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Existence of a competitive equilibrium when all goods are indivisible

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Abstract

We study a production economy where all consumption goods are indivisible at the individual level but perfectly divisible at the overall economy level. In order to facilitate the exchange in this setting, we introduce a perfectly divisible parameter that does not enter into consumer preferences (fiat money). When consumption goods are indivisible, a Walras equilibrium does not necessarily exist. We introduce a rationing equilibrium concept and proof its existence. Unlike the standard Arrow-Debreu model, fiat money can always have a strictly positive price at the rationing equilibrium. In our set up a rationing equilibrium is a Walras equilibrium, provided that the initially endowed of fiat money is dispersed.

Keywords: competitive equilibrium, indivisible goods.

JEL Classification: C62, D51, E41.

1 Introduction

Most economic models assume that consumption goods are perfectly divisible. The rationale behind this assumption is that commodities are usually considered to be almost perfectly divisible, in the sense that the minimal unit is sufficiently insignificant for its indivisibility to be neglected. According to this approach, one should be able to approximate an economy with a sufficiently small level of goods indivisibility, by some idealized economy where they are perfectly divisible. Consequently, it would be reasonable to expect that a competitive equilibrium in this idealized economy should be an approximation of some competitive outcome of the economy with indivisible goods. From Henry [15] we already know that this competitive outcome, if it exists is not necessarily a Walras equilibrium.[1] So the question arises, what competitive outcome of an economy with

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[1] A Walras equilibrium may fail to exist when consumption goods are indivisible (Henry [15]), and even the core of the economy may be an empty set (Shapley and Scarf [28]).

indivisible goods would be approximated by a Walras equilibrium of an economy with perfectly divisible goods?

In order to address the question above, we present a model where indivisibility is negligible at the overall economy level, but relevant at the individual level. This is achieved by assuming that a continuum of agents (consumers and producers) participate in the economy such that individual exchanges are carried out with only indivisible goods. In order to facilitate the exchange in such a setting, like Drèze and Müller [10], we add a continuum parameter as part of the fundamentals of the economy. This parameter might be interpreted as fiat money. It has no intrinsic value whatsoever, since it does not enter into consumers' preferences.[2]

Indivisibility of consumption goods implies that the Walras demand may fail to be upper hemi - continuous. This leads us to introduce a regularized notion of demand, called weak demand. Under the assumptions we consider, the weak demand will always be an upper hemi-continuous correspondence, coinciding with the Walras demand when consumption goods are perfectly divisible. Based on the weak demand, we will then define a rationing equilibrium, where the consumers demand is a refinement of the auxiliary weak demand. At a rationing equilibrium, in order to formulate their demand, consumers will need in addition to the prices an aggregate knowledge on the demand supply imbalance in the market summarized by an endogenously determined pointed cone. This cone indicates the net trade directions for which rationing can occur. We will then be interested in the situations where agents only prefer points in their budget set which would require the execution of a net trade in the cone, for which rationing can occur. The cone is pointed, implying that if there exists a somehow excessive demand for a certain net-trade direction, there occurs no rationing in the exact inverse net-trade direction.

The main result of this paper is the proof of the existence of a rationing equilibrium, with a strictly positive price for fiat money (Theorem 4.1). However, when consumer's initial endowment in fiat money is dispersed, rationing occurs at most for a null set of the consumers, and then the rationing equilibrium reduces to a Walras equilibrium. Hence, the present paper also establishes a Walras equilibrium existence result for the case where all consumption goods are indivisible. The efficiency and core equivalence properties of a rationing equilibrium are studied in Florig and Rivera [12]. There it is proven that rationing equilibria satisfy the First and Second Welfare theorems for weak Pareto optimality, and that they coincide with the rejective core proposed by Konovalov [22].

So far, we have not mentioned the vast literature on indivisible goods, which can be roughly

[2]In the convex case, when the non-satiation assumption does not necessarily hold for consumers, one may establish the existence of a competitive equilibrium by allowing for the possibility that some agents spend more than the value of their initial endowment. The corresponding generalization of the Walras equilibrium is called dividend equilibrium or equilibrium with slack (see Aumann and Drèze [4], Balasko [5], Makarov [23] and Mas-Colell [25] among others). This concept was first introduced in a fixed price setting by Drèze and Müller [10]. Kajii [19] shows that this dividend approach is equivalent to considering a Walras equilibrium with an additional commodity called fiat money. If local non-satiation holds, fiat money is worthless and we are back in the standard Arrow-Debreu setting. However, if the satiation problem occurs, fiat money may have a positive price at equilibrium. If a consumer does not want to spend his entire income on goods, he can satisfy his budget constraints with equality by buying fiat money, provided that fiat money has a positive price. In our set up, since all goods are indivisible local non-satiation cannot hold at any point.

divided into two different approaches. The first approach follows Shapley and Scarf [28], who model a market without perfectly divisible goods, assuming only one commodity per agent. Under suitable conditions, they prove that the core of the economy is non-empty and that a competitive equilibrium exists. Subsequent extensions of their results can be found in Inoue [17, 18], Konishi et ál [21], Sönmez [29] and Wako [31]. For these models, the existence of a competitive equilibrium depends strongly on the number of agents and/or the number of indivisible goods existing in the economy.

The second approach follows Henry [15], and considers an economy with indivisible commodities but at least one perfectly divisible commodity, which might be interpreted as commodity money (see Bikhchandani and Mamer [6], Broome [8], Khan and Yamazaki [20], Mas-Colell [24], Quinzii [26], van der Laan et al. [30]; see Bobzin [7] for a survey). All these works suppose that money satisfies overriding desirability, i.e., it is so desirable by the agents that an adequate amount of money could replace the consumption of any bundle of indivisible goods. Under such an assumption they can prove the non-emptiness of the core and the existence of a Walras equilibrium.

The approach we follow is similar to the one developed by Dierker [9], who established the existence of a quasi-equilibrium for exchange economies without perfectly divisible consumption goods. However in that approach, agents do not necessarily receive an individually rational commodity bundle at a quasi-equilibrium, a drawback that a rationing equilibrium overcomes.

This paper is organized as follows. Mathematical preliminaries used throughout this paper are presented in Section 2. The economic model, as well as the equilibrium notions, are introduced in Section 3. Section 4 is devoted to present the equilibrium existence results. Most of the proofs are established in the appendix, i.e. Section 5.

2 Mathematical notations

In the following, x^{t} is the transpose of a vector $x \in \mathbb{R}^m$, $x \cdot y = x^{\mathrm{t}} y$ the inner product between $x, y \in \mathbb{R}^L$, $\|x\|$ the Euclidean norm of x, and $x^{\perp} = \{p \in \mathbb{R}^m,\, p \cdot x = 0\}$ is the hyperplane in $\mathbb{R}^m$ orthogonal to x. The origin of $\mathbb{R}^m$ is 0_m, and the open ball with center x and radius $\varepsilon > 0$ is $\mathbb{B}(x, \varepsilon)$. Additionally, $\operatorname{cl} K$, $\operatorname{int} K$ and $\operatorname{conv} K$ denote, respectively, the closure, interior and the convex hull of subset $K \subseteq \mathbb{R}^m$, while its positive hull is

$$\operatorname{pos} K = \left\{ \sum_{i=1}^{s} \mu_i x_i \mid \ \mu_i \geq 0,\ x_i \in K,\, i = 1, \ldots, s,\ s \in \mathbb{N} \right\}.$$

A cone $K \subset \mathbb{R}^m$ is "pointed" when $K \cap -K = \{0_m\}$, and the set of pointed cones of $\mathbb{R}^m$ is denoted $\mathcal{C}_m$. Furthermore, for a couple of sets K_1, $K_2 \subseteq \mathbb{R}^m$, $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^m$, we denote $\xi K_1 = \{\xi x,\, x \in K_1\}$, $p \cdot K_1 = \{p \cdot x,\, x \in K_1\}$ and $K_1 \pm K_2 = \{x_1 \pm x_2, x_1 \in K_1,\, x_2 \in K_2\}$, while the set-difference between K_1 and K_2 is $K_1 \setminus K_2$.

By denoting

$$\mathbb{N}_\infty = \{\mathrm{N} \subseteq \mathbb{N} \,|\, \mathbb{N} \setminus \mathrm{N} \text{ is finite}\} \quad \text{and} \quad \mathbb{N}^*_\infty = \{\mathrm{N} \subset \mathbb{N} \,|\, \mathrm{N} \text{ is infinite}\},$$

the outer limit of sequence of subsets $\{K_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^m$ is the set

$$\limsup_{n\to\infty} K_n \;=\; \{x \in \mathbb{R}^m \,|\, \exists \mathrm{N} \in \mathbb{N}^*_\infty,\ \exists x_n \in K_n,\ n \in \mathrm{N}, \text{ with } x_n \to_{\mathrm{N}} x\},$$

while the inner limit is the set

$$\liminf_{n\to\infty} K_n \;=\; \{x \in \mathbb{R}^m \,|\, \exists \mathrm{N} \in \mathbb{N}_\infty,\ \exists x_n \in K_n, n \in \mathrm{N}, \text{ with } x_n \to_{\mathrm{N}} x\}.$$

We say the sequence converges in the sense of Kuratowski - Painlevé to $K \subseteq \mathbb{R}^m$ if

$$\limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n = K.$$

The outer limit of a correspondence $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^m$ is

$$\limsup_{x\to\bar{x}} F(x) \;=\; \bigcup_{\{x_n\to\bar{x}\}} \limsup_{n\to\infty} F(x_n), \tag{1}$$

and this correspondence is said outer semicontinuous at $\bar{x}$ if $\limsup_{x\to\bar{x}} F(x) = F(\bar{x})$. We note the outer semicontinuity F at $\bar{x}$ is not equivalent to the upper hemi-continuity of F at that point.[3] However, when F is closed valued, by Theorem 5.19 in Rockafellar and Wets [27] the equivalence holds under the condition that F is locally bounded at $\bar{x}$.

