Type I ancient compact solutions of the Yamabe flow

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A B S T R A C T

We construct new ancient compact solutions to the Yamabe flow. Our solutions are rotationally symmetric and converge, as $t \to -\infty$, to two self-similar complete non-compact solutions to the Yamabe flow moving in opposite directions. They are type I ancient solutions.

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1. Introduction

Let (M, g_0) be a compact manifold without boundary of dimension $n \geq 3$. If $g = \bar{u}^{\frac{4}{n-2}} g_0$ is a metric conformal to g_0, the scalar curvature R of g is given in terms of the scalar curvature R_0 of g_0 by

$$R = \bar{u}^{-\frac{n+2}{n-2}} \left(-\bar{c}_n \Delta_{g_0} \bar{u} + R_0 \, \bar{u} \right)$$

where Δ_{g_0} denotes the Laplace Beltrami operator with respect to g_0 and $\bar{c}_n = 4(n-1)/(n-2)$.

In 1989 R. Hamilton introduced the *Yamabe flow*

$$\frac{\partial g}{\partial t} = -R \, g \tag{1.1}$$

as an approach to solve the *Yamabe problem* on manifolds of positive conformal Yamabe invariant. It is the negative L^2-gradient flow of the total scalar curvature, restricted to a given conformal class. This was shown by S. Brendle [1,2] (up to a technical condition in dim $n \geq 6$). Significant earlier works in this directions

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include those by R. Hamilton [9], B. Chow [3], R. Ye [17], H. Schwetlick and M. Struwe [14] among many others. The Yamabe conjecture, was previously shown by R. Shoen via elliptic methods in his seminal work [13].

In the special case where the background manifold M_0 is the sphere S^n and g_0 is the standard spherical metric g_{S^n}, the Yamabe flow evolving a metric $g = \bar{u}^{\frac{4}{n-2}}(\cdot, t) g_{S^n}$ takes (after rescaling in time by a constant) the form of the *fast diffusion equation*

$$\left(\bar{u}^{\frac{n+2}{n-2}}\right)_t = \Delta_{S^n}\bar{u} - c_n\bar{u}, \quad c_n = \frac{n(n-2)}{4}. \tag{1.2}$$

Starting with any smooth metric g_0 on S^n, it follows by the results in [3,17,7] that the solution of (1.2) with initial data g_0 will become singular at some finite time $t < T$ and v becomes spherical at time T, which means that after a normalization, the normalized flow converges to the spherical metric. In addition, v becomes extinct at T.

A metric $g = \bar{u}^{\frac{4}{n-2}} g_{S^n}$ may also be expressed as a metric on $\mathbb{R}^n$ via stereographic projection. It follows that if $g = \hat{u}^{\frac{4}{n-2}}(\cdot, t) g_{\mathbb{R}^n}$ (where $g_{\mathbb{R}^n}$ denotes the standard metric on $\mathbb{R}^n$) evolves by the Yamabe flow (1.1), then $\hat{u}$ satisfies (after a rescaling in time) the fast diffusion equation on $\mathbb{R}^n$

$$(\hat{u}^p)_t = \Delta\hat{u}, \quad p := \frac{n+2}{n-2}. \tag{1.3}$$

Observe that if $g = \hat{u}^{\frac{4}{n-2}}(\cdot, t) g_{\mathbb{R}^n}$ represents a smooth solution when lifted to S^n, then $\hat{u}(\cdot, t)$ satisfies the growth condition

$$\hat{u}(y, t) = O(|y|^{-(n-2)}), \quad \text{as } |y| \to \infty.$$

Definition 1.1 (*Type I and Type II Ancient Solutions*). The solution $g = u(\cdot, t)^{\frac{4}{n-2}} g_0$ to (1.1) is called ancient if it exists for all time $t \in (-\infty, T)$, where $T < \infty$. We will say that the ancient solution g is of type I, if its Riemannian curvature satisfies

$$\limsup_{t \to -\infty}\left(|t| \max_{M_0} |\mathrm{Rm}|(\cdot, t)\right) < \infty.$$

An ancient solution which is not of type I, will be called of type II.

The simplest example of an ancient solution to the Yamabe flow on S^n is the *contracting spheres*. They are special solutions $\bar{u}$ of (1.2) which depend only on time t and satisfy the ODE

$$\frac{d\bar{u}^{\frac{n+2}{n-2}}}{dt} = -c_n\,\bar{u}.$$

They are given by

$$\bar{u}_S(p, t) = \left(\frac{4}{n-2}\,c_n\,(T-t)\right)^{\frac{n-2}{4}}, \quad p \in S^n \tag{1.4}$$

and represent a sequence of round spheres shrinking to a point at time $t = T$. They are shrinking solitons and type I ancient solutions.

King solutions: They were discovered by J.R. King [10]. They can be expressed on $\mathbb{R}^n$ in closed form (after stereographic projection), namely $g = \hat{u}_K(\cdot, t)^{\frac{4}{n-2}} g_{\mathbb{R}^n}$, where $\hat{u}_K$ is the radial function

$$\hat{u}_K(y, t) = \left(\frac{a(t)}{1 + 2b(t)\,|y|^2 + |y|^4}\right)^{\frac{n-2}{4}}, \quad y \in \mathbb{R}^n \tag{1.5}$$

and the coefficients $a(t)$ and $b(t)$ satisfy a certain system of ODEs. The King solutions are *not solitons* and may be visualized, as $t \to -\infty$, as two Barenblatt self-similar solutions "glued" together to form a compact solution to the Yamabe flow. They are type I ancient solutions.

