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Ambiguity and Long-Run Cooperation

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Ambiguity and Long-Run Cooperation

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Abstract

This paper studies the effects of ambiguity on long-run cooperation, by analyzing the infinitely repeated Prisoner's Dilemma and its application to Cournot's duopoly model. We show that ambiguity decreases the likelihood of cooperation in the infinitely repeated Prisoner's Dilemma, regardless the level of optimism. In the economic application, we find that ambiguity is positively related with static equilibrium quantities and negatively related with the probability of sustaining a tacit collusion, i.e. positively related with competition. In fact, the critical discount factor associated with the probability of achieving a collusive equilibrium can be even higher than one for some parametric combinations. Nevertheless, depending on the level of optimism, a discontinuity can arise when ambiguity is too high, emerging a situation where collusion can be implemented as a short-run equilibrium. That is due to the fact that, for some parametric combinations, the economic application stops being a particular case of the Prisoner's Dilemma and start behaving as different games in which cooperation can be achieved as a short-run pure Nash equilibrium. Finally, an alternative interpretation suggests an equivalence result: a Cournot's duopoly with high ambiguity and relatively pessimist players behaves as a coordination game with exogenous payoffs.

Keywords: Ambiguity, Neo-capacities, Prisoner's Dilemma, Long-run Cooperation, Cournot Duopoly Model, Tacit Collusion.

1. Introduction

The concept of ambiguity extends the notion of risk, stating that not only the realization of future states is unknown, but also the probabilities assigned to them. In economic theory, ambiguity has become an important topic as it has been able to explain some facts that the standard theory has failed to.[2] Therefore, many economic topics which rely on choice theory have been analyzed to know whether their results hold or not when an ambiguity setting is justified. In concrete, several applications have been made in game theory, where it has been studied how standard theories and the outcomes of different games are affected when assigning probabilities to uncertain events is not a credible situation. This paper seeks to contribute to this literature by studying the effects of

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[2]See, for example, Ellsberg (1961) and Chen and Epstein (2002).

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ambiguity on long-run cooperation.

When examining strategic interactions and potential long-run cooperative equilibria, standard theory assumes that once individuals agree on cooperating, they cooperate forever. Therefore, deviations from a cooperative equilibrium takes individuals always by surprise. However, it is reasonable to think that individuals may anticipate that even when they agree on cooperating, the other player may deviate. Likewise, when players do not cooperate, an individual may internalize that her counterpart may deviate to a cooperative equilibrium; for example, due to signalling reasons or simply by mistake. This might have differential impacts on how individuals decide to behave in the long-run. In concrete, this perturbs the decision of cooperating, as the expected payoffs of the different options are affected by ambiguity.

In this work, we follow Eichberger, Kelsey, and Schipper (2009) strategy to model ambiguity in strategic games[3] to analyze the infinitely repeated Prisoner's Dilemma and its application to Cournot's duopoly model. This framework considers how players may expect that their counterparts might not exactly behave in the way they are *supposed* to. This is internalized by the players, who distrust their own beliefs about other player's actions and place themselves in the best and worst cases off the *supposed* actions. Therefore, the setting allows us to study the effects of ambiguity on the infinitely repeated Prisoner's Dilemma outcome, given that the presence of uncertainty on the other player's decision affects the payoffs related to the decision. In particular, we find that ambiguity increases the minimum discount factor needed for sustaining a cooperative equilibrium, i.e. ambiguity decreases the probability of sustaining a cooperative equilibrium in the long-run. Moreover, under this setting, the critical discount factor could be even higher than one, making impossible the implementation of a cooperative equilibrium in some situations.

By making the application to Cournot's model, we find that ambiguity and optimism increase static equilibrium quantities. Also, for the majority of the cases considered, ambiguity decreases the probability of sustaining a collusion, even making it impossible for some parametric combinations (critical discount factors higher than one). Therefore, in general, competition increases with ambiguity. Nevertheless, in the presence of ambiguity aversion, i.e. *low* levels of optimism, a discontinuity arises when analyzing the decision of colluding. Given that in this setting payoffs are endogenous (they depend on some game parameters), for some parametric combinations the game stops being a particular case of the Prisoner's Dilemma and, interestingly, the new games that emerge may sustain collusion even as a short-run equilibrium.

The discontinuity can be read as follows. Given a *low* level of optimism, the higher the ambiguity, the higher the critical discount factor and, therefore, the lesser the probability of colluding in the long run. Above certain level of ambiguity, the critical discount factor is higher than one, which implies that tacit collusion is impossible as a long-run equilibrium. Curiously, a second threshold appears, wherein for higher levels of ambiguity, the game stops behaving as the Prisoner's Dilemma and, conversely, the new payoffs conform games where cooperation appears as a possible short-run equilibrium. This result is very interesting as it suggests that depending on the value of the parameters, i.e. of the levels of ambiguity and optimism, collusion could be achieved in the short-run as a pure Nash equilibrium or, by contrast, it might be even impossible to be sustained as a long-run

[3]Based on Chateauneuf, Eichberger, and Grant (2007).

equilibrium. We finish with an alternative interpretation that suggests an equivalence result: a Cournot's duopoly with high ambiguity and relatively pessimist players behaves as a coordination game with exogenous payoffs.

This paper contributes to the existing literature by shedding light on the long-run implications of short-term ambiguity for this kind of games. Indeed, when considering long-run competition in an oligopolistic market, which is a particular case of the repeated Prisoner's Dilemma, it is an important matter to find to what extent a highly concentrated market is likely to start up and sustain a collusion. As it was mentioned above, if it is assumed that potential deviations from the expected equilibrium may be internalized by the players even when agreeing on cooperating, ambiguity appears as a useful tool to study these cases. Moreover, as it will be discussed in the following sections, the specific strategy followed in this work to model ambiguity offers many advantages for studying strategic interactions.

This paper is structured as follows. After this brief introduction, Section 2 reviews the related literature. Section 3 specifies the strategy that is used in this work to model ambiguity. Section 4 solves the infinitely repeated Prisoner's Dilemma under this setting, while Section 5 makes the concrete application to the Cournot's duopoly model. A brief discussion about some empirical issues is made on Section 6. Section 7 suggests an alternative interpretation of our main result. Finally, Section 8 concludes.

2. Related Literature

In this section, we review the main ambiguity applications made in game theory and industrial organization. For a broad review on ambiguity research, see Etner, Jeleva, and Tallon (2012).