Given $\mathrm{N} \in \mathbb{N}^*_\infty$ and $\{z_n\}_{n\in\mathrm{N}}$ a sequence of elements in $\mathbb{R}^m$, we denote by

$$\mathrm{acc}\,\{z_n\}_{n\in\mathrm{N}} \;=\; \{z \in \mathbb{R}^m | \exists \mathrm{N}' \subset \mathrm{N},\, \mathrm{N}' \in \mathbb{N}^*_\infty,\, z_n \to_{\mathrm{N}'} z\}$$

the accumulation points of $\{z_n\}_{n\in\mathrm{N}}$.

We now remind the integral of a correspondence $F : K_1 \rightrightarrows K_2$, where $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subseteq \mathbb{R}^\ell$. For the aim of this paper, it is sufficient to consider that K_1 is a compact set, and K_2 is a closed set. Provided they are non-empty sets, using the standard Lebesgue measure, the set of Lebesgue integrable functions from K_1 to K_2 is $L^1(K_1, K_2)$, and the Lebesgue integral of $f \in L^1(K_1, K_2)$ is denoted by $\int_{K_1} f(t)dt$. Following Aubin and Frankowska [2], §8.6, we define

$$\int_{K_1} F(t)dt \;=\; \left\{ \int_{K_1} f(t)dt \,|\, f \in L^1(K_1, K_2), f(t) \in F(t) \text{ for a.e. } t \in K_1 \right\}.$$

[3] In this paper we use the notion of upper hemi - continuity as stated in Hildenbrand [16]: for every open set O such that $F(\bar{x}) \subset O$ there is a neighborhood V of $\bar{x}$ such that $F(x) \subset O$ for every $x \in V$.

3 The model

3.1 The economy and assumptions

By abuse of notation, we denote by $L = \{1, \ldots, L\}$, $I = \{1, \ldots, I\}$ and $J = \{1, \ldots, J\}$ the finite sets of consumption goods, consumers and firms, respectively. We assume that each type of agent $i \in I$ and $j \in J$ corresponds to a continuum of identical individuals indexed by compacts subsets $T_i \subset \mathbb{R}$, $i \in I$, and $T_j \subseteq \mathbb{R}$, $j \in J$, pairwise disjoint. Given that, the set of consumers and firms is respectively denoted by

$$\mathcal{I} = \bigcup_{i \in I} T_i \qquad \text{and} \qquad \mathcal{J} = \bigcup_{j \in J} T_j.$$

The type of producer $t \in \mathcal{J}$ is $j(t) \in J$, and each firm of type $j \in J$ is characterized by a production set $Y_j \subset \mathbb{R}^L$. The aggregate production set for firms of type $j \in J$ is the convex hull of $\lambda(T_j)Y_j$, where $\lambda(\cdot)$ is the standard Lebesgue measure in $\mathbb{R}$. A production plan for a firm $t \in \mathcal{J}$ is denoted by $y(t) \in Y_{j(t)}$, and the set of admissible production plans is

$$Y = \left\{ y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y(t) \in Y_{j(t)} \text{ a.e. } t \in \mathcal{J} \right\}.$$

The type of consumer $t \in \mathcal{I}$ is $i(t) \in I$, and each consumer of type $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$, an initial endowment of resources $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i : X_i \rightrightarrows X_i$. A consumption plan of an individual $t \in \mathcal{I}$ is denoted as $x(t) \in X_{i(t)}$, and the set of admissible consumption plans is

$$X = \left\{ x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x(t) \in X_{i(t)} \text{ a.e. } t \in \mathcal{I} \right\}.$$

The total initial resources of the economy is $e = \sum_{i \in I} \lambda(T_i) e_i \in \mathbb{R}^L$, and for $(i, j) \in I \times J$, $\theta_{ij} \in [0, 1]$ is the consumer of type i's share in firms of type j. As usual, we assume for every $j \in J$, $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$. In addition, we also assume that each consumer $t \in \mathcal{I}$ is initially endowed with an amount of fiat money $m(t) \in \mathbb{R}_+$, assuming $m \in L^1(\mathcal{I}, \mathbb{R}_+)$. Note that consumers t, t' of the same type may be initially endowed with different amounts of fiat money.

An economy $\mathcal{E}$ is a collection

$$\mathcal{E} \;=\; \left((X_i, P_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J}, m, \{T_i\}_{i \in I}, \{T_j\}_{j \in J}\right),$$

and the feasible consumption-production plans of $\mathcal{E}$ are the elements of

$$A(\mathcal{E}) \;=\; \left\{ (x, y) \in X \times Y | \int_{\mathcal{I}} x(t)dt = \int_{\mathcal{J}} y(t)dt + e \right\}.$$

The following assumptions will be used at different points in this paper. The strongest condition we will use is the finiteness of consumption and production sets (i.e. the number of their elements is finite). The rest of our assumptions are relatively weak. Indeed, we will not need a strong survival condition, that is our consumers may not initially own a strictly positive quantity of every good, and the interior of the convex hull of the consumption sets may even be an empty-set (compare with Arrow and Debreu [1]).

Assumption F. *For all $i \in I$, and for all $j \in J$, the sets X_i and Y_j are finite.*

Assumption C. *For all $i \in I$, P_i is irreflexive and transitive.*[4]

Assumption S. *For all $i \in I$, $e_i \in \operatorname{conv} X_i - \sum_{j \in J} \theta_{ij} \operatorname{conv}(\lambda(T_j) Y_j)$.*

Assumption M. *The function $m : \mathcal{I} \to I\!\!R_+$ is bounded and for a.e. $t \in \mathcal{I}$, $m(t) > 0$.*

Assumption D. *For all $M \in \mathbb{R}$, $\lambda(\{t \in \mathcal{I} \,|\, m(t) = M\}) = 0$.*

3.2 Supply, demand and the equilibrium concepts

The main goal at this part is to introduce two new equilibrium concepts. The first, called weak equilibrium, is auxiliary, while the second, called rationing equilibrium, is our main concept.

For $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$, $K \in \mathcal{C}_L$ and $j \in J$, we set

$$\pi_j(p) = \lambda(T_j) \sup_{z \in Y_j} p \cdot z, \quad S_j(p) = \arg\max_{z \in Y_j} p \cdot z,$$

$$\sigma_j(p, K) = \{z \in S_j(p) \,|\, p \neq 0_L \Rightarrow (Y_j - \{z\}) \cap K = \{0_L\}\},$$

respectively, the profit, the Walras supply and the rationing supply[5] of type $j \in J$ firms. Observe that, by definition, $\sigma_j(p, K) \subset S_j(p)$. Moreover, when $p \neq 0_L$, for

$$K(p) \quad = \quad \{0_L\} \cup \{z \in \mathbb{R}^L |\, p \cdot z > 0\} \in \mathcal{C}_L, \tag{2}$$

we have $\sigma_j(p, K(p)) = S_j(p)$.

The income of consumer $t \in \mathcal{I}$ is denoted by

$$w_t(p, q) = p \cdot e_{i(t)} + q m(t) + \sum_{j \in J} \theta_{i(t)j} \pi_j(p)$$

and the resulting budget set is

$$B_t(p, q) = \left\{\xi \in X_{i(t)} |\, p \cdot \xi \leq w_t(p, q)\right\}.$$

Given that,

$$d_t(p, q) = \left\{\xi \in B_t(p, q) |\, B_t(p, q) \cap P_{i(t)}(\xi) = \emptyset\right\}, \quad D_t(p, q) = \limsup_{(p', q') \to (p, q)} d_t(p', q'),$$

[4]That is, for each $i \in I$ and $x, y, z \in X_i$, $x \notin P_i(x)$ and if $x \in P_i(y)$ and $y \in P_i(z)$ then $x \in P_i(z)$.

[5]Note that we define the rationing supply in a less restrictive way than in Florig and Rivera [12]. However, in the proofs given therein, it is only the property as defined here which is used.

and

$$\delta_t(p,q,K) = \left\{\xi \in D_t(p,q) | \ \left(P_{i(t)}(\xi) - \{\xi\}\right) \subset K\right\},$$

are, respectively, the Walras, weak and rationing demand for consumer $t \in \mathcal{I}$.

