

STIT Process and Trees

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Abstract

We study several constructions of the STIT tessellation process in a window of \mathbb{R}^ℓ and supply an exact formula for its transition probability.

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1 Introduction

We will describe the STIT tessellation process $Y \wedge W$ on a window W in \mathbb{R}^ℓ as it was defined for the first time in [6].

A STIT process is a particular cell division process, and within a bounded window it is a pure jump Markov process. Hence, it can be considered from two aspects. One aspect is that each cell has a random lifetime, and at the end of its lifetime the cell is divided and two new cells are born. Since the lifetimes of the cells run simultaneously, this approach can appropriately be described using binary rooted trees where the nodes represent the cells of the tessellation. The other aspect is that the STIT process in a bounded window has a random holding time in a state and when this time elapsed it jumps into another state. This jump is performed by first a random selection of a cell that has to be divided and then dividing this cell. In the present paper we consider both these aspects in detail and we relate them to each other.

The main result is a new construction of $Y \wedge W$ using a sequence of random hyperplanes generated with a random measure. This is done in Section 9. This method allows to gain in the efficiency of simulation of STIT, because all the hyperplanes are used in the construction in contrast to other constructions using

a rejection method where one must attend that a random hyperplane cuts a prescribed cell.

Our construction uses a sequence of hyperplanes with a random distribution which depends on the current tessellation (and hence it has to be adapted after each cell division). This is different from the construction done in Section 4 in [5] that uses a Poisson point process of hyperplanes with the fixed intensity, but it requires to be corrected because the process underestimates the rate of apparition of hyperplanes in STIT. On the other hand, in [4] the mean length of segments is computed for mixtures of tessellations in the case $\ell = 2$, and a Poisson tessellation process is constructed with an appropriate intensity measure. This construction differs from the one we make in the last paragraph of Section 9, since our construction has a random intensity measure and holds for any dimension.

We use rooted binary (dyadic) trees for the description of the STIT process in a window. This is natural since STIT is a cell division process. In Section 5 we define this class of trees, and define finite trees and its leaves as appropriate graph objects in our study.

The construction of STIT is formally done in Section 6, and the rooted binary tree helps to write a simple algorithm. The root of the tree represents the window and each node stands for a cell which appears in the cell division procedure. When a cell is divided, the corresponding node in the tree has two children, representing the two new cells and denoted by '+' or '-', each symbol indicating the half-space of the dividing hyperplane. The lifetimes of cells are independent and exponentially distributed. Even if Proposition 6.1 and Proposition 6.2 are not new, we recall them to provide explicit conditions characterizing the STIT process and to identify the stability-under-iteration property as one close to what is called branching property of a fragmentation chain in [1].

In Section 7 we use the tree representation of STIT to give a formula for the marginal distribution of $(Y \wedge W)_t$ at a fixed time t by considering all the possible paths on a binary rooted tree.

In Section 8 we revisit the construction of STIT in a window, and the order of choosing the random objects is kept: first one selects the cell that will be broken, and conditioned to it one chooses the hyperplane that cuts it. We summarize the results of this construction in Proposition 8.1, that only serves to order the elements but with no novel elements. We emphasize that this construction is not optimal for simulations because the time of retrieving such an hyperplane can be highly-time consuming since it is based on a rejection method.

In Section 9 we modify the above construction with a different order of choosing the random objects: first one generates a random hyperplane with a distribution depending on the whole tessellation in the time of up-dating. Then the cell to be divided is chosen with equal probability among all the cells being intersected (this is formula (44)). This is done in detail in Theorem 9.1, which

to our view gives a novel approach to construct STIT. We point out that a relation similar to (44) appears in [9] page 9, but with a different hyperplane measure.

In Section 2 we give some useful facts in probability, mainly on conditional independence and Lebesgue probability spaces. The basic notions and notations for tessellations are supplied in Section 3 and in Section 4 we summarize the main elements of the random law of random hyperplanes and supply its main properties which ensure that it serves to define a STIT tessellation processes.

2 Preliminaries on Probability

Let $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ be a probability space which is the basis for the construction of all the random objects we will use.

To describe relations among the random objects it is useful to introduce some notions and notation. Let (D, \mathcal{D}) be a measurable space. $g : \Omega \rightarrow D$ is a random variable if $g^{-1}(\mathcal{D}) \subseteq \mathcal{B}$ and we put $\sigma(g) = g^{-1}(\mathcal{D})$. If h is another random variable we put $h \in \sigma(g)$ if $\sigma(h) \subseteq \sigma(g)$. Here g and h can also be countable sequences of random variables.

If g and h are two random variables, we write $g \perp h$ when $\sigma(g)$ and $\sigma(h)$ are $(\mathbb{P}-)$ independent. If g, h are two random variables and $\mathcal{A} \subseteq \mathcal{B}$ is a sub σ -field, we express by $(g \perp h) | \mathcal{A}$ that g and h are conditionally independent given \mathcal{A} , that is $\mathbb{P}(D' \cap D'' | \mathcal{A}) = \mathbb{P}(D' | \mathcal{A})\mathbb{P}(D'' | \mathcal{A})$ (a.s.) for all $D' \in \sigma(g), D'' \in \sigma(h)$. Also if z is a random variable we put $(g \perp h) | z$ for $(g \perp h) | \sigma(z)$.

Let $g : \Omega \rightarrow D$ be a random variable and F be a probability measure on (D, \mathcal{D}) , by $g \sim F$ we mean that g is distributed as F . If z is a random variable and $F(z)$ is a random distribution depending on z , we write $g | z \sim F(z)$ to express that the conditional distribution of g given $\sigma(z)$ is $F(z)$.

Let $(D, \mathcal{B}(D), P)$ be a probability space such that: D is a complete separable metric space, $\mathcal{B}(D)$ is its Borel σ -field completed with respect to the probability measure P . Also assume that P is non-atomic. Then, $(D, \mathcal{B}(D), P)$ is a Lebesgue probability space, see [2]. This means that $(D, \mathcal{B}(D), P)$ and $([0, 1], \mathcal{B}[0, 1], \lambda)$ are isomorphic, where λ is the Lebesgue measure. That is, there exists an isomorphism $v : [0, 1] \rightarrow D$, which is a bimeasurable function such that $\lambda(v^{-1}(A)) = P(A)$ for all $A \in \mathcal{B}(D)$.

Let $(D, \mathcal{B}(D), P)$ be a Lebesgue probability space. Let Q be a probability measure equivalent to P with Radon-Nikodym derivate $f = dQ/dP > 0$ P -a.s. Then also $(D, \mathcal{B}(D), Q)$ is isomorphic to $([0, 1], \mathcal{B}[0, 1], \lambda)$. So $(D, \mathcal{B}(D), P)$ and $(D, \mathcal{B}(D), Q)$ are isomorphic: there exists a bimeasurable function $\Xi : D \rightarrow D$ that satisfies $P(\Xi^{-1}(A)) = Q(A)$ for all $A \in \mathcal{B}(D)$. We will mainly consider Lebesgue probability spaces.

3 Tessellations

For a set $B \subseteq \mathbb{R}^\ell$ we denote respectively by ∂B and $\text{Int } B$ the *boundary* and the *interior* of B . A *polytope* K is the convex hull of a finite point set. A polytope with nonempty interior will be called *a cell* or *a window*. This distinction will depend on the context, usually, we reserve the name cell when the polytope belongs to a tessellation.

A tessellation T in \mathbb{R}^ℓ is a locally finite class of cells with disjoint interiors and covering the Euclidean space. The locally finiteness property means that each bounded subset of \mathbb{R}^ℓ is intersected by only finitely many cells. So, the set of cells of a tessellation T is necessarily countably infinite. We put $C \in T$ for a cell C of the tessellation T . A tessellation can as well be considered as the closed subset $\partial T = \bigcup_{\{C \in T\}} \partial C$ which is the union of the cell boundaries. There is an obvious one-to-one relation between both ways of description of a tessellation, and their measurable structures can be related appropriately, see [3, 8]. We denote by \mathbb{T} the set of all tessellations of \mathbb{R}^ℓ .

Let \mathcal{C} be the set of all compact subsets of \mathbb{R}^ℓ . We endow \mathbb{T} with the Borel σ -algebra $\mathcal{B}(\mathbb{T})$ of the Fell topology, namely

$$\mathcal{B}(\mathbb{T}) = \sigma(\{\{T \in \mathbb{T} : \partial T \cap A = \emptyset\} : A \in \mathcal{C}\}) .$$

(As usual, for a class of sets \mathcal{I} we denote by $\sigma(\mathcal{I})$ the smallest σ -algebra containing \mathcal{I} .)