The first important work was done by Dow and Werlang (1994), who defined a Nash equilibrium concept under Knightian uncertainty for normal-form games with two players.[4] In addition to showing that the equilibrium exists for any level of uncertainty aversion, the work demonstrates that backward induction does not hold in the twice repeated Prisoner's Dilemma under this setting. Therefore, the study illustrates that classical game theory results may no longer be valid if uncertainty is treated in a different way. Eichberger and Kelsey (2000) extend the work of Dow and Werlang (1994) by demonstrating the existence of that equilibrium in a n-player setup and establishing that for high levels of ambiguity, the equilibrium under uncertainty differs from the classical Nash equilibrium and approximates a max-min behavior.

On the other hand, Marinacci (2000) introduces *ambiguous games*, which are a modified version of normal form games that permits the presence of ambiguity in terms of uncertainty on beliefs on other players' choices. The paper defines an equilibrium concept and provides a demonstration of existence, in addition with many examples that illustrate how the outcomes of classical games, as for example the *Stag Hunt Game* or the *Game of Deference*, may change under this new scenario. Also, the author states that ambiguity attitudes may be irrelevant in some cases. In fact, he concludes that, given the dominant strategy nature of the static Prisoner's Dilemma, ambiguity has

[4]It is said that individuals face Knightian uncertainty, when they face risks that are immeasurable.

no effect on the outcome.

In the context of game theory and industrial organization, an important contribution is given by Eichberger, Kelsey, and Schipper (2009), who model ambiguity in strategic games by using neo-capacities (Chateauneuf, Eichberger, and Grant, 2007) and, once again, show how the introduction of ambiguity may have effects on the outcomes of different applied examples. In particular, they find that in a static Cournot (Bertrand) duopoly model, given a *high* level of optimism, ambiguity increases (decreases) quantities and decreases (increases) market prices, thus claiming that ambiguity may have different impact on the level of competition on a given market depending on the attitude that players have toward it.

The articles cited above are silent about the long-run implications of short-term ambiguity. In that sense, we contribute to the literature by studying the effects of ambiguity on a particular issue: the possibility of achieving a cooperative long-run equilibrium. While Marinacci (2000) argues that ambiguity has no effect on the static Prisoners Dilemma outcome, we find that it has a negative effect on the probability of sustaining a cooperative equilibrium in the long-run. Also, while Eichberger, Kelsey, and Schipper (2009) find different effects of ambiguity on competition depending on the level of optimism, we find that *in general* ambiguity increases competition regardless the level of optimism, in the sense that it increases static equilibrium quantities and decreases the probability of sustaining a collusive equilibrium. Finally, we find a discontinuity on the likelihood of a collusive potential equilibrium that seems novel to this literature: marginally changing the value of some key parameters of the model, the Cournot's duopoly may move from a situation in which cooperation is not possible in the long-run, to a situation in which it can be achieved even as a short-run equilibrium, also suggesting an equivalence result of the static game. This is discussed with more detail in the following sections.

3. Preliminaries: Ambiguity and Strategic Games

We follow Eichberger, Kelsey, and Schipper (2009) strategy to model ambiguity in strategic games. The authors propose a game in the form $G = \langle (S_i, u_i)_{i=1,2} \rangle$, where S_i and u_i are the strategies space and the utility function of player i, respectively. Then, the expected utility function under ambiguity of player i, is defined by

$$v(s_i; \delta, \alpha, \pi) := \delta(\alpha M_i(s_i) + (1-\alpha) m_i(s_i)) + (1-\delta) E_\pi u_i(s_i, s_{-i}), \tag{1}$$

where $s_i \in S_i$ is the strategy played by player i, s_{-i} is the strategy played by her counterpart, $\delta \in [0,1]$ is the degree of ambiguity, $\alpha \in [0,1]$ is the degree of optimism, π is a probability distribution over s_{-i}, E_π is the expectation induced by π , $M_i(s_i) = \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ and $m_i(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$. Therefore, in this context, ambiguity is understood as the uncertainty an individual faces with respect to the probability of the other player's decisions. It can be seen that in a context of no ambiguity ($\delta = 0$), the functional form is reduced to the standard Expected Utility model (Savage, 1954).

In particular, this representation of ambiguity has three good properties. First, it has a clear intuition. The individual faces a subjective additive probability measure, π, but he does not trust it fully. The ambiguity level parameter measures the degree of distrust on π. Then, the *unassigned* probability is mapped to the best and worst possible outcomes, depending on the degree of optimism of the individual. Second, it does not assume the existence of total ambiguity aversion, i.e. $\alpha = 0$, as it is done in other models, and it can therefore represent both optimistic and pessimistic individuals. Third, in the context of this paper, the setting fits well on the strategic games modeling. Namely, it makes sense to assume the existence of π, because it can be derived endogenously from the game's equilibrium.

This functional form is not an *ad-hoc* representation of preferences under ambiguity. In fact, it is an application of the Choquet integral induced by neo-capacities, which is axiomatized as a choice criterion under ambiguity by Chateauneuf, Eichberger, and Grant (2007). For a brief discussion about the technical details, see Appendix A.

4. Ambiguity and the Infinitely Repeated Prisoner's Dilemma

Consider the normal form representation of the Prisoner's Dilemma

	C	N
C	(c, c)	(e, d)
N	(d, e)	(n, n)

where C and N stand for *Cooperate* and *Non Cooperate*, $d > c > n > e$ and $\frac{d+e}{2} < c$.[5] This parametric relation imposes benefits associated with deviation from cooperation, being (N, N) the only Nash equilibrium in the static version of the game. In the infinite repetition of the static game, cooperation may eventually be sustained as a long-run equilibrium under certain setting and parametric conditions. In fact, given that $c > n$, individuals sufficiently patient may chose to cooperate by computing the discounted payoffs of their decisions.

Our analysis consists on evaluating, given an initial scenario of cooperation, under which parametric conditions cooperation can be a stable long-run equilibrium. We assume a *trigger* punishment scheme (Friedman, 1971): an individual plays C until the other player deviates, punishing him by playing N forever.[6] Therefore, there exists a cooperative equilibrium if, given a discount factor, the sum of the discounted benefits when playing C is greater than the sum of the discounted benefits when deviating from that equilibrium. The novelty in this work is that these discounted benefits internalize the possibility that the other player might choose an action different from the expected equilibrium, i.e. deviating from the cooperative equilibrium in future periods or playing a cooperative action given a non-cooperative equilibrium.

We assume that individuals face an utility function $U = \sum_{t \geq 0} \beta^t v_t$ with $\beta \in [0, 1]$ the exogenous discount factor and v_t the payoff modeled as (1), where the best (worst) scenario is the other individual to play C (N).[7] We focus on a symmetric equilibrium, in the sense that both individuals

[5]This condition is imposed for preventing that alternating cooperation and non cooperation is preferred than mutual cooperation in the long-run.

[6]The choice of this punishment scheme is standard in the theoretical analysis of this kind of problems. It can be argued that in some specific situations, other reactions to unexpected deviations may be better suited. We do not deal with that issue.

