UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA INDUSTRIAL

DYNAMIC EQUILIBRIUM IN MULTIPLE MARKETS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL INDUSTRIAL

ÍTALO TOMÁS RIARTE CAMPILLAY

PROFESOR GUÍA:
ALEJANDRO BERNALES SILVA

MIEMBROS DE LA COMISIÓN:
ALEJANDRO BERNALES SILVA
MARCELA VALENZUELA BRAVO
PATRICIO VALENZUELA AROS

SANTIAGO DE CHILE
2016

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE INGENIERO CIVIL INDUSTRIAL
POR: ÍTALO TOMÁS RIARTE CAMPILLAY
FECHA: ENERO 2016
PROF. GUÍA: SR. ALEJANDRO BERNALES SILVA

DYNAMIC EQUILIBRIUM IN MULTIPLE MARKETS

Se presenta un modelo dinámico de múltiples mercados financieros, organizados como *limit order markets*, en el cual agentes endógenamente toman decisiones óptimas para maximizar el valor esperado de sus ganacias. Los agentes toman sus decisiones considerando incentivos propios, condiciones de mercado, potenciales decisiones de *trading* futuras y diferentes estrategias adoptadas por otros agentes.

Para efectos de la presente investigación, se compara el escenario de un único mercado financiero ('single market') con un escenario de dos mercados interconectados que compiten por el flujo de órdenes ('multi markets'). Los resultados indican que la posibilidad de transar en múltiples mercados, beneficia ampliamente a agentes sin valoración privada por el activo, ya que buscan oportunidades de transar en ambos mercados, mientras que perjudica el bienestar de agentes con motivación intrínseca para transar, dado que obtienen peores condiciones de negociación. Por otro lado, se observa una reducción en varias medidas de liquidez en multi markets, lo que sugiere la existencia de externalidades positivas asociadas a mercados consolidados.

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE INGENIERO CIVIL INDUSTRIAL
POR: ÍTALO TOMÁS RIARTE CAMPILLAY
FECHA: ENERO 2016
PROF. GUÍA: SR. ALEJANDRO BERNALES SILVA

DYNAMIC EQUILIBRIUM IN MULTIPLE MARKETS

We develop a dynamic model of multiple limit order markets, in which traders make endogenously optimal decisions to maximize their expected payoffs, taking into account intrinsic incentives, markets conditions, potential future trading decision and different strategies adopted by other agents.

We study two main scenarios. First, a scenario with a single exchange ('single market') and then, a scenario with multiple markets interconnected that compete for the order flow ('multi markets'). Our results indicate that possibility to trade in multiple markets, widely benefits agents without private valuation for the asset, because they are looking for trading opportunities in both markets. Conversely, agents with intrinsic motives to trade obtain worse terms of trade, which involves a decline in their welfare. Besides, we observe a reduction in several liquidity measures in multi markets, which suggests positive externalities of consolidated markets.

A todas las personas importantes en mi vida y que fueron parte de este largo proceso.

Acknowledgments

Me gustaría agradecer por el apoyo financiero para la realización de esta memoria provenientes del proyecto Fondecyt # 11140628 y el Instituto Milenio para la Investigación en Imperfecciones de Mercado y Políticas Públicas ICM IS130002.

Table of Contents

1. Introduction **1**

2. Model **6**

2.1. General model description . . . 6
2.2. The traders' dynamic maximization problem . . . 8
2.3. Numerical parameterization of the trading game . . . 9

3. Results **11**

3.1. Welfare . . . 11
3.2. Market Quality . . . 14
3.3. Trading Behavior . . . 17
3.4. Market Dynamics . . . 22
3.5. Non-Symmetric Case . . . 24

4. Conclusion **26**

Bibliography **29**

Appendix **30**

Index of Tables

1. Welfare per trader type 12
2. Average payoff, waiting cost and money transfer per trader type 14
3. Market quality measures 15
4. Trading behavior measures 18
5. Strategies per trader type 20
6. Traders' behavior differentiated by private value 21
7. Average proportion of traders who switch from one market to another 22
8. Average proportion of agents who trade in market A or B 24
9. Market quality measures 25

1. Introduction

In the last decade, traders have experienced a growth in the opportunities to trade simultaneously in multiple exchanges, which has triggered changes in financial regulation in USA (the Regulation National Market System, RegNMS) and Europe (the Market in Financial Instruments Directive, MiFID), encouraging the proliferation of new trading venues. Indeed, trading activity in USA is dispersed across 11 exchanges and approximately 44 alternative trading systems[1]. On the other hand, several electronic platforms such as BATS Chi-X and Turquoise have spurred competition in Europe. These facts open the debate about the costs and benefits of market fragmentation.

In spite of this financial trend, in recent literature there have been limited attempts to model agents' behavior dynamically in multiple markets. The main reason is due to the difficulties of characterizing this dynamic decision mechanism, including the market microstructure features, which endogenously affects agents' behaviour. Thus, previous works present some restrictive assumptions in the market microstructure setups to make models analytically tractable. Our goal is to fill this lack in the literature by presenting a dynamic and continuous time model of multiple limit order markets, which incorporates most relevant microstructure characteristics of financial exchanges.[2]

Our framework allow us to understand trading behavior of market participants in several scenarios and analyse the impact of this financial tendency on market quality and global welfare. The knowledge of potential advantages and disadvantages of multiple markets has tremendous relevance for global economy, since this is one of the most important financial trends at the moment and it is not entirely understood.

In our dynamic model, agents make endogenously sequential optimal decisions in continuous time, to maximize their expected payoffs, taking into account markets conditions, potential future trading decisions, and different strategies adopted by other agents. We model two limits order markets with a single and common financial asset. This asset has a fundamental value v_t that follows a compound Poisson process with fixed jump size and can be interpreted as the present value of the future cash flows. As in real limit order markets, limit orders books are characterized by a set of discretized prices at which traders can submit orders that respect time and price priority for execution of limit orders. Risk neutral agents

[1]See May 11, 2015 Public Statement 'U.S. Equity Market Structure: Making Our Markets Work Better for Investors' U.S. Securities and Exchange Commission (SEC).

[2]For better understanding of trading mechanisms in financial markets see Hasbrouck (2007) and Parlour and Seppi (2008).

arrive randomly and decide whether to put (or not) an order (limit or market order) to buy or sell one share of the asset in one of two markets. A trader may revisit the markets multiple times to revise or modify previous trading decisions. However, due to cognition limits, agents cannot immediately modify their previous orders after a change in the market conditions (i.e. they cannot monitor the market all the time). Therefore, a trader re-enters the market multiple times, according to a Poisson process until his order executes.

In order to consider heterogeneity across traders, each one has a private valuation for the asset[3], denoted by α drawn from a distribution F_α. For instance, traders with $\alpha > 0$ have a positive valuation for the asset and therefore they are more likely to be buyers. Analogously, traders with $\alpha < 0$ are more likely to be sellers. Agents with $\alpha = 0$ do not have private valuation for the asset and therefore they are willing to be buyers or sellers depending on trading possibilities.

Since the model incorporates so many realistic features of limit order markets, the equilibrium does not admit an analytic and closed form expression. For that reason, we obtain the equilibrium numerically using the algorithm introduced by Pakes and McGuire (2001), originally proposed for industrial organization problems with sequential decisions. This algorithm provides a Markov-perfect equilibrium which has been successfully implanted into dynamics models for limit order markets by Goettler et al. (2009), Goettler et al. (2005) and Bernales and Daoud (2014), although without considering the possibility to trade in multi markets.

As first approach to study multi markets trends, we test two main scenarios. First, we study a scenario in which there is a consolidated, single limit order market (henceforth, single market), similar to Goettler et al. (2009) and second, we test a scenario, in which there are two limit order markets with the same financial asset to trade and the same market structures (henceforth, multi markets). Thereby, we can measure in terms of welfare the impact of multiple market and determine which traders' type increase their welfare when they can trade in both venues. Furthermore, we can study market quality considering several liquidity measures, such as quoted spread, effective spread, depth at and away the quotes, and the effect of microstructure noise in both scenarios. Finally, we can understand traders' behavior considering main strategies they select, given their own characteristic and incentives.

We find a slightly, but statistically significant global welfare reduction as result of new trading dynamics of fragmentation. This can be explained, first, because there exist a transfer of profits among trader types. For instance, agents with no intrinsic motives to trade (i.e., $\alpha = 0$) take advantages of opportunities in both exchanges, effecting 'queue-jumping' strategies between markets, which involves higher profits. Conversely, agents with positive (or negative) valuation for the asset, are willing to pay the cost for a quicker trade, although this entails worse benefit from trade. Furthermore, the composition of profits change in comparison with single market. Agents with extreme valuation for the asset have lower waiting cost and worse money transfer, which are not compensated with a sufficient increase in money transfer for agents with zero private value, who make profits through liquidity provision in both markets.

In terms of market quality, we obtain different results if we examine the effect of frag-

[3]Private value represents idiosyncratic motives for trade (such as wealth shocks, hedging needs, tax exposures, differences in investment horizons, among others.

mentation in each market separately (local markets) or in aggregated terms. We observe that multi market trends entail wider spreads and lower market depth in each local venue interconnected when comparing the same measures with single market. These two facts show a harmful effect in market quality for each exchange. Wider spreads are consistent with the intuition that in consolidated markets, liquidity providers compete more aggressively, thus more efficient prices are achieved,[4] and liquidity is absorbed in a single venue, which explain higher depth in single market. Despite previous measures of each venue, for multi markets scenario we can consider an aggregated measure: the consolidated market depth (i.e., the aggregate depth in both markets). We observe that consolidated market depth is larger than market depth in single market scenario. This aggregated improvement represents a positive externality of multi market although not an overall liquidity enhancement, because at the same time, market quality measures throughout each trading venue show negative effects.