Let \mathbb{F} be the family of closed sets of \mathbb{R}^ℓ . When \mathbb{F} is endowed with the Fell topology (for definitions and properties see [8]), it is a compact Hausdorff space with a countable base, so it is metrizable. Also the class of nonempty closed sets $\mathbb{F}' = \mathbb{F} \setminus \{\emptyset\}$ endowed with the restricted Fell topology is a complete separable metric space, so for any nonatomic probability measure P on $(\mathbb{F}', \mathcal{B}(\mathbb{F}'))$, the completed probability space $(\mathbb{F}', \mathcal{B}(\mathbb{F}'), P)$ is Lebesgue. Each tessellation $T \in \mathbb{T}$, as a countable collection of polytopes is a closed set in \mathbb{F}' . Furthermore in Lemma 10.1.2. in [8] it was shown that $\mathbb{T} \in \mathcal{B}(\mathbb{F}')$, so for any nonatomic probability measure P on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, the completed probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), P)$ is Lebesgue. (For more detailed arguments see [3], Section 1.3.)

Let W be a window in \mathbb{R}^ℓ . The tessellations of W are defined similarly and the class of them is denoted by $\mathbb{T} \wedge W$. If $T \in \mathbb{T}$ we denote by $T \wedge W = \{C \cap W : C \in T\}$ the induced tessellation on W . The tessellation $T \wedge W$ has a finite number of cells because T is locally finite and we put

$$\#(T \wedge W) : \text{number of cells of } T \wedge W.$$

The boundary of $T \wedge W$ is $\partial(T \wedge W) = (\partial T \cap W) \cup \partial W$. We introduce the following σ -algebra,

$$\mathcal{B}(\mathbb{T} \wedge W) = \sigma(\{\{T \in \mathbb{T} \wedge W : \partial T \cap A = \emptyset\} : A \subseteq W, A \in \mathcal{C}\}) .$$

Also, for any nonatomic probability measure P on $(\mathbb{T} \wedge W, \mathcal{B}(\mathbb{T} \wedge W))$, the completed probability space $(\mathbb{T} \wedge W, \mathcal{B}(\mathbb{T} \wedge W), P)$ is Lebesgue.

We note that for another window $W' \subseteq W$ we have $T \wedge W' = (T \wedge W) \wedge W'$.

4 Hyperplanes

Let \mathcal{H} be the set of all hyperplanes in \mathbb{R}^ℓ , we will define a parameterization of it. Let $\|\cdot\|$ be the Euclidean norm, $\langle \cdot, \cdot \rangle$ be the inner product, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{S}_+^{\ell-1} = \{x \in \mathbb{R}^\ell : \|x\| = 1\} \cap (\mathbb{R}^{\ell-1} \times \mathbb{R}_+)$ be the upper half unit hypersphere in \mathbb{R}^ℓ . Define

$$H(\alpha, u) = \{x \in \mathbb{R}^\ell : \langle x, u \rangle = \alpha\}, \quad \alpha \in \mathbb{R}, u \in \mathbb{S}_+^{\ell-1}, \quad (1)$$

which is the hyperplane with normal direction u and signed distance (in direction u) α from the origin. Thus we can write

$$\mathcal{H} = \{H(\alpha, u) : (\alpha, u) \in \mathbb{R} \times \mathbb{S}_+^{\ell-1}\} \quad (2)$$

and on \mathcal{H} we use the σ -algebra that is induced from the Borel σ -algebra on the parameter space. Any hyperplane generates two closed half-spaces

$$H^-(\alpha, u) = \{x \in \mathbb{R}^\ell : \langle x, u \rangle \leq \alpha\} \quad \text{and} \quad H^+(\alpha, u) = \{x \in \mathbb{R}^\ell : \langle x, u \rangle \geq \alpha\}.$$

For an hyperplane H the above notions are written by H^- and H^+ for short. We define

$$[B] = \{H \in \mathcal{H} : H \cap B \neq \emptyset\} \text{ for } B \in \mathcal{B}(\mathbb{R}^\ell).$$

Now, let Λ be a (non-zero) measure on the space of hyperplanes \mathcal{H} .

4.1 Assumptions on Λ

We assume:

- (i) Λ is translation invariant;
- (ii) Λ possesses the following locally finiteness property:

$$\Lambda([B]) < \infty, \quad \text{for all bounded sets } B \in \mathcal{B}(\mathbb{R}^\ell); \quad (3)$$

- (iii) the support of Λ is such that there is no line in \mathbb{R}^ℓ with the property that all the hyperplanes of the support are parallel to it.

The image of a non-zero, locally finite and translation invariant measure Λ with respect to the parameterization (1), (2), can be written as the product measure

$$\gamma \cdot \lambda \otimes \theta, \quad (4)$$

where $\gamma > 0$ is a constant, λ is the Lebesgue measure on \mathbb{R} and θ is a probability measure on $\mathbb{S}_+^{\ell-1}$ (cf, e.g. [8], Theorem 4.4.1 and Theorem 13.2.12).

From the properties of Λ there is no one-dimensional subspace L_1 of \mathbb{R}^ℓ such that the support of θ equals $L_1^\perp \cap \mathbb{S}_+^{\ell-1}$ (where L_1^\perp denotes the orthogonal complement of L_1). This property allows to obtain a.s. bounded cells in STIT tessellations, cf. [8], Theorem 10.3.2, which can also be applied to STIT.

From (3) we find that the space $(\mathcal{H}, \mathcal{B}(\mathcal{H}), \Lambda)$ is σ -finite. In fact, for an increasing sequence of windows $(W_n : n \in \mathbb{N} = \{0, 1, \dots\})$ covering \mathbb{R}^ℓ (that is $\mathbb{R}^\ell = \bigcup_{n \in \mathbb{N}} W_n$) we have $\mathcal{H} = \bigcup_{n \in \mathbb{N}} [W_n]$ and $\Lambda([W_n]) < \infty$ for all $n \in \mathbb{N}$.

Let W be a window. Since $\text{Int } W \neq \emptyset$ we get $\Lambda([W]) > 0$. Then, $0 < \Lambda([W]) < \infty$ and we can define

$$\widehat{\Lambda}_{[W]} = \Lambda([W])^{-1} \Lambda_{[W]}$$

the (normalized) probability measure associated to $\Lambda_{[W]}$, the restriction of Λ to $[W]$. Since Λ is translation invariant we have that $\widehat{\Lambda}_{[W]}$ is non-atomic, see [3]. Hence, regarding the properties of the parameter space, which are inherited by the space of hyperplanes we have that

$$([W], \mathcal{B}, \widehat{\Lambda}_{[W]}) \text{ is a Lebesgue probability space.} \quad (5)$$

For $T \wedge W \in \mathbb{T} \wedge W$ we define

$$\zeta(T \wedge W) = \sum_{C \in T \wedge W} \Lambda([C]). \quad (6)$$

5 A tree structure

Let us introduce the rooted binary trees. First we set $\mathbb{N}^* = \{1, 2, \dots\}$. Let $\mathcal{E} = \{-, +\}$ be a two symbol alphabet and for $k \in \mathbb{N}^*$ let \mathcal{E}^k be the set of sequences (or words) of length k . They describe paths in the tree starting from the root. We take $\mathcal{E}^0 = \{o\}$ a singleton where o is the empty word. We define $\mathcal{E}^* = \bigcup_{k \in \mathbb{N}} \mathcal{E}^k$. By \vec{e} we denote an element of \mathcal{E}^* and we say it has level k if $\vec{e} \in \mathcal{E}^k$.

Let $\vec{e} = (e_1, \dots, e_k) \in \mathcal{E}^k$ for $k \in \mathbb{N}$ (so $\vec{e}_0 = (o)$). The successors of \vec{e} are the two elements in $\text{Succ}(\vec{e}) = \{(e_1, \dots, e_k, e_{k+1}) : e_{k+1} \in \mathcal{E}\}$ and \vec{e} is called the predecessor of each of its successors. For $\vec{e} \in \mathcal{E}^* \setminus \{o\}$ we denote by $\text{Pred}(\vec{e})$ its predecessor. Note that $\text{Succ}(o) = \mathcal{E}$.

For simplicity, we will often omit brackets and commas and write $\vec{e} = e_1 \dots e_k$ for $\vec{e} = (e_1, \dots, e_k)$.

It is useful to have a total order \leq on \mathcal{E}^* compatible with the levels. We fix (\mathcal{E}^*, \leq) as the totally ordered set that satisfies:

$$\begin{aligned} \vec{e} \in \mathcal{E}^j, \vec{e}' \in \mathcal{E}^k, j < k &\Rightarrow \vec{e} < \vec{e}'; \\ \text{on } \mathcal{E} : - < +; \text{ inducing } \leq \text{ on } \mathcal{E}^k, k \in \mathbb{N}^*, &\text{ the lexicographical order.} \end{aligned} \quad (7)$$

In particular the empty word is the minimal one: $o < \vec{e}$ for all $\vec{e} \in \mathcal{E}^*$ with $\vec{e} \neq o$; and $\vec{e} - < \vec{e} +$ is the order between the successors of \vec{e} .

Now we put all paths of a binary tree into ordered tuples R . These tuples will later be useful for the description of STIT processes. Note that there can be different such tuples referring to the same tree. These tuples can be interpreted as a protocol in which order the edges of the tree 'grow'. It is assumed that the pairs of edges to the two successors of a node appear simultaneously, but different pairs cannot 'grow' simultaneously.