We observe that, by definition, $d_t(p,q) \subset D_t(p,q)$ and $\delta_t(p,q,K) \subset D_t(p,q)$. For $p \neq 0_L$ and the cone given by (2), $d_t(p,q) \subset \delta_t(p,q,K(p))$. Moreover, for $\xi \in \delta_t(p,q,K(p))$ such that $p\cdot\xi = w_t(p,q)$, we have $\xi \in d_t(p,q)$.

Definition 3.1. *Given $(x,y,p,q) \in A(\mathcal{E}) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K \in \mathcal{C}_L$,*

(a) we call (x,y,p,q) a Walras equilibrium with money of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in d_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,

(c) we call (x,y,p,q) a weak equilibrium of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in D_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,

(c) we call (x,y,p,q,K) a rationing equilibrium of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in \delta_t(p,q,K)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in \sigma_t(p,K)$.

Remark 3.1. Note that perfect divisibility of fiat money is not enough to guarantee that the Walras demand $d_t(p,q)$ is upper-hemi continuous when goods are indivisible.[6] However, when $d_t(p,q)$ is closed valued and locally bounded (which we will ensure), its outer regularization $D_t(p,q)$, is upper-hemi continuous (cf Section 2).

The following proposition is proven in Section 5. It has important implications concerning the relationships among the equilibrium concepts we have defined.

Proposition 3.1. *Suppose Assumption* **F** *holds and $q\,m(t) > 0$, then*

$$D_t(p,q) = \left\{\xi \in B_t(p,q) \mid \inf\left\{p \cdot P_{i(t)}(\xi)\right\} \geq w_t(p,q),\ \xi \notin \operatorname{conv} P_{i(t)}(\xi)\right\}.$$

Therefore, if the assumptions of the proposition hold true, then:

$$\delta_t(p,q,K) = \left\{\xi \in B_t(p,q) | \inf\{p \cdot P_{i(t)}(\xi)\} \geq w_t(p,q),\ \left(P_{i(t)}(\xi) - \{\xi\}\right) \subset K\right\}.$$

Furthermore, note that in the standard case of convex consumption sets, provided the budget set has a non-empty interior and preferences are continuous, the weak and rationing demand would then be the standard Walras demand. Florig and Rivera [13] show that weak or rationing equilibria converge under certain assumptions to standard competitive equilibria, when the consumption and production sets converge to convex polyhedra.

[6]For example, assume $J = \emptyset$, and suppose that the preference relation is defined by the utility function $u(x,y) = 2x + y$, the initial endowment is $e = (0,1)$, $m = 1$ and the consumption set is $X = \{0,1\} \times \{0,1\}$; for $(p^n,q^n) = ((1+1/n,1),1/n^2) \to (p,q) = ((1,1),0)$ we have that $\lim_{n\to\infty} d(p^n,q^n) = (0,1)$, and $d(p,q) = (1,0)$. This implies that the Walras demand correspondence is not upper hemi-continuous at $p = (1,1),\ q = 0$.

The following example should motivate the introduction of our main equilibrium concept, and illustrates why the weak equilibrium might only be viewed as an auxiliary concept. Moreover, it illustrates the important role of fiat money in our framework. In this example we consider a finite set of consumers. However, we could replace each consumer by a continuum of identical consumers, with a constant initial endowment of fiat money per type and the same Lebesgue measure for each type, without altering the conclusions.

Example 3.1. Consider an exchange economy with consumers indexed by $I = \{1, 2, 3, 4\}$, and three goods. The consumption sets are given by $X_i = \{0, 1, 2, 3\}^3$ for $i \in I$ and $m_i \geq 0$ is the endowment of fiat money for this agent $i \in I$. Preferences and endowment of resources are given by:

$$\begin{cases} u_1(x,y,z) = 2x + y + z & e_1 = (0,1,1), \\ u_2(x,y,z) = x + 2y + z & e_2 = (1,0,0), \\ u_3(x,y,z) = x + y + 2z & e_3 = (1,0,0), \\ u_4(x,y,z) = x + y + 2z & e_4 = (1,0,0). \end{cases}$$

When for some $m \geq 0$, $m_i = m$, $i \in I$, then there does not exist a Walras equilibrium.[7] On the other hand, assuming $m_4 < m_3$, it is easy to see that there exist a unique Walras equilibrium (x, p, q) (in terms of the allocation), where

$$p = (m_1 + m_2 + m_3 + m_4)(1,1,1) + (0, m_2, m_3) \in \mathbb{R}^3, \qquad q = 1,$$

and

$$x_1 = (2,0,0),\ x_2 = (0,1,0),\ x_3 = (0,0,1),\ x_4 = (1,0,0). \tag{3}$$

We notice, however, when $m_2 = m_3 \geq m_4$ there also exists a weak equilibrium allocation x^*, supported by the same price, with

$$x_1^* = (2,0,0),\ x_2^* = (0,0,1),\ x_3^* = (0,1,0),\ x_4^* = (1,0,0).$$

A situation like this would be, in a certain sense, "unstable" with respect to the information (prices) available by the consumers. For instance, if consumers 2 and 3 knew each others preferences and allocations, they could continue exchanging leading to allocation $x = (x_1, x_2, x_3, x_4)$.

[7] At equilibrium. allocations must be individually rational. At an equilibrium allocation candidate, each consumer consumes at least one unit of one good, and thus by feasibility one consumer consumes two units. Assume (x, p, q) is a Walras equilibrium, then consumers $2, 3, 4$ have all the same budget set. Thus consumers 3 and 4 must obtain the same level of utility. If consumer 1 consumes only one unit, then individual rationality imposes $x_1 = (1, 0, 0)$. In this case it must be 3 (or 4) consuming $(0, 0, 1)$ and then 4 (or 3) would need to consume $(1, 1, 0)$ to attain the same level of utility. Then nothing would be left for consumer 2 to consume. Thus consumer 1 consumes at equilibrium two units of goods. At equilibrium $(0, 0, 1)$ cannot be in the budget set of consumer 3 or 4, since otherwise one of them would consume it and the other would need to consume $(1, 1, 0)$ so that both maximize utility, but only one consumer consumes two units of goods. Then however $p_1 > p_3$ and at equilibrium consumer 1 consumes good three, and this would contradict utility maximization for him.

However, the weak equilibrium (x^*, p, q) cannot be supported by a rationing equilibrium. Otherwise, defining

$$K_i(x_i^*) \quad = \quad \{\xi - x_i^* \,|\, \xi \in X_i,\ u_i(\xi) > u_i(x_i^*)\},\ i \in I, \tag{4}$$

we would need the existence of a pointed cone K such that $K_i(x_i^*) \subseteq K$, for all $i \in I$. However, as $0_3 \neq x_3^* - x_2^* \in K_2(x_2^*)$ and $-(x_3^* - x_2^*) = (x_2^* - x_3^*) \in K_3(x_3^*)$, it follows that such a cone K cannot be pointed.

Using the allocation in (3) and the definition in (4), by setting

$$K(x) = \text{pos}\left(\cup_{i\in I} K_i(x_i)\right),$$

then it is easy to see that $(x, p, q, K(x))$ is a rationing equilibrium.[8] Note that for all $i \in I \setminus \{4\}$, x_i is a maximal element in the budget set, and thus for those consumers the cone $K(x)$ –which was in fact introduced in order to define a concept of demand less restrictive than the Walras demand, but more restrictive than the weak demand– is somehow irrelevant. Consumer $i = 4$ could just about afford $x_4' = (0, 0, 1)$ by selling all his initial endowment in goods and fiat money. The vector $z = x_4' - x_4 = (-1, 0, 1)$ is in $K(x)$, and we could think of z –or including the amount of money of consumer $i = 4$, $(z, -m_4)^{\text{t}}$– as a net-trade direction (in terms of goods and paper money) for which consumers might be rationed. In fact, in the present example it is the only affordable net-trade direction in $K(x)$ which matters here.

Remark 3.2. The Cone K in the rationing equilibrium definition will be determined endogenously as part of the equilibrium, and summarizes the information that each consumer needs to have in addition to market prices (and their own characteristics) in order to formulate a demand, leading to a stable economic situation, in the sense that no further trading can take place making all participants in a second round of trading strictly better off (see Florig and Rivera [12]).[9] It seems natural to us to impose that K is pointed, since if there is some rationing for a net-trade direction $z \in \mathbb{R}^L$, say due to some sort of excessive-demand, then it should be easy to find counterpart for the opposite net-trade direction, $-z$.

4 Existence of equilibrium

The next theorem is the main result of this paper. The proof is given in Section 5, the Appendix.

Theorem 4.1. *If Assumptions* **F**, **C**, **S** *and* **M** *hold, then there exists a rationing equilibrium* (x, y, p, q, K) *with a strictly positive price for fiat money.*

[8] We observe that cone $K(x)$ is pointed, since it is the positive hull of a finite number of points $z \in \mathbb{R}^3$ satisfying $p \cdot z > 0$.

[9] When studying the existence of a Walras equilibrium without transfers, Hammond [14] uses an exogenous closed convex set containing zero to restrict the possible trades among consumers. We do not follow this approach because we do not have ex ante information regarding the directions of trade that we would like to favor in order to exclude unstable allocations a weak equilibrium allocation could produce.