Let us make the analogy with the Ricci flow on S^2. The two explicit compact ancient solutions to the two dimensional Ricci flow are the contracting spheres and the King–Rosenau solutions [10–12]. The latter ones are the analogues of the King solution (1.5) to the Yamabe flow. The difference is that the King–Rosenau solutions are type II ancient solutions to the Ricci flow while the King solution above is of type I.

It has been shown by Daskalopoulos, Hamilton and Sesum [4] that the spheres and the King–Rosenau solutions are the only compact ancient solutions to the two dimensional Ricci flow. The natural question to raise is whether the analogous statement holds true for the Yamabe flow, that is, whether the contracting spheres and the King solution are the only compact ancient solutions to the Yamabe flow. This occurs not to be the case as the following discussion shows.

Indeed, in [15] the existence of a new class of type II ancient radially symmetric solutions of the Yamabe flow (1.2) on S^n was shown. These new solutions, as $t \to -\infty$, may be visualized as two spheres joined by a short neck. Their curvature operator changes sign. We will refer to them as *towers of moving bubbles*.

Since the towers of moving bubbles are shown to be type II ancient solutions, while the contracting spheres and the King solutions are of type I, one may still ask whether the latter two are the only ancient compact type I solutions of the Yamabe flow on S^n, Eq. (1.2). In this work we will observe that this is not the case, as will show the existence of other ancient compact type I solutions on S^n.

It is simpler to construct these new solutions in cylindrical coordinates, so let us first describe the coordinate change. Let $g = \hat{u}^{\frac{4}{n-2}}(\cdot, t) \, g_{\mathbb{R}^n}$ be a radially symmetric solution of (1.3). For any $T > 0$ the cylindrical change of variables is given by

$$u(x, \tau) = (T - t)^{-\frac{1}{p-1}} r^{\frac{2}{p-1}} \, \hat{u}(y, t), \quad x = \ln |y|, \ \tau = -\ln(T - t). \tag{1.6}$$

In this language Eq. (1.3) becomes

$$(u^p)_\tau = u_{xx} + \alpha u^p - \beta u, \quad \beta = \frac{(n-2)^2}{4}, \quad \alpha = \frac{p}{p-1} = \frac{n+2}{4}. \tag{1.7}$$

By suitable scaling we can make the two constants α and β in (1.7) equal to 1, so that from now on we will consider the equation

$$(u^p)_\tau = u_{xx} + u^p - u. \tag{1.8}$$

Indeed, one can see that

$$u(x, \tau) = \left(\frac{\alpha}{\beta} \right)^{\frac{1}{p-1}} \tilde{u}\left(\frac{x}{\sqrt{\beta}}, \frac{\tau}{\alpha} \right) \tag{1.9}$$

solves (1.8) iff the $\tilde{u}$ solves (1.7).

It is well known (we refer the reader to the book by J.L. Vazquez [16], Section 3.2.2) that for any given $\lambda \geq 0$ Eq. (1.8) admits an one parameter family of traveling wave solutions of the form $u_\lambda(x, t) = v_\lambda(x - \lambda t)$ with the behavior

$$v_\lambda(x) = O(e^x), \quad \text{as } x \to -\infty. \tag{1.10}$$

It follows that $v := v_\lambda$ satisfies the equation

$$v_{xx} + \lambda (v^p)_x + v^p - v = 0. \tag{1.11}$$

The solutions v_λ define Yamabe shrinking solitons which correspond to smooth self-similar solutions of (1.3) when expressed as metrics on $\mathbb{R}^n$ (the smoothness follows from condition (1.10)). It was shown in [6] that they are type I ancient solutions.

Solutions of (1.11) with $\lambda = 0$ correspond to the steady states of Eq. (1.7) and are given in closed form as the one parameter family,

$$v_0(x) = \left(\frac{k_n \, c \, e^{\gamma x}}{1 + c^2 \, e^{2\gamma x}} \right)^{\frac{n-2}{2}}, \quad c > 0 \tag{1.12}$$

with $\gamma = \frac{2}{n-2}$ and $k_n = \left(\frac{4n}{n-2} \right)^{1/2}$. They represent geometrically the standard metric on the sphere.

When $\lambda > 0$, solutions to (1.11) with behavior (1.10) define smooth complete and non-compact Yamabe solitons (shrinkers) which all have *cylindrical behavior at infinity*, namely

$$v_\lambda(x) = 1 + o(1), \quad \text{as } x \to +\infty.$$

In [5] the asymptotic behavior, up to second order, of these solutions was shown. In particular, it follows from Theorem 1.1 in [5], that for any $\lambda \geq 1$ there exists a unique solution v_λ of (1.8) which satisfies

$$v_\lambda(0) = \frac{1}{2} \tag{1.13}$$

and has the asymptotic behavior

$$v_\lambda(x) = O(e^x), \quad \text{as } x \to -\infty \quad \text{and} \quad v_\lambda(x) = 1 - C_\lambda \, e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}), \quad \text{as } x \to +\infty \tag{1.14}$$

for some constants $\gamma_\lambda > 0$ and $C_\lambda > 0$ (depending on λ). For values of λ in the range $0 < \lambda < 1$, the behavior of the solutions v_λ was also studied in [5] and differs for dimensions $3 \leq N \leq 6$ and $N \geq 6$.

Remark 1.1. For the convenience of the reader let us point out that the proof of Theorem 1.1 in [5] is given in Chapter 3 where the solution v in cylindrical coordinates satisfies equation

$$\bar{\alpha}^{-1} v_{xx} + \lambda(p-1)\, v^{p-1}\, v_x + v^p - v = 0 \tag{1.15}$$

for a parameter $\lambda > 0$ and

$$\bar{\alpha} := \frac{(n-2)^2}{4} = \frac{4}{(p-1)^2}.$$

If

$$\bar{v}(x) = v(\gamma x), \quad \gamma = \bar{\alpha}^{-1/2} = \frac{p-1}{2}$$

then $\bar{v}$ satisfies (1.11) with $\lambda = \beta(p-1)/(p\gamma) = 2\beta/p$. Hence $\beta = \beta_1 := p/2$ in Theorem 1.1 in [5] corresponds to $\lambda = 1$ in our case.