We also find that fragmentation does not impact on the strategies adopted by traders in terms of who act as liquidity supplier and who as liquidity demander, but it does change the behavior of agents with zero private value since they exploit the opportunities of multiple markets obtaining better profits than in single market scenarios. Contrarily, agents with higher valuation for the asset, rarely access to both markets due to their have high waiting costs, which involves worse profits essentially as result of worse market quality measures of local markets.

As second approach to market fragmentation, we study a multi market scenario considering two interconnected markets with different market structures (i.e. non-symmetric markets) and their respective versions of single market scenarios in order to exploit flexibility of our model. The first market consider a transaction cost $c_t > 0$, while the second assume that cost as zero. Besides, both markets have different latency restrictions, which means that in the first market, agents can return to modify their order with different speed than the second one. We find that most order flow concentrates in most liquid exchange, which is intuitive since the one of the two venues do not consider any transaction costs and it is less restricted in terms of latency. We also find that market quality implications depend on whether we are interested: aggregated or local measures. While local measures show a detrimental effect on market liquidity, there are aggregated measures as consolidated depth that show liquidity improvements.

Our study is related to the microstructure literature and specifically with the effect of market fragmentation in comparison with market consolidation. Theoretical work of Mendelson (1987) shows that fragmentation reduces the expected quantity traded, reduces gains from trade, contrary to consolidation of order flow, that can reduce spreads and thus improve liquidity. Pagano (1989) considers a two-period stock market economy and shows that in absence of differential transaction costs, all order flow concentrates in a single market (the more liquid one), unless the two markets are identical. Chowdhry and Nanda (1991) develop a model with a single asset that is traded at multiple locations and show that a fragmented market can exist as equilibrium. Biais (1993) shows that centralized limit order markets and fragmented dealer markets coexist in presence of risk-averse market participants. Madhavan (1995) based on a dealer market model of Glosten and Milgrom (1985), shows that heterogeneity of traders allow the existence of fragmented markets which do not necessary gravitate

[4]By 'efficient prices' we mean prices less deviated from the fundamental value of the asset.

to a single market and also shows that fragmentation results in violations of price efficiency.

More recent work of Hendershott and Mendelson (2000) study competition between dealer markets and crossing networks (where agents trade directly with one another). They find that crossing networks exhibit both positive and negative externalities. Parlour and Seppi (2003) present a microstructure model of competition for order flow between a pure limit order market and a hybrid market and conclude that coexistence is possible and competition between exchanges can increase as well as reduce liquidity. Foucault and Menkveld (2008) extend the framework of Parlour and Seppi (2003) to study competition between two pure limit order markets and shows that positive global externalities can be obtained from fragmentation, although this effect could differ throughout each trading venue. They also test their model empirically studying the entry of the London Stock Exchange into the Dutch equity market with the launch of EuroSETS (an electronic limit order market).

There are several empirical studies that compare market quality before and after entry or consolidation of markets. The evidence on market fragmentation has mixed conclusions. For example, related to reduction in fragmentation, Arnold et al. (1999) find that merging exchanges attract order flow and experience a decrease in bid-ask spreads. Amihud et al. (2003) study the reduction in fragmentation of Tel Aviv Stock Exchange and find that stock liquidity improves. Gajewski and Gresse (2007) compare the fragmented hybrid order-driven segment of the London Stock Exchange with the centralised electronic order book of Euronext, and find that spreads in the centralized order book are lower. On the other hand, Boehmer and Boehmer (2004) examine the impact of NYSE entry in the market for exchange traded funds (ETFs) and find improvement in market quality in the entire market and also in individual market centres. Hengelbrock and Theissen (2009) examine the market entry of Turquoise and reveal results somewhat ambiguous but pointing in the direction of positive impacts on market quality. O'Hara and Ye (2011) study how fragmentation is affecting market quality in US equity markets and find that fragmentation does not harm liquidity. However, they relate a specific feature of US equity market and state 'our results support the conclusion that while US equity markets are spatially fragmented, they are, in fact, virtually consolidated into single market with many points of entry' (O'Hara and Ye, 2011, p. 460-461), which explains why positive externalities that stems from consolidated trading are observed in US equity markets.

Our framework is a natural continuation of the model developed by Goettler et al. (2009), Goettler et al. (2005) that consider a single limit order market, where the equilibrium is found numerically with the algorithm introduced by Pakes and McGuire (2001). We use an extension of their framework to derive a dynamic model with multiple limit order markets interconnected, which allow us to present a more developed model that can be used to respond all questions raised previously. To our knowledge, this the first dynamic model of multi-limit order markets that considers so many features of real markets, such as: full record and evolution of multiple limit order books in continuous time; different market configuration among markets in terms of ticksize, transparency, cost and re-entry restrictions; heterogeneity across traders in terms of differences in their private valuation for the asset, or differences in their skills given asymmetric information regimes; a volatile asset that reflect changes in economies; the re-entries and endogenous adjustments to previous orders submitted; the endogenous decision, that consists in selecting the optimal action among all feasible decisions across all trading venues, and so many other features that model real frictions in limit order markets.

This paper is organized as follows. Section 2 presents the model and the numerical parameterization of the trading game. We present main results and data output in Section 3. Section 4 concludes. Details of the numerical algorithm of our model appear in Appendix.

2. Model

We develop a dynamic continuous-time model of two pure limit order markets[5], with a single financial asset that can be traded in both venues. The model is an asynchronous dynamic trading game in which there are risk-neutral agents, who arrive at the market randomly. Traders have one share to trade and can submit limit orders and market orders. They also can revise and modify their unexecuted limit orders multiple times before execution. In this section, we provide a complete description of the model.

2.1. General model description

Limit order books: Each market $m \in \{1, 2\}$ has a respective order book. As in real limit order markets, the limit order book relative to the market m at time t, $L_{m,t}$ is described by a discrete set of feasible prices, denoted as $\{p^i_m\}^{\infty}_{i=-\infty}$, where the tick size, d_m is the distance between any two consecutive prices. There are backlogs of outstanding orders to buy or to sell in the market m, $l^i_{m,t}$, at prices p^i_m. A positive (negative) number in $l^i_{m,t}$ denotes buy (sell) orders, and it represents the depth at price p^i_m. Therefore, for the book $L_{m,t}$, the bid price is $B(L_{m,t}) = \sup\{p^i_j | l^i_{m,t} > 0\}$, while the ask price is $A(L_{m,t}) = \inf\{p^i_j | l^i_{m,t} < 0\}$, and if the order book is empty on the buy side or on the sell side, $B(L_{m,t}) = -\infty$ or $A(L_{m,t}) = \infty$.[6]

Each limit order book respects the price and time priorities for the execution of limit orders. Buy (sell) orders at higher (lower) prices are executed first, and limit orders submitted earlier have priority in the queue when they have the same price. In addition, when a trader submits an order, the order price identifies whether the order is a market order or a limit order. This means that an order to buy (sell) at a price above (below) the ask (bid) price is executed immediately at the ask (bid) price; and thus this order is a market order.

Asset: The financial asset has a fundamental value v_t at the time t, and follows a Poisson process at rate λ_v. Whenever a change occurs, the fundamental value rises or reduces its value in a fixed amount f, each with the same probability. Differently to Goettler et al. (2009), when a change occurs we do not assume that the fundamental value coincides with one feasible price to trade, therefore the fixed amount f does not depend on the tick sizes

[5] Note that single market is a particular configuration of multi markets, when all trading is restricted to the first market.

[6] We use a similar notation to Goettler et al. (2009) regarding the microstructure features of the model for the dynamic order book market.

d_m, $m \in \{1,2\}$. This is important since each market could have different tick sizes (and so different price grids to trade) but v_t can take any value according to the Poisson process described above.

Traders: They are risk-neutral agents, who arrive at the market randomly following a Poisson process with intensity λ. Each agent can trade one share[7] and has to select the best action given the current state of economy. To this end a trader has to make four main trading decisions after arriving at the market:

i) To submit an order or to wait until the market conditions change.

ii) To trade in $L_{1,t}$ or $L_{2,t}$.

iii) To buy or to sell the asset.

iv) To choose the price at which he will submit the order, which implies the decision to submit a market order or a limit order, depending on whether the price is above or below the quotes

Despite the fact that traders arrive following a Poisson process with intensity λ, the submission rate is different as agents can decide to submit or not an order, which depends endogenously on the market conditions at the time t.

Traders can re-enter the market and monitor their previous unexecuted limit orders. However, agents can not immediately modify their previous limit orders after a change in the market conditions, mainly due to cognition limits.[8] Traders re-enter at the market according to a Poisson processes at rate λ_r and have to make additional trading decisions after re-entering the market:

i) Whether to cancel an unexecuted limit order or retain the order without changes.

ii) If he decides to cancel the order, whether to submit a new order for the asset or wait for different market conditions in the future.

iii) If he decides to submit a new order after a cancellation, he has to choose to which market send the new order, the type of order (buy or sell) and its price.

Thus, agents have to take the possibility of re-entry into account in the utility maximization problem.

Once a trader submits a limit order, he remains part of the trading game by revising her order until it is executed; however, after execution the trader exits the market forever. Consequently, there are a random number of active market participants at each instant who are monitoring their previous limit orders.

Traders have to pay a cancellation fee c_m when they cancel an unexecuted submitted limit order in $L_{m,t}$, $m \in \{1,2\}$. In the case of a re-entry, a trader can leave the order without changes, which has the benefit of keeping her priority time in the respective queue and

[7] We can include additional shares per agent in the trading decision. However, similarly to Goettler et al. (2009), we assume one share per trader to make the model computationally tractable.