[7]This comes directly from the payoffs' structure. In the next section, when this relation does not necessarily holds for every parametric combination, we reinterpret α as the probability assigned to other individual playing C.

face the same δ and α parameters.[8] In this context, π is defined as the probability of the other player cooperating, which can be computed endogenously using the game's equilibria. Following a standard treatment of this kind of problems, we can derive Proposition I.

Proposition I: (C, C) is a long-run equilibrium if

$$\beta \quad \geq \quad \beta^*(\delta, \alpha) = \gamma_1 \frac{(d-c)}{(d-n)} + \gamma_2 \frac{(n-e)}{(d-n)}, \tag{2}$$

where $\gamma_1 = \frac{(1-\delta(1-\alpha))}{(1-\delta)}$ and $\gamma_2 = \frac{\delta(1-\alpha)}{(1-\delta)}$.

Proof: See Appendix B.

In the no ambiguity case ($\delta = 0$), the critical value corresponds to

$$\beta^*(0, \alpha) \quad = \quad \frac{d-c}{d-n}.$$

Then, $\beta^*(\delta, \alpha)$ can be interpreted as a linear combination between the critical value in the absence of ambiguity and a second term describing the benefits associated with non-cooperating internalized by the functional form chosen. In fact, $(n - e)$ represents the gains of non-cooperating with respect to the case in which players agree on cooperating but the other individual deviates, which starts to be a plausible case in the ambiguity context considered.[9]

Generally, the literature assumes that β is exogenously distributed to individuals. Therefore, the smaller the critical value is, the higher the probability to achieve cooperation as a long-run equilibrium is. Since with $\delta \neq 0$, we have that $\gamma_1 > 1$ and $\gamma_2 > 0$, we can conclude that the critical value is always larger in a context of ambiguity with respect to a non-ambiguity context.[10] Therefore, ambiguity decreases the probability of achieving a cooperative equilibrium.

5. Application: Cournot's Duopoly Model

One particular case in the analysis of the probability of achieving a cooperative equilibrium in the infinitely repeated Prisoner's Dilemma, is the determination of the likelihood of sustaining a tacit collusion between two firms, given an imperfect competition framework. Thus, we focus on studying this problem in a Cournot competition scenario.

[8] Symmetry plays no role in the key result of this section (Proposition I). In fact, by allowing the players to have specific δ and α parameters, a different critical discount factor is computed for each player, which depends only on her own parameters in the same way the symmetric discount factor does ($\beta^*(\delta_1, \alpha_1)$ and $\beta^*(\delta_2, \alpha_2)$). Then, the relevant critical discount factor is $\max\{\beta^*(\delta_1, \alpha_1), \beta^*(\delta_2, \alpha_2)\}$, which behaves equivalently to the symmetrical discount factor.

[9] It can be seen that when $\alpha = 1$, i.e. the individual is fully optimistic, $\gamma_2 = 0$ and therefore the second term becomes irrelevant. This is consistent with the intuition described, as fully optimistic individuals do not consider the possibility of the other player deviating from a cooperative equilibrium already agreed.

[10] More precisely, γ_1 can be equal to 1 and γ_2 can be equal to 0, but the equalities cannot be held simultaneously, as $\gamma_1 = 1$ implies $\alpha = 0$ and $\gamma_2 = 0$ implies $\alpha = 1$. Therefore, $\beta^*(\delta, \alpha) > \beta^*(0, \alpha)$ holds for every parametric combination with $\delta \neq 0$.

We assume the existence of two firms competing in quantities, producing a homogeneous product with constant marginal cost k and facing an inverse demand function $P(Q) = A - bQ$, where $A > k$ and $b > 0$. In this framework, given the existence of ambiguity on the other player's action, firms consider the possibility of the other firm choosing a quantity different from the expected action, i.e. the standard Cournot equilibrium, when they choose their own optimal production. Then, in the light of equation (1) and given the starting point of no cooperation, the objective function for firm i is defined by

$$\begin{aligned} \max_{q^N} V_N &= (1-\delta)(A - b(q^N + q_j))q^N + \delta\left[\alpha(A - b(q^N + q^M))q^N\right. \\ &\quad \left. + (1-\alpha)(A - b(q^N + q_j))q^N\right] - kq^N, \end{aligned} \tag{3}$$

where q_j is the quantity optimally produced by the other firm. Here, we consider the possibility of an *optimistic* firm to think that the other firm may produce the collusive quantity, q^M. At the same time, the collusive quantity comes from maximizing the following problem

$$\begin{aligned} \max_{q^M} V_M &= (1-\delta)(A - 2bq^M)q^M + \delta\left[\alpha(A - 2bq^M)q^M\right. \\ &\quad \left. + (1-\alpha)(A - b(q^M + q^D))q^M\right] - kq^M, \end{aligned} \tag{4}$$

where *pessimistic* firms consider the possibility of the other firm deviating from equilibrium and optimally choosing q^D, which in turn comes from the following problem

$$\begin{aligned} \max_{q^D} V_D &= (1-\delta)(A - b(q^D + q^M))q^D + \delta\left[\alpha(A - b(q^D + q^M))q^D\right. \\ &\quad \left. + (1-\alpha)(A - b(q^D + q_j))q^D\right] - kq^D, \end{aligned} \tag{5}$$

where a *pessimistic* firm considers the possibility of the other firm taking the same decision simultaneously. Then, by taking first order conditions an applying symmetry, we can derive the reaction functions to then compute the equilibrium quantities.[11] Proposition II summarizes those results.

Proposition II: In the scenario considered above, the equilibrium quantities are given by

$$\begin{aligned} q^N(\delta,\alpha) &= \frac{(A-k)}{b(3-\delta\alpha)}\left[\frac{2 + (1-\delta(1-\alpha))(6-\delta(3-\alpha))}{2 + 3(1-\delta(1-\alpha))(2-\delta(1-\alpha))}\right], \\ q^M(\delta,\alpha) &= \frac{2(A-k)(1-\delta(1-\alpha))}{b(2 + 3(1-\delta(1-\alpha))(2-\delta(1-\alpha))}, \\ q^D(\delta,\alpha) &= \frac{(A-k)(2-\delta(1-\alpha))(3-\delta(1-\alpha))}{b(2-\delta(1-\alpha))(2 + 3(1-\delta(1-\alpha))(2-\delta(1-\alpha)))}. \end{aligned}$$

Proof: See Appendix C.

[11] In this specific case, the symmetry assumption is taken to simplify the analytical derivation of equilibrium quantities, as reaction functions depend on the other firm quantities, which in turn depend on the other firm parameters. Nevertheless, if heterogeneity in the ambiguity and optimism parameters is assumed, it is no clear how the collusion should behave; for example, in terms of splitting the production. Thus, further assumptions must be considered if the symmetry assumption is going to be lifted.