Remark 1.2. Since Eq. (1.11) is translation invariant the solution v_λ generates an one parameter family $v_{\lambda,h}$ of solutions of Eq. (1.11) given by $v_{\lambda,h} = v_\lambda(x+h)$ which satisfy $v_{\lambda,h}(-h) = 1/2$.

Linearizing Eq. (1.11) around the constant solution $v = 1$ (which corresponds to the cylinder in geometric terms) we obtain the equation

$$\tilde{v}_{xx} + \lambda p\, \tilde{v}_x + (p-1)\, \tilde{v} = 0. \tag{1.16}$$

Hence, assuming that $v \approx 1 - C\, e^{-\gamma_\lambda x}$, as $x \to +\infty$, it follows that γ_λ satisfies the equation

$$\gamma^2 - \lambda p \gamma + (p-1) = 0 \tag{1.17}$$

and its roots are non-complex (which corresponds to non-oscillating solutions $\hat{v}$ of (1.11)) iff

$$\lambda \geq \frac{2\sqrt{p-1}}{p}.$$

All such solutions were studied in [5], however here we will restrict ourselves to the case

$$\lambda \geq 1.$$

It has been shown in [5] (Theorem 1.1) that when $\lambda \geq 1$, the solution v_λ of (1.11) is monotone increasing and satisfies (1.14) with

$$\gamma_\lambda = \frac{\lambda p - \sqrt{\lambda^2 p^2 - 4(p-1)}}{2} > 0, \tag{1.18}$$

which corresponds to the smallest of the roots of (1.17). When $\lambda = 1$, Eq. (1.11) admits the explicit one parameter family of Barenblatt solutions

$$v_{1,c} = \left(\frac{1}{1 + c\, e^{-(p-1)\, x}} \right)^{1/(p-1)}, \quad c > 0$$

where we recall that $p - 1 = 4/(n-2)$ and one may choose $c = c_p$ so that

$$v_1 = \left(\frac{1}{1 + c\, e^{-(p-1)\, x}} \right)^{1/(p-1)}$$

satisfies the condition $v_1(0) = 1/2$. It follows, that in this case

$$v_1 = 1 - C_1\, e^{-(p-1)\, x} + o(e^{-(p-1)\, x}), \quad \text{as } x \to +\infty \tag{1.19}$$

for a constant $C_1 = C_1(p) > 0$. Notice that when $\lambda = 1$ the roots of (1.17) are given by

$$\gamma = \frac{p \mp |p - 2|}{2}$$

and $\gamma_1 := (p - 1)$ in (1.14) (as it follows from (1.19)), hence it satisfies

$$\gamma_1 = \begin{cases} \dfrac{p - |p - 2|}{2}, & \text{if } p \leq 2 \\[2mm] \dfrac{p + |p - 2|}{2}, & \text{if } p > 2. \end{cases}$$

In other words, when $p > 2$ (corresponding to $n < 6$) the Barenblatt solution ($\lambda = 1$) satisfies (1.14) where γ_1 is now the largest of the roots of (1.17).

Next we introduce an *ansatz* for the new type I solutions of (1.8), which will be the main focus in this work. Let us assume that

$$u_{\lambda,h}(x, \tau) := v_\lambda(x - \lambda\tau + h) \tag{1.20}$$

is a traveling wave solution of (1.8) for given parameters $\lambda \geq 1$ and $h \in \mathbb{R}$. Since Eq. (1.8) is invariant under the reflection $x \to -x$, it follows that

$$\hat{u}_{\lambda,h}(x, \tau) := u_{\lambda,h}(-x, t) = v_\lambda(-x - \lambda\tau + h) \tag{1.21}$$

is a solution to (1.8). It corresponds to another traveling wave of (1.11) which travels in the opposite direction than $u_{\lambda,h}$. It follows from (1.14) that $u_{\lambda,h}$ and $\hat{u}_{\lambda,h}$ satisfy the asymptotics

$$u_{\lambda,h}(x, \tau) = O(e^x), \quad \text{as } x \to -\infty \quad \text{and} \quad \hat{u}_{\lambda,h}(x, \tau) = O(e^{-x}) \quad \text{as } x \to +\infty. \tag{1.22}$$

In addition, we have

$$u_{\lambda,h}(x, \tau) = 1 - C_\lambda\, e^{-\gamma_\lambda(x - \lambda\tau + h)} + o(e^{-\gamma_\lambda(x - \lambda\tau + h)}), \quad \text{as } x - \lambda\tau + h \to +\infty \tag{1.23}$$

and also

$$\hat{u}_{\lambda,h}(x, \tau) = 1 - C_\lambda\, e^{-\gamma_\lambda(-x - \lambda\tau + h)} + o(e^{-\gamma_\lambda(-x - \lambda\tau + h)}), \quad \text{as } x + \lambda\tau - h \to -\infty \tag{1.24}$$

with γ_λ given by (1.18) and $C_\lambda > 0$ depends only on λ.