[8] Rosu (2009) proposes a dynamic model of a limit order book in which traders can modify instantaneously their previous orders.

avoiding a cancellation fee.[9] The negative side of leaving an order in the book is that the asset value could move in directions that affect future payoffs. For instance, in the scenario of a growth in the asset value, some limit sell orders could be priced too low, and a quick trader could make profits from the difference. This possibility represents an implicit transaction cost of being 'picked off' when the fundamental value change unexpectedly after limit orders have been submitted. Conversely, when the asset value decreases, a sell limit order has the risk of not resulting in a trade since could be priced too high. To take into account the risk that a limit order may not result in a trade, we include a cost of 'delaying' by a discount rate ρ, which is common for both markets and reflects the cost of not executing immediately. This 'delaying' cost does not represent the time value of money; instead ρ reflects opportunity costs and the cost of monitoring the market until a limit order is executed. Thus, the payoffs of order executions are discounted back to the order submission time at rate ρ.

Heterogeneity: Each trader has a private value α for the asset, which is drawn from a discrete distribution F_α and known before making any trading decision. The private value can be interpreted as a personal valuation of the asset and it is constant to each agent. This private value gives additional heterogeneity to the different agents in the dynamic trading game. For instance, traders with zero private value (and hence with no intrinsic benefits to trade) are indifferent in taking either side of the market and hence maximize their benefits depending on the available trading possibilities; consequently they are likely liquidity suppliers since they will probably submit limit orders.[10] Conversely, traders with higher absolute values in their intrinsic benefits to trade are likely to be liquidity demanders and will probably submit market orders.

2.2. The traders' dynamic maximization problem

Suppose that a trader arrives at the trading game and observes state s of the market. Let the entry time be equal to t. The state s that a given trader observes includes:

i) A set of variables of limit order books $L_{1,t}$ and $L_{2,t}$ that result from previous trading activity in each market, such as quotes, depths, last transaction, depending on the transparency of each book;
ii) The current fundamental value of the asset v_t;
iii) His private value α and delay rate ρ
iv) The status of her previous action in the case that the trader has already submitted an earlier limit order to any of the markets, which includes the original submission price, the book where the order was submitted, the current priority in the book, and if the order was a buy or a sell.

Define $\Gamma(s)$ as the set of possible of actions that a trader can take given the state s (e.g. to send an order to the limit order book $L_{1,t}$ or $L_{2,t}$, to wait until market conditions

[9] It is important to point out that the order priority could have changed, depending on the shape of the book, which should be taken into account in the decision to cancel and re-submit.

[10] Traders with zero private value are equivalent to liquidity suppliers as in Jovanovic and Menkveld (2011).

change, to cancel and submit a new order, among others). Formally, we define an action as $\tilde{a} = (\tilde{b}, \tilde{p}, \tilde{x}, \tilde{q})$ where $\tilde{b}$ is a variable representing the optimal limit order book chosen, $\tilde{p}$ is the price of the order, $\tilde{x}$ is $+1$ or -1 when the order is a buy or sell and 0 if there is no order, and $\tilde{q} \geq 0$ the priority associated to this order, which is determined by $\tilde{p}, \tilde{x}$ and $L_{\tilde{b},t}$. Let $\eta(t|\tilde{a}, s)$ be the probability that an order is executed at time t given that the trader takes the action $\tilde{a} \in \Gamma(s)$ when he faces the state s. It is important to notice that $\eta(\cdot)$ incorporates all possible future states and strategic actions adopted by other traders until t. If the decision $\tilde{a}$ is the submission of a market order, $\eta(0|\tilde{a}, s) = 1$, while $\eta(h|\tilde{a}, s)$ converges asymptotically to zero when the trader decides to submit a limit order with a price far away from the fundamental value (not an aggressive order). In addition, let $\gamma(v|t, s)$ be the density function of v at time t given the state s. Therefore, the expected value of an order that is executed prior to a re-entry at time t_r is:

$$\pi(t_r, \tilde{a}, s) = \int_0^{t_r} \int_{-\infty}^{\infty} e^{-\rho t} \left((\alpha + v - \tilde{p})\tilde{x}\right) \cdot \eta(t|\tilde{a}, s) \cdot \gamma(v|t, s) dv dt \tag{1}$$

Here, $(\alpha + v - \tilde{p})\tilde{x}$ is the instantaneous payoff of the order where $\tilde{p}$ is the submission price which is part of the decision $\tilde{a}$; while $\tilde{x}$ is also a component of the decision $\tilde{a}$ and reflects whether the trader decides to submit a buy order ($\tilde{x} = 1$), to submit a sell order ($\tilde{x} = -1$) or to submit no order ($\tilde{x} = 0$). This payoff is transformed to a present value at the rate ρ which is the cost of 'delaying' previously defined in this section.

Let $R(\cdot)$ be the probability distribution of the re-entry time which is exogenous and follows an exponential distribution at rate λ_r. In addition, let $\psi(s_{t_r}|\tilde{a}, s, t_r)$ be the probability that the state s_{t_r} takes place at time t_r given the previous state s and the action $\tilde{a}$, which also includes all potential states and strategic decisions followed by other traders until t_r. Therefore, the value to an agent of arriving at the state s, $V(s)$, is given by the Bellman equation of the trader's optimization problem:

$$V(s) = \max_{\tilde{a} \in \Gamma(s)} \int_0^{\infty} \left[\pi(t_r, \tilde{a}, s) + e^{-\rho t_r} \int_{s_{t_r} \in \mathcal{S}} (V(s_{t_r}) - \tilde{z}_{s_{t_r}} c_m) \cdot \psi(s_{t_r}|\tilde{a}, s, t_r) ds_{t_r} \right] dR(t_r) \tag{2}$$

where $\tilde{z}_{s_{t_r}} = 1$ if the optimal decision in the state s_{t_r} is a cancellation and $\tilde{z}_{s_{t_r}} = 0$ in any other case, $m \in \{1, 2\}$ is an indicator for the market selected previously (i.e., $L_{1,t}$ or $L_{2,t}$, respectively), and $\mathcal{S}$ is the set of possible states in re-entries. The first term is defined in equation (1); while the second term reflects the subsequent payoff in the case of re-entries.

2.3. Numerical parameterization of the trading game

For our numeric simulations, we set parameters taking into account the existing empirical literature. For instance, in single market scenarios, we follow parameters proposed by Goettler et al. (2009), while in multi markets, we adjust some parameters to make both scenarios properly comparable.

Single Market Parameterization

- We allow agents to trade only in the book $L_{1,t}$.

- We normalize the rate of the Poisson process for new trader arrivals, λ, to one. This means that a unit of time in our simulations represent the average time between new trader arrivals. On average, a trader reenters the market after four units of time, thus the rate of the Poisson process for reentries is, $\lambda_r = \frac{1}{4}$. This reentry rate is assumed constant across all traders.
- A tick in the simulation corresponds to one-eighth of a dollar.
- We assume that the distribution of private value F_α is discrete with support $\{-8, -4, 0, 4, 8\}$ and cumulative distribution $\{0.15, 0.35, 0.65, 0.85, 1.0\}$. This distribution is based in finding of Hollifield et al. (2006) who estimate the distributions of private values for three stocks on the Vancouver exchange.
- Innovations in the fundamental value of the asset, v, occur according to a Poisson process at rate $\lambda_v = \frac{1}{8}$, which means that the fundamental value jumps, on average every eight units of model time. Whenever v changes, it increases or decreases in a fixed amount $f = 1$, each with the same probability.
- We assume $\rho = 0.05$. We experimented with lower values and found results qualitatively similar.
- When a trader submits a limit order at time t, the price is restricted in a range $[v_t - k, v_t + k] \cap \mathcal{P}$, where $\mathcal{P}$ is the set of feasible prices. This assumption is made for computational tractability, since we need finite set of prices. The amount k is chosen large enough that it does not affect equilibrium. For the simulations we choose $k = 6$.[11]
- The market is transparent, since each trader can observe quotes, respective depth at quotes, depth at both sides of the markets and the last transaction, which includes the price, direction and optimal book for that order.
- We set the cancellation fee to zero. This allow us to explore the trading process with less frictions.

Multi Markets Parameterization

- We allow agents to trade in the books $L_{1,t}$ and $L_{2,t}$
- We set the arrival rate $\lambda = 2$ to inject liquidity with the same intensity to each venue in comparison with single market.
- In our model, each market has its own characteristics, such as, cancellation fee, transparency, ticksize, among others. For our case study we set all market parameters identical to each other, making it comparable with single market scenario.

Since our model implementation is flexible enough, we can test both, single and multi markets scenarios.[12] Certainly, multi markets scenarios involve higher levels of complexity in comparison with single market. On the one hand, the state space is extremely increased if consider two markets instead of one. On the other hand, we need to track the evolution of two limit order books in continuous time, which represents an additional difficulty for that scenario. We provide more details about the model implementation for two markets in Appendix.

[11]We experimented with higher values of k and noticed that traders rarely send orders so far from the fundamental value.

[12]Indeed, many scenarios can be tested, for instance, we can consider different market features between the two markets or different traders characteristics.

3. Results

This section presents results derived by the model and the numerical parameterization presented in previous section. We simulate a scenario with single limit order market (henceforth, single market), and a scenario with two limit order markets (henceforth, multi markets). We organize main results in three sections. Section 3.1 considers global welfare for both scenarios and the change on gains for each trader type. Section 3.2 shows liquidity variables to examine the impact of multi markets on market quality. Section 3.3 presents the trading behaviour of market participants which allows us to understand integrally all previous results. In Section 3.4, we study interaction between markets in order to understand the coexistence of multiple exchanges in equilibrium. Finally, in Section 3.5 we present a new scenario with two interconnected market with different characteristics and briefly discuss about primordial findings.

3.1. Welfare

Do the connection and competition of multiple markets improve global welfare? Which traders' type take advantages of multi markets dynamics? We examine the effect on welfare for both single and multi markets scenarios to answer the previous questions. Consider a trader with a private value α and delaying discount rate ρ who arrives to the market at time t and executes at time t' as consequence of a market order submission or the execution of his previous limit order. Then the gross payoff or profit[13] for the trader is calculated as:

$$\Pi = \tilde{x} \cdot (\alpha + v_{t'} - \tilde{p}) \cdot e^{-\rho(t'-t)} \tag{3}$$

where $\tilde{x} = 1$ if the order was a buy and $\tilde{x} = -1$ if the order was a sell, $v_{t'}$ is the fundamental value of the asset at the time of execution t' and $\tilde{p}$ is the price of the transaction. We determine the global welfare in a given scenario as mean of Π, for all traders who execute.

