.

Since y' is doubly transitive, a minimal transition block (w, n, M) of π_1 occurs in $y'_{[k,k+|w|)} = \vec{y}_{[k,k+|w|)} = w$ for some $k \geq j$. Let the point x' be routable through a symbol $a \in M$ at time $k + n$. There is a point $\vec{x}$ in $\pi_1^{-1}(\vec{y})$ which is also routable through a at time $k + n$. Reset $\vec{x}_{[k,\infty)}$ to have $\vec{x}_{k+n} = a$ and $\vec{x}_t = x'_t$ for all $t \geq k + |w|$.

Since y is doubly transitive, the minimal transition block (w, n, M) also occurs in $y_{[-l,-l+|w|)} = \vec{y}_{[-l,-l+|w|)} = w$ for some $l \geq |w|$. Let the point $\vec{x}$ be routable through a symbol $b \in M$ at time $-l + n$. By Lemma 4.1 and Theorem 4.4, there is exactly one $x^{(i)}$ among $x^{(1)}, \dots, x^{(c_1)}$ which is routable through b at time $-l + n$. Reset $\vec{x}_{(-\infty,-l+|w|)}$ to have $\vec{x}_{-l+n} = b$ and $\vec{x}_t = x^{(i)}_t$ for all $t \leq -l$. Therefore, we have that $\vec{x}$ is left asymptotic to $x^{(i)}$ and right asymptotic to x', and $\pi(\vec{x}) = \pi_2(\vec{y}) = z$. Corollary 4.9 implies that $x' \sim x^{(i)}$ with respect to π. It follows that $\pi_1^{-1}(C)$ contains at most c_1 doubly transitive π-preimages of z which are not equivalent to each other with respect to π.

Now let $C_1, C_2, \dots, C_{c_2}$ be all the transition classes over z with respect to π_2. Note that, by Corollary 3.6, the class degree of π is the maximal number of doubly transitive points in $\pi^{-1}(z) = \bigcup_{j=1}^{c_2} \pi_1^{-1}(C_j)$ which are not equivalent to each other with respect to π. By the above argument, each $\pi_1^{-1}(C_j)$ contains at most c_1 doubly transitive points which are not equivalent to each other with respect to π. Therefore we have $c_\pi \leq c_1 c_2$. □

We finish with the following question, which can be regarded as a measure-theoretical version of Theorem 3.4.

Question 6.6. Let (X, Y, π) be an irreducible factor triple and ν an ergodic measure on Y. Given a right transitive point $y \in Y$ which is ν-generic, does each transition class over y contain a generic point of a measure of relative maximal entropy over ν?

Note that the class $C \in \mathscr{C}(y)$ may not contain generic points for *different* measures of relative maximal entropy over ν. For example, consider the factor code π_1 on the full 2-shift in Example 6.4. Then $c_{\pi_1} = 2$ and π_1 maps the $(1/3, 2/3)$- and $(2/3, 1/3)$-Bernoulli measures to the same measure on X. However, each transition class over a point in X is a singleton.

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