In this work, we will show the existence of a four parameter class of ancient solutions $u_{\lambda,\lambda',h,h'}$ of Eq. (1.8) with $\lambda, \lambda' > 1$ and $h, h' \in \mathbb{R}$, which as $t \to -\infty$ may be visualized as two traveling wave solutions, $u_{\lambda,h}$ (traveling on the left) and $\hat{u}_{\lambda',h'}$ (traveling on the right). In fact, we will show in the next section that $u_{\lambda,\lambda',h,h'}$ is given by

$$u_{\lambda,\lambda',h,h'} = v_{\lambda,\lambda',h,h'} - w_{\lambda,\lambda',h,h'} \tag{1.25}$$

with

$$v_{\lambda,\lambda',h,h'} := \min\left(u_{\lambda,h}(\cdot,\tau), \hat{u}_{\lambda',h'}(\cdot,\tau)\right)$$

and $w_{\lambda,\lambda',h,h'} > 0$ an error term which is small in an appropriate norm.

Let $g_{\lambda,\lambda',h,h'} := u_{\lambda,\lambda',h,h'}^{\frac{4}{n-2}} g_{cyl}$ denote the metric on the cylinder $\mathbb{R} \times S^{n-1}$ defined by the solution u_λ of (1.8). Here $g_{cyl} := dx^2 + g_{S^{n-1}}$ denotes the standard cylindrical metric. We have seen that (1.8) is equivalent to $g_{\lambda,\lambda',h,h'}$ satisfying the rescaled Yamabe flow $g_t = -(R-1)g$. In addition we will show that each $u_{\lambda,\lambda',h,h'}$, when lifted on S^n, defines a *smooth ancient type I* solution to the Yamabe flow on $S^n \times (-\infty, T)$, in the sense that the norms of its curvature operators are uniformly bounded in time. This exactly means that the corresponding solution to the unrescaled Yamabe flow (1.1) is a type I ancient solution in the sense of Definition 1.1. Next we state our main result.

Theorem 1.1. *For any* $(\lambda, \lambda', h, h') \in \mathbb{R}^4$ *such that* $\lambda, \lambda' > 1$ *there exists an ancient solution* $u_{\lambda,\lambda',h,h'}$ *of* (1.8) *defined on* $\mathbb{R} \times (-\infty, T)$, *for some* $T = T_{\lambda,\lambda',h,h'} \in (-\infty, +\infty]$ *and satisfies*

$$0 < u_{\lambda,\lambda',h,h'} \leq v_{\lambda,\lambda',h,h'}, \quad \text{for all } (x,\tau) \in \mathbb{R} \times (-\infty, T).$$

In addition, the metric $g_{\lambda,\lambda',h,h'} := u^{\frac{4}{n-2}} g_{cyl}$ *when lifted on* S^n *defines a smooth ancient solution of the rescaled Yamabe flow* $g_t = -(R-1)\, g$, *on* $S^n \times (-\infty, T)$. *This is a type I ancient solution in the sense that the norms of its curvature operators are uniformly bounded in time (which exactly means the corresponding solution to the unrescaled Yamabe flow* (1.1) *is a type I ancient solution in the sense of Definition 1.1).*

The rest of this paper is organized as follows: in Section 2 we prove Theorem 2.1 which is the existence of a four parameter family of ancient solutions u_f. In particular, we show that each of them is exponentially close in the integral sense to a given approximating solution which depends on the four parameters λ, λ', h, h'. In Section 3 we show that all our constructed solutions are Type I ancient solutions, as stated in Theorem 3.1. Theorem 1.1 is a direct consequence of Theorems 2.1 and 3.1.

Remark 1.3 (*KPP Equation and the Work of Hamel–Nadirashvili [8]*)**.** Eq. (1.8) resembles the well known semilinear KPP equation

$$u_t = u_{xx} + f(u) \tag{1.26}$$

for a nonlinearity $f(u)$ which satisfies appropriate growth conditions (cf. in [8]). It is well known that Eq. (1.26) possesses a family of traveling wave solutions v_λ, $\lambda \geq \lambda_*$ with similar behavior as those of Eq. (1.8) described above. F. Hamel and N. Nadirashvili, in [8], constructed ancient solutions $u_{\lambda,\lambda',h,h'}$ of Eq. (1.26). The main idea in [8] is to exploit the semilinear character of Eq. (1.26) and estimate the error of approximation $w_{\lambda,\lambda',h,h'}$ as in (1.25) by the solution to the linear equation

$$\nu_t = \nu_{xx} + f'(0)\,\nu.$$

This allows them to estimate the error of the approximation $w_{\lambda,\lambda',h,h'}$ pointwise in a rather precise manner. However, the same method cannot be applied to our quasilinear equation (1.8), which actually becomes *singular* as $x \to \pm\infty$ (where the approximating supersolution $v_{\lambda,\lambda',h,h'}$ vanishes). In this work we need to depart from the methods in [8] and we have chosen to use integral methods in order to estimate the error term $w_{\lambda,\lambda',h,h'}$.

2. The construction of merging traveling waves

For fixed $\lambda, \lambda' \geq 1$, $h, h' \in \mathbb{R}$, let $u_{\lambda, h}$ and $\hat{u}_{\lambda', h'}$ be the two traveling wave solutions of Eq. (1.8) as introduced in the previous section. We define the *approximate supersolution* $v := v_{\lambda, \lambda', h, h'}$ as

$$v_{\lambda, \lambda', h, h'}(\cdot, \tau) = \min\big(u_{\lambda, h}(\cdot, \tau), \hat{u}_{\lambda', h'}(\cdot, \tau)\big), \quad \tau \in (-\infty, +\infty). \tag{2.1}$$

Using the definitions of $u_{\lambda, h}$ and $\hat{u}_{\lambda', h'}$ we have

$$v_{\lambda, \lambda', h, h'}(\cdot, \tau) = \min\big(v_\lambda(x - \lambda \tau + h), v_{\lambda'}(-x - \lambda' \tau + h')\big). \tag{2.2}$$

We will show in this section that for any $(\lambda, \lambda', h, h') \in \mathbb{R}^4$ such that $\lambda, \lambda' > 1$, there exists a solution $u_{\lambda, \lambda', h, h'}$ which is close in certain sense to the approximate supersolution $v_{\lambda, \lambda', h, h'}$, as stated next.