The next result is a direct consequence of Theorem 4.1.

Corollary 4.1. *If Assumptions* **F**, **C**, **S**, **M** *and* **D** *hold, then there exists a Walras equilibrium* (x, y, p, q) *with a strictly positive price for fiat money.*

Proof. Let (x, y, p, q) be a weak equilibrium with $q > 0$ implied by Theorem 4.1. If (x, y, p, q) is not a Walras equilibrium, there would exists $i \in I$ and $T_i' \subseteq T_i$, with $\lambda(T_i') > 0$, such that for all $t \in T_i'$, $\inf\{p \cdot P_{i(t)}(x(t))\} = w_t(p, q)$. By the finiteness of X_i, we can choose T_i' such that $x(t)$ is constant on T_i'. Therefore there exists $\xi \in X_i$, such that for all $t \in T_i'$, $\xi \in P_i(x(t))$ and $p \cdot \xi = w_t(p, q)$. It follows that

$$m(t) = \frac{1}{q}\left(p \cdot \xi - p \cdot e_{i(t)} - \sum_{j \in J} \theta_{i(t)j} \pi_j(p)\right)$$

is constant on T_i', contradicting Assumption **D**. □

5 Appendix

5.1 Proof of Proposition 3.1

Proof. For $t \in \mathcal{I}$ and $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$, let

$$a_t(p, q) = \left\{\xi \in B_t(p, q) \mid \inf\left\{p \cdot P_{i(t)}(\xi)\right\} \geq w_t(p, q),\ \xi \notin \operatorname{conv} P_{i(t)}(\xi)\right\},$$

and by definition (cf Section 2)

$$D_t(p, q) \quad = \quad \left\{\xi \in \mathbb{R}^L \mid \exists (p_n, q_n) \to (p, q),\ \exists \xi_n \to \xi \text{ with } \xi_n \in d_t(p_n, q_n)\right\}. \tag{5}$$

Let $\xi \in D_t(p, q)$, and for $n \in \mathbb{N}$, let ξ_n, p_n and q_n from the identity in (5). For all $n \in \mathbb{N}$, and all $x' \in P_{i(t)}(\xi_n)$,

$$p \cdot x' > w_t(p_n, q_n) \geq p_n \cdot \xi_n,$$

and therefore $\xi_n \notin \operatorname{conv} P_{i(t)}(\xi_n)$. Assumption **F** implies trivially that $P_{i(t)}$ has an open graph in $X_{i(t)}$ and therefore $\xi \notin \operatorname{conv} P_{i(t)}(\xi)$. Moreover, as w_t is well defined and continuous in (p, q), $\inf\left\{p \cdot P_{i(t)}(\xi)\right\} \geq w_t(p, q)$, and therefore $D_t(p, q) \subseteq a_t(p, q)$.

For the opposite inclusion, let $\xi \in a_t(p, q)$. If $p \cdot \xi < w_t(p, q)$, then for $\varepsilon > 0$ small enough, $\xi \in d_t(p, q - \varepsilon)$, and therefore $\xi \in D_t(p, q)$. Thus, assume $p \cdot \xi = w_t(p, q)$. Since $\xi \notin \operatorname{conv} P_{i(t)}(\xi)$, from the Hahn-Banach separation theorem (see, for example, Rockafellar and Wets [27]), and the finiteness of $P_{i(t)}(\xi)$, there exists $\bar{p} \in \mathbb{R}^L$ such that $\inf\left\{\bar{p} \cdot P_{i(t)}(\xi)\right\} > \bar{p} \cdot \xi$. For $\varepsilon > 0$ small enough, define $p_\varepsilon = p + \varepsilon\, \bar{p}$ and

$$q_\varepsilon = \frac{p_\varepsilon \cdot (\xi - e_{i(t)}) - \sum_{j \in J} \theta_{i(t)j} \pi_j(p_\varepsilon)}{m(t)}.$$

Note that $\lim_{\varepsilon \to 0}(p_\varepsilon, q_\varepsilon) = (p, q)$. Moreover, for small $\varepsilon > 0$, $\inf\left\{p_\varepsilon \cdot P_{i(t)}(\xi)\right\} > p_\varepsilon \cdot \xi = w_t(p_\varepsilon, q_\varepsilon)$ and therefore $\xi \in d_t(p_\varepsilon, q_\varepsilon)$. Hence, $\xi \in D_t(p, q)$. □

5.2 Existence of rationing equilibria

The proof of Theorem 4.1 rests on the existence of a weak equilibrium (see Definition 3.1), which is given in Section 5.2.2. For this result we need some technical lemmata provided in Section 5.2.1. The proof of the Theorem 4.1 is given in Section 5.2.3.

5.2.1 Technical results

This first lemma is a straightforward extension of the well-known Debreu-Gale-Nikaido lemma.

Lemma 5.1. *Let $\varepsilon \in]0,1]$ and $\varphi : \mathbb{B}(0_L,\varepsilon) \rightrightarrows \mathbb{R}^L$ be an upper hemi-continuous correspondence, with nonempty, convex, compact values. If for some $b > 0$,*

$$\forall p \in \mathbb{B}(0_L,\varepsilon), \quad \|p\| = \varepsilon \quad \Longrightarrow \quad \sup_{p\in\mathbb{B}(0_L,\varepsilon)} p\cdot\varphi(p) \leq b(1-\varepsilon),$$

then there exists $\bar{p} \in \mathbb{B}(0_L,\varepsilon)$ such that, either (i) $0_L \in \varphi(\bar{p})$ or (ii) $\|\bar{p}\| = \varepsilon$ and $\exists\ \xi \in \varphi(\bar{p})$ such that ξ and $\bar{p}$ are collinear, with $\|\xi\| \leq b\,\frac{1-\varepsilon}{\varepsilon}$.

Proof. From the properties of φ, we can select a convex compact subset $K \subset \mathbb{R}^L$ such that $\varphi(p) \subset K,\ p \in \mathbb{B}(0_L,\varepsilon)$. Consider now the correspondence $F : \mathbb{B}(0_L,\varepsilon)\times K \rightrightarrows \mathbb{B}(0_L,\varepsilon)\times K$ such that

$$F(p,z) = \left\{q \in \mathbb{B}(0_L,\varepsilon) \mid \forall q' \in \mathbb{B}(0_L,\varepsilon),\ q\cdot z \geq q'\cdot z\right\} \times \varphi(p).$$

From Kakutanis' Fixed Point Theorem, F has a fixed point, $(\bar{p},\xi)$. If $\|\bar{p}\| < \varepsilon$, then $\xi = 0_L$. If $\|\bar{p}\| = \varepsilon$, then from the definition of F, $\bar{p}$ and ξ are collinear. Therefore, $\|\xi\| \leq b\,\frac{1-\varepsilon}{\varepsilon}$. □

Hereinafter, by convenience regarding the notation, vectors of $\mathbb{R}^m$ are supposed to be a columns, and for $r \in \mathbb{N}$, $[\psi_1,\dots,\psi_r] \in \mathbb{R}^{m\times r}$ is the matrix with columns $\psi_1,\dots,\psi_r \in \mathbb{R}^m$. For $x \in \mathbb{R}^m$, $[\psi_1,\dots,\psi_r]^{\mathrm{t}}x = (\psi_1\cdot x,\dots,\psi_r\cdot x)^{\mathrm{t}} \in \mathbb{R}^r$, and for $K \subset \mathbb{R}^m$, we set

$$[\psi_1,\dots,\psi_r]^{\mathrm{t}}K = \left\{[\psi_1,\dots,\psi_r]^{\mathrm{t}}\,x \,|\, x \in K\right\}.$$

Furthermore, for $m \in \mathbb{N}$, the lexicographic order on $\mathbb{R}^m$ is denoted by $\leq_{lex}$.[10] The maximum and the argmax with respect to this order are denoted by $\max_{lex}$ and $\operatorname{argmax}_{lex}$, respectively.

Definition 5.1. *For a positive integer $k \leq m$, a set of two-by-two orthonormal vectors $\{\psi_1,\dots,\psi_k\} \subseteq \mathbb{R}^m$ coupled with sequences $\varepsilon_r : \mathbb{N} \to \mathbb{R}_{++}$, $r \in \{1,\dots,k\}$, is called a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$, if there exists $\mathrm{N} \in \mathbb{N}^*_\infty$ (see Section 2) such that following hold:*

(a) for all $r \in \{1,\dots,k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$,

(b) for all $n \in \mathrm{N}$, $\psi(n) = \sum_{r=1}^k \varepsilon_r(n)\psi_r$.

[10] We recall, for $(s,t) \in \mathbb{R}^m\times\mathbb{R}^m$, $s \leq_{lex} t$, if $s_r > t_r$, $r \in \{1,\dots,m\}$ implies that $\exists\rho \in \{1,\dots,r-1\}$ such that $s_\rho < t_\rho$. We write $s <_{lex} t$ if $s \leq_{lex} t$, but not $t \leq_{lex} s$.