Observation 1

(i) Global welfare of the economy slightly decreases when there are two limits order markets, instead of one.

(ii) Agents with no intrinsic motives to trade take advantages in multi markets scenarios and consequently increase their profits.

[13]We already discussed in previous section that $\tilde{x} \cdot (\alpha + v_{t'} - \tilde{p}.)$ is the instantaneous payoff, which can be discounted back to obtain the gross payoff

Table 1 reports the global average payoff in both scenarios and shows a slightly reduction on global welfare from 3.742 ticks (single market) to 3.721 (multi markets), hence welfare decreases in 0.021 ticks. Table 1 also displays the standard errors for the mean payoffs for each trader type and mean of payoffs across all trader types. Since we have a large number of new arrivals (20 million), the standard errors on the sample means are less than 0.00065 for single market and 0.00057 for multi markets, which are sufficient low such that all differences in means on the order of 10^{-2} or higher are significantly different from zero. Hence, in what follow, we no longer report the standard error.

Table 1: Welfare per trader differentiated by private value: This table shows the welfare for each trader type calculated as the average payoff in ticks. Since the model is symmetric, we combine positive and negative values of α with the same absolute value. Each row of the table is a different scenario. The first row reports single market scenario. In the second we report multi markets scenario. The average payoffs are determined as mean of payoffs over 20 million market new entries in equilibrium following equation (3). Standard errors of average payoffs are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Scenario	Private Value $\|\alpha\|$			Total
	0	4	8	
Single Market	0.575	3.497	7.236	3.742
Multi Markets	0.738	3.511	6.984	3.721

The reduction on global welfare is explained mainly for the transfer of gains among trader types, in each scenario. Agents with no intrinsic motives for trade (i.e. $\alpha = 0$) are the major beneficiaries of multi markets dynamics since they make profits directly from trading activity, exploiting their opportunities and sending orders to both markets depending on markets conditions. Further we will show that traders with $\alpha = 0$ are more likely to be liquidity suppliers. Therefore, in single market scenarios they have to place more aggressive orders[14] to 'jump the queue' (the outstanding limits orders) and raise their probability of execution, otherwise they have to wait until execution of all orders according to price and time rules for limit order markets. This fact generates competition on price due to the consolidation of all order flow in a single trading venue.

Contrarily, in presence of multiple markets, traders not necessarily have to put an aggressive order for raising their probability of execution, since they can choose the market to trade, 'queue-jumping' allows traders to jump ahead of the outstanding limit orders in one market by submitting an order to other market, so the effect competition on price is reduced and consequently agents with $\alpha = 0$ obtain better terms of trade.[15] On the other hand, agents with higher valuation for the asset act as liquidity demanders, and pay the cost associated to higher profit for agents with $\alpha = 0$ in multi markets scenarios. Agents with nonzero private value want to trade quickly and most times they do not access to all trading venues because it

[14]By aggressive order we mean to an order with a submission price closer to the respective quote.
[15]The absence of time priority across markets is key for 'queue-jumping'.

is costly. Later we show that fragmentation has a detrimental effect on local liquidity, hence, agents with nonzero private value face a less liquid market in comparison when it operates alone, so they are willing to submit more aggressive orders than in single market scenarios, which entails worse terms from trade. The final outcome between traders who are better off and worse off ends in a reduction of global welfare, since we consider a smaller proportion of agents without valuation for the asset.

In order to understand different elements of trading profits, we decompose the agents' payoffs to analyse gains and losses from the trading process, similar to Bernales and Daoud (2014). Suppose a trader with private value α arrives at the market at time t and execute at time t'. Define $\Delta t = t' - t$. As we describe previously his discounted realized payoff is given by $\Pi = x \cdot ((\alpha + v_{t'} - p) \cdot e^{-\rho \Delta t}$. Suppose that $x = 1$, i.e. the order executed was a limit or market buy. Hence, we can rewrite expression (3) as:

$$\Pi = \alpha + \alpha(e^{-\rho \Delta t} - 1) + (v_{t'} - p)e^{-\rho \Delta t} \tag{4}$$

where we define,

- Private value: α
- Waiting Cost: $\alpha(e^{-\rho \Delta t} - 1)$
- Money Transfer: $(v_{t'} - p)e^{-\rho \Delta t}$

We can consider the sell side, i.e. $x = -1$ and rewrite the discounted realized payoff expression (3) for a sell order in a similar fashion.

In general, a trader cannot execute inmediatly to gain his intrinsic private value α. Instead, traders have to wait, either because there is a lack of liquidity, poor market conditions, because the optimal action is to submit a limit order or even not an order. This waiting cost is reflected by $\alpha(e^{-\rho \Delta t} - 1)$.[16] Additionally, when a execution occurs, the trader gains (or losses) some money product of the difference between the transaction price p and the fundamental value at the moment of the trade $v_{t'}$, that can be discounted back. Consequently, this is a money transfer and it is expressed by $(v_{t'} - p)e^{-\rho \Delta t}$.

Observation 2

(i) In multi markets scenarios, agents with intrinsic motives to trade decrease the absolute value of their waiting cost in exchange of worse terms on money transfer.

(ii) Agents with no intrinsic motives to trade increase their money transfer with multi markets trends, which entails better payoff.

Table 2 presents the average payoffs (as in Table 1), average waiting cost and average money transfer[17] for each transaction in the market. Agents with intrinsic motives to trade (i.e. $|\alpha| > 0$) exhibit a reduction in waiting cost (in absolute value). For instance, agents

[16] The expression for the waiting cost is inspired in Hollifield et al. (2006)

[17] Note that in Table 2, the column labeled 'Total' does not report a zero as result, because the difference $v_{t'} - p$ is discounted back at time Δt, where Δt is different for the trader who submit the market order and the trader who submit the corresponding limit order which is being matched with that market order, since new arrivals are asynchronous. Instead of that, if we were considering the instantaneous money transfer (i.e., not discounted back) it should report a zero as result.

with private value $|\alpha| = 8$ have a waiting cost of -0.359 ticks in single market scenarios and -0.338 in multi markets scenarios. Besides, agents with private value $|\alpha| = 4$ have a waiting cost of -0.173 ticks in single market and -0.127 in multi markets. These results suggest that agents with $|\alpha| > 0$ remain less time until execution, but as a consequence obtain worse terms of transaction, which is reflected in money transfer. For example, agents with private value $|\alpha| = 8$ decrease money transfer from -0.591 ticks in single market to -0.889 in multi markets which involves a harmful effect on their welfare, because the reduction in waiting cost does not compensate the loss related with the significant decline in money transfer.

Conversely, agents with $\alpha = 0$ obtain better benefit per trading and thus, raise their money transfer from 0.575 ticks to 0.738 in multi markets scenarios, which is exactly equivalent to their welfare, since their gains are directly from the trading activity. The overall result is a slightly reduction in terms of global welfare in multi markets scenarios.

Table 2: Average payoff, waiting cost and money transfer per trader differentiated by private value (all three measures in ticks). Payoffs determined are as in Table 1, while waiting costs and money transfers are described equation (4). The first row reports single market scenario. In the second we report multi markets scenario. The average payoffs, waiting cost and money transfer per trader are determined as mean of payoffs over 20 million market new entries in equilibrium. Standard errors for measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Scenario	Average payoff per trader				Waiting cost per trader				Money transfer per trader			
	Private Value $\|\alpha\|$			Total	Private Value $\|\alpha\|$			Total	Private Value $\|\alpha\|$			Total
	0	4	8		0	4	8		0	4	8	
Single Market	0.575	3.497	7.236	3.742	0.000	-0.359	-0.173	-0.195	0.575	-0.144	-0.591	-0.062
Multi Market	0.738	3.511	6.984	3.721	0.000	-0.338	-0.127	-0.173	0.738	-0.151	-0.889	-0.106

3.2. Market Quality

What is the impact of multi markets on market quality? In the previous section we discussed about the transfer in gains between single and multi markets scenarios and we observe a slightly decrease on global welfare. In this section we discuss about the impact of multi markets on market quality presenting measures that allow us to analyse market liquidity across each venue. Table 3 shows different liquidity measures, such as bid-ask spread, the effective spread, average number of limit orders at the ask quote (total and effective traded), average number of limit orders on the sell side of the book (total and effective traded) and microstructure noise. The quoted spread is calculated observing the market every 10 units of time whenever bid and ask exist. The effective spread is calculated with all transactions as mean of $|p - m|$, where p is the transaction price, and m is the midpoint between the bid and ask quotes. Since the model is symmetric we show liquidity measures for one of two markets, because both markets have the same features.

Observation 3

(i) Multi markets impairs liquidity in each local market. This is reflected in wider spreads and lower market depth.

(ii) Despite liquidity reduction in local markets, consolidated depth is higher in multi markets which represent a positive global effect of market fragmentation.