With this result, we can see how equilibrium quantities of the static competitive equilibrium vary with the level of ambiguity, δ, and the level of optimism, α. Figure 1 shows that, for different given levels of optimism, the quantity produced in a non cooperative equilibrium increases monotonically with the degree of ambiguity.[12] This result differs from Eichberger, Kelsey, and Schipper (2009), who conclude that in a Cournot duopoly, rises in ambiguity only increase output given *high* levels of ambiguity. Likewise, Figure 2 shows that, for different given levels of ambiguity, rises in the degree of optimism are also accompanied by an increase in output.[13] This result is consistent with the cited work. Then, our exercise suggests that ambiguity increases static competition.

Figure 1: Quantities of the Competitive Equilibrium: α fixed

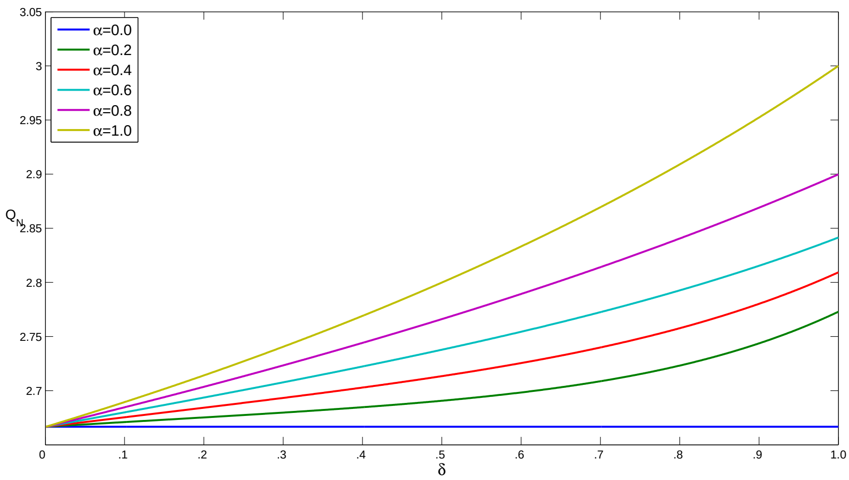
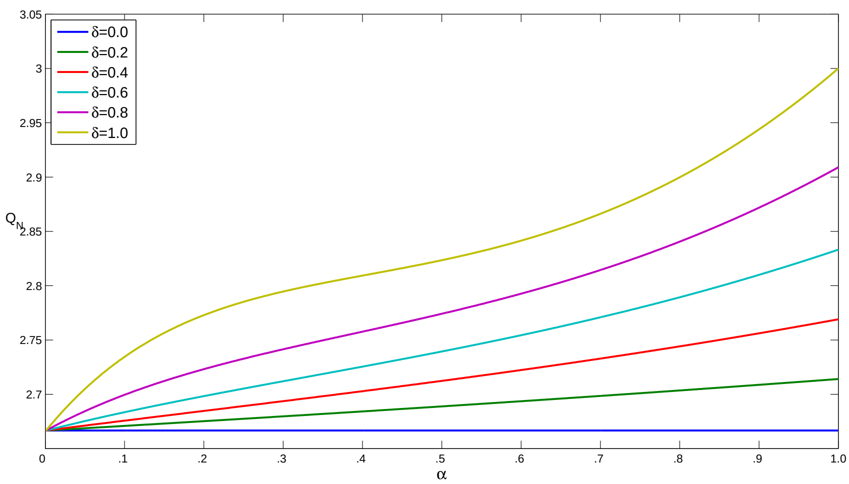


Figure 2: Quantities of the Competitive Equilibrium: δ fixed



[12] The exception is $\alpha = 0$, as firms totally pessimistic will always think that their counterparts will compete, regardless the level of ambiguity.

[13] This does not hold for $\delta = 0$, as in that case there is no ambiguity and therefore optimism plays no role at all.

The differences with respect to Eichberger, Kelsey, and Schipper (2009) arise because different best and worst outcomes are considered here. In concrete, the best and worst scenarios contemplated by us come from an optimization problem, which is aligned with the case we are studying (the potential long-run cooperative equilibrium). By contrast, Eichberger, Kelsey, and Schipper (2009) consider the best scenario when the other player produces zero and the worst scenario the when the counterpart produces high enough to set the price to zero.

The next step is to compute the static payoffs of the game to make the long-run analysis. Specifically, we have that

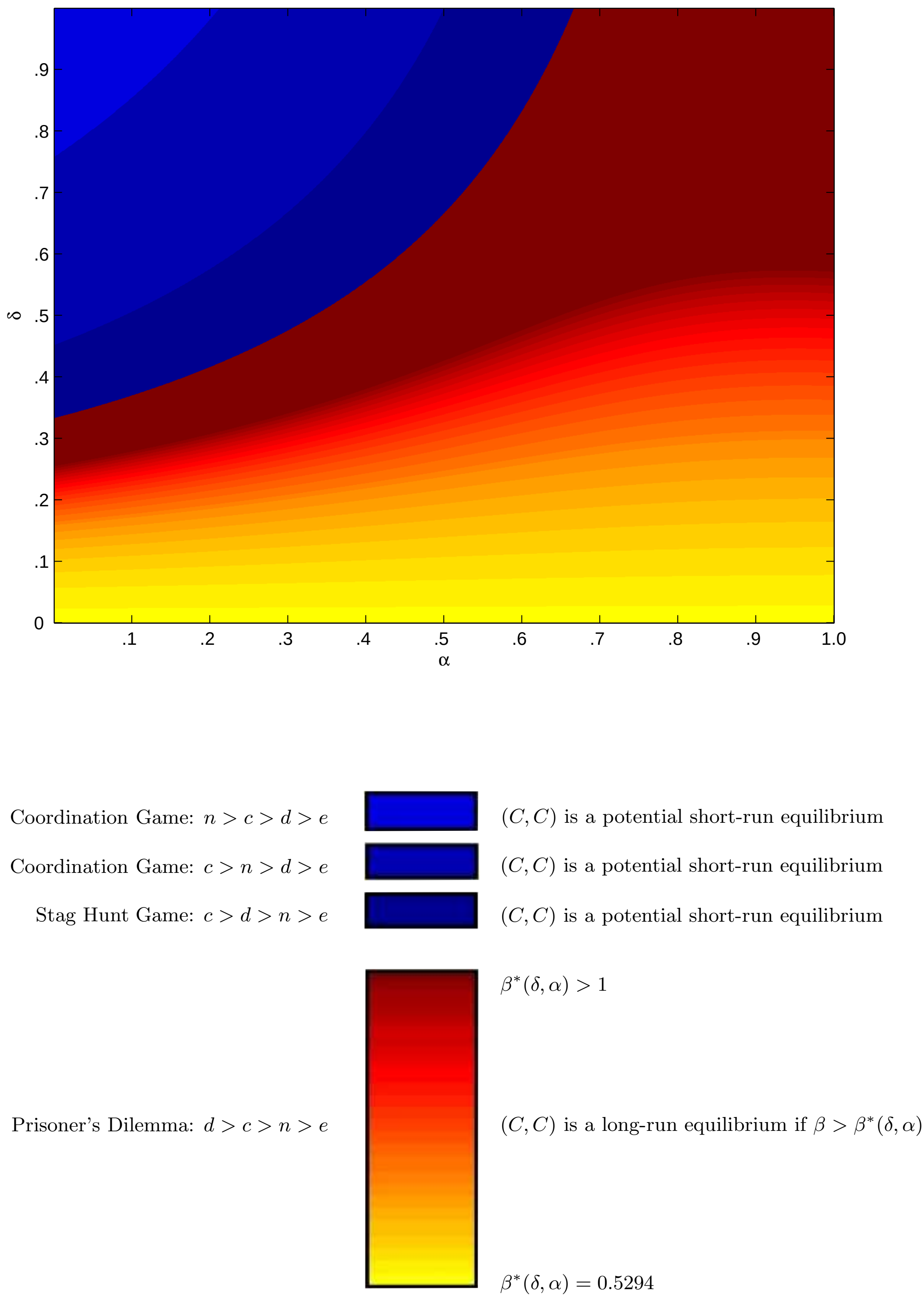
$$
\begin{aligned}
d = d(\delta, \alpha) &= (A - bq^M - bq^D - k)q^D, \\
c = c(\delta, \alpha) &= (A - 2bq^M - k)q^M, \\
n = n(\delta, \alpha) &= (A - 2bq^N - k)q^N, \\
e = e(\delta, \alpha) &= (A - bq^M - bq^D - k)q^M.
\end{aligned}
$$