In the following, a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$ as before is denoted by $\{\{\psi_r, \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$.

Lemma 5.2. *Every sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ admits a lexicographic decomposition.*

Proof. By setting $\widehat{\psi}_1(n) = \psi(n)$, $n \in \mathbb{N}$, there are $\psi_1 \in \mathbb{R}^m$, with $\|\psi_1\| = 1$, and $\mathrm{N}_1 \in \mathbb{N}^*_\infty$, such that

$$\frac{\widehat{\psi}_1(n)}{\|\widehat{\psi}_1(n)\|} \to_{\mathrm{N}_1} \psi_1.$$

Recursively, for $r \in \{1, \dots, m-1\}$, given $\psi_r \in \mathbb{R}^m$, $\|\psi_r\| = 1$, and $\mathrm{N}_r \in \mathbb{N}^*_\infty$, we define $\mathcal{H}^r = \psi_r^\perp$ and we set

$$\widehat{\psi}_{r+1}(n) = \mathrm{proj}_{\mathcal{H}^r}\left(\widehat{\psi}_r(n)\right), \; n \in \mathrm{N}_r.$$

If there exists $\mathrm{N}' \subset \mathrm{N}_r$ such that $\widehat{\psi}_{r+1}(n) = 0_m$ for all $n \in \mathrm{N}'$, then we set $\mathrm{N} = \mathrm{N}'$, otherwise choose $\mathrm{N}_{r+1} \subset \mathrm{N}_r$ such that

$$\frac{\widehat{\psi}_{r+1}(n)}{\|\widehat{\psi}_{r+1}(n)\|} \to_{\mathrm{N}_{r+1}} \psi_{r+1} \in \mathbb{R}^m,$$

and define $\mathcal{H}^{r+1} = \psi_{r+1}^\perp$. Therefore, by construction, for each $r \in \{1, \dots, m-1\}$, the subset $\left\{\widehat{\psi}_{r+1}(n),\, n \in \mathrm{N}_{r+1}\right\}$ is contained in a subspace of dimension $m-r$. Thus, there are $k \in \{1, \dots, m\}$ and $\mathrm{N}' \in \mathbb{N}^*_\infty$, $\mathrm{N}' \subset \mathrm{N}_k$, such that $\widehat{\psi}_{k+1}(n) = 0_m$ and $\widehat{\psi}_k(n) \neq 0_m$ on N'. We set $\mathrm{N} = \mathrm{N}'$, and then, by construction, we have that the vectors of $\{\psi_1, \dots, \psi_k\}$ are two-by-two orthonormal. For all $n \in \mathrm{N}$ and all $r \in \{1, \dots, k\}$, ψ_r is orthogonal to $\widehat{\psi}_{r+1}(n)$, and the following holds for each $r \in \{1, \dots, k-1\}$:

$$\widehat{\psi}_r(n) = \widehat{\psi}_{r+1}(n) + (\widehat{\psi}_r(n) \cdot \psi_r)\psi_r \quad \text{and} \quad \frac{\| \widehat{\psi}_{r+1}(n) \|}{\| \widehat{\psi}_r(n) \|} \to_{\mathrm{N}} 0.$$

Therefore

$$\widehat{\psi}_r(n) \cdot \psi_r = \left(\| \widehat{\psi}_r(n) \|^2 - \| \widehat{\psi}_{r+1}(n) \|^2\right)^{\frac{1}{2}},$$

from which we have $\psi(n) = \sum_{r=1}^k \varepsilon_r(n)\ \psi_r$, with $\varepsilon_r(n) = \widehat{\psi}_r(n) \cdot \psi_r > 0$. Developing $\varepsilon_r(n)$, we have

$$\frac{\varepsilon_{r+1}(n)}{\varepsilon_r(n)} = \left(\frac{\left\|\widehat{\psi}_{r+1}(n)\right\|^2 - \left\|\widehat{\psi}_{r+2}(n)\right\|^2}{\left\|\widehat{\psi}_r(n)\right\|^2 - \left\|\widehat{\psi}_{r+1}(n)\right\|^2}\right)^{1/2} = \frac{\left\|\widehat{\psi}_{r+1}(n)\right\|}{\left\|\widehat{\psi}_r(n)\right\|}\left(\frac{1 - \frac{\left\|\widehat{\psi}_{r+2}(n)\right\|^2}{\left\|\widehat{\psi}_{r+1}(n)\right\|^2}}{1 - \frac{\left\|\widehat{\psi}_{r+1}(n)\right\|^2}{\left\|\widehat{\psi}_r(n)\right\|^2}}\right)^{1/2},$$

and therefore $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$. □

Lemma 5.3. *Let $\{\{\psi_r, \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$ be a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ and let $z \in \mathbb{R}^m$. There exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:*

$$[\psi_1, \dots, \psi_k]^{\mathrm{t}} z \leq_{lex} 0_k \quad \Longleftrightarrow \quad \psi(n) \cdot z \leq 0.$$

Proof. For $r \in \{0, 1, \dots, k\}$ we set $\Psi(r) = [\psi_1, \dots, \psi_r]^{\mathrm{t}}$ if $r > 0$, and $\Psi(r) = 0_m^{\mathrm{t}}$ when $r = 0$. Let $z \in \mathbb{R}^m$. If $\Psi(k)z = 0_k$, then $\psi(n) \cdot z = 0$, $n \in \mathrm{N}$.

For the sequel assume $\Psi(k)z \neq 0_k$. Then, there exists $s \in \{1, \dots, k\}$ such that

$$\Psi(s-1)z = 0^{\mathrm{t}}_{\max\{1,s-1\}} \quad \text{and} \quad \psi_s \cdot z = \delta \neq 0.$$

Since

$$\psi(n) \cdot z = \sum_{r=1}^{k} \varepsilon_r(n)\ \psi_r \cdot z,$$

we have that

$$\frac{1}{\varepsilon_s(n)} \psi(n) \cdot z = a_n + b_n,$$

with

$$a_n = \frac{1}{\varepsilon_s(n)} \sum_{r=1}^{s} \varepsilon_r(n)\, \psi_r \cdot z \quad \text{and} \quad b_n = \frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \sum_{r=s+1}^{k} \frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)}\, \psi_r \cdot z.$$

By the properties above, it is clear that for all $n \in \mathrm{N}$, $a_n = \delta$ and b_n converges to 0, which implies that for all large $n \in \mathrm{N}$,

$$\frac{1}{\varepsilon_s(n)} \psi(n) \cdot z \in [\delta/2, 2\delta].$$

Therefore, on the one hand, if $\Psi(k)z <_{lex} 0_k$, then $\delta < 0$ implying that for all large $n \in \mathrm{N}$, $\psi(n) \cdot z < 0$, and, on the other hand, if $\Psi(k)z >_{lex} 0_k$, then $\delta > 0$ and for all large $n \in \mathrm{N}$, $\psi(n) \cdot z > 0$, establishing the converse statement. □

Lemma 5.4. *Let Z be a finite subset of $\mathbb{R}^m$, and let $\{\{\psi_r, \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$ be a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$. Then, there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:*

$$\operatorname{argmax}_{lex} [\psi_1, \dots, \psi_k]^{\mathrm{t}} Z \quad = \quad \operatorname{argmax}\ \psi(n) \cdot Z.$$

Proof. For $n \in \mathrm{N}$, denote $F(n) = \operatorname{argmax}\ \psi(n) \cdot Z$, and $F = \operatorname{argmax}_{lex} [\psi_1, \dots, \psi_k]^{\mathrm{t}}\, Z$. Assume now that there exists $\mathrm{N}_0 \in \mathbb{N}^*_\infty$, $\mathrm{N}_0 \subseteq \mathrm{N}$, such that for all $n \in \mathrm{N}_0$, $F(n) \neq F$. Since Z is a finite set, we can chose $\overline{\mathrm{N}} \in \mathbb{N}^*_\infty$, $\overline{\mathrm{N}} \subseteq \mathrm{N}_0$, such that $F(n)$ is constant on $\overline{\mathrm{N}}$.[11] Given that, let $\bar{\xi} \in F$ and $\xi' \in F(n)$ for $n \in \overline{\mathrm{N}}$. If $\xi' \notin F$ then there exists $\rho \in \{1, \dots, k\}$, such that $\psi_\rho \cdot \xi' < \psi_\rho \cdot \bar{\xi}$, and for $r \in \{1, \dots, \rho - 1\}$, $\psi_r \cdot \xi' = \psi_r \cdot \bar{\xi}$. As for all $r \in \{1, \dots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$, this contradicts the fact that $\psi(n) \cdot \xi' \geq \psi(n) \cdot \bar{\xi}$, $n \in \overline{\mathrm{N}}$. Therefore $\xi' \in F$, and then

$$[\psi_1, \dots, \psi_k]^{\mathrm{t}} \xi' = [\psi_1, \dots, \psi_k]^{\mathrm{t}} \bar{\xi}.$$

Furthermore, this last identity implies that for all $n \in \mathrm{N}$, $\psi(n) \cdot \bar{\xi} = \psi(n) \cdot \xi'$, and then $\bar{\xi} \in F(n)$ for all $n \in \overline{\mathrm{N}}$, a contradiction. Therefore, there exist $\bar{n}$ such that for all $n \in \mathrm{N}$, $n > \bar{n}$, $F(n) = F$. □

[11] If $Z = \{\zeta_1, \dots, \zeta_f\}$, then for every $i \in \{1, \dots, f\}$, either there exists $n_i \in \mathrm{N}_0$ such that for all $n \in \mathrm{N}_0$, $n > n_i$, $\zeta_i \notin F(n)$, in which case we define $\mathrm{N}_i = \mathrm{N}_{i-1} \cap \{n \in \mathbb{N} \,|\, n > n_i\}$, or for some $\mathrm{N}_i \in \mathbb{N}^*_\infty$, $\mathrm{N}_i \subseteq \mathrm{N}_{i-1}$, $\zeta_i \in F(n)$ for each $n \in \mathrm{N}_i$. Then, $F(n)$ is constant for $n \in \overline{\mathrm{N}} = \mathrm{N}_f$.