Below for a finite sequence $R = (r_0, \dots, r_{2k}) \in (\mathcal{E}^*)^{2k+1}$ we denote by $\{R\} = \{r_i : i = 0, \dots, 2k\}$ the set of its components and by $|R|$ the cardinality of $\{R\}$. So, $|R| = 2k + 1$ means that the values in R are pairwise different.

We shall define the following set Θ :

$$\begin{aligned} \Theta &= \bigcup_{k \in \mathbb{N}} \Theta_k \text{ with } \Theta_k \subset (\mathcal{E}^*)^{2k+1} \text{ and } R = (r_0, \dots, r_{2k}) \in (\mathcal{E}^*)^{2k+1} \text{ satisfies} \\ R \in \Theta_k &\Leftrightarrow r_0 = o, |R| = 2k + 1, \text{ and} \\ \forall k \geq 1 \forall l \in \{1, \dots, k\} \exists j_l \in \{0, \dots, 2l-2\} : &\{r_{2l-1}, r_{2l}\} = \text{Succ}(r_{j_l}), r_{2l-1} < r_{2l}. \end{aligned} \quad (8)$$

Note that

$$\left[\{r_{2l-1}, r_{2l}\} = \text{Succ}(r_{j_l}), r_{2l-1} < r_{2l} \right] \Leftrightarrow \left[(r_{2l-1}, r_{2l}) = (r_{j_l} -, r_{j_l} +) \right].$$

For $R \in \Theta$ and $\vec{e} \in \mathcal{E}^*$ we have that:

$$\text{Succ}(\vec{e}) \cap \{R\} = \emptyset \text{ or } \text{Succ}(\vec{e}) \cap \{R\} = \text{Succ}(\vec{e}).$$

We define the set of leaves of R by

$$\mathcal{L}(R) = \{r \in \{R\} : \text{Succ}(r) \cap \{R\} = \emptyset\}. \quad (9)$$

This is the set of elements in $\{R\}$ such that both successors are not in $\{R\}$. When $R = (o)$ we have $\mathcal{L}(R) = \{o\}$. For $k > 0$, and $R = (r_0, \dots, r_{2k}) \in \Theta_k$ we have in particular

$$\{r_{2k-1}, r_{2k}\} \subseteq \mathcal{L}(R).$$

If $|R| = 3$ we have $R = (o, -, +)$ and $\mathcal{L}(R) = \{-, +\}$. For $|R| = 5$ we have $R = (o, -, +, --, -+)$ or $R = (o, -, +, +-, ++)$, in the first case $\mathcal{L}(R) = \{+, --, -+\}$ and in the second one $\mathcal{L}(R) = \{-, +-, ++\}$.

Take $R = (r_0, \dots, r_{2k}) \in \Theta_k$. For all $s \in \{0, \dots, k\}$ we define $R^{(s)} = (r_0, \dots, r_{2s}) \in \Theta_s$. E.g., if $R = (o, -, +)$ we have $R^{(0)} = (o)$ and $R^{(1)} = R$.

For $k > 0$ and $s \in \{0, \dots, k-1\}$ there exists $j_{s+1} \leq 2s$ such that $\text{Succ}(r_{j_{s+1}}) = \{r_{2s+1}, r_{2(s+1)}\}$. We denote $r_s^* = r_{j_{s+1}}$ and then we have

$$\forall k > 0, s \in \{0, \dots, k-1\} : \mathcal{L}(R^{(s+1)}) = (\mathcal{L}(R^{(s)}) \setminus \{r_s^*\}) \cup \{r_{2s+1}, r_{2s+2}\}, \quad (10)$$

that is $r_s^* \in \mathcal{L}(R^{(s)})$ is substituted by its successors. Then,

$$\forall R = (r_0, \dots, r_{2k}) \in \Theta_k : |\mathcal{L}(R)| = k + 1 \text{ and } |\mathcal{L}(R^{(s)})| = s + 1 \text{ for } s = 0, \dots, k. \quad (11)$$

Note that we always have $r_0^* = o$.

6 Construction of STIT tessellations in a window: main properties

A STIT tessellation is defined as a homogeneous (i.e. spatially stationary) tessellation with a distribution that is invariant under rescaled iteration (or nesting) of tessellations. A precise definition was given in [6] where also the existence was shown (by construction) as well as the uniqueness of its law if a hyperplane measure Λ is given. Meanwhile, several equivalent constructions of STIT tessellations in a bounded window are published. Here we start with one of these constructions.

On every window W and for every hyperplane measure Λ satisfying the assumptions (i), (ii), (iii), formulated in Section 4, there is a STIT tessellation process $Y \wedge W = ((Y \wedge W)_t : t \geq 0)$ associated to $\Lambda_{[W]}$, that is now constructed.

Let us take two independent families of independent random variables $(Z(\vec{e}) : \vec{e} \in \mathcal{E}^*)$ and $(G_n : n \in \mathbb{N}^*)$, with $Z(\vec{e}) \sim \text{Exponential}(1)$ and $G_n \sim \widehat{\Lambda}_{[W]}$. So $\mathbb{P}(Z(\vec{e}) > t) = e^{-t}$ for all $t \geq 0$. We note that $\lambda^{-1}Z(\vec{e}) \sim \text{Exponential}(\lambda)$ for $\lambda > 0$.

Now we define cells $C(\vec{e})$ which are later used to describe the states of the STIT tessellation process.

Step I: $C(o) = W$.

Step II: Define

$$\begin{aligned} \forall \vec{e} \in \mathcal{E}^* : H(\vec{e}) = G_{\kappa(\vec{e})} \text{ where } \kappa(o) = 1 \text{ and for } \vec{e} \neq o : \quad (12) \\ \kappa(\vec{e}) = \inf \{n : G_n \in [C(\vec{e})], n > \max\{\kappa(\vec{e}') : \vec{e}' < \vec{e}\}\}. \end{aligned}$$

So $(H(\vec{e}) : \vec{e} \in \mathcal{E}^*)$ and $(\kappa(\vec{e}) : \vec{e} \in \mathcal{E}^*)$ are well-defined a.s.

Step III: For $\vec{e} \in \mathcal{E}^*$, define $C(\vec{e}-) = C(\vec{e}) \cap H^-(\vec{e})$, $C(\vec{e}+) = C(\vec{e}) \cap H^+(\vec{e})$.

Step IV: $C(o)$ is born at time $t_b(o) = 0$, its lifetime is $t_l(o) \sim \Lambda([C(o)])^{-1}Z(o)$ and so dies at $t_l(o)$, i.e. at that time it is divided by $H(o) = G_1$. A cell $C(\vec{e})$, with $\vec{e} \in \mathcal{E}^*$, $\vec{e} \neq o$, is born at time $t_b(\vec{e}) = t_b(\text{Pred}(\vec{e})) + t_l(\text{Pred}(\vec{e}))$, has lifetime $t_l(\vec{e}) \sim \Lambda([C(\vec{e})])^{-1}Z(\vec{e})$ and dies at time $t_b(\vec{e}) + t_l(\vec{e})$. At that time it is divided by $H(\vec{e}) = G_{\kappa(\vec{e})}$ into $C(\vec{e}-)$ and $C(\vec{e}+)$.

In step II a rejection method is applied where random hyperplanes are thrown onto the window until the first time a hyperplane hits $C(\vec{e})$. Note that in (12), the sequence $(\kappa(\vec{e}) : \vec{e} \in \mathcal{E}^*)$ is increasing and $(H(\vec{e}) : \vec{e} \in \mathcal{E}^*)$ is an independent family conditioned to $H(\vec{e}) \cap [C(\vec{e})] \neq \emptyset$. Moreover,

$$\forall \vec{e} \in \mathcal{E}^* : H(\vec{e}) \sim \widehat{\Lambda}_{[C(\vec{e})]}. \quad (13)$$

Remark 6.1 Let $\vec{e} \neq o$. In the sequence $(G_n : n > \max\{\kappa(\vec{e}') : \vec{e}' < \vec{e}\})$ of independent identically distributed random hyperplanes with common law $\widehat{\Lambda}_{[W]}$, the first of these hyperplanes which intersects $[C(\vec{e})]$ is distributed as $\widehat{\Lambda}_{[C(\vec{e})]}$. The random time of attending such an hyperplane depends on the inverse of the Λ -measure of $[C(\vec{e})]$, which depends on the size but also on the shape of the cell $C(\vec{e})$.

It is easy to see that at any time a.s. at most one cell dies and so a.s. at most only two cells are born.

At each time $t \geq 0$ we define $(Y \wedge W)_t$ as the class of cells $C(\vec{e})$ which are alive at time t , that is

$$(Y \wedge W)_t = \{C(\vec{e}) : \vec{e} \in E_t\} \text{ where } E_t = \{\vec{e} \in \mathcal{E}^* : t_b(\vec{e}) \leq t < t_b(\vec{e}) + t_l(\vec{e})\}.$$

Since $C(\vec{e}) = C(\vec{e}-) \cup C(\vec{e}+)$ and $\emptyset = \text{Int } C(\vec{e}-) \cap \text{Int } C(\vec{e}+)$, it is easy to see that $(Y \wedge W)_t$ is a tessellation of W . On the other hand, it can be checked that $E_t = \mathcal{L}(R)$, the set of leaves of some $R \in \Theta$ defined in Section 5. Note that such an R is not necessarily unique.