Considering that the payoffs are endogenous, i.e that depend on δ and α (and in a high non-linear way), it is not obvious that the parametric relation that defines the Prisoner's Dilemma holds for every combination of δ and α. Thus, for combinations that preserve the Prisoner's Dilemma ordering, we can use (2) to compute the critical discount factor that describes the possibility of sustaining a collusive agreement. On the other hand, if there are parametric combinations that induce different games, it is relevant to ask whether cooperation is feasible in these new structures.

Figure 3 summarizes those calculations. The yellow-to-brown colored region represents the parametric combinations of δ and α that preserve the Prisoner's Dilemma relation, i.e. parametric combinations in which $d > c > n > e$. The three blue colored regions represent combinations that induce different game structures. Therefore, the first conclusion is that, given the presence of ambiguity, it is not clear that the Cournot's duopoly model can always be interpreted as a particular case of the Prisoner's Dilemma. This can be interpreted as an equivalence result, which is discussed with more detail in Section 7.

Moreover, the different blue areas represent game structures in which cooperation can be implemented as a short-run Nash equilibrium. In particular, the three ones represent games where there are two Nash equilibria in pure strategies, namely, (C, C) and (N, N). The payoffs of the darkest area are ordered as $c > d > n > e$, then mimicking the *Stag Hunt* game. The two remaining areas, from darker to lighter, order their payoffs as $c > n > d > e$ and $n > c > d > e$, then mimicking *Pure Coodination* games where multiple Nash equilibria exist. Thus, given low levels of optimism (understood as the belief a firm has on the other firm of playing C), high levels of ambiguity could induce game structures that allow the possibility of sustaining cooperation even as a short-run Nash equilibrium.

Figure 3: Critical Discount Factors



The yellow-to-red colored area shows how the critical value describing the possibility of cooperating varies with changes on the key parameters. In concrete, when $\delta = 0$, we have the critical value associated with the no ambiguity case, i.e. $\beta^*(0, \alpha) = 0.5294$. As the areas start getting darker (redder), we have that the critical value starts to increase, which can be interpreted as lower probabilities to sustain a collusive equilibrium. In general, it can be seen that regardless the level of optimism, higher levels of ambiguity decrease the probability of collusion as long as the parametric relation in the Prisoner's Dilemma holds, i.e. increase the potential competition in the long-run. It does so up to the brown area in the center, where collusion is no longer possible: in that area, the critical value is higher than one. This result is surprising, as it suggests that regardless the patience of the firms, some parametric combinations may induce payoffs that impede the possibility of collusion. Of course, the degree of optimism affects the level of uncertainty needed to arrive to this zone. Nevertheless, it remains impressive that this zone exists for every level of optimism.

Summarizing, Figure 3 can be read as follows. Given a level of optimism, there is a chance of cooperation for *low* levels of ambiguity. However, as ambiguity increases, the possibility of a collusive equilibrium decreases, given the rise in the critical discount factor associated. This is true until a certain point, where beyond it is no longer possible to achieve a collusive equilibrium as the critical discount factor goes above one. Despite this, given *low* levels of optimism, cooperation can arise again if this vagueness is *too high* and, moreover, it can potentially be implemented as a short-run Nash equilibrium. This discontinuity in the probability of sustaining a tacit collusion is possibly the most remarkable result, as it states that given *low* levels of optimism, *high* levels of ambiguity reduce the probability of collusion, up to making it impossible, but then reopening the possibility of cooperation but now as a potential short-run equilibrium.

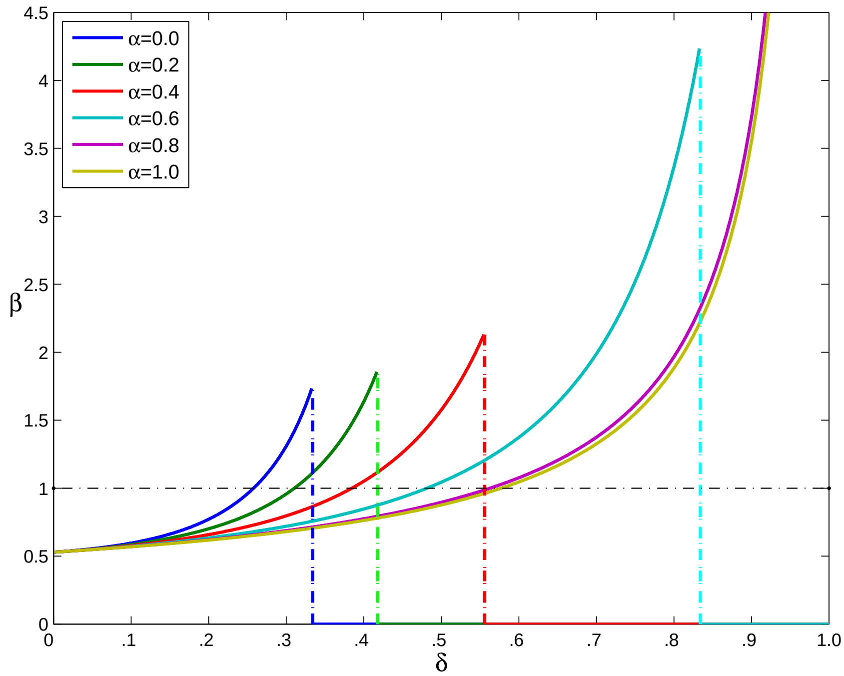
What lies behind this discontinuity? The numerical analysis suggests that all payoffs decrease with ambiguity. But given the high non-linear relation between the payoffs and the key parameters, the decline rates vary between the payoffs, along different levels of optimism.