Table 3: This table shows different market quality measures for one of the two markets such as Bid-Ask Spread, Effective Spread, Number of limit orders at the ask (total and effectively traded), Number of limit orders on the sell side of the book (total and effectively traded) and absolute mean and standard deviation of microstructure noise. Each column is a different scenario, the first represent single market and the second, multi markets. The mean and standard deviation of microstructure noise are calculated with transactions as transaction price (p_t) minus the fundamental value (v_t) in ticks. All market quality measures are determined as mean of 20 million market new entries in equilibrium for one of the two market. Since both markets have identical characteristic, we do not need to report all measures for the second market. Standard errors for all Market quality measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Single Market	Multi Markets
Bid-ask spread	
1.620	3.121
Effective spread	
0.715	1.204
N. of limit orders at the ask	
1.487	0.942
N. of limit orders at the ask (effectively traded)	
0.586	0.525
N. of limit orders on the sell side of the book	
2.595	1.392
N. of limit orders on the sell side of the book (effectively traded)	
0.823	0.598
Microstructure noise: Mean $\|v_t - p_t\|$	
0.662	0.848
Microstructure noise: Std. Dev. $(v_t - p_t)$	
0.866	1.061

As Table 3 reports, for single market scenario, the quoted spread is 1.62 ticks and the effective spread is 0.715 ticks, while in multi markets these measures rise to 3.121 and 1.204 ticks, respectively. These two results suggest that liquidity in local markets is lower when another venue compete for order flow. This can be explained due to orders' aggressiveness; when a consolidated market absorb all order flow, liquidity suppliers have to place aggressive orders to improve their probabilities of execution, otherwise they have to wait in queue respecting price and time priorities, which entails narrower spreads. This effect is founded by Gajewski and Gresse (2007), Bennett and Wei (2006), and Amihud et al. (2003) that observe improvements on liquidity (in terms of spreads) for a consolidated market. In contrast, in multi markets scenarios, orders sometimes executes at 'inefficient' prices[18], because agents not

[18] By inefficient price we mean price more deviated from the fundamental value of the asset. As we will discuss this is reflected in the increment of microstructure noise for multi markets scenarios.

necessarily have to place a more aggressive order for raise their probabilities of execution, instead of that they can 'jump the queue' placing an order in the other market which is reflected in wider spreads.

Related to market depth, Table 3 also reports the number of limit orders at ask quote (or 'depth at ask') and number of limit order at the sell side of the book (or 'depth at sell side of the book'). The results are consistent with worse liquidity in local markets when they are interconnected, because show a reduction on depth at ask and depth on the sell side of that market. For instance, depth at ask is reduced from 1.487 to 0.942, and a similar effect is observed with depth on the sell side, i.e., in general we will observe less liquidity supply in a market interconnected compared with the same market unconnected. This is closely related to order flow fragmentation between two markets, since part of the order flow executes against limit orders posted in one market and the rest in the other market, therefore, other thing equal, execution probabilities are smaller for a particular market interconnected and, as a consequence each market attracts less limits orders in comparison when it operates alone. van Kervel and Menkveld (2015) show that liquidity providers duplicate their limit orders across multiple markets. However, traders in our model cannot adopt this strategy because they have a single share to trade, thus liquidity provision fragments between two markets.

In spite of detrimental effect of fragmentation on market quality for local venues separately, we can consider aggregated measures between the two markets, such as consolidated depth at ask (i.e. depth at ask of first venue + depth at ask of second venue) and consolidated depth on the sell side (defined in a similar fashion). We observe an improvement in aggregated liquidity, which is reflected in both consolidated depth at ask quote and consolidated depth on the sell side of the book. For example, consolidated depth at ask, that can be deduced[19] from Table 3 as $2 \cdot 0.942 = 1.884 > 1.487$ (which is the depth at ask when the market operates alone). This result is consistent with Foucault and Menkveld (2008) that study competition between two pure limit order market and observe the same effect on consolidated depth. Besides, Degryse et al. (2014) find empirical evidence for Dutch Stocks pre and post fragmentation and concludes that more lit fragmentation entails more consolidated liquidity but lower local liquidity. The previous fact implies that traders who do not or cannot access all trading venues might be worse off, because they face a less liquid market.

Consolidated depth is an aggregated market quality measure that suggest positive global externalities of multi markets, although this is not an overall liquidity improvement, because at the same time, local measures such quoted and effective spreads widens, which involves detrimental effects on liquidity of each trading center separately.

Observation 4

(i) Microstructure noise is incremented with market fragmentation.

The microstructure characteristic of financial markets may induce frictions that make the transaction price p_t deviate from the fundamental value v_t. Therefore, the transaction price can be written as the fundamental value v_t plus microstructure noise ξ_t, hence $p_t = v_t + \xi_t$. In a perfect scenario without frictions, the transaction price should be identical to the fundamental

[19]Recall that both market are symmetric, therefore consolidated depth can be estimate as double of depth reported in Table 3.

value, so ξ_t should be zero, but in real markets with frictions it can be an important component of prices. For instance, in our model, we consider discrete prices given a ticksize strictly positive, we have traders with intrinsic motives to trade who are willing to trade at prices even lower (or higher) than the fundamental value to obtain their private valuation for the asset, we also include some cognition limits of market participants who cannot respond to changes in markets conditions inmediatly, among others, which adds more frictions to the model.

In Table 3, we calculate the absolute mean and standard deviation of microstructure noise to introduce a measure of the trading frictions in the market. We observe that both, absolute mean and standard deviation of microestructure noise increment their value in multi markets scenarios. For example, the standard deviation increases from 0.866 ticks (single market) to 1.061 ticks (multi markets). This result suggest that price volatility is higher with fragmentation, thus less information is impounded in prices as stated Madhavan (1995). The increment in the microstructure noise is intuitive, since a single market structure already has disturbances that deviates price from fundamental value (which is induced by the characteristics of the market), then considering multiple markets is somehow incorporate more frictions to traders, for instance, traders have to consider larger amounts of information between both market in order to select their optimal action, given the observed state they face.

3.3. Trading Behavior

How do traders optimally act? Who act as liquidity supplier? Who as liquidity demander? Do traders submit more aggressive orders in multi markets scenarios? The optimal strategies of traders involve high complexity, because they take into account large amounts of information. On the one hand, traders can submit market orders and execute inmediatly, since these orders do not have associated waiting costs, however, traders usually have to pay an immediacy cost per a quick trade, given by the difference between the fundamental value and the respective quote. On the other hand, traders can submit limit orders, which have associated waiting costs according to a delay rate ρ. A trader who submits a limit order plays the role of 'speculator'[20], since he is not certain about future changes in the fundamental value, which could lead to unexpected gains. With an accurate limit order, agents can get good terms from trade, but also there is a inherent risk of being picked off when the fundamental value moves in unfavourable direction. For instance, if the fundamental value increases, some of the outstanding limit sell orders could be priced too low and other agents could make profits from the difference, which can lead to place less aggressive prices to get protection behind other orders, and reduce the picking off risk.

Table 4 presents some metric for trading behavior, following Bernales and Daoud (2014). We report percentage of limit orders executed per trader, probability of being picked off after submitting a limit order, average number of limit order submitted per trader, average number of limit order cancellations per trader, the probability of submitting a limit sell at the ask quote, and some measures of time as average time to execution and average time for

[20] Indeed, Goettler et al. (2009) denote agents with zero-private value as 'speculators', since they mainly submit limit orders

execution of limit orders. Since the model is symmetric, we focus our analysis on the sell side of the market, because similar results is obtained for the buy side of the market.

Table 4: Trading behavior. This table shows different measures of trader behavior for different scenarios, such as, the percentage of limit orders executed among all limit orders submitted, the probability of being 'picked-off' after submitting a limit order, the number of limit orders submitted per trader, the number of limit order cancellations per trader, the average time between the instant in which a trader arrives and his execution (in time units of our model), the time between the instant in which a trader arrives and the execution of his limit order (in time units of our model) and the probability of submitting a limit sell order at the ask price (i.e., an aggressive limit sell order). Since the model is symmetric on both sides of the book it is not necessary to also report the probability of submitting a limit buy order at the bid price. The probability of being 'picked-off' is calculated with executed limit orders: we take the number of limit sell (buy) orders that are executed when their execution price is below (above) the fundamental value of the asset, which is divided by all the limit orders executed in the market. All trading behavior measures are determined as mean of 20 million market new entries in equilibrium. Standard errors for all trader behavior measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Single Market	Multi Markets
Percentage of limit orders 'executed' per trader	
31.50 %	43.02 %
Prob. of being picked-off after submitting a limit order	
21.86 %	8.69 %
Number of limit orders submitted per trader	
1.587	1.162
Number of limit order cancellations per trader	
1.087	0.662
Time between the instant in which a trader arrives and his execution	
4.545	4.108
Time between the instant in which a trader arrives and the execution of his limit order	
7.806	7.104
Prob. of submitting a limit sell order at the ask price (an aggressive order)	
38.63 %	28.16 %

Observation 5

(i) In multi markets scenarios agents execute faster than in single market.

(ii) On average, each trader submits and cancels lower amounts of limit orders in multi markets scenarios.

Multi markets scenarios lead to shorter execution times, as is shown in Table 4. On the one hand, the average time between a trader arrives and his execution[21] when there is a single market is 4.545 units of time, while in multi markets scenarios this average time is reduced to 4.108. Recall that a single unit of time in our model represents the average time until a new trader arrives to the market, which means that approximately, there is one transaction every 4 or 5 new traders arrivals in both scenarios. In Table 4, we can observe the same

[21]Note that a trader can execute submitting a market order or being counterpart of a market order submitted by another trader. The time reported here considers both possibilities of execution.

behavior for the time between the instant in which a trader arrives and the execution of his limit order, since it decreases from 7.806 units of time in single market to 7.104 units in multi markets.[22].

While it is true that time to execution is linked to traders' behavior, this quantity is also used in the literature as another market quality indicator, because for some traders, speed is more important than spread. In general, faster markets are viewed as higher quality. Since we calculate time to execution across all transactions in both markets, this an aggregate liquidity measure that support Observation 3(ii), because it is an aggregated liquidity improvement as result of fragmentation.

The intuition behind faster execution times in multi markets scenarios is related with a liquidity issue. In section 3.2 we found that consolidated depth is higher than depth when the market operates alone. On the other hand, a trader with high private valuation for the asset has incentives to capture their private values as soon as possible making a transaction. Naturally, if this trader arrives to the market and observes a greater amount of limit orders, the probability of submitting a market order for this trader is higher and as a consequence the speed of execution for the entire economy might be faster.