Let $t \geq 0, s > 0$. Let us show that $(Y \wedge W)_{t+s}$ is conditionally independent from $((Y \wedge W)_v, v < t)$, conditioned on $(Y \wedge W)_t$. From definition

$$(Y \wedge W)_{t+s} \cap \left\{ C(\vec{e}) : \vec{e} \in \left(\bigcup_{v < t} E_v \right) \setminus E_t \right\} = \emptyset.$$

On the other hand

$$(Y \wedge W)_{t+s} \subseteq \bigcup_{\vec{e} \in E_t} \{C(\vec{e}') : \vec{e}' \in \text{Succ}^*(\vec{e})\} \text{ where}$$

$$\text{Succ}^*(\vec{e}) = \{\vec{e}' : \exists k \geq 1, \exists \vec{e}^1, \dots, \vec{e}^{k-1}, \forall j = 1, \dots, k : \vec{e}^j \in \text{Succ}(\vec{e}^{j-1}), \vec{e}^0 = \vec{e}, \vec{e}^k = \vec{e}'\}.$$

Then, the memoryless property of the exponential distribution, implies that $Y \wedge W$ is a Markov processes,

$$\mathbb{P}((Y \wedge W)_{t+s} \in \cdot \mid (Y \wedge W)_v, v \in [0, t]) = \mathbb{P}((Y \wedge W)_{t+s} \in \cdot \mid (Y \wedge W)_t).$$

From (6),

$$\zeta((Y \wedge W)_t) = \sum_{C \in (Y \wedge W)_t} \Lambda([C]). \quad (14)$$

Let

$$\tau_t = \inf\{s > 0 : (Y \wedge W)_{t+s} \neq (Y \wedge W)_t\}.$$

be the holding time at t . The memoryless property of the exponential distribution (and the property that the minimum of finitely many independent exponentially distributed random variables is again exponentially distributed where its parameter is the sum of the parameters of the variables) implies,

$$\tau_t \mid \sigma((Y \wedge W)_t) \sim \text{Exponential}(\zeta((Y \wedge W)_t)). \quad (15)$$

At time $t + \tau_t$ a (a.s.) unique cell $C_t^* \in (Y \wedge W)_t$ dies, we put $C_t^* = C(\bar{e}^*)$ with $\bar{e}^* \in E_t$ a random index. Then, two new cells $\{C_t^* \cap H^-(\bar{e}^*), C_t^* \cap H^+(\bar{e}^*)\}$ are born at this time, so

$$(Y \wedge W)_{t+\tau_t} = \{C : C \in (Y \wedge W)_t, C \neq C_t^*\} \cup \{C_t^* \cap H^-(\bar{e}^*), C_t^* \cap H^+(\bar{e}^*)\}. \quad (16)$$

Hence,

$$Y_{t+\tau_t} \text{ is uniquely defined from } [(Y \wedge W)_t, C_t^* \in (Y \wedge W)_t, H(\bar{e}^*)]. \quad (17)$$

To any realization of the process $((Y \wedge W)_t : 0 \leq t \leq t_0)$ we associate a sequence $(R^s \in \Theta : s = 0, \dots, s_0)$ as follows. Define $R^{(0)} = (o)$. Now let $\tau^0 = 0$ and for an integer $s \geq 0$ define

$$\tau^{s+1} = \inf\{t > 0 : \tau^s + t \leq t_0, (Y \wedge W)_{\tau^s+t} \neq (Y \wedge W)_{\tau^s}\},$$

where as usual $\infty = \inf \emptyset$. Then a.s. there exists $s_0 = \sup\{s : \tau^s \leq t_0\}$. Then, we define $R = R^{(s_0)}$ by induction as follows: for all $0 \leq s < s_0$ define,

$$R^{(s+1)} = (r_0, \dots, r_{2s}, r^*-, r^*+), \quad (18)$$

where $r^* \in \{r_0, \dots, r_{2s}\}$ is the unique element in $R^{(s)}$, such that $C_{\tau^s}^* = C(r^*)$ according to (16).

In next result we characterize the Markov process $Y \wedge W$ by supplying its holding times, the jump rates and their conditional independence. We follow Section 1.1.1. in [1].

Proposition 6.1 (I) $Y \wedge W$ is a pure jump Markov process and satisfies

$$\begin{aligned} \forall C(\bar{e}) \in (Y \wedge W)_t, \forall H \in [C(\bar{e})], s > 0 : \\ \mathbb{P}(H(\bar{e}) \in dH, C_t^* = C(\bar{e}), \tau_t \in ds \mid (Y \wedge W)_t) = \Lambda_{[C(\bar{e})]}(dH) e^{-\zeta((Y \wedge W)_t) s} ds. \end{aligned} \quad (19)$$

(II) We have the consistency property, namely

$$\forall \text{ windows } V \subseteq W : (Y \wedge W) \wedge V \sim Y \wedge V \text{ where } (Y \wedge W) \wedge V = ((Y \wedge W)_t \wedge V : t \geq 0). \quad (20)$$

(III) Moreover, $Y \wedge W$ satisfies the following regeneration property: For all fixed $t_0 \geq 0$ and $C \in Y_{t_0}$ the processes $(Y \wedge W) \wedge C = ((Y \wedge W)_t \wedge C : t \geq t_0)$, satisfy

$$\begin{aligned} ((Y \wedge W) \wedge C : C \in Y_{t_0}) \text{ are conditionally independent, given } Y_{t_0} \text{ and} \\ (Y \wedge W) \wedge C \text{ is a STIT process on } C, \text{ associated to } \Lambda. \end{aligned} \quad (21)$$

Proof: From (13) and Step III, we have that

$$\forall C(\bar{e}) \in (Y \wedge W)_t, \forall H \in [C(\bar{e})] : \mathbb{P}(H(\bar{e}) \in dH \mid C_t^* = C(\bar{e}), (Y \wedge W)_t) = \widehat{\Lambda}_{[C(\bar{e})]}(dH).$$

Hence, (19) will follow once we prove

$$\forall C \in (Y \wedge W)_t : \mathbb{P}(C_t^* = C, \tau_t \in ds \mid (Y \wedge W)_t) = \Lambda([C]) e^{-\zeta((Y \wedge W)_t) s} ds.$$

The proof of this last relation is based upon the following fact applied to the lifetime variables $Z(\vec{e})$ of $C(\vec{e}) \in (Y \wedge W)_t$. Let $(Z_i : i = 1, \dots, k)$ be independent random variables with $Z_i \sim \text{Exponential}(q_i)$ and $Z = \min\{Z_i : i = 1, \dots, k\}$. Then for all $z > 0$,

$$\mathbb{P}(Z_j = Z, Z \in dz) = \mathbb{P}(Z_i > z, i \neq j, Z_j \in dz) = \frac{q_j}{\sum_{i=1}^k q_i} \mathbb{P}(Z \in dz) = q_j e^{-(\sum_{i=1}^k q_i)z} dz.$$

The consistency property (20) was already shown in [6].

The proof of the regeneration property (21) follows from the consistency and the memoryless property of the exponential distribution applied to $C(\vec{e}') \in Y_{t_0} \wedge W$, which is

$$\forall t \geq t_0 : \mathbb{P}(t_b(\vec{e}) + t_l(\vec{e}) > t \mid t_b(\vec{e}) \leq t_0 < t_b(\vec{e}) + t_l(\vec{e})) = e^{-\Lambda([C(\vec{e})])(t-t_0)}.$$

For $t \geq t_0$ and $C(\vec{e}') \in Y_{t_0} \wedge W$, $(Y \wedge W) \wedge C(\vec{e}') = ((Y \wedge W)_t \wedge C(\vec{e}') : t \geq t_0)$ satisfies

$$(Y \wedge W)_t \wedge C(\vec{e}') = \{C(\vec{e}) : \vec{e} \in \text{Succ}^*(\vec{e}'), t_b(\vec{e}) \leq t_0 \leq t < t_b(\vec{e}) + t_l(\vec{e})\}.$$

From the memoryless property we find that $(Y \wedge W) \wedge C(\vec{e}')$ is a STIT process. Finally, the interiors of the cells $C \in \bigcup_{t \geq 0} (Y \wedge W)_t \wedge C(\vec{e}')$ are contained in $\text{Int } C(\vec{e}')$, and so they are pairwise disjoint as $C(\vec{e}')$ varies in $Y_{t_0} \wedge W$. We deduce that the processes $(Y \wedge W) \wedge C(\vec{e}')$, $C(\vec{e}') \in Y_{t_0} \wedge W$, are conditionally independent. \square

Note that (19) implies, $\tau_t \mid (Y \wedge W)_t \sim \text{Exponential}(\zeta((Y \wedge W)_t))$, and that for all $C(\vec{e}) \in (Y \wedge W)_t$, $H \in [C(\vec{e})]$ holds

$$\mathbb{P}(H(\vec{e}) \in dH, C_t^* = C(\vec{e}) \mid (Y \wedge W)_t) = \hat{\Lambda}_{[C(\vec{e})]}(dH) \frac{\Lambda([C(\vec{e})])}{\zeta((Y \wedge W)_t)}, \quad (22)$$

in particular $\mathbb{P}(C_t^* = C \mid (Y \wedge W)_t) = \Lambda([C]) / \zeta((Y \wedge W)_t)$.