Thus, for *low* levels of optimism, the payoff d decreases faster than c, which in turn falls faster than n. The reason is that for pessimist players, ambiguity plays almost no role in the payoff of competing in quantities, as the expected *worst* scenario is the same as the no ambiguity case. In contrast, pessimist players internalize that even when agreeing on cooperating, the other player may deviate and therefore, they collude on a higher quantity, receiving a lower benefit. Given that, deviating from a cooperative equilibrium for pessimistic individuals is less attractive, even more if their pessimism suggests them that the other player may deviate simultaneously. This is shown in Figure 3, as the brown area is reached at higher values of δ when α increases. Then, as d is falling faster than c, there is a point where the relation between them is reverted and starts holding $c > d$ (two darkest blue areas). This has two potential stable Nash equilibria: (C, C) and (N, N). Finally, as c is falling faster than n, there is also a point where their relation reverts and $n > c$ starts holding (lightest blue area). Under this scenario of high ambiguity, the potential cooperative agreement may be Pareto dominated by the non-cooperative equilibrium.

On the other hand, for *high* levels of optimism, d and c fall slower than n. In fact, optimistic players produce higher equilibrium quantities when competing, as they think that their counterparts may deviate from the non-cooperative equilibrium, obtaining lower benefits from competition. In contrast, they collude on smaller quantities, as they think their counterparts will comply the agreement, hence, driving higher the benefits from deviating. Then, the payoffs will never change their order and, more interesting, deviation is so profitable in the short-run that beyond certain level of ambiguity there will be no possibility of sustaining a tacit collusion.

Figure 4 presents an alternative way of understanding the discontinuity already discussed. It shows how the critical discount factor, $\beta(\delta, \alpha)^*$, behaves as the level of ambiguity varies for given values of optimism. It can be seen that, for pessimistic individuals, the discount factor needed to sustain a tacit collusion increases monotonically with the level of ambiguity, reaching values even higher than one, up to a point that patience is no longer needed to agree on colluding. The thresholds on the level of ambiguity for a given value of optimism in which the critical discount factor starts being higher than one and then shifts to zero, are the same ones in which the areas in Figure 3 become brown and blue, respectively.[14] On the other hand, it can be seen that for optimistic individuals the critical discount factor grows monotonically with ambiguity, even to values higher than one, but never shifts to zero. In terms of Figure 3, once players enter the brown area, they stay there for all higher ambiguity levels.[15]

Figure 4: Critical Discount Factors



[14]When we say that the critical discount factor shifts to zero, we are referring to the fact that in the new game structure conformed by the change in payoff's ordering, there is no need for a certain discount factor to agree on colluding. In fact, the collusion can be reached even without any weight on future payoffs, i.e. $\beta = 0$, as a short-run Nash equilibrium.

[15]The y-axis of Figure 4 is truncated in 4.5 for a matter of presentation. The curves associated with $\alpha = 0.8$ and $\alpha = 1$ grow monotonically for all values of δ.

6. Empirical Discussion

The previous section shows an impressive result which holds only for certain parametric combinations. Are those combinations empirically plausible? This section shortly addresses this question, by reviewing some empirical literature, specially related with the optimism parameter.[16]

Calibration of the ambiguity parameter, δ, certainly depends on the specific situation. In fact, it represents the distrust over the expected action taken by the other player, hence, it will be affected by the degree of familiarity between players, the specific economic context, and so on. In that sense, δ seems to be *situation* specific. It is reasonable to think that *high* levels of ambiguity can be seen in certain economic situations related with our economic application, for example, in infant industries or highly regulated contexts, where communication between competitors is restricted. Therefore, it is possible to find real-life situations characterized by different degrees of ambiguity along the $[0, 1]$ interval.

On the other hand, the optimism parameter, α, seems to be *player* specific. Thus, several empirical studies, mostly experimental, have analyzed whether individuals tend to be ambiguity averse (seekers), i.e. pessimists (optimists). The broad picture of the literature suggests that the evidence is mixed, finding heterogeneity of ambiguity aversion across individuals and situations. Therefore, the critical zones described in the previous section may be feasible for some individuals and situations, and hence, relevant. Either the blueish zone, when emerges a new game where cooperation can potentially be implemented as a short-run equilibrium, or the brown zone, when cooperation is unfeasible even in the long-run, can be compatible with some real economic situations. In fact, Cohen, Jaffray, and Said (1985) find diverse attitudes toward ambiguity, ranging from pessimism to optimism. Moreover, according to Camerer and Weber (1992), heterogeneity exists not only between individuals, but also within them when facing different situations. The findings of Abdellaoui, Baillon, Placido, and Wakker (2011) also support the existence of heterogeneity across individuals.

Going more in detail, some literature have found evidence in favor of ambiguity aversion, see for example Yates and Zukowski (1976), Curley and Yates (1985) and Cohen, Tallon, and Vergnaud (2009). For instance, Ahn, Choi, Gale, and Kariv (2014) evaluate four ambiguity models in a laboratory experiment and find that, while there exists heterogeneity in preferences, most individuals exhibit some degree of pessimism. Borghans, Golsteyn, Heckman, and Meijers (2009) argue that there are differences in attitudes towards ambiguity among genders, with women being more pessimistic than men in situations with high levels of ambiguity.

Other works argue that optimism can be found in some given contexts. For example, Viscusi and Chesson (1999) argue that attitude toward ambiguity depends on benchmark probabilities. Then, when faced with *high* probabilities of *winning*, individuals tend to behave pessimistically, while when faced with *low* probabilities of *winning*, they tend to be optimistic. Likewise, Heath and Tversky (1991) find that when individuals feel competent in the context considered, they tend to be optimistic.[17]

[16] Most of the works presented here, are taken from Etner, Jeleva, and Tallon (2012) survey.