Lemma 5.5. *Let* $\{(p_n, q_n) \in \mathbb{R}^m \times \mathbb{R}\}_{n\in\mathbb{N}}$ *a sequence for which* $\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1,\dots,k}, \mathrm{N}\}$ *is a lexicographic decomposition. Assume that* $(\mathrm{q}_1, \dots, \mathrm{q}_k)^{\mathrm{t}} \neq 0_k$*, and let* ρ *the smallest* $r \in \{1, \dots, k\}$ *such that* $\mathrm{q}_r \neq 0$*. For* $n \in \mathrm{N}$*, define*

$$(\bar{p}_n, \bar{q}_n) = \sum_{r=1}^{\rho} \varepsilon_r(n)(\mathrm{p}_r, \mathrm{q}_r) \in \mathbb{R}^m \times \mathbb{R}.$$

Let $z \in \mathbb{R}^m$ *and for* $T \subset \mathbb{R}$*, let* $\mu : T \to \mathbb{R}$*, such that for all* $t \in T$*,*

$$[\mathrm{p}_1, \dots, \mathrm{p}_k]^{\mathrm{t}} z \quad \leq_{lex} \quad \mu(t)\,(\mathrm{q}_1, \dots, \mathrm{q}_k)^{\mathrm{t}}$$

and $\mathrm{q}_\rho\, \mu(t)$ *is bounded from below on* T*. Then there exists* $\bar{n} \in \mathrm{N}$*, such that for all* $t \in T$ *and all* $n \in \mathrm{N}$*,* $n > \bar{n}$*,*

$$\bar{p}_n \cdot z \leq \bar{q}_n\, \mu(t).$$

Proof. For $r \in \{1, \dots, k\}$, we set $\mathcal{P}(r) = [\mathrm{p}_1, \dots, \mathrm{p}_r]^{\mathrm{t}}$ and $\mathcal{Q}(r) = (\mathrm{q}_1, \dots, \mathrm{q}_r)^{\mathrm{t}}$. Using this, we notice $\mathcal{P}(k)z \leq_{lex} \mu(t)\, \mathcal{Q}(k)$ implies that $\mathcal{P}(\rho)\, z \leq_{lex} \mu(t)\, \mathcal{Q}(\rho)$. We will split the remainder into two cases.

Case A. $\mathcal{P}(\rho - 1)\, z = \mu(t)\, \mathcal{Q}(\rho - 1)$.

In this case $\mathcal{P}(\rho)\, z \leq_{lex} \mu(t)\, \mathcal{Q}(\rho)$ implies that for all $r \in \{1, \dots, \rho\}$, $\mathrm{p}_r \cdot z \leq \mathrm{q}_r\, \mu(t)$, from which we have that for all $n \in \mathrm{N}$ and all $t \in T$, $\bar{p}_n \cdot z \leq \bar{q}_n\, \mu(t)$.

Case B. There exists $s \in \{1, \dots, \rho-1\}$ *such that* $\mathrm{p}_s \cdot z < \mathrm{q}_s\, \mu(t) = 0$ *and* $\mathcal{P}(s-1)\, z = \mu(t)\, \mathcal{Q}(s-1)$.

Let $\delta = -\mathrm{p}_s \cdot z > 0$ and let

$$\underline{\mu} \;=\; -\inf_{t\in T}\, \mathrm{q}_\rho\, \mu(t).$$

For $n \in \mathrm{N}$ we set

$$a_n = \frac{1}{\varepsilon_s(n)} \sum_{r=1}^{s} \varepsilon_r(n)\,(\mathrm{p}_r \cdot z - \mathrm{q}_r\, \mu(t)) = -\delta,$$

and

$$b_n = \frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \sum_{r=s+1}^{\rho} \frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)} \left(\mathrm{p}_r \cdot z + \underline{\mu}\right).$$

By the fact that $\frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \to_{\mathrm{N}} 0$, and for $r = s+1, \dots, \rho$, $\frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)}$ converges to zero for $n \in \mathrm{N}$, or is identically equal to 1, then we have $b_n \to_{\mathrm{N}} 0$. Hence, there exists $\bar{n}$ such that for all $n \in \mathrm{N}$, $n > \bar{n}$, $a_n + b_n < -\delta/2 < 0$, and then, for all $t \in T$, $\bar{p}_n \cdot z - \bar{q}_n\, \mu(t) \leq \varepsilon_s(n)\,(a_n + b_n) < 0$ for n as stated. □

5.2.2 Existence of weak equilibrium

Proposition 5.1. *If Assumptions* **F***,* **C***,* **S** *and* **M** *hold, then there exists a weak equilibrium for economy* $\mathcal{E}$*, with a strictly positive price for fiat money.*

Proof. The proof will be split into different steps.

Step 1. *Perturbed equilibrium.*

For $n \in \mathbb{N}$, $n > 1$, we set $\varepsilon_n = 1 - 1/n$. Given that, for $t \in \mathcal{I}$ and $j \in J$, the set-valued mappings

$$\operatorname{conv} D_t : \mathbb{B}\left(0_L, \varepsilon_n\right) \rightrightarrows \operatorname{conv} X_{i(t)} \quad | \quad p \mapsto \operatorname{conv} D_t(p, 1 - \|p\|)$$

and

$$\operatorname{conv} S_j : \mathbb{B}\left(0_L, \varepsilon_n\right) \rightrightarrows \operatorname{conv} Y_j \quad | \quad p \mapsto \operatorname{conv} S_j(p),$$

are upper hemi-continuous, nonempty, and compact valued, we define the excess of demand correspondence

$$\varphi : \mathbb{B}\left(0_L, \varepsilon_n\right) \rightrightarrows \sum_{i \in I} \operatorname{conv}\left(\lambda(T_i) X_i\right) - \sum_{j \in J} \operatorname{conv}\left(\lambda(T_j) Y_j\right) - \{e\}$$

such that

$$\varphi(p) \;\; = \;\; \int_{\mathcal{I}} D_t(p, 1 - \|p\|) dt - \sum_{j \in J} \operatorname{conv}\left(\lambda(T_j) S_j(p)\right) - e,$$

which is nonempty, convex and compact valued, and upper hemi-continuous. Furthermore, for each $n \in \mathbb{N}$, $n > 1$, and $p \in \mathbb{B}\left(0_L, \varepsilon_n\right)$, we have

$$p \cdot \varphi(p) \;\; \leq \;\; (1 - \|p\|) \int_{\mathcal{I}} m(t) dt,$$

and therefore we can use Lemma 5.1 to conclude that for each $n \in \mathbb{N}, n > 1$, there exists

$$(x_n, y_n, p_n, q_n) \in \prod_{i \in I} L^1(T_i, X_i) \times \prod_{j \in J} L^1(T_j, S_j(p_n)) \times \mathbb{B}\left(0, \varepsilon_n\right) \times \mathbb{R}_{++} \tag{6}$$

such that for a.e. $t \in \mathcal{I}$, $x_n(t) \in D_t(p_n, q_n)$, and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}(p_n)$, with $q_n = 1 - \|p_n\|$,

$$\int_{\mathcal{I}} x_n(t) dt - \int_{\mathcal{J}} y_n(t) dt - e \in \varphi(p_n)$$

and (see (ii) in Lemma 5.1)

$$\left\| \int_{\mathcal{I}} x_n(t) dt - \int_{\mathcal{J}} y_n(t) dt - e \right\| \leq \frac{1}{n-1} \int_{\mathcal{I}} m(t) dt.$$

By Lemma 5.1, if for some n, $p_n < 1 - 1/n$, then $\varphi(p_n) = 0_L$, and (x_n, y_n, p_n, q_n) would therefore be a weak equilibrium with $q_n > 0$. For the sequel we will assume that $q_n = 1/n$.