In [6] it was shown that the STIT process $Y \wedge W$ has no explosion. In fact the process of number of cells $\#\{C : C \in (Y \wedge W)_t\}$ is stochastically dominated by a birth chain $(M(t) : t \geq 0)$ starting from $M(0) = 1$ with linear birth rates $b_n = n \Lambda([W])$. Since $(M(t) : t \geq 0)$ does not explode we deduce that the process of the number of cells does not explode too. This also follows straightforwardly from Lemma 1.1 in [1].

The consistency property (20) implies the existence of a probability measure on $\mathbb{T}^{\mathbb{R}_+}$ endowed with the product σ -field, and it defines the distribution of a process Y , which is Markov and it satisfies $Y_t \wedge W \sim (Y \wedge W)_t$ for all $t \in \mathbb{R}_+$ and all windows W . (See [6]).

A global construction for a STIT process Y was provided in [5].

Proposition 6.2 *The process $Y = (Y_t : t > 0)$ is a Markov process which is the STIT tessellation process on \mathbb{R}^ℓ associated to Λ . Its marginals Y_t take values in \mathbb{T} and $(Y_t \wedge W : t \geq 0) \sim ((Y \wedge W)_t : t \geq 0)$.*

Moreover Y satisfies the regeneration property: For all $t_0 > 0$ and $C \in Y_{t_0}$ the processes $Y \wedge C = (Y_t \wedge C : t \geq t_0)$, satisfy

$$(Y \wedge C : C \in Y_{t_0}) \text{ are conditionally independent given } Y_{t_0}, \text{ and} \\ Y \wedge C \text{ is a STIT process on } C \text{ associated to } \Lambda. \quad (23)$$

The proof of the regeneration property (23) is straightforward from (21). We notice that this regeneration property (23) is equivalent to the stable-under-iteration property. For this last property see ([3, 9]). On the other hand (23) is, once written appropriately, the branching property of a fragmentation chain described in Proposition 1.2 (i) [1].

7 The marginal distribution of STIT tessellations in a window

In (18) we used the tuples R defined for a rooted binary tree in Section 5 to index the sequences of tessellations associated constructed in Steps I and III in Section 6.

For each $R = (r_0, \dots, r_{2k}) \in \Theta_k$ we have defined a sequence $R^{(s)} = (r_0, \dots, r_{2s}) \in \Theta_s$ for $s = 0, \dots, k$. We will associate to each $R = (r_0, \dots, r_{2k}) \in \Theta_k$ a sequence of tessellations of W , denoted by $T(R) = (T(R^{(s)})) : s = 0, \dots, k$. We will do it by describing the family of cells of each $T^{(s)}$, by using induction on $s = 0, \dots, k$.

We define

$$T^{(s)}(R) = \{C(r) : r \in \mathcal{L}(R^{(s)})\}.$$

Note that $T^{(0)}(R) = \{C(o)\}$ because $R^{(0)} = (o)$. Let $k \geq 1$. From (10) we get for $s = 0, \dots, k-1$:

$$T^{(s+1)}(R) = \{C \in T^{(s)}(R) \setminus C(r_s^*)\} \cup \{C(r_{2s+1}), C(r_{2s+2})\}.$$

That is the tessellation $T^{(s+1)}(R)$ results from dividing the cell $C(r_s^*) \in T^{(s)}(R)$ into the two cells associated to its successors (here, the dividing hyperplane $H(r_s^*)$, see (12), is not indicated in the notion).

We have $\#(T^{(s)}(R)) = s+1$, see (11), and $\zeta(T^{(s)}(R)) = \sum_{C \in T^{(s)}(R)} \Lambda([C])$, see (14).

Let $A \in \mathcal{B}(\mathbb{T})$. The STIT process $Y \wedge W = ((Y \wedge W)_t : t \geq 0)$ satisfies

$$\mathbb{P}((Y \wedge W)_t \in A) = \mathbb{P}((Y \wedge W)_t = \{W\}, \{W\} \in A) + \sum_{k \in \mathbb{N}^*} \mathbb{P}(\#(Y \wedge W)_t = k+1, (Y \wedge W)_t \in A).$$

To describe the summands for $k \in \mathbb{N}^*$, we must take into account that at any time s , the tessellation $(Y \wedge W)_s$ attends a random time τ_s for the division of one

of its cells, and this time satisfies $\tau_s \mid (Y \wedge W)_s \sim \text{Exponential}(\zeta((Y \wedge W)_s))$ and the cell C_s^* of $(Y \wedge W)_s$ divided at time $s + \tau_s$ is chosen by $\mathbb{P}(C_t^* = C \mid (Y \wedge W)_s) = \Lambda([C]) / \zeta((Y \wedge W)_s)$. Therefore, using (19), we obtain

Proposition 7.1 *For $k \in \mathbb{N}^*$*

$$\begin{aligned} & \mathbb{P}(\#(Y \wedge W)_t = k+1, (Y \wedge W)_t \in A) \\ &= \sum_{R \in \Theta_k} \int d\widehat{\Lambda}_{[C(r_0^*)]}(H_1) \int d\widehat{\Lambda}_{[C(r_1^*)]}(H_2) \dots \int d\widehat{\Lambda}_{[C(r_{k-1}^*)]}(H_k) \int_0^t dw_1 \int_0^{t-w_1} dw_2 \dots \int_0^{t-\sum_{s=1}^{k-1} w_s} dw_k \\ & \times \left(\prod_{s=0}^{k-1} \Lambda([C(r_s^*)]) \right) \exp \left[- \sum_{s=0}^{k-1} \zeta(T^{(s)}(R)) w_{s+1} \right] \exp \left[- \zeta(T^{(k)}(R)) \left(t - \sum_{s=1}^k w_s \right) \right] \mathbf{1}_A(T^{(k)}(R)). \end{aligned} \quad (24)$$

8 Revisiting the STIT tessellation process in a window

Let us give another equivalent construction of the STIT process $Y \wedge W$ that will be useful to understand the construction done in the next Section.

Let us consider three independent sequences $(U_n : n \in \mathbb{N}^*)$, $(V_n : n \in \mathbb{N}^*)$, $(G_n : n \in \mathbb{N}^*)$, of independent identically distributed random variables, such that $U_n \sim \text{Uniform}[0, 1]$, $V_n \sim \text{Exponential}(1)$, $G_n \sim \widehat{\Lambda}_{[W]}$. We start with a construction of a sequence $(\mathcal{Y}_n, n \in \mathbb{N})$ of tessellations in W .

The algorithm of the construction is:

Step $n = 0$: $\mathcal{Y}_0 = \{C(o)\}$ with $C(o) = W$. Let $\zeta(\mathcal{Y}_0) = \Lambda([C(o)])$ and $\kappa_0 = 0$.

Step $n + 1$:

Assume $\mathcal{Y}_n = \{C(\vec{e}) : \vec{e} \in E_n\}$, $n \geq 0$, has been defined with E_n a set of the form $E_n = \mathcal{L}(R)$ with some $R \in \Theta_n$, and so $|E_n| = n + 1$. We also assume $\kappa_n \in \mathbb{N}^*$ has been defined. Let

$$\zeta(\mathcal{Y}_n) = \sum_{C \in \mathcal{Y}_n} \Lambda([C]) = \sum_{\vec{e} \in E_n} \Lambda([C(\vec{e})]).$$

Since E_n is totally ordered by (7), also the class of cells $\{C : C \in \mathcal{Y}_n\}$ is totally ordered. We define a partition of $[0, 1]$ by

$$[0, 1] = \bigcup_{\vec{e} \in E_n} [a_{\vec{e}}^n, b_{\vec{e}}^n] \text{ with } b_{\vec{e}}^n - a_{\vec{e}}^n = \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})]), \quad (25)$$

where the intervals $[a_{\vec{e}}^n, b_{\vec{e}}^n]$ and $[a_{\vec{e}'}, b_{\vec{e}'})$ are consecutive when \vec{e}' is the element following \vec{e} in E_n with respect to the total order \leq . We define a random cell $C_n^* \in \mathcal{Y}_n$ by

$$\forall \vec{e} \in E_n : C_n^* = C(\vec{e}) \Leftrightarrow U_{n+1} \in [a_{\vec{e}}^n, b_{\vec{e}}^n]. \quad (26)$$

Hence

$$\forall \vec{e} \in E_n : \mathbb{P}(C_n^* = C(\vec{e}) \mid \mathcal{Y}_n) = \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})]). \quad (27)$$

We denote by $\vec{e}^* \in E_n$ the random index such that $C(\vec{e}^*) = C_n^*$. Note that

$$C_n^* \in \sigma(\mathcal{Y}_n, U_{n+1}) \quad (28)$$

We define the random hyperplane H_{n+1} in a similar way as in (12), so

$$H_{n+1} = G_{\kappa_{n+1}} \text{ where } \kappa_{n+1} = \min\{j > \kappa_n : G_j \in [C_n^*]\}. \quad (29)$$

By definition $H_{n+1} \sim \widehat{\Lambda}_{[C_n^*]}$. Note that $H_1 = G_1$. Obviously $(\kappa_n : n \in \mathbb{N}^*)$ is an increasing sequence of random times. We note that κ_{n+1} is a stopping time with respect to the filtration $(\sigma(G_j, \kappa_n, C_n^*) : j \in \mathbb{N}^*)$.