[17] Some studies have dealt with attitudes towards ambiguity within specific populations. For example, Cabantous

Finally, Ludwig and Zimper (2013) use neo-capacities to model subjective life expectancy estimations. Their empirical estimates of the optimism parameter are around 0.4. Kelsey and le Roux (2014) also model ambiguity with neo-capacities and find, in an experimental study of a cooperative game, that individuals tend to behave as ambiguity averse players instead of playing the Nash equilibria. An important contribution of that paper is that no correlation is found between the behavior in coordination games and the behavior in single experiments, then supporting the idea that attitude toward ambiguity varies with the situation even when considering a single individual.

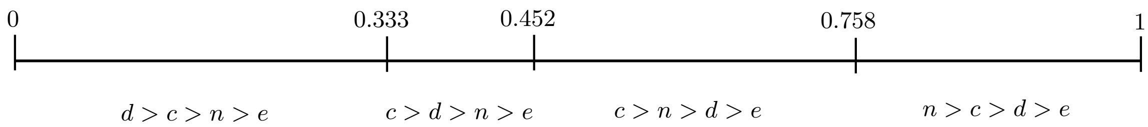
7. Alternative Interpretation: An Equivalence Result

Section 5 showed that when individuals are relatively pessimistic, and ambiguity is sufficiently high, the Courtnot's duopoly stops behaving as a particular case of the Prisoner's Dilemma and; by contrast, it starts behaving as different games in which cooperation can be implemented as a short-run equilibrium. This section argues that this fact can be interpreted as an equivalence result.

We refer to equivalence result a situation in which, given a specific context, there is more than one model than can account for or explain the observed behavior, i.e. a case in which different settings yield indistinguishable outcomes. A classical example in the industrial organization literature is Kreps and Scheinkman (1983). They showed that a model with quantity precommitment and Bertrand competition produce equivalent results to a model with Cournot competition.

In that line, we argue that a Cournot duopoly with *relatively* pessimistic firms facing a *relatively* high degree of ambiguity may behave equivalently to games with exogenous payoffs in which cooperation is one of the potential short-run Nash equilibria. To illustrate the idea, Figure 5 shows how the game's payoffs are ordered for different levels of ambiguity, when firms are completely pessimistic ($\alpha = 0$). It can be seen that the outcome of the Cournot duopoly with ambiguity and pessimistic firms is equivalent to a Stag Hunt game with exogenous payoffs when $\delta \in [0.333, 0.452)$; to a Coordination Game with exogenous payoffs and (C, C) the Pareto optimal equilibrium when $\delta \in [0.452, 0.758)$; and to a Coordination Game with exogenous payoffs and (N, N) the Pareto optimal equilibrium when $\delta \in [0.758, 1]$.

Figure 5: Games for Different Values of δ when $\alpha = 0$



(2007) finds that insurance professionals tend to be ambiguity averse, and that the level of pessimism varies with the source of ambiguity. Other examples are given by Salmon and Kozhan (2008), who report evidence on pessimism in foreign exchange markets, or Potamites and Zhang (2012), who develop an experiment with Chinese investors, suggesting the existence of heterogeneity across individuals.

The same analysis can be made for every given value of α, where each level of optimism can induce different thresholds for δ. In particular, when $\alpha \in [0, 0.212)$, there are four possible games (Prisoner's Dilemma, Stag Hunt Game and two Different Coordination Games) and therefore three different equivalence results. When $\alpha \in [0.212, 0.5)$ there are three possible games (Prisoner's Dilemma, Stag Hunt Game and one Coordination Game) and therefore two different equivalence results. When $\alpha \in [0.5, 0.666)$ there are two possible games (Prisoner's Dilemma and Stag Hunt Game) and therefore one equivalence result. Finally, when $\alpha \in [0.666, 1]$ there is only one possible game (Prisoner's Dilemma) and therefore, there is no equivalence result. Hence, the parametric combinations that conform the three blue areas of Figure 3 determine specific equivalence results, with specific thresholds.

This interpretation is interesting, as the cases characterizing the blue areas are well known games with easy analytic tractability. Therefore, by being aware of this equivalence, other analysis of the Cournot duopoly with ambiguity could be undertaken with greater simplicity.

8. Conclusions

Ambiguity has become an interesting topic, as it has been able to model decision making in contexts where it is not possible to assume that probability distributions are known, or alternatively, when there are reasons to distrust some existing subjective beliefs. Among other topics, several applications have been made in game theory and industrial organization. Our work relates with this literature by analyzing the effects of ambiguity on long-run cooperation.

By studying the infinitely repeated Prisoner's Dilemma and its application to Cournot's duopoly model, we find that ambiguity decreases the probability of sustaining a cooperative equilibrium by showing that the minimum subjective discount factor needed to achieve cooperative agreements increases with the level of ambiguity. In the specific case of the Cournot duopoly, ambiguity increases the static equilibrium quantities. Therefore, the results suggest that ambiguity increases competition in the short and in the long-run, given that it increases the static equilibrium quantities and decreases the likelihood of sustaining a tacit collusion.

Despite that, we find that a discontinuity may arise in the Cournot's analysis. In concrete, for some parametric combinations, Cournot's duopoly stops being a particular case of the Prisoner's Dilemma and becomes behaving as other games in which collusion can be achieved even as a short-run Nash equilibrium. Therefore, different parametric combinations suggest different predictions about the potential cooperative behavior: the higher the ambiguity, the lesser the probability of cooperating in the long-run. This is true unless players are sufficiently pessimistic and ambiguity is sufficiently high, to start playing within a different setting where collusion might be implemented as a short-run equilibrium. As it was argued before, this can be alternatively interpreted as an equivalence result between a Cournot duopoly with ambiguity and pessimistic firms, and different well-known static games with exogenous payoffs, which are characterized by a higher tractability than the initial structure. This result can be exploited to make different analyses of Cournot duopolies in the presence of ambiguity.

Beyond the specific results showed in this paper, this work is also helpful in illustrating how standard theories may be affected when ambiguity is justifiably incorporated. In many economic situations, it is assumed that individuals face a probability distribution to deal with unknown events. Nevertheless, in many cases it is reasonable to consider that individuals may distrust their distributions or even it might not be possible to assume their existence. Hence, economic theory should continue exploring whether classical results still hold if uncertainty is treated in a different way. There is a large body of related literature, yet much remains to be done.

Appendix A. Ambiguity and Neo-Capacities

Ambiguity, defined simply as the existence of uncertainty about beliefs on future states, has been modeled in several ways, usually by using *capacities* to represent beliefs. Given a finite space X and its correspondent power set 2^X, a capacity $v : 2^X \to \mathbb{R}_+$ is a function that satisfies,

$$\begin{aligned} v(\phi) &= 0, \\ v(A) &\leq v(B) \quad \text{if } A \subseteq B, \\ v(X) &= 1. \end{aligned}$$

A capacity is said to be convex if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ (concave if the relation holds with $\geq$). Hence, capacities not necessarily comply the additivity law of probabilities, constituting a generalization of the concept of probabilities.