The second part of Observation 5 is also shown in Table 4. For example, the average number of limit orders submitted per trade is 1.587 for single market scenarios and 1.162 for multi markets. This is closely related with the first part of Observation 5, about reduction in execution times with fragmentation. Since liquidity demanders have incentives to execute faster in multi markets, liquidity suppliers do not need to submit too many limit orders (and hence, they do not cancel so much), because it is highly probable the order executes. Indeed, Table 4. shows that 31.503 % of limit orders submitted in single market scenario ended in execution, while in multi markets this percentage increases to 43.016 %.

Observation 6

(i) Picking off risk is reduced with multi markets, since liquidity providers are more conservative.

Observation 6 is also shown in Table 4. The picking off risk is related with traders who take advantages picking limit orders that were placed at prices too low (high) for sell (buy) orders, mainly due to cognition limits of agents who can not respond inmediatly to changes, for instance, in the fundamental value. This effect is measured with transactions, counting how often a limit sell (buy) is executed at a price below (above) the fundamental value, among all limit orders executed in the market. Results in Table 4 indicate that the picking off risk declines from 21.864 % in single market to 8.689 % in multi markets which is a direct implication of the orders' aggressiveness. For instance, the probability of submitting a limit sell order at the ask price (reported in Table 4) is lower in multi markets. Furthermore, in Table 6 we show that average price of submitted limit sell orders moves from 1.1 ticks above the fundamental value in single market to 1.36 in multi markets, so prices are more conservative, which explains a lower probability that a limit sell ends in a execution with a price below the asset value, and therefore a lower probability of being picked off after

[22]Note that this second time metric now focuses on time to execution being counterpart of a market order submitted by another trader, so the quantity is meant to be higher than the previous time metric

submission of limit orders.

Observation 7

(i) Agents with intrinsic motives to trade act as liquidity demander.
(ii) Agents with no intrinsic motives to trade act as liquidity suppliers.

In the following, we report the average behavior of traders by type. Specifically we compare agents with different private valuation α for the asset.

Supplying liquidity to the markets requires posting limit orders at competitive prices. To examine liquidity provision we consider Table 5. This table shows the average strategies adopted by each trader type in terms of order submission. Among all limit orders submitted in single market scenario, a 67.9 % were submitted by traders with $\alpha = 0$, 26.1 % by agents with $|\alpha| = 4$ and the remaining 6.1 % were submitted by traders with $|\alpha| = 8$. For multi markets scenarios these result are quantitative and qualitative similar, which means that fragmentation does not change main strategies adopted by traders. Agents with no intrinsic motive to trade submit most of limit orders, which is explained mainly by two reasons. First, agents with $\alpha = 0$ 'speculate' in both markets submitting and sometimes cancelling and re submitting new limits order given markets conditions. Second, these traders are aware that other agents (for example, with intrinsic motives to trade) are willing to pay the cost of a quick trade by demanding liquidity, which incentives traders with $\alpha = 0$ to be liquidity suppliers.

Table 5: Strategies per trader type. This table reports the proportion of market orders submitted by each trader type as percentage of all market orders submitted, the proportion of limit orders submitted by each trader type as percentage of all limit orders submitted, and proportion of non order by each trader type as percentage of all traders that choose not an order in equilibrium. All percentages are determined as mean of 20 million market new entries in equilibrium. Standard errors for all trader strategies are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Scenario	Order Type	Private Value $\|\alpha\|$		
		0	4	8
Single Market	Limit Orders	67.9 %	26.1 %	6.1 %
	Market Orders	13.4 %	38.8 %	47.7 %
	Non Orders	80.3 %	15.5 %	4.2 %
Multi Markets	Limit Orders	68.0 %	26.8 %	5.2 %
	Market Orders	5.8 %	43.4 %	50.9 %
	Non Orders	74.8 %	21.7 %	3.5 %

As Table 5 reports, most market orders are submitted by agents with $|\alpha| = 8$. Indeed, a 47.7 % of market orders are submitted by them in single market scenario, while in multi markets scenarios this quantity increase to 50.9 %, which goes along with faster executions in multi markets scenarios. [23] These results are supported by Goettler et al. (2009) who analyse liquidity supply and demand for a isolated market obtaining similar results.

Observation 8

(i) Agents with no intrinsic motives to trade exploit multi markets potential more than other market participants. This effect decreases in the absolute value of α.

All traders type have different incentives in the trading process that make them to play specific roles in the market. For instance, agents with $\alpha = 0$ supply liquidity to the market, agents with extreme valuation ($|\alpha| = 8$) are more likely to demand liquidity, and agents with $|\alpha| = 4$ sometimes behave like agents with $\alpha = 0$ and sometimes like agents with extreme valuation making an equilibrium between liquidity supply and demand given market conditions.

Observation 8 is reflected in several part of our study. We already discussed that agents with $\alpha = 0$ obtain better profit in multi markets scenarios. This effect also can be observed in Table 6. Besides the fact of better profit, we can analyse the terms of trading. Consider a trader who submits a market sell at price p when the fundamental value is $v_{t'}$. This market order is matched with its respective counterpart that is a previous limit buy. The trader who places the market order, sells the asset at price p when it has a value $v_{t'}$, thus he obtains $p - v_{t'}$. On the other hand, the trader who submitted the limit order, obtain $v_{t'} - p$. A higher amount $p - v_{t'}$, signifies an improvement in the terms of trade for the agent who submits the market sell, and as consequence worse terms of trade for the agent who submitted the limit buy.

Table 6: Traders' behavior differentiated by private value. This table shows statistics of traders differentiated by private values both single market and multi market scenarios. We report time between the instant in which a trader arrives and his execution (Time to execution in Table 6), price of submitted limit sell orders and mean price of all executed sell orders, market and limit sell (since the model is symmetric we do not need to report the price of buy orders). Finally, for multi market scenarios we also report the percentage of traders who switch from one to another book. All trader behavior measures are determined as mean of 20 million market entries in equilibrium. Standard errors for all trader behavior measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each case.

Scenario	Time to execution				Price of submitted limit sells				Price of all executed sell orders			
	Private Value $\|\alpha\|$			Total	Private Value α			Total	Private Value α			Total
	0	4	8		0	4	8		0	4	8	
Single Market	13.03	3.60	2.06	4.55	1.71	0.00	-0.82	1.10	0.84	-0.16	-0.61	0.00
Multi Market	10.83	3.05	1.20	4.11	1.80	0.55	-0.36	1.36	1.10	-0.15	-0.90	0.00

[23] See Table 6, Time to execution per trader type.

In Table 6, we report the price of all executed sell orders [24] (market and limit), relative to the fundamental value at the time of execution, $v_{t'}$, which is precisely the benefit from trade. We can observe that agents with $\alpha = 0$ obtain better terms from trading than traders with intrinsic motives to trade, in both single and multi markets scenarios. This effect is even greater in multi markets scenarios increasing their benefit from trade from 1.71 to 1.8 ticks above the fundamental value. If we consider the fact of faster execution times from Observation 5 (i), it is consistent with higher profits in multi markets as we discussed in Observation 1 (ii). As expected, for agents with extreme valuation for the asset ($\alpha = -8$), opposite results can be found since in single market they are willing to sell, on average -0.61 ticks below the fundamental value, while in multi markets they are willing to sell, on average, -0.90 ticks below the fundamental value, which is explained by two main reasons. First, the decline in liquidity across each market discussed in section 3.2, and the reduced level of participation in multiples markets which is shown in Table 7. For example, just 1.7 % of traders with $|\alpha| = 8$ submit orders in both markets, and the rest just trade in one market until their execution. Hence, traders with intrinsic motives to trade do not care too much about multi markets. This effect is strongly incremented for agents with $\alpha = 0$, since 46.6 % of them submit at least one order to each market before their execution, which shows how they exploit opportunities of multiple markets in order to obtain better profits.

Table 7: Average proportion of traders who switch from one market to another. this table report Average proportion of traders who switch from one market to another for multi markets scenario. The percentage is found as follows. For each trader type, we determine the number of trader that sent orders to both markets, among all traders that submitted at least one order. This percentage is determined as mean of 20 million market entries in equilibrium. Standard errors are small enough since we use a large number of simulated events.

Scenario	Private Value $\|\alpha\|$			Total
	0	4	8	
Multi Markets	42.23 %	10.92 %	1.66 %	17.54 %

3.4. Market Dynamics

Is there any dynamic relationship or pattern between interconnected markets? In previous sections we discussed about consequences derived by the model in terms of welfare, market quality and trading behavior, taking advantage of the symmetry between the interconnected exchanges, but we did not focus on the dynamics related to the markets. As a starting point, consider the amount of transactions in the first and second market. Naturally, for large numbers it is expected symmetry between the amount of transactions in each market, but in the trading process both markets evolves according to the order flow absorbed by them, for instance, could exist periods in which one market has higher liquidity than the other, or

[24] Recall that as the model is symmetric, sometimes we combine positive and negative α with the same absolute value or we focus on the sell side of the market since the buy side is analogue.