The tessellation \mathcal{Y}_{n+1} is formed from \mathcal{Y}_n by the division of the random cell C_n^* of \mathcal{Y}_n by H_{n+1} , giving

$$\mathcal{Y}_{n+1} = (\{C \in \mathcal{Y}_n\} \setminus \{C_n^*\}) \cup \{C_n^* \cap H_{n+1}^-, C_n^* \cap H_{n+1}^+\}. \quad (30)$$

So \mathcal{Y}_{n+1} is indexed by $E_{n+1} = (E_n \setminus \{\vec{e}^*\}) \cup \text{Succ}(\vec{e}^*) \subset \mathcal{E}^*$, and so $E_{n+1} = \mathcal{L}(R_{n+1})$ for some (uniquely determined) $R_{n+1} \in \Theta_{n+1}$. This shows $(\mathcal{Y}_n : n \in \mathbb{N})$ is well-defined. Notice that $\mathcal{Y}_{n+1} \in \sigma(\mathcal{Y}_n, U_{n+1}, H_{n+1})$ and then by recursion we get

$$\mathcal{Y}_{n+1} \in \sigma(U_k, H_k : k \leq n+1). \quad (31)$$

Now, use (27) and (29) to get that for all $\mathcal{K} \in \mathcal{B}([W])$ we have,

$$\begin{aligned} \mathbb{P}(H_{n+1} \in \mathcal{K}, C_n^* = C(\vec{e}) \mid \mathcal{Y}_n) &= \mathbb{P}(H_{n+1} \in \mathcal{K} \mid C_n^* = C(\vec{e}), \mathcal{Y}_n) \mathbb{P}(C_n^* = C(\vec{e}) \mid \mathcal{Y}_n) \\ &= \mathbb{P}(H_{n+1} \in \mathcal{K} \cap [C(\vec{e})]) \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})]) = \frac{\Lambda(\mathcal{K} \cap [C(\vec{e})])}{\Lambda([C(\vec{e})])} \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})]) \\ &= \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})] \cap \mathcal{K}). \end{aligned} \quad (32)$$

Since $\sigma(U_n, G_n : n \in \mathbb{N}) \perp\!\!\!\perp \sigma(V_n : n \in \mathbb{N})$, from (28), (29) and (31) the above random objects satisfy the relation

$$\sigma(\mathcal{Y}_n, U_n, G_n, H_n : n \in \mathbb{N}) \perp\!\!\!\perp (V_n : n \in \mathbb{N}). \quad (33)$$

Define the sequence of jump times by

$$\forall n \in \mathbb{N}^* : \pi_n = (\zeta(\mathcal{Y}_{n-1}))^{-1} V_n,$$

which are conditionally distributed as,

$$\pi_n \mid \mathcal{Y}_{n-1} \sim \text{Exponential}(\zeta(\mathcal{Y}_{n-1})). \quad (34)$$

Since $(V_n : n \in \mathbb{N}^*)$ is a sequence of independent random variables, $(\pi_n : n \in \mathbb{N}^*)$ is conditionally independent given $\sigma(\zeta(\mathcal{Y}_n) : n \in \mathbb{N})$. From (33) and (34) we get

$$(\pi_{n+1} \perp\!\!\!\perp \sigma(C_n^*, H_{n+1})) \mid \mathcal{Y}_n. \quad (35)$$

Define the sequence of times

$$\mathcal{S}_0 = 0 \text{ and } \mathcal{S}_n = \mathcal{S}_{n-1} + \pi_n = \sum_{j=1}^n \pi_j \text{ for } n \in \mathbb{N}^*.$$

The proof ensuring that $\lim_{n \rightarrow \infty} \mathcal{S}_n = \infty$ \mathbb{P} -a.s., is the same as the one where we proved that $Y \wedge W$ has no explosion. In fact, $\#(\mathcal{Y}_n) = n + 1$ implies $\zeta(\mathcal{Y}_n) \leq (n + 1)\Lambda([W])$. Hence, the process $(N(t) : t \geq 0)$ given by

$$N(t) = \sup\{n \in \mathbb{N} : \mathcal{S}_n \leq t\}, \quad (36)$$

is stochastically dominated by a birth chain $(M(t) : t \geq 0)$ starting from $M(0) = 1$ with linear birth rates $b_n = n\Lambda([W])$, so $(N(t) : t \geq 0)$ does not explode.

By using the sequences $(\mathcal{Y}_n : n \in \mathbb{N})$ and $(\mathcal{S}_n : n \in \mathbb{N})$ we define the tessellation process $\underline{Y} \wedge W = ((\underline{Y} \wedge W)_t : t \geq 0)$ by

$$(\underline{Y} \wedge W)_t = \mathcal{Y}_n \text{ when } t \in [\mathcal{S}_n, \mathcal{S}_{n+1}), n \in \mathbb{N}. \quad (37)$$

From $\lim_{n \rightarrow \infty} \mathcal{S}_n = \infty$ \mathbb{P} -a.s. we get that $\underline{Y} \wedge W$ is well-defined for all times $t \geq 0$ \mathbb{P} -a.s. We also have $(\underline{Y} \wedge W)_{\mathcal{S}_n} = \mathcal{Y}_n$ for all $n \in \mathbb{N}$ and so $(\mathcal{S}_n : n \in \mathbb{N}^*)$ is the sequence of times of jumps of $\underline{Y} \wedge W$.

Proposition 8.1 *The process $\underline{Y} \wedge W$ is a STIT process associated to $\Lambda_{[W]}$.*

Proof: Since (I) in Proposition 6.1 completely characterizes the law of a STIT process associated to Λ , it is sufficient to show those properties.

Let us prove $\underline{Y} \wedge W$ satisfies the Markov property. For all $t \geq 0$ and $s > 0$ we have

$$\mathbb{P}((\underline{Y} \wedge W)_{t+s} | (\underline{Y} \wedge W)_u, u \leq t) = \mathbb{P}((\underline{Y} \wedge W)_{t+s} | (\underline{Y} \wedge W)_t = \mathcal{Y}_{N(t)}, \mathcal{S}_{N(t)}).$$

The memoryless property of the exponential distribution implies $(\mathcal{S}_n - t : n > N(t)) \perp \mathcal{S}_{N(t)} | \mathcal{Y}_{N(t)}$, and so $(\underline{Y} \wedge W)_{t+s} \perp \mathcal{S}_{N(t)} | \mathcal{Y}_{N(t)}$. Then

$$\mathbb{P}((\underline{Y} \wedge W)_{t+s} | (\underline{Y} \wedge W)_u, u \leq t) = \mathbb{P}((\underline{Y} \wedge W)_{t+s} | (\underline{Y} \wedge W)_t = \mathcal{Y}_{N(t)}),$$

so the Markov property is satisfied.

The process $\underline{Y} \wedge W$ is a jump process. Let us compute the distribution of the holding time $\tau_t = \inf\{s > 0 : (\underline{Y} \wedge W)_{t+s} \neq (\underline{Y} \wedge W)_t\}$, Again by the memoryless property of the exponential distribution we get

$$\mathbb{P}(\tau_t > s | \mathcal{Y}_{N(t)}) = \mathbb{P}(\pi_{N(t)+1} > s | \mathcal{Y}_{N(t)}) = e^{-\zeta(\mathcal{Y}_{N(t)})s},$$

and so $\tau_t | \mathcal{Y}_{N(t)} \sim \text{Exponential}(\zeta(\mathcal{Y}_{N(t)}))$. Now, from (35) we deduce the conditional independence relation,

$$\tau_t \perp \sigma(C_{N(t)}^*, H_{N(t)+1}) | \mathcal{Y}_{N(t)}, \quad (38)$$

and so, from (27) and (29), we get for all $C(\vec{e}) \in \mathcal{Y}_{N(t)}$, $H \in [C(\vec{e})]$ and $s > 0$:

$$\mathbb{P}(H(\vec{e}) \in dH, C_t^* = C(\vec{e}), \tau_t \in ds | \mathcal{Y}_{N(t)}) = \widehat{\Lambda}_{[C(\vec{e})]}(dH) \Lambda([C(\vec{e})]) e^{-\zeta(\mathcal{Y}_{N(t)})s} ds.$$

We have proven relation (19), so the result follows. \square

9 A new construction of STIT tessellations in a window, point processes

Fix a window W .