Step 2 . *Lexicographic price decomposition.*

Since $q_n > 0$, by Lemma 5.2 there exists $\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1,\ldots,k}, \mathrm{N}\}$, a lexicographic decomposition of $\{(p_n, q_n) \in \mathbb{R}^L \times \mathbb{R}\}_{n\in\mathbb{N}}$. Given that, we set

$$(\bar{p}_n, \bar{q}_n) = \sum_{r=1}^{\rho} \varepsilon_r(n)(\mathrm{p}_r, \mathrm{q}_r),$$

with ρ being the smallest $r \in \{1, \ldots, k\}$ such that $\mathrm{q}_r \neq 0$. As $q_n > 0$, such $\rho \leq k$ exists and $\mathrm{q}_\rho > 0$.

In the sequel, without loss of generality we assume $\mathrm{N} = \mathbb{N}$, and for $r \in \{1, \ldots, k\}$ we denote

$$\mathcal{P}(r) = [\mathrm{p}_1, \ldots, \mathrm{p}_r]^{\mathrm{t}} \text{ and } \mathcal{Q}(r) = (\mathrm{q}_1, \ldots, \mathrm{q}_r)^{\mathrm{t}}.$$

Step 3. *Supply: There exists n_J and for all $j \in J$, there exist $A_j \subseteq B_j \subseteq Y_j$, such that for all $n > n_J$ with $n \in \mathbb{N}$ and all $j \in J$, $S_j(p_n) = A_j \subseteq B_j = S_j(\bar{p}_n)$.*

Applying Lemma 5.4 twice for each $j \in J$, we have $n_j \in \mathbb{N}$ such that for all $n > n_j$,

$$S_j(p_n) = \mathrm{argmax}_{lex}\, \mathcal{P}(k)\, Y_j \qquad \text{and} \qquad S_j(\bar{p}_n) = \mathrm{argmax}_{lex}\, \mathcal{P}(\rho)\, Y_j.$$

Note that $\mathrm{argmax}_{lex}\mathcal{P}(k)\, Y_j \subset \mathrm{argmax}_{lex}\mathcal{P}(\rho)\, Y_j$. Set $n_J = \max_{j\in J} n_j$.

Step 4. *Income: For all $i \in I$, there exists $z_i \in \mathbb{R}^L$ such that for all $t \in T_i$ and $n > n_J$ with $n \in \mathbb{N}$*

$$w_t(p_n, q_n) = p_n \cdot z_i + q_n m(t) \quad \text{and} \quad w_t(\bar{p}_n, \bar{q}_n) = \bar{p}_n \cdot z_i + \bar{q}_n\, m(t).$$

For all $j \in J$, let $\zeta_j \in \mathrm{argmax}_{lex}\mathcal{P}(k)Y_j$. Step 3 the establishes the result by setting for all $i \in I$

$$z_i = e_i + \sum_{j\in J} \lambda(T_j)\theta_{ij}\, \zeta_j.$$

Step 5. *There exists $\bar{n} > n_J$ such that for all $t \in \mathcal{I}$, $\xi \in X_{i(t)}$ and $n > \bar{n}$ with $n \in \mathbb{N}$,*

$$\mathcal{P}(k)(\xi - z_{i(t)}) \ \leq_{lex}\ m(t)\, \mathcal{Q}(k) \quad \Rightarrow \quad \bar{p}_n \cdot (\xi - z_{i(t)}) \leq \bar{q}_n m(t)$$

$$\mathcal{P}(k)(\xi - z_{i(t)}) \ \geq_{lex}\ m(t)\, \mathcal{Q}(k) \quad \Rightarrow \quad \bar{p}_n \cdot (\xi - z_{i(t)}) \geq \bar{q}_n m(t).$$

Let $i \in I$ and $\xi \in X_i$. Consider the partition of T_i defined by the subsets:

$$T_i^1(\xi) = \left\{t \in T_i \,|\, \mathcal{P}(k)(\xi - z_{i(t)}) \ \leq_{lex}\ m(t)\, \mathcal{Q}(k)\right\}$$

$$T_i^2(\xi) = \left\{t \in T_i \,|\, \mathcal{P}(k)(\xi - z_{i(t)}) \ \geq_{lex}\ m(t)\, \mathcal{Q}(k)\right\}.$$

Since $m(\cdot)$ is bounded on every subset of $\mathcal{I}$, we can apply Lemma 5.5 to the vector $(\xi - z_{i(t)})$ for the set $T_i^1(\xi)$ leading to the existence of $n_i^1(\xi)$ such that for all $n > n_i^1(\xi)$,

$$\bar{p}_n \cdot (\xi - z_{i(t)}) \leq \bar{q}_n \, m(t)$$

on $T_i^1(\xi)$. Moreover, we can apply as well the same lemma to the vector $-(\xi - z_{i(t)})$ coupled with $\widetilde{m} : T_i^2(\xi) \to \mathbb{R}$ with $\widetilde{m}(t) = -m(t)$, leading to the existence of $n_i^2(\xi)$ such that for all $n > n_i^2(\xi)$,

$$\bar{p}_n \cdot (\xi - z_{i(t)}) \geq \bar{q}_n m(t)$$

on $T_i^2(\xi)$. As I is finite and the consumption sets are finite, by choosing

$$\bar{n} = \max_{\xi \in X_i, \, i \in I} \left\{ n_i^1(\xi), n_i^2(\xi), n_J \right\} \in \mathbb{N}$$

we can establish the desired result.

Step 6. *Budget: For all $n > \bar{n}$, for all $t \in \mathcal{I}$, $\limsup_{\nu \to \infty} B_t(p_\nu, q_\nu) \subseteq B_t(\bar{p}_n, \bar{p}_n)$.*

Let $t \in \mathcal{I}$ and $\xi \in \limsup_{\nu \to \infty} B_t(p_\nu, q_\nu)$. Then there are $\{\xi_\nu\}_{\nu \in \mathbb{N}}$ with $\xi_\nu \in B_t(p_\nu, q_\nu) \subset X_{i(t)}$ for all $\nu \in \mathbb{N}$, and $\mathrm{N}_\xi(t) \in \mathbb{N}_\infty^*$, such that $\xi_\nu \to_{\mathrm{N}_\xi(t)} \xi$. As $X_{i(t)}$ is finite, we can choose $\mathrm{N}_\xi(t)$ such that for all $\nu \in \mathrm{N}_\xi(t)$, $\xi_\nu = \xi$ and $\nu > \bar{n}$. Hence, for all $\nu \in \mathrm{N}_\xi(t)$, $p_\nu \cdot (\xi - z_i) \leq q_\nu \, m(t)$, and then, by Lemma 5.3, we have that $\mathcal{P}(k)(\xi - z_{i(t)}) \leq_{lex} m(t) \, \mathcal{Q}(k)$. This implies by Step 5 that for all $n \in \mathbb{N}$, $n > \bar{n}$, $\xi \in B_t(\bar{p}_n, \bar{q}_n)$.

Step 7. *Demand: For all $n > \bar{n}$, for all $t \in \mathcal{I}$ with $m(t) > 0$, $\limsup_{\nu \to \infty} D_t(p_\nu, q_\nu) \subseteq D_t(\bar{p}_n, \bar{p}_n)$.*

Let $t \in \mathcal{I}$ with $m(t) > 0$ and $\xi \in \limsup_{\nu \to \infty} D_t(p_\nu, q_\nu)$. Then, similar to previous step, there are $\{\xi_\nu\}_{n \in \mathbb{N}}$ with $\xi_\nu \in D_t(p_\nu, q_\nu) \subset B_t(p_\nu, q_\nu)$ for all $\nu \in \mathbb{N}$, and $\mathrm{N}_\xi(t) \in \mathbb{N}_\infty^*$, such that $\xi_\nu \to_{\mathrm{N}_\xi(t)} \xi$. Since $X_{i(t)}$ is finite, we can choose $\mathrm{N}_\xi(t)$ such that for all $\nu \in \mathrm{N}_\xi(t)$, $\xi_\nu = \xi$ and $\nu > \bar{n}$.

As $\nu \in \mathrm{N}_\xi(t)$, $q_\nu m(t) > 0$ and $\xi \in D_t(p_\nu, q_\nu)$. Then, Proposition 3.1 implies, on the one hand, that $\xi \notin \mathrm{conv} P_{i(t)}(\xi)$ and, on the other hand, that for all $\nu \in \mathrm{N}_\xi(t)$ and all $\bar{\xi} \in P_{i(t)}(\xi)$, $p_\nu \cdot (\bar{\xi} - z_i) \geq q_\nu \, m(t)$. Thus, by Lemma 5.3, we have that $\mathcal{P}(k)(\bar{\xi} - z_{i(t)}) \geq_{lex} m(t) \, \mathcal{Q}(k)$. This implies by Step 5 that for all $n \in \mathbb{N}$, $n > \bar{n}$, $w_t(\bar{p}_n, \bar{q}_n) \leq \bar{p}_n \cdot \bar{\xi}$. Since $\xi \in \limsup_{\nu \to \infty} B_t(p_\nu, q_\nu)$ we have also that for all $n \in \mathbb{N}$, $n > \bar{n}$, $\bar{p}_n \cdot \xi \leq w_t(\bar{p}_n, \bar{q}_n)$. Therefore, for all $t \in \mathcal{I}$ with $m(t) > 0$, all $n \in \mathbb{N}$, $n > \bar{n}$, and all $\xi \in \limsup_{\nu \to \infty} D_t(p_\nu, q_\nu)$, we have $\bar{p}_n \cdot \xi \leq w_t(\bar{p}_n, \bar{q}_n) \leq \inf \bar{p}_n \cdot P_{i(t)}(\xi)$ and $\xi \notin \mathrm{conv} P_{i(t)}(\xi)$. Hence, by Proposition 3.1 it follows that $\xi \in D_t(\bar{p}_n, \bar{p}_n)$, for $n > \bar{n}$.