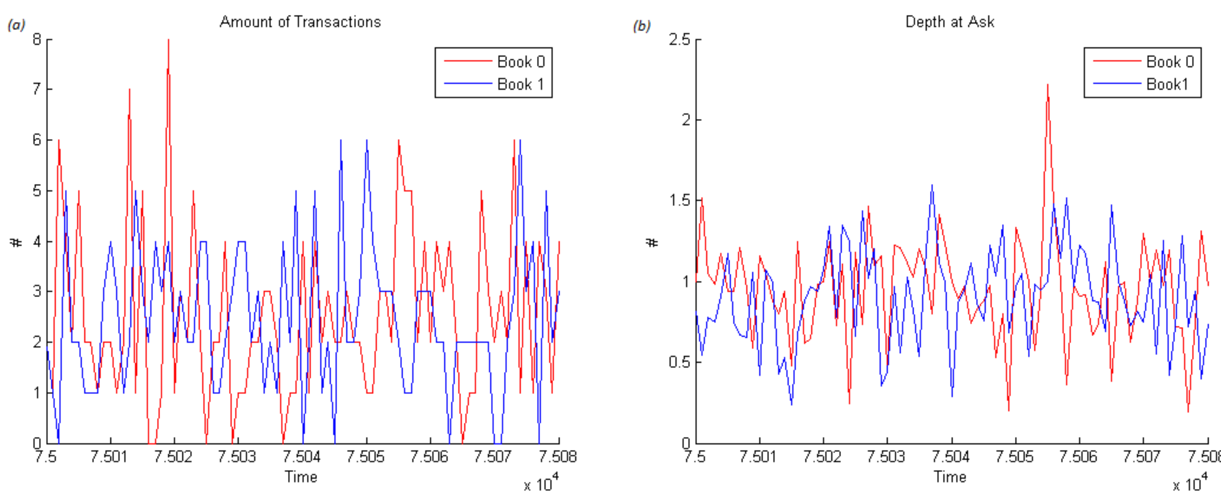
periods in which both market have a similar liquidity level. The evolution of each market is completely recorded in their respective limit order books, therefore, we can calculate several measures for understanding order flow oscillations between the two markets.

We study two measures related with the evolution of each market: the amount of transactions and depth at ask quote. Other measures could be considered as well, but for reporting purposes we limit our analysis to that. Our goal is to understand the relation between markets. For that purpose, we examine the market every 10 units of model time (which is equivalent to 5 minutes in reality) and determine the mean for each one of the two measures every 5 minutes. Instinctively, we do not show the complete history (20 million of new entries) because we are not interested in the average behavior. Instead of that, we want to explore small pieces of the trading game to understand how both markets evolve across time.

Observation 9

(i) Liquidity tend to oscillate from one market to another, which allows coexistence between multiple limit order markets.

Figure 1 is divided in two graphics that report the evolution of the mean of our two measures for both markets. We center our analysis for the amount of transactions (i.e Figure 1 (a)), since the analysis for depth at ask is analogue. We can observe oscillatory movements that suggest that there are periods of time, in which the first market absorb more order flow than the second and hence, it is preferred to send market orders there, since it exhibits better liquidity.[25] This effect is also supported by correlation coefficient between the amount of transactions in the first and the second market is negative (-0.208), i.e. the evolution of number of market orders tend to lie on opposite sides of their respective means.



An intuitive question that arises is why liquidity fluctuates from one market to another, rather than consolidate in the more liquid one? The main intuition behind that question is in the absence of time priority across markets, which allows traders to avoid the price priority in one market by submitting orders in the other venue. That fact allow coexistence between

[25]Correspondingly, there are periods of time, in which the second market absorb more order flow than the first one

two market, given that both exchanges have similar features.[26]

3.5. Non-Symmetric Case

Does market structure determine the level of market fragmentation? Certainly, market fragmentation is a financial trend not yet well understood since markets can experience a great variety of fragmentation forms, given the features of exchanges. In previous section we study fragmentation in detail and derive several implication with a framework of two symmetric markets. In this section we analyse a non-symmetric case of market fragmentation considering different transaction costs and different latency restriction between the two interconnected markets. The first market (henceforth 'market A') has both, transaction cost and latency restriction higher than the second exchange (henceforth, 'market B'). We also simulate the market A and B, when they operate alone (single market scenario) for comparative purposes.[27]

Observation 10

(i) Market B absorbs a great part of order flow available.

(ii) Transactions in market A involves mainly traders with non-zero private value.

Table 8: This table report average proportion of agents who trade in market A or B separated by private value for multi markets scenario. For market A, the percentage is found as follows. For each trader type, we determine the number of times a trader type is involved in a transaction in the market A, among all transaction in that market. For market B, the respective percentage is determined similarly. This percentage is determined as mean of 20 million market entries in equilibrium. Standard errors are small enough since we use a large number of simulated events.

Multi Markets	Private Value $\|\alpha\|$		
	0	4	8
Market A	12.45 %	48.19 %	39.36 %
Market B	33.06 %	38.56 %	28.39 %

Naturally, if we analyse both markets independently, market B seems to be 'better' than market A, because it provides less frictions to traders. Hence, traders prefer to act in market B over market A. In fact 85.2 % of all transactions were submitted in market B and the remain 14.8 % in market A.[28]. In table 8, for all transaction we report the proportion of agent who trade in market A and B, separated by type, either for the submission of a market order or

[26] Indeed, in our simulation both markets are symmetric.

[27] It is important to state that simulation of the non-symmetric case does not reach the equilibrium due to lack of time, Therefore we do not give the same emphasis of previous sections. Instead of that this pre-eliminar simulation is briefly included with most intuitive result as starting point

[28] Recall that market B is more attractive since do not consider transaction cost, and allow traders to monitor the market more frequently than market A.

being counterpart of a market order submission. We observe that traders with zero-private value are involved in transactions, just a 12.45 % of the times in market A, while in market B this quantity increases to 33.06 %. These results imply that agents with non zero private value are involved in the remaining 87.55 % of the transactions and suggest that in the market A, liquidity provision is not managed by agents with not intrinsic motives to trades, since their participation is to low in order to supply liquidity in that market. Conversely, agents with $|\alpha| > 0$ are more involved in transactions in market A, if compares with the same metrics for market B. Recall that agents with zero private value are 'speculators' and the market A, have associated latency restrictions that does not incentive their massive participation in that market, instead of market B that allows trader to modify their orders quickly.

Observation 11

(i) Fragmentation entails wider spreads and lower depth in each local venue, but in aggregated terms there exist a liquidity improvement on consolidated depth at ask.

Table 9: This table shows different market quality measures for both market configurations such as Bid-Ask Spread, Effective Spread, Number of limit orders at the ask and Number of limit orders on the sell side of the book. Each column is a different scenario, the first two represent single market scenarios and the two seconds, multi markets scenarios. All market quality measures are determined as mean of 20 million market new entries in equilibrium for each scenario. Since both markets do not have identical characteristic, we need to report all measures for the second market. Standard errors for all Market quality measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Single Market		Multi Markets	
Market A	Market B	Market A	Market B
Bid-ask spread			
2.170	2.087	4.840	3.110
Effective spread			
0.939	0.960	1.688	1.296
N. of limit orders at the ask			
1.315	1.569	0.414	1.273
N. of limit orders on the sell side of the book			
3.793	2,700	0.510	1.273

In table 9 we present four liquidity measures, such as bid-ask spread, effective spread, depth at ask, and depth on the sell side, we observe that in comparison with markets that operates without connection(or single market scenarios) liquidity for local markets is reduced for both configurations, either A or B. Similar to Section 3.2, we obtain different results that depend if we consider aggregated or local liquidity measures. If we compare the local market A interconnected and the local market A isolated, we can observe a drastic reduction in every market quality measure in multi market scenarios, which is consistent with the fact that the other market is absorbing almost all order flow. While it is true that each local market interconnected shows low levels of liquidity, the consolidated depth at ask, $0.414 + 1.273 = 1.688$, is higher than the same quantity for any of the two market configurations when they operate alone (1.315 or 1.569), which represents an aggregated liquidity improvement.

4. Conclusion

We provide a framework using a dynamic model to explore the connection between two symmetric pure limit order markets in which agents can trade the same financial asset. The model includes the main features of real limit order market and generate complete evolution of both limit order books, which represents an additional contribution in our study. Consequently, we can analyse the effect of multiple markets on several aspects as welfare, market quality and trading behavior.

Our simulations allow us to show that multi market dynamics do not change main trading strategies of agents. For instance, traders without intrinsic motives to trade are suppliers of liquidity in both, single and multi markets scenarios, but when their provision is fragmented between two markets, different measures of liquidity for each market present a detrimental effect on market quality for a specific trading center. In spite of that, if we consider consolidated depth as a global liquidity measure, we find a positive effect on aggregated market quality, which is consistent with Foucault and Menkveld (2008). In other words, liquidity improvement or reduction depends if we focus in aggregated or local performance. We find that agents with intrinsic motives to trade do not exploit all opportunities of multi markets and in most cases they send orders to one of two markets, contrary to agents with zero private value who exploits trading opportunities in both markets, making them major beneficiaries of this financial tendency.

About the coexistence of two symmetric limit order markets in equilibrium, we concludes that coexistence it is possible given the oscillatory dynamics of markets in small pieces of the trading game. Finally, while it is true that coexistence in a non-symmetric case of fragmentation is possible, we show that in our non-symmetric scenario, most trading activity turns to the most liquid market.

Future research could consider other interesting financial trends. For example:

- The effect of high-frequency traders, which has been recently studied for Bernales and Daoud (2014) and Rojcek and Ziegler (2015) but without considering multiple markets.
- The interplay between markets with different microstructure characteristics, as hidden limit order markets (or dark pools), which is the case of Euronex that allows agents to submit hidden orders. Although several empirical research have studied the effect of hidden limit orders markets, there is a lack on dynamical models that compare opaque markets to transparent ones.

Different features of real markets such as disparity in information among traders, several assets to trade in different venues, diverse operating hours of each market, among others, have been simplified in our model to make it computationally tractable, but future research could consider them in order to obtain better understanding of the trading process.

Bibliography

Amihud, Y., Lauterbach, B., and Mendelson, H. (2003). The value of trading consolidation: Evidence from the exercise of warrants. *Journal of Financial and Quantitative Analysis*, 39:829–846.

Arnold, T., Hersch, P., Mulherin, J., and Netter, J. (1999). Merging markets. *Journal of Finance*, 54:1083–1107.

Bennett, P. and Wei, L. (2006). Market structure, fragmentation, and market quality. *Journal of Financial Markets*, 9:49–78.