Intuitively, capacities can represent ambiguity in the sense that, given their non-additivity, the sum of the probabilities of all possible states does not necessarily sum one. Then, for example, the weight assigned to the union of two excluding acts may be greater than the sum of the weights assigned to each act individually, when the individual faces ambiguity aversion (i.e., if her beliefs are represented by convex capacities). In this setting, integral of a function $f : X \to \mathbb{R}$ with respect to a capacity v (the analogous of the expectation in the additive probability framework) is made by Choquet integrals (Choquet, 1954).

Many authors have axiomatized choice under ambiguity models, deriving choice criteria based on Choquet integrals.[18] Chateauneuf, Eichberger, and Grant (2007) axiomatized a choice criterion based on the use of neo-capacities to represent beliefs, where *neo* comes from *non-extreme outcome*. Given a finite space X, a neo-capacity v is a particular capacity defined by

$$v(A) \quad := \quad (1 - \delta)\pi(A) + \delta\mu_\alpha^{\mathcal{N}}(A),$$

for all $A \subset X$, where $\delta \in [0, 1]$ is the degree of ambiguity, π is an additive probability distribution defined over X and $\mu_\alpha^{\mathcal{N}}$ is a Hurwicz capacity exactly congruent with $\mathcal{N} \subset X$ with an $\alpha \in [0, 1]$ degree of optimism, defined by

$$\mu_\alpha^{\mathcal{N}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{N}, \\ \alpha & \text{if } A \notin \mathcal{N} \text{ and } S \setminus A \notin \mathcal{N}, \\ 1 & \text{if } S \setminus A \in \mathcal{N}, \end{cases} \tag{A.1}$$

[18] See, for example, Schmeidler (1989), Gilboa and Schmeidler (1989) and Ghirardato, Maccheroni, and Marinacci (2004).

where S is the set of all possible states and $\mathcal{N} \subset X$ is the set of *null* events, i.e. the set of the states that it is impossible to occur.

Chateauneuf, Eichberger, and Grant (2007) showed that, given a space of actions X, the Choquet integral of the function $f : X \to \mathbb{R}$ with respect to the neo-capacity $v : 2^X \to \mathbb{R}_+$ is defined by

$$\int f dv \quad = \quad \delta(\alpha M + (1-\alpha)m) + (1-\delta)E_\pi f,$$

where E_π is the expectation induced by the additive probability distribution π, $M = \max_{x\in X} f(x)$ and $m = \min_{x\in X} f(x)$. The functional form used by Eichberger, Kelsey, and Schipper (2009) is built on this Choquet integral, which was axiomatized as a choice criterion under ambiguity by Chateauneuf, Eichberger, and Grant (2007).

Appendix B. Proof of Proposition 1

In order to obtain the critical discount factor, we must compare, given an initial scenario of cooperation, the present value of continuing cooperation against the present value of deviating from cooperation, i.e. find β such that $PV_{\text{Coop}} \geq PV_{\text{NoCoop}}$. To calculate the present values we need to know the expected payoffs of playing the different strategies. The one associated with cooperating corresponds to

$$v(C;\delta,\alpha,\pi(p)) = \delta\alpha c + \delta(1-\alpha)e + (1-\delta)(pc + (1-p)e). \tag{B.1}$$

Given an initial scenario of cooperation, we have that $p = 1$ (probability suggested by the game's outcome). Then, (B.1) reduces to

$$v(C;\delta,\alpha,\pi(p=1)) = \delta\alpha c + \delta(1-\alpha)e + (1-\delta)c. \tag{B.2}$$

An individual who decides to commit expects to receive (B.2) in every period and thus PV_{Coop} corresponds to

$$PV_{\text{Coop}} = \frac{\delta\alpha c + \delta(1-\alpha)e + (1-\delta)c}{1-\beta}. \tag{B.3}$$

On the other hand, if the individual decides to deviate from cooperation, in the first period she receives

$$v(N;\delta,\alpha,\pi(p=1)) = \delta\alpha d + \delta(1-\alpha)n + (1-\delta)d,$$

and in all following periods, given the *trigger* punishment scheme, she receives

$$v(N;\delta,\alpha,\pi(p=0)) = \delta\alpha d + \delta(1-\alpha)n + (1-\delta)n.$$

It is worth noting that in the last expression, $p = 0$ as the punishment is known to all players, and therefore once any individual has deviated, the game's equilibrium suggest that both expect the other to no longer cooperate. Thus,

$$PV_{\text{NoCoop}} = \delta\alpha d + \delta(1-\alpha)n + (1-\delta)d + \beta\frac{\delta\alpha d + \delta(1-\alpha)n + (1-\delta)n}{1-\beta}. \tag{B.4}$$

Putting together (B.3) and (B.4) and solving for β, we obtain (2). ■

Appendix C. Proof of Proposition 2

We can reorder terms in (3), (4) and (5), to obtain

$$\begin{aligned}
\max_{q^N} V_N &= \left(A - bq^N - b\left[(1-\alpha\delta)q_j + \alpha\delta q^M\right] - k\right) q^N, \\
\max_{q^M} V_M &= \left(A - bq^M - b\left[(1-\delta(1-\alpha))q^M + \delta(1-\alpha)q_j\right] - k\right) q^M, \\
\max_{q^D} V_D &= \left(A - bq^D - b\left[(1-\delta(1-\alpha))q^M + \delta(1-\alpha)q_j\right] - k\right) q^D.
\end{aligned}$$

Then, the corresponding first order conditions are:

$$\begin{aligned}
A - bq^N - b((1-\alpha\delta)q_j + \alpha\delta q^M) - k - bq^N = 0, \\
A - bq^M - b((1-\delta(1-\alpha))q^M + (1-\alpha)\delta q_j) - k - b(1 + (1-\delta(1-\alpha))q^M) = 0, \\
A - bq^D - b((1-\delta(1-\alpha))q^M + (1-\alpha)\delta q_j) - k - bq^D = 0.
\end{aligned}$$

Applying symmetry, we can derive the reaction functions

$$\begin{aligned}
q^N\left(q^M\right) &= \frac{A - k - b\delta\alpha q^M}{b(3-\delta\alpha)}, \\
q^M\left(q^D\right) &= \frac{A - k - b\delta(1-\alpha)q^D}{2b(2-\delta(1-\alpha))}, \\
q^D\left(q^M\right) &= \frac{A - k - b(1-\delta(1-\alpha))q^M}{b(2-\delta(1-\alpha))}.
\end{aligned}$$

Finally, solving the system yields the final expressions for q^N, q^M and q^D. It is important to note that, given Vives (1990) result, we can guarantee the existence of a symmetric Nash equilibrium. ■

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