Theorem 2.1. *For any $(\lambda, \lambda', h, h') \in \mathbb{R}^4$ such that $\lambda, \lambda' > 1$ there exists an ancient solution $u_{\lambda, \lambda', h, h'}$ of (1.8) defined on $\mathbb{R} \times (-\infty, T_{\lambda, \lambda', h, h'})$ for some $T_{\lambda, \lambda', h, h'} \in (-\infty, +\infty)$ which satisfies*

$$0 < u_{\lambda, \lambda', h, h'} \leq v_{\lambda, \lambda', h, h'}, \quad \text{for all } (x, \tau) \in \mathbb{R} \times (-\infty, T_{\lambda, \lambda', h, h'}).$$

In addition, for $\tau \ll 0$, the solution $u_{\lambda, \lambda', h, h'}$ is close to the approximate supersolution $v_{\lambda, \lambda', h, h'}$ in the sense that

$$\int_{\mathbb{R}} |v_{\lambda, \lambda', h, h'}^p - u_{\lambda, \lambda', h, h'}^p|(\cdot, \tau) \, dx \leq D_{\lambda, \lambda', h, h'} \, e^{d\tau}$$

where $d = \frac{\gamma_\lambda \gamma_{\lambda'} + (p-1)}{p}$ and $D_{\lambda, \lambda', h, h'}$ is a positive constant depending only on the dimension n and λ, λ', h, h'. Moreover, if $\lambda, \lambda', h, h' \neq \bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'$, then $u_{\lambda, \lambda', h, h'} \neq u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'}$.

We have seen in the introduction that $u_{\lambda, h}$ and $\hat{u}_{\lambda', h'}$ satisfy conditions (1.22)–(1.24). It follows that for each τ there is a unique *intersection point* $x(\tau)$ for which $u_{\lambda, h}(x(\tau), \tau) = \hat{u}_{\lambda', h'}(x(\tau), \tau)$.

Lemma 2.1. *The intersection point $x(\tau)$ of $u_{\lambda, h}$ and $u_{\lambda', h'}$ satisfies, as $\tau \to -\infty$, the asymptotic behavior*

$$x(\tau) = \frac{\gamma_\lambda - \gamma_{\lambda'}}{p} \tau + \frac{1}{\gamma_\lambda + \gamma_{\lambda'}} \left(\ln \frac{C_\lambda}{C_{\lambda'}} + h' \gamma_{\lambda'} - h \gamma_\lambda \right) + o(1). \tag{2.3}$$

In addition at $x = x(\tau)$ we have

$$u_{\lambda, h}(x(\tau), \tau) = \hat{u}_{\lambda', h'}(x(\tau), \tau) = 1 - C_{\lambda, \lambda', h, h'} \, e^{d\tau} + o(e^{d\tau}) \tag{2.4}$$

with

$$d := \frac{\gamma_\lambda \gamma_{\lambda'} + (p - 1)}{p} \tag{2.5}$$

and $C_{\lambda, \lambda', h, h'}$ depending on λ, λ', h, h'. Also, it follows that

$$(u_{\lambda, h})_x(x(\tau), \tau) = \gamma_\lambda \, C_{\lambda, \lambda', h, h'} \, e^{d\tau} + o(e^{d\tau}), \qquad (\hat{u}_{\lambda', h'})_x(x(\tau), \tau) = -\gamma_{\lambda'} \, C_{\lambda, \lambda', h, h'} \, e^{d\tau} + o(e^{d\tau}). \tag{2.6}$$

Proof. Using the asymptotics behavior (1.23) and (1.24) it follows that at $x = x(\tau)$ we have

$$C_\lambda e^{-\gamma_\lambda (x - \lambda \tau + h)} + o(e^{-\gamma_\lambda (x + h - \lambda \tau)}) \approx C_{\lambda'} e^{-\gamma_{\lambda'} (-x - \lambda' \tau - h')} + o(e^{-\gamma_{\lambda'} (-x' - \lambda' \tau - h')}).$$

Solving for x readily implies that

$$x(\tau) = \frac{\lambda \gamma_\lambda - \lambda' \gamma_{\lambda'}}{\gamma_\lambda + \gamma_{\lambda'}} \tau + \frac{1}{\gamma_\lambda + \gamma_{\lambda'}} \left(\ln \frac{C_\lambda}{C_{\lambda'}} + h' \gamma_{\lambda'} - h \gamma_\lambda \right) + o(1).$$

Using Eq. (1.17), we may eliminate the parameters λ, λ' from the above expression substituting

$$\lambda\gamma_\lambda = \frac{\gamma_\lambda^2 + (p-1)}{p}, \qquad \lambda'\gamma_{\lambda'} = \frac{\gamma_{\lambda'}^2 + (p-1)}{p}$$

and obtain (2.3). With this choice of $x(\tau)$ we have

$$\gamma_\lambda\big(x(\tau) - \lambda\,\tau\big) = \gamma_\lambda\left(\frac{\gamma_\lambda - \gamma_{\lambda'}}{p} - \lambda\right)\tau + c_{\lambda,\lambda',h,h'}$$

for some constant $c_{\lambda,\lambda',h,h'}$ depending on λ, λ', h, h' and eliminating λ as above we obtain

$$\gamma_\lambda\big(x(\tau) - \lambda\,\tau\big) = -\frac{\gamma_\lambda\gamma_{\lambda'} + (p-1)}{p}\tau + c_{\lambda,\lambda',h,h'}.$$

Setting $d := \frac{\gamma_\lambda\gamma_{\lambda'}+(p-1)}{p}$, we conclude using (1.23) that

$$u_{\lambda,h}(x(\tau),\tau) = 1 - C_{\lambda,\lambda',h,h'}\,e^{d\tau} + o(e^{d\tau})$$

for a constant $C_{\lambda,\lambda',h,h'} > 0$ depending on λ, λ', h, h'. Since $u_{\lambda,h}(x(\tau),\tau) = u_{\lambda',h'}(x(\tau),\tau)$, (2.4) follows.