Bernales, A. and Daoud, J. (2014). How fast can you trade? high-frequency trading in dynamic limit order markets. Working paper Banque de France.

Biais, B. (1993). Price formation and equilibrium liquidity in fragmented and centralized markets. *Journal of Finance*, 48:157–185.

Boehmer, B. and Boehmer, E. (2004). Trading your neighbor's etfs: Competition or fragmentation. *Journal of Banking and Finance*, 27:1667–1703.

Chowdhry, B. and Nanda, V. (1991). Multimarket trading and market liquidity. *The Review of Financial Studies*, 4:483–511.

Degryse, H., Jong, F. D., and Kervel, V. V. (2014). The impact of dark trading and visible fragmentation on market quality. *Review of Finance*, pages 1–36.

Foucault, T. and Menkveld, A. J. (2008). Competition for order flow and smart order routing systems. *Journal of Finance*, 63:119–158.

Gajewski, J. and Gresse, C. (2007). Centralised order books versus hybrid order books: A paired comparison of trading costs on nsc (euronext paris) and sets (london stock exchange). *Journal of Banking and Finance*, 31:2906–2924.

Glosten, L. and Milgrom, P. (1985). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 14:71–100.

Goettler, R., Parlour, C. A., and Rajan, U. (2005). Equilibrium in a dynamic limit order market. *Journal of Finance*, 60:2149–2192.

Goettler, R., Parlour, C. A., and Rajan, U. (2009). Informed traders and limit order markets.

Journal of Financial Economics, 93:67–87.

Hasbrouck, J. (2007). *Empirical Market Microstructure*. Oxford.

Hendershott, T. and Mendelson, H. (2000). Crossing networks and dealer markets: Competition and performance. *Journal of Finance*, 55:2071–2115.

Hengelbrock, J. and Theissen, E. (2009). Fourteen at one blow: The market entry of turquoise. Available at SSRN 1570646.

Hollifield, B., Miller, R., Sandas, P., and Slive, J. (2006). Estimating the gains from trade in limit-order markets. *Journal of Finance*, 61:2753–2804.

Jovanovic, B. and Menkveld, A. J. (2011). Middlemen in limit-order markets. Working Paper, New York University and VU University of Amsterdam.

Madhavan, A. (1995). Consolidation, fragmentation and the disclosure of trading information. *The Review of Financial Studies*, 8:579–603.

Mendelson, H. (1987). Consolidation, fragmentation and market performance. *Journal of Financial and Quantitative Analysis*, 22:189–207.

O'Hara, M. and Ye, M. (2011). Is market fragmentation harming market quality? *Journal of Financial Economics*, 100:459–474.

Pagano, M. (1989). Trading volume and asset liquidity. *The Quarterly Journal of Economics*, 104:255–274.

Pakes, A. and McGuire, P. (2001). Stochastic algorithms, symmetric markov perfect equilibrium, and the 'curse' of dimensionality. *Econometrica*, 69:1261–1281.

Parlour, C. and Seppi, D. (2003). Liquidity-based competition for order flow. *Review of Financial Studies*, 63:329–355.

Parlour, C. and Seppi, D. (2008). Limit order markets: A survey. *Handbook of Financial Intermediation and Banking*, 5:63–95.

Rojcek, J. and Ziegler, A. (2015). High-frequency trading in limit order markets: Equilibrium impact and regulation. Research Paper Series N 15-23.

Rosu, I. (2009). A dynamic model of the limit order book. *Review of Financial Studies*, 22:4601–4641.

van Kervel, V. and Menkveld, A. (2015). High-frequency trading around large institutional orders. *Available at SSRN 2619686*.

Appendix

1. Details of the numerical algorithm

In this section, we describe several assumption to make our simulations computationally tractable. First, the fundamental value evolves across time, thus the set of prices feasible for trade is huge (although finite since our simulations are finite). However, traders only care about the difference between the price and the fundamental value v_t, so we can write prices relatively to v_t. Furthermore, historical prices can be also expressed in terms of v_t which significantly reduces the state space in our numerical simulations. Ideally, we would like to condition agents strategies across all information available in the order books, but this is computationally intractable. Instead, we restrict the state space for a trader as follows. Let s_t denote the state observed by an agent at the time t, then:

$$s_t = \{L_{1,t}, L_{2,t}, \delta_t, v_t, \alpha, \rho, \tilde{a}_t\} \tag{5}$$

where $L_{m,t}, m \in \{1, 2\}$ is a set of variables of the book m at time t that depends on the transparency of each book. For our simulation we consider the same transparency for both books. Accordingly the variables in $L_{m,t}$ includes:

- Bid and Ask prices $(B_{m,t}, A_{m,t})$
- Depths at these prices $(l^B_{m,t}, l^A_{m,t})$
- Depth on buy and sell side of the market $\sum\{l^i_{m,t} > 0\}$ and $\sum\{l^i_{m,t} < 0\}$.

Besides, δ_t indicates the most recent transaction which includes the book, the price and if the last transaction was a buy or sell, v_t is the fundamental value, α and ρ are respectively agents' private value and delay rate, and $\tilde{a} = (\tilde{b}, \tilde{p}, \tilde{x}, \tilde{q})$ represent the status of his previous order. In order to summarize, we consider 3 exogenous events:

- Changes in the fundamental value: Every time the fundamental value v changes, it increases or decreases its value with probability $\frac{1}{2}$ and appropriately shift all orders in the books relative to the new value of v
- New trader arrivals: Every time a new trader arrives to the market, he observes the current state s and takes the optimal action a^*. If a^* is a market order, he executes and leaves the market forever, if he takes any other order we draw a random time for his re-entry.
- Re-entry of 'old' traders who have not yet executed: Every time a trader returns to the market, he observes the current state s, which includes the status of his previous order.

Then, he takes an optimal action (which could include retaining his previous order). If the order he selects is a market order, he executes and leaves the market forever, if he takes any other order we draw a new return time for his re-entry.

2. Learning Process

In the algorithm, at the time t, each action[29] a in the state s has an associated expected payoff $U_t(a|s)$, which is a real number and represent current belief about the payoff from this action at state s. Consequently, a trader selects the action a^* in the state s at time t iff $a^* \in \underset{a\in\Gamma(s)}{\text{argmax}}\, U_t(a|s)$. Then, the optimal value of state s from Bellman equation for the the traders' dynamic maximization problem is determined as $V(s) = U_t(a^*|s)$.

Each pair (a, s) has a initial belief $U_0(a|s)$. Note that any $U_0(\cdot)$ can lead to an equilibrium, so we choice it for reaching the equilibrium quickly, similar to Goettler et al. (2009). These initial beliefs are updated with the progress of the algorithm. Suppose the optimal action a^* is not a market order, that is, it is either a limit order or no order. Further that, suppose at some future time t' the trader re-enters the market and observes a state s' with an optimal value $V(s')$. Then, the belief $U_t(a^*|s)$ is updated as:

$$U_{t'}(a^*|s) = \frac{n}{n+1}U_t(a^*|s) + \frac{1}{n+1}e^{-\rho(t'-t)}V(s') \tag{6}$$

where $n = n(a^*, s)$ is a positive integer that is incremented by one each time action a^* is chosen in state s. Similar to Goettler et al. (2009), periodically during the simulation, we restart n to n_0 to obtain quicker convergence.

Similarly, suppose a trader submit a limit sell (denoted by a^*) when he faces the state s at time t, and this order executes against a market order submitted by another trader at time t'. In this case we update the belief $U_t(a^*|s)$ with the realized payoff associated with the transaction[30] as:

$$U_{t'}(a^*|s) = \frac{n}{n+1}U_t(a^*|s) + \frac{1}{n+1}e^{-\rho(t'-t)}x(\alpha + v_{t'} - p) \tag{7}$$

In the simulations, if all traders traders take the optimal action given their current beliefs, there is a possibility that the algorithm would be in a sub-optimal equilibrium, maybe because traders have not learned payoffs of other actions. To ensure that beliefs are updated for all actions in every state, with a small probability ε a trader trembles over all suboptimal limits orders with same probability, which allows the trader to visit states that he never would have been able to visit without trembles.

[29]Recall that given a state s, each action a has a finite action set feasible, denoted by $\Gamma(s)$

[30]We always know the instantaneous payoff associated to a market order, which is $x(\alpha + v - p)$

3. Convergence Criteria

The simulation has three main phases:

(i) As a first phase, we update the beliefs of agents for sufficient time to ensure equilibrium is reached (two billion of new arrivals for single market scenarios and five billion for multi market).

(ii) After first phase, we check that the algorithm converges properly.

(iii) If convergence criteria is satisfied means the algorithm reaches the equilibrium. We fix the belief of agents (i.e., there is not more learning process), disallow trembles (i.e., $\varepsilon = 0$) and simulate the model another 300 million of new arrivals to obtain our results in equilibrium.

Related to (ii), we use a similar convergence criteria than Goettler et al. (2009). We take a snapshot of beliefs at the end of (i) and simulate several millions of additional new arrivals. Then, we take another snapshot of beliefs. The intuition suggest if both snapshots are similar enough, then the algorithm has converged.

Formally, we evaluate every 600 million new trader arrivals, the weighted relative difference among the beliefs at the end of (i), $U_{t_1}(\cdot)$ and the belief at the end of the current 600 million new trader arrivals, $U_{t_2}(\cdot)$. We require the weighted relative difference not to exceed 1 %, which can be expressed as:

$$\sum_{(a,s)\in\mathcal{X}} \frac{U_{t_1}^{k_1}(a|s) - U_{t_2}^{k_2}(a|s)}{(k_2 - k_1)U_{t_1}^{k_1}(a|s)} < 0.01 \tag{8}$$

where $\mathcal{X}$ is the set of all actions a selected in the state s during the first phase, k_1 is the number of times the action a has been taken in state s at the end of (i) and $k_2 \geq k_1$ the number of times it has been chosen at the end of current 600 millions of new traders arrivals.