Step 8. *Equilibrium allocation.*

From (6), using Artstein's [3] version of Fatou's lemma, there exists $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$, $x^*(t) \in \mathrm{acc}\{x_n(t)\}_{n \in \mathbb{N}}$ and $y^*(t') \in \mathrm{acc}\{y_n(t')\}_{n \in \mathbb{N}}$.

Step 9. *Conclusion: For all $n \in \mathbb{N}$ with $n > \bar{n}$, $(x^*, y^*, \bar{p}_n, \bar{q}_n)$ is a weak equilibrium with $\bar{q}_n > 0$.*

Indeed, let $n \in \mathbb{N}$ with $n > \bar{n}$. By Step 3, we have for all $t' \in \mathcal{J}$, $\mathrm{acc}\{y_\nu(t')\}_{\nu \in \mathbb{N}} \subset S_{j(t')}(\bar{p}_n)$. By Step 7, we have for all $t \in \mathcal{I}$ with $m(t) > 0$, $\mathrm{acc}\{x_\nu(t)\}_{\nu \in \mathbb{N}} \subset D_t(\bar{p}_n, \bar{q}_n)$. Of course $\bar{q}_n > 0$ and the last property is valid for a.e. $t \in \mathcal{I}$ since, by Assumption **D**, $\lambda(\{t \in \mathcal{I} \,|\, m(t) = 0\}) = 0$.

□

5.2.3 Proof of Theorem 4.1: existence of a rationing equilibrium

Proof. Let (x_0, y_0, p_0, q_0) be a weak equilibrium of $\mathcal{E}$ with $q_0 > 0$. If $p_0 = 0_L$, then for a.e. $t \in \mathcal{I}$, $P_{i(t)}(x_0(t)) = \emptyset$ and then, for $K = \{0_L\}$, (x_0, y_0, p_0, q_0, K) is a rationing equilibrium. Otherwise let $m^1 : \mathcal{I} \to \mathbb{R}_{++}$ be a mapping strictly increasing and bounded. As consumption sets and the number of type of consumers are finite, we can define a finite set of types of consumers $A = \{1, \ldots, A\}$ satisfying the following:

(i) $\{T_a\}_{a\in A}$ is a finer partition of $\mathcal{I}$ than $\{T_i\}_{i\in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_0(t) = x_a$.

For $a \in A$ we set

$$X_a^1 = (P_a^1(x_a) \cup \{x_a\}) \cap (\{x_a\} + p_0^{\perp}).$$

In a similar manner as done for consumers, we can define a finite set of types of producers $B = \{1, \ldots, B\}$ satisfying the following:

(i) $\{T_b\}_{b\in B}$ is a finer partition of $\mathcal{J}$ than $\{T_j\}_{j\in J}$,

(ii) for every $b \in B$, there exists y_b such that for every $t \in T_b$, $y_0(t) = y_b$.

For $b \in B$, let

$$Y_b^1 = (Y_b - \{y_b\}) \cap p_0^{\perp},$$

and then, denoting $e_a^1 = x_a$, we define the following auxiliary economy

$$\mathcal{E}^1 = \left((X_a^1, P_a^1, e_a^1)_{a\in A}, (Y_b^1)_{b\in B}, (\theta_{ab})_{(a,b)\in A\times B}, m^1, \{T_a\}_{a\in A}, \{T_b\}_{b\in B}\right),$$

where m^1 defines the initial endowments of fiat money for each consumer, and θ_{ab} satisfying the conditions for the privately ownership of the firms and P_a^1 is the restriction of P_a to X_a^1.

Clearly the economy $\mathcal{E}^1$ satisfies the assumptions of Proposition 5.1, hence there exists a weak equilibrium for this economy, with the price of fiat money strictly positive. Moreover, since m^1 satisfies Assumption **D**, from Corollary 4.1 there exists a Walras equilibrium (with fiat money) for the economy $\mathcal{E}^1$, which is denoted by (x_1, y_1, p_1, q_1), with $q_1 > 0$.

In the following, we denote

$$\mathcal{P} = [p_0, p_1]^{\mathrm{t}} \in \mathbb{R}^{2\times L},$$

and for $t \in \mathcal{I}$ we set $w_t = (w_t^0, w_t^1) \in \mathbb{R}^2$, with

$$w_t^0 = p_0 \cdot e_{i(t)} + q_0 m(t) + \sum_{j\in J} \theta_{i(t)j}\, \pi_j(p_0)$$

and

$$w_t^1 = p_1 \cdot e_{a(t)}^1 + q_1 m^1(t) + \sum_{b \in B} \theta_{a(t)b}\, \lambda(T_b) \max p_1 \cdot Y_b^1,$$

where $a(t) \in A$ such that $t \in T_{a(t)}$.

Claim 5.1. *For a.e.* $t \in \mathcal{I}$*,* $\mathcal{P}x_1(t) \leq_{lex} w_t$*,.*

By definition of X_a^1, $a \in A$, $p_0 \cdot x_0(t) = p_0 \cdot x_1(t)$, a.e. $t \in \mathcal{I}$. Since for every $r \in \{0, 1\}$, $p_r \cdot x_r(t) \leq w_t^r$, we conclude $\mathcal{P}x_1(t) \leq_{lex} w_t$, for a.e. $t \in \mathcal{I}$.

Claim 5.2. *For a.e.* $t \in \mathcal{I}$*,* $\xi(t) \in P_{i(t)}(x_1(t))$ *implies* $\mathcal{P}x_1(t) <_{lex} \mathcal{P}\xi(t)$*.*

Let $\xi(t) \in P_{i(t)}(x_1(t))$. By construction, for a.e. $t \in \mathcal{I}$, $x_1(t) \neq x_0(t)$ implies $x_1(t) \in P_{i(t)}(x_0(t))$ and then, by transitivity of the preferences, $\xi \in P_{i(t)}(x_0(t))$. Therefore, for a.e. $t \in \mathcal{I}$, $\xi \in P_{i(t)}(x_0(t))$ and $p_0 \cdot x_1(t) = p_0 \cdot x_0(t) \leq p_0 \cdot \xi$. For $t \in \mathcal{I}$, the claim is thus satisfied for $\xi(t) \in P_{i(t)}(x_1(t))$ such that $p_0 \cdot x_1(t) < p_0 \cdot \xi$. For $t \in \mathcal{I}$ and $\xi(t) \in P_{i(t)}(x_1(t))$ such that $p_0 \cdot x_1(t) = p_0 \cdot \xi$ we have $\xi \in X_{a(t)}^1$ and as (x_1, y_1, p_1, q_1) is a Walras equilibrium of $\mathcal{E}^1$, it follows that $p_1 \cdot x_1(t) < p_1 \cdot \xi$ for a.e. $t \in \mathcal{I}$ satisfying $p_0 \cdot x_1(t) = p_0 \cdot \xi$.

We will show that

$$(\bar{x}, \bar{y}, \bar{p}, \bar{q}) = (x_1, y_0 + y_1, p_0, q_0)$$

coupled with the pointed cone

$$K = \{0_L\} \cup \{\xi \in \mathbb{R}^L \mid 0_2 <_{lex} \mathcal{P}\xi\}$$

is a rationing equilibrium. Let

$$\mathcal{I}' = \left\{t \in \mathcal{I} \mid \mathcal{P}x_1(t) <_{lex} \min_{lex} \mathcal{P} P_{i(t)}(x_1(t))\right\},$$

and note that by Claim 5.2 $\lambda(\mathcal{I} \setminus \mathcal{I}') = 0$. Again by Claim 5.2, K contains[12]

$$K' \quad = \quad \{0_L\} \cup \left\{ \bigcup_{t \in \mathcal{I}'} P_{i(t)}(\bar{x}(t)) - \{\bar{x}(t)\} \right\}.$$

By the construction of K, it follows that for a.e. $t \in \mathcal{I}'$, $\bar{x}(t) \in \delta_t(\bar{p}, \bar{q}, K)$. Finally, by the construction of the iteration of weak equilibria, for a.e. $t \in \mathcal{J}$ and all $z \in Y_j$, $\mathcal{P}(z - \bar{y}(t)) \leq_{lex} 0_2$ and therefore

$$\{0_L\} = \left(Y_{j(t)} - \{\bar{y}(t)\}\right) \cap K.$$

From this we can conclude that for a.e. $t \in \mathcal{J}$, $\bar{y}(t) \in \sigma_t(\bar{p}, \bar{q}, K)$. □

[12] As the consumption sets are finite, the set K' generates a closed cone contained in K. Thus we could impose the cone in the rationing equilibrium to be closed.

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