It remains to show (2.6). Recall that $u_{\lambda,h}(x,\tau) = v_\lambda(x - \lambda\tau + h)$. First we claim that

$$\lim_{x\to+\infty} (v_\lambda)_x = 0. \tag{2.7}$$

To prove this fact note that by (1.11) we have

$$\big((v_\lambda)_x + \lambda v_\lambda^p\big)_x = v_\lambda - v_\lambda^p \geq 0,$$

since $v_\lambda \leq 1$, implying there exists a finite limit $\lim_{x\to+\infty}((v_\lambda)_x + \lambda v_\lambda^p)$ and hence the $\lim_{x\to+\infty}(v_\lambda)_x = c$. We claim that $c = 0$. Indeed, if $c > 0$, there would exist an x_0 so that for all $x \geq x_0$ we would have $(v_\lambda)_x \geq c/2$. This would imply that

$$v_\lambda(x) = v_\lambda(x_0) + \int_{x_0}^x (v_\lambda)_x\,dx \geq \frac{c}{2}(x - x_0), \quad x \geq x_0$$

contradicting that the $\lim_{x\to+\infty} v_\lambda(x) = 1$. Using that $v_\lambda > 0$ we argue similarly in the case we assume $c < 0$. We will prove next more precise asymptotics on the derivatives of v_λ, which will yield (2.6). By (1.11) we have

$$\big((v_\lambda)_x + \lambda v_\lambda^p\big)_x = v_\lambda - v_\lambda^p.$$

On the other hand, by (1.14) we have

$$v_\lambda - v_\lambda^p = C_\lambda\,(p-1)\,e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}), \quad \text{for } x \gg 1$$

and hence,

$$\big((v_\lambda)_x + \lambda v_\lambda^p\big)_x = C_\lambda\,(p-1)\,e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}).$$

Integrating this relation from x to $+\infty$ and using (2.7) and that the $\lim_{x\to+\infty} v_\lambda(x) = 1$ yields

$$(v_\lambda)_x = \lambda - \lambda v_\lambda^p - \frac{C_\lambda\,(p-1)}{\gamma_\lambda}e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}).$$

Asymptotics (1.14) implies $v_\lambda^p = 1 - p\,C_\lambda e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x})$, and therefore,

$$(v_\lambda)_x = C_\lambda\,e^{-\gamma_\lambda x}\left(p\lambda - \frac{p-1}{\gamma_\lambda}\right) + o(e^{-\gamma_\lambda x})$$

$$= C_\lambda\gamma_\lambda e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}), \quad \text{as } x \to +\infty, \tag{2.8}$$

where we have used that $p\lambda\gamma_\lambda = \gamma_\lambda^2 + (p-1)$. Finally, since $x(\tau) - \lambda\tau + h \gg 1$ for $\tau \ll -1$, we get (2.6) by substituting $x(\tau) - \lambda\tau + h$ in (2.8). $\square$

Denote briefly by $v := v_{\lambda,\lambda',h,h'}$. Then we have the following integral identity.

Lemma 2.2. *We have*

$$\frac{d}{d\tau} \int_{\mathbb{R}} v^p \, dx = \int_{\mathbb{R}} v^p \, dx - \int_{\mathbb{R}} v \, dx + (\gamma_\lambda + \gamma_{\lambda'}) \, C_{\lambda,\lambda',h,h'} \, e^{d\tau} + o(e^{d\tau}). \tag{2.9}$$

Proof. For simplicity set $u_1 := u_{\lambda,h}$ and $u_2 := u_{\lambda',h'}$. Then $u_1(\cdot,\tau), u_2(\cdot,\tau)$ are solutions to (1.8) on $(-\infty, x(\tau))$, $(x(\tau), +\infty)$ respectively and by definition we have $v = u_1$ on $(-\infty, x(\tau))$ and $v = u_2$ on $(x(\tau), +\infty)$. In addition, because of (1.22) we have

$$\lim_{x \to -\infty} (u_1)_x(x,\tau) = \lim_{x \to +\infty} (u_2)_x(x,\tau) = 0.$$

Note this can be proved in the same way as we have proved (2.8), just using the asymptotics of our solitons at $x \to -\infty$ instead of $x \to +\infty$. Hence, integrating Eq. (1.8) for u_1 on $(-\infty, x(\tau))$ and Eq. (1.8) for u_2 on $(x(\tau), +\infty)$ we obtain

$$\frac{d}{d\tau} \int_{-\infty}^{x(\tau)} u_1^p \, dx = \int_{-\infty}^{x(\tau)} u_1^p \, dx - \int_{-\infty}^{x(\tau)} u_1 \, dx + (u_1)_x(x(\tau),\tau) + x'(\tau) \, u_1^p(x(\tau),\tau)$$

and

$$\frac{d}{d\tau} \int_{x(\tau)}^{+\infty} u_2^p \, dx = \int_{x(\tau)}^{+\infty} u_2^p \, dx - \int_{x(\tau)}^{+\infty} u_2 \, dx - (u_2)_x(x(\tau),\tau) - x'(\tau) \, u_2^p(x(\tau),\tau).$$

Since $u_1(x(\tau),\tau) = u_2(x(\tau),\tau)$, adding the last two equalities yields

$$\frac{d}{d\tau} \int_{\mathbb{R}} v^p \, dx = \int_{\mathbb{R}} v^p \, dx - \int_{\mathbb{R}} v \, dx + (u_1)_x(x(\tau),\tau) - (u_2)_x(x(\tau),\tau).$$

Combining this with (2.6), we readily get (2.9). $\quad\square$

For any $m \in \mathbb{N}$, let u_m denote the solution of the initial value problem

$$\begin{cases} (u^p)_\tau = u_{xx} - u + u^p & x \in \mathbb{R}, \ \tau > -m \\ u(\cdot,-m) = v_{\lambda,\lambda',h,h'}(\cdot,-m) & x \in \mathbb{R} \end{cases} \tag{2.10}$$

with exponent $p = \frac{n+2}{n-2} > 1$.