Let us consider three independent sequences $(U_n : n \in \mathbb{N}^*)$, $(V_n : n \in \mathbb{N}^*)$, $(G_n : n \in \mathbb{N}^*)$, of independent identically distributed random variables, such that $U_n \sim \text{Uniform}[0, 1)$, $V_n \sim \text{Exponential}(1)$, $G_n \sim \widehat{\Lambda}_{[W]}$.

We will construct the STIT tessellation process $Y \wedge W$ by using these three independent sequences. By using $(G_n : n \in \mathbb{N})$ and regarding the current state of the tessellation process, we will construct a sequence of random hyperplanes $(H_n : n \in \mathbb{N})$ on $[W]$ and all the hyperplanes H_n will be effectively used in constructing the STIT, contrary to the rejection procedure of previous sections where we must wait until a random hyperplane cuts a prescribed cell. The $(U_n : n \in \mathbb{N})$ are used to choose the cell to be divided, out of the set of cells which are intersected by the H_n .

The construction will be done in an iterative way. For $n \in \mathbb{N}$, \mathcal{Y}_n is a random tessellation of $[W]$ and Γ_n is a random measure on $([W], \mathcal{B}([W]))$ defined by

$$\Gamma_n = \sum_{C \in \mathcal{Y}_n} \Lambda_{[C]}.$$

Note that Γ_n is absolutely continuous with respect to $\Lambda_{[W]}$ and its Radon-Nikodym derivate

$$\xi_n(H) = \frac{d\Gamma_n}{d\Lambda_{[W]}}(H) \text{ satisfies } \xi_n(H) = \#\{C \in \mathcal{Y}_n : H \in [C]\}. \quad (39)$$

This follows from the partition:

$$[W] = \bigcup_{j=1}^{\#\mathcal{Y}_n} \widehat{\mathcal{K}}_j \text{ with } \widehat{\mathcal{K}}_j = \{H \in [W] : \#\{C \in \mathcal{Y}_n : H \in [C]\} = j\}. \quad (40)$$

So, the random measures Γ_n can be described by the functions ξ_n , which belong to $L^1(\Lambda)$. (For an explicit form of the corresponding density on the parameter space see further in Section 9.1.) Since this is a metric complete separable space we are in the framework of measurability described in Section 2. By definition,

$$\sigma(\Gamma_n) \subseteq \sigma(\mathcal{Y}_n). \quad (41)$$

Consider the probability distribution $\widehat{\Gamma}_n = (\Gamma_n([W]))^{-1}\Gamma_n = (\zeta(\mathcal{Y}_n))^{-1}\Gamma_n$ on $([W], \mathcal{B}([W]))$. From (5) and from Section 2 we know that there is a bimeasurable function $\Xi_n : [W] \rightarrow [W]$ such that $\widehat{\Lambda}_{[W]} \circ \Xi_n^{-1} = \widehat{\Gamma}_n$.

The algorithm of the construction is:

Step $n = 0$: $\mathcal{Y}_0 = \{W\}$ and $\Gamma_0 = \Lambda_{[W]}$. So $\xi_0 \equiv 1$ and $\Xi_0 = \text{Identity}_{[W]}$.

Step $n + 1$: Assume \mathcal{Y}_n , $n \geq 0$, has been defined. We take

$$H_{n+1} = \Xi_n \circ G_{n+1}. \quad (42)$$

Its conditional distribution satisfies

$$H_{n+1} | \mathcal{Y}_n \sim \widehat{\Gamma}_n \text{ i.e. } \forall \mathcal{K} \in \mathcal{B}[W] : \mathbb{P}(H_{n+1} \in \mathcal{K} | \mathcal{Y}_n) = \widehat{\Gamma}_n(\mathcal{K}) = \frac{\Gamma_n(\mathcal{K})}{\Gamma_n([W])}. \quad (43)$$

The tessellation \mathcal{Y}_{n+1} is formed from \mathcal{Y}_n by the division of the random cell C_n^* of \mathcal{Y}_n , chosen with the help of H_{n+1} and U_{n+1} . All the cells of \mathcal{Y}_n which are hit by H_{n+1} have the same probability $\xi_n(H_{n+1})^{-1}$ to be chosen for division. Formally, divide the unit interval $[0, 1)$ into $\xi_n(H_{n+1})$ intervals of equal length, namely

$$[0, 1) = \bigcup_{\substack{\vec{e} \in E_n \\ H_{n+1} \in [C(\vec{e})]}} [a_{\vec{e}}^n, b_{\vec{e}}^n) \text{ with } b_{\vec{e}}^n - a_{\vec{e}}^n = \xi_n(H_{n+1})^{-1} \quad (44)$$

if $\mathcal{Y}_n = \{C(\vec{e}) : \vec{e} \in E_n\}$ where $E_n = \mathcal{L}(R_n)$ is a set of leaves, for some $R_n \in \Theta_{n+1}$. As in (25), the intervals $[a_{\vec{e}}^n, b_{\vec{e}}^n)$ and $[a_{\vec{e}'}^n, b_{\vec{e}'}^n)$ are consecutive when \vec{e}' is the element following \vec{e} in E_n . Now, when

$$U_{n+1} \in [a_{\vec{e}}^n, b_{\vec{e}}^n) \quad (45)$$

we take $C_n^* = C(\vec{e})$ and divide it by H_{n+1} . Thus \mathcal{Y}_{n+1} is defined as

$$\mathcal{Y}_{n+1} = (\{C \in \mathcal{Y}_n\} \setminus \{C_n^*\}) \cup \{C_n^* \cap H_{n+1}^+, C_n^* \cap H_{n+1}^-\}. \quad (46)$$

and indexed by

$$E_{n+1} = (E_n \setminus \{\vec{e}\}) \cup \text{Succ}(\vec{e}) \subset \mathcal{E}^*.$$

Define the sequence of jump times π_n by

$$\forall n \in \mathbb{N}^* : \pi_n = \zeta(\mathcal{Y}_n)^{-1} V_n,$$

such that their conditional distributions satisfy,

$$\pi_n | \zeta(\mathcal{Y}_n) \sim \text{Exponential}(\zeta(\mathcal{Y}_n)). \quad (47)$$

As $(V_n : n \in \mathbb{N}^*)$ is a sequence of independent random variables, $(\pi_n : n \in \mathbb{N}^*)$ is conditionally independent given $\sigma(\Gamma_n : n \in \mathbb{N})$. Define the sequence of times

$$\mathcal{S}_0 = 0 \text{ and } \mathcal{S}_n = \mathcal{S}_{n-1} + \pi_n = \sum_{j=1}^n \pi_j \text{ for } n \in \mathbb{N}^*.$$

We have $\lim_{n \rightarrow \infty} \mathcal{S}_n = \infty$ \mathbb{P} -a.s. because $(N(t) : t \geq 0)$ defined by

$$N(t) = \sup\{n \in \mathbb{N} : \mathcal{S}_n \leq t\}, \quad (48)$$

is stochastically dominated by a birth chain $(M(t) : t \geq 0)$, with $M(0) = 1$ and linear birth rates $b_n = n \Lambda([W])$.

Define the tessellation process $\widehat{Y} \wedge W = ((\widehat{Y} \wedge W)_t : t \geq 0)$ taking values on $\mathbb{T} \wedge W$, by

$$(\widehat{Y} \wedge W)_t = \mathcal{Y}_n \text{ when } t \in [\mathcal{S}_n, \mathcal{S}_{n+1}), n \in \mathbb{N}, \quad (49)$$

where $(\mathcal{Y}_n : n \in \mathbb{N})$ is the sequence of tessellations defined in the algorithm. Hence, $\widehat{Y} \wedge W$ is well-defined for all times $t \geq 0$ \mathbb{P} -a.s. We also have

$$\forall n \in \mathbb{N} : (\widehat{Y} \wedge W)_{\mathcal{S}_n} = \mathcal{Y}_n,$$

and so $(\mathcal{S}_n : n \in \mathbb{N}^*)$ is the sequence of jump times of $\widehat{Y} \wedge W$.

Theorem 9.1 $\widehat{Y} \wedge W$ is the STIT process associated to $\Lambda_{[W]}$.

Proof: The proof of the Markov property of $\widehat{Y} \wedge W$ is analogous to the one made in Proposition 8.1.