Lemma 2.3 (*Uniform Barrier from Above*)**.** *The solution u_m exists for all time $-m \le \tau < +\infty$ and satisfies*

$$u_m \le v_{\lambda,\lambda',h,h'}. \tag{2.11}$$

Proof. The bound (2.11) simply follows from the comparison principle. Since $u_m(\cdot,-m) \le u_{\lambda,h}(\cdot,-m)$ and $u_m(\cdot,-m) \le \hat{u}_{\lambda',h'}(\cdot,-m)$ we have $u_m \le u_{\lambda,h}$ and $u_m \le \hat{u}_{\lambda',h'}$ for $\tau \ge -m$, concluding that $u_m \le v_{\lambda,\lambda',h,h'}(\cdot,\tau) := \min(u_{\lambda,h}(\cdot,\tau), \hat{u}_{\lambda',h'}(\cdot,\tau))$. The bound (2.11) and standard arguments on quasilinear parabolic PDE imply that the solution u_m exists for all $-m \le \tau < +\infty$. $\quad\square$

Remark 2.1. In what follows we show a bound from below for u_m, which is uniform in m and will guarantee that the solutions u_m will stay positive for $-m \le \tau < T$, for some uniform in m time T.

Lemma 2.4 (*The Profile for u_m*)**.** *There exists a number $T_m \le +\infty$ such that $u_m > 0$ on $\mathbb{R} \times [-m, T_m)$ and in the cases where $T_m < +\infty$, $u_m \equiv 0$ for $\tau \ge T_m$. In addition, for all $\tau \in [-m, T_m)$, $u_m(\cdot,t)$ satisfies the asymptotic behavior*

$$u_m(x,\tau) = O(e^x), \quad as\, x \to -\infty \quad and \quad u_m(x,\tau) = O(e^{-x}), \quad as\, x \to +\infty. \tag{2.12}$$

Moreover, the function $u_m(x,\tau)$ is decreasing in τ, for all $\tau > -m$.

Proof. The first two claims in this lemma follow immediately from well known results for fast-diffusion equations and the Yamabe flow on S^n, since $g = u_m(\cdot, \tau)^{\frac{4}{n-2}} g_{cyl}$ corresponds to a solution of the Yamabe flow and the behavior (2.12) is equivalent to saying that g can be lifted to a smooth metric on S^n.

We will next show the monotonicity in τ of the solutions u_m. It follows from (1.8) that the function $w_1(x) = v_\lambda(x + \lambda m + h)$ satisfies

$$
\begin{aligned}
w_1'' + w_1^p - w_1 &= v_\lambda''(x + \lambda m + h) + v_\lambda^p(x + \lambda m + h) - v_\lambda(x + \lambda m + h) \\
&= -\lambda \, v_\lambda'(x + \lambda m + h) < 0
\end{aligned}
$$

and similarly the function $w_2(x) = v_{\lambda'}(-x - \lambda' m + h')$ satisfies

$$
\begin{aligned}
w_2'' + w_2^p - w_2 &= v_{\lambda'}''(-x - \lambda' m + h) + v_{\lambda'}^p(-x - \lambda' m + h) - v_{\lambda'}(-x - \lambda' m + h) \\
&= -\lambda' \, v_{\lambda'}'(-x - \lambda' m + h) < 0
\end{aligned}
$$

since $v_\lambda' > 0$ for all $\lambda \geq 1$. It follows that $f_m := \min(w_1, w_2)$ is a supersolution, namely it satisfies

$$ f_m'' + f_m^p - f_m < 0 $$

in the distributional sense. This implies the function $u_m(x, \tau)$ is decreasing in τ for any $\tau > -m$, $x \in \mathbb{R}$. Hence, the result follows by a simple approximation argument. $\quad\square$

Remark 2.2. Each solution $u_{\lambda,h}$ satisfies $(u_{\lambda,h})_\tau \leq 0$, since $(u_{\lambda,h})_\tau = -\lambda \, v_\lambda'(x - \lambda t + h) < 0$, because $v_\lambda' < 0$.

Remark 2.3. The inequality $(u_m)_\tau \leq 0$ implies that the scalar curvature R_m of the corresponding metric defined by the solution u_m is nonnegative. Recall that for a solution u of (1.8), $R \geq 0$ corresponds to $(u^p)_\tau \leq u^p$.

We will next show that each u_m is sufficiently close to $v_{\lambda,\lambda',h,h'}$ in certain sense and this happens uniformly in m. This will assure that the limit as $m \to +\infty$ is a non-trivial solution of (1.8). We begin with the following crucial for our purposes estimate which is a consequence of Lemma 2.2.

Proposition 2.1. *We have*

$$ Q_m(\tau) := \int_{\mathbb{R}} (v^p - u_m^p) \leq D_{\lambda,\lambda',h,h'} \, e^{d\tau} \tag{2.13} $$

for a constant $D_{\lambda,\lambda',h,h'} > 0$ depending only on λ, λ', h, h'.