The tessellation \mathcal{Y}_{n+1} is formed from \mathcal{Y}_n by the division of the random cell C_n^* of \mathcal{Y}_n , chosen with the help of H_{n+1} and U_{n+1} , so

$$C_n^* \in \sigma(\mathcal{Y}_n, H_{n+1}, U_{n+1}) \quad (50)$$

As said, to define \mathcal{Y}_{n+1} we need to define the random cell C_n^* which is divided by hyperplane H_{n+1} . Hence, $\mathcal{Y}_{n+1} \in \sigma(\mathcal{Y}_n, U_{n+1}, H_{n+1})$ and by recursion we get

$$\mathcal{Y}_{n+1} \in \sigma(U_k, H_k : k \leq n+1). \quad (51)$$

From (42), (41) and (51) and also by recursion we find,

$$H_{n+1} \in \sigma(\mathcal{Y}_n, G_{n+1}) \subseteq \sigma(U_k : k \leq n; G_k : k \leq n+1). \quad (52)$$

Since $\sigma(U_n, G_n : n \in \mathbb{N}) \perp\!\!\!\perp \sigma(V_n : n \in \mathbb{N})$, we obtain

$$\sigma(\mathcal{Y}_n, U_n, G_n, H_n : n \in \mathbb{N}) \perp\!\!\!\perp (V_n : n \in \mathbb{N}). \quad (53)$$

Note also that (51) and (52) imply that

$$\mathcal{Y}_n \perp\!\!\!\perp G_{n+1} \text{ and } H_{n+1} \perp\!\!\!\perp U_{n+1} \mid \mathcal{Y}_n. \quad (54)$$

From $\pi_{n+1} \mid \mathcal{Y}_n \sim \text{Exponential}(\zeta(\mathcal{Y}_n))$ we get that the distribution of the holding time at t , noted $\widehat{\tau}_t$, satisfies,

$$\widehat{\tau}_t \mid \mathcal{Y}_{N(t)} \sim \text{Exponential}(\zeta(\mathcal{Y}_n)). \quad (55)$$

Now, from (53) and (47) we get the conditional independence relation (35), this is $\pi_{n+1} \perp\!\!\!\perp \sigma(C_n^*, H_{n+1}) \mid \mathcal{Y}_n$. So,

$$\widehat{\tau}_t \perp\!\!\!\perp \sigma(C_{N(t)}^*, H_{N(t)+1}) \mid \mathcal{Y}_{N(t)}. \quad (56)$$

(I) Let us prove that for $\vec{e} \in E_n$, the conditional probability given \mathcal{Y}_n , that $C_n^* = C(\vec{e})$ is the cell divided at time \mathcal{S}_{n+1} satisfies (27), that is we must show that

$$\mathbb{P}(C_n^* = C(\vec{e}) | \mathcal{Y}_n) = \frac{\Lambda([C(\vec{e})])}{\zeta(\mathcal{Y}_n)}. \quad (57)$$

Note that (43) can be written $H_{n+1} | \mathcal{Y}_n \sim (\zeta(\mathcal{Y}_n))^{-1} \Gamma_n$. Hence, by using this relation together with (45) and (44) we get,

$$\begin{aligned} \mathbb{P}(C_n^* = C(\vec{e}) | \mathcal{Y}_n) &= \zeta(\mathcal{Y}_n)^{-1} \int_{[W]} \mathbb{P}(U_{n+1} \in [a_{\vec{e}}^n, b_{\vec{e}}^n] | H_{n+1}) d\Gamma_n(H_{n+1}) \\ &= \zeta(\mathcal{Y}_n)^{-1} \int_{[W]} \xi_n(H_{n+1})^{-1} \mathbf{1}_{\{H_{n+1} \in C(\vec{e})\}} d\Gamma_n(H_{n+1}) \\ &= \zeta(\mathcal{Y}_n)^{-1} \Lambda_{[C(\vec{e})]}([W]) \\ &= \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})]). \end{aligned} \quad (58)$$

Hence (57) follows.

(II) We claim that the distribution of the random hyperplane H_{n+1} , conditional to $C_n^* = C(\vec{e})$ and \mathcal{Y}_n , is $\widehat{\Lambda}_{[C(\vec{e})]}$. To prove it we use that $H_{n+1} = \Xi_n \circ G_{n+1}$, that Ξ_n only depends on Γ_n and so on \mathcal{Y}_n , and that

$$\mathcal{Y}_n \perp G_{n+1}, H_{n+1} \in \sigma(\mathcal{Y}_n, G_{n+1}) \text{ and } H_{n+1} \perp U_{n+1} | \mathcal{Y}_n.$$

These relations allow to get for all $\mathcal{K} \in \mathcal{B}([W])$,

$$\begin{aligned} \mathbb{P}(H_{n+1} \in \mathcal{K}, C_n^* = C(\vec{e}) | \mathcal{Y}_n) &= \int_{\mathcal{K}} \mathbb{P}(dH_{n+1}, C_n^* = C(\vec{e}) | \mathcal{Y}_n) \\ &= \int_{\mathcal{K}} \mathbb{P}(dH_{n+1}, U_{n+1} \in [a_{\vec{e}}^n, b_{\vec{e}}^n] | \mathcal{Y}_n) = \int_{\mathcal{K}} \xi_n(H_{n+1})^{-1} \mathbf{1}_{\{H_{n+1} \in C(\vec{e})\}} \mathbb{P}(dH_{n+1}) \\ &= \int_{C(\vec{e}) \cap \mathcal{K}} \zeta(\mathcal{Y}_n)^{-1} d\Gamma_n(H) = \zeta(\mathcal{Y}_n)^{-1} \Lambda_{[C(\vec{e})]}(\mathcal{K}) \\ &= \zeta(\mathcal{Y}_n)^{-1} \Lambda([C(\vec{e})] \cap \mathcal{K}). \end{aligned} \quad (59)$$

From (57) we get desired distribution:

$$\mathbb{P}(H_{n+1} \in \mathcal{K} | C_n^* = C(\vec{e}), \mathcal{Y}_n) = (\Lambda([C(\vec{e})])^{-1} \Lambda([C(\vec{e})] \cap \mathcal{K}), \quad (60)$$

Then, from (55), (60) and (57), we get that for all $C(\vec{e}) \in \mathcal{Y}_{N(t)}$, $H \in [C(\vec{e})]$ and $s > 0$ it is satisfied

$$\mathbb{P}(H(\vec{e}) \in dH, C_t^* = C(\vec{e}), \widehat{\tau}_t \in ds | \mathcal{Y}_{N(t)}) = \widehat{\Lambda}(dH) \Lambda([C(\vec{e})]) e^{-\zeta(\mathcal{Y}_{N(t)}) s} ds.$$

We have shown (19) and so $\widehat{Y} \wedge W$ given by (49) is a STIT tessellation associated to $\Lambda_{[W]}$. \square

With respect to relation (44): we have first selected an hyperplane with a probability measure $\widehat{\Gamma}_n$ (so depending on the tessellation), and the equiprobability relation (44) for the cells which are intersected by the hyperplane, is

nothing but an explicit computation of $(d\Lambda_{[C(e)]}/d\Gamma_n)(H)$. The equiprobability relation appearing in [9] page 9, is in a different context and the hyperplane is chosen with probability measure Λ .

Example. As an illustrative example let us see what happens in the case $n = 2$. The hyperplane H_1 divides C in two cells $C(+)$ and $C(-)$, and so $\Gamma_1 = \Lambda_{[C(+)]} + \Lambda_{[C(-)]}$.

If $\{H_2\} \cap [C(+)] \cap [C(-)] = \emptyset$, then $C_1^* = C(e)$ when $H_2 \in [C(e)]$.

If $\{H_2\} \cap [C(+)] \cap [C(-)] \neq \emptyset$ then $\xi_1(H) = \frac{1}{2}$. Following the order on E_2 the decision is: if $U_1 \in [0, 1/2)$ then $C_1^* = C(-)$ and if $U_1 \in [1/2, 1)$ then $C_1^* = C(+)$. We note that $[C(+)] \cap [C(-)] = [C(+)] \cap C(-)$, and so $H_2 \cap [C(+)] \cap [C(-)] \neq \emptyset$ is equivalent to $H_2 \cap [C(+)] \cap C(-) \neq \emptyset$.

9.1 Simulation of random hyperplanes

Because it is not obvious from (42) how to generate H_{n+1} , we provide here a description of its density (39) which may be used in a simulation. Denote by $\pi_u C$ the orthogonal projection of C onto the one-dimensional linear subspace (of \mathbb{R}^ℓ) spanned by $u \in \mathbb{S}_+^{\ell-1}$, and by $\lambda(\pi_u C)$ the length of this projection, which is also called the width or breadth of C in direction u . If the image of Λ on the parameter space is given by (4), then the density $\tilde{\xi}_n$ of the parametric representation of H_{n+1} is given by

$$\tilde{\xi}_n(\alpha, u) \lambda(d\alpha) \theta(du) = \frac{\sum_{C \in \mathcal{Y}_n} 1_{\pi_u C}(\alpha)}{\sum_{C \in \mathcal{Y}_n} \lambda(\pi_u C)} \lambda(d\alpha) \frac{\sum_{C \in \mathcal{Y}_n} \lambda(\pi_u C)}{\sum_{C \in \mathcal{Y}_n} \lambda(\pi_{u'} C)} \theta(du'). \quad (61)$$

Of course, the sum $\sum_{C \in \mathcal{Y}_n} \lambda(\pi_u C)$ can be canceled out, but the given form shows better the decomposition of the joint density of the two parameters into a probability density w.r.t. θ for the direction $u \in \mathbb{S}_+^{\ell-1}$ and a conditional probability density, given u , for $\alpha \in \mathbb{R}$, w.r.t. the Lebesgue measure. Note that $\sum_{C \in \mathcal{Y}_n} 1_{\pi_u C}(\alpha) = \xi_n(H(\alpha, u)) = \#\{C \in \mathcal{Y}_n : H(\alpha, u) \in [C]\}$ (see (39)), and hence the conditional density of α is a step function.

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