Proof. Since u_m satisfies (1.8), integrating this equation on $\mathbb{R}$ readily yields

$$ \frac{d}{d\tau} \int_{\mathbb{R}} u^p \, dx = \int_{\mathbb{R}} u^p \, dx - \int_{\mathbb{R}} u \, dx. $$

Here we used that

$$ \lim_{x \pm \infty} (u_m)_x(x, \tau) = 0 $$

which easily follows from (2.12) and the fact that the metric $u_m^{\frac{4}{n-2}} g_{cyl}$ when lifted to a sphere defines a smooth metric. If we combine this with (2.9) we obtain

$$ \frac{d}{d\tau} \int_{\mathbb{R}} (v^p - u_m^p) \, dx \leq \int_{\mathbb{R}} (v^p - u_m^p) \, dx - \int_{\mathbb{R}} (v - u_m) \, dx + \bar{C}_{\lambda,\lambda',h,h'} \, e^{d\tau} \tag{2.14} $$

with $\bar{C}_{\lambda,\lambda',h,h'} := (\gamma_\lambda + \gamma_{\lambda'}) C_{\lambda,\lambda',h,h'} + 1$. Next set $w_m := v - u_m$ and observe that since $u_m \leq v$ we have $w_m \geq 0$. Since

$$(v^p - u_m^p) = a\,(v - u_m), \quad \text{where } a := p \int_0^1 (s\,v + (1-s)\,u_m)^{p-1}\,ds$$

we may write (2.14) as

$$\frac{d}{d\tau} \int_{\mathbb{R}} a\,w_m\,dx = \frac{p-1}{p} \int_{\mathbb{R}} a\,w_m\,dx + \frac{1}{p} \int (a-p)w_m\,dx + \bar{C}_{\lambda,\lambda',h,h'}\,e^{d\tau} + o(e^{d\tau}).$$

Note that, since both $v \leq 1$ and $u_m \leq 1$, we have

$$a - p = p \left(\int_0^1 (s\,v + (1-s)\,u_m)^{p-1}\,ds - 1 \right) \leq 0.$$

Hence, using also that $w_m \geq 0$, we conclude

$$\frac{d}{d\tau} \int_{\mathbb{R}} a\,w_m\,dx \leq \frac{p-1}{p} \int_{\mathbb{R}} a\,w_m\,dx + \bar{C}_{\lambda,\lambda',h,h'}\,e^{d\tau}.$$

Setting

$$Q_m(\tau) := \int_{\mathbb{R}} (v^p - u_m^p)(\cdot,\tau)\,dx = \int_{\mathbb{R}} a\,(v - u_m)(\cdot,\tau)\,dx$$

we obtain

$$\frac{d}{d\tau} Q_m(\tau) \leq \frac{p-1}{p} Q_m(\tau) + \bar{C}_{\lambda,\lambda',h,h'}\,e^{d\tau}.$$

Equivalently, if

$$\hat{Q}_m(\tau) := e^{-\frac{(p-1)\tau}{p}} Q_m(\tau)$$

and $\mu := d - \frac{p-1}{p}$ we have

$$\frac{d}{d\tau} \hat{Q}_m(\tau) \leq \bar{C}_{\lambda,\lambda',h,h'}\,e^{\mu\tau} + o(e^{\mu\tau}).$$

Next observe that by (2.5) we have $d > \frac{p-1}{p}$, hence $\mu > 0$. Also, since $w_m = 0$ at $\tau = -m$, we have $\hat{Q}_m(-m) = 0$. Hence, the above differential inequality yields the bound

$$\hat{Q}_m(\tau) \leq \mu^{-1}\bar{C}_{\lambda,\lambda',h,h'}\,e^{\mu\tau}, \quad \tau > -m$$

from which the bound (2.13) readily follows. □

Proposition 2.2 (*Passing to the Limit*). *After passing to a subsequence, the sequence $\{u_m\}$ converges, uniformly on compact subsets of $\mathbb{R} \times (-\infty, +\infty)$, to an ancient solution $u = u_{\lambda,\lambda',h,h'}$ of* (1.8). *It is positive, $u > 0$, on $\mathbb{R} \times (-\infty, T_{\lambda,\lambda',h,h'})$ for some $T_{\lambda,\lambda',h,h'}$, depending only on λ, λ', h, h' and the dimension n. In addition, $u(x,\tau)$ is decreasing in τ for all $(x,\tau) \in \mathbb{R} \times (-\infty, +\infty)$ and satisfies conditions* (2.12).

Proof. The uniform bound $u_m \leq v$ implies that the sequence of solutions $\{u_m\}$ is uniformly bounded on compact subsets of $\mathbb{R} \times (-\infty, +\infty)$, hence by standard estimates it is equicontinuous. Hence, passing to a subsequence it converges to a limit $u = u_{\lambda,\lambda',h,h'}$ and $u(x,\tau)$ is decreasing in τ for all $x \in \mathbb{R}$, since the same holds for each u_m be the previous lemma.

We will next show that the limit u is nontrivial. Since $u_m(\cdot,\tau) \leq v(\cdot,\tau)$, $v^{p-1} \leq a \leq p\,v^{p-1}$ and $v(\cdot,\tau) \leq C(\tau) \min(e^x, e^{-x})$, we can pass to the limit $m \to +\infty$ in (2.13) and using the dominated convergence theorem we obtain the bound

$$Q(\tau) := \int_{\mathbb{R}} \hat{a}\,(v - u)(\cdot,\tau)\,dx \leq D_{\lambda,\lambda',h,h'}\,e^{d\tau}, \tag{2.15}$$

for

$$\hat{a} = p \int_0^1 (s\,v + (1-s)\,u)^{p-1}\, ds = \frac{v^p - u^p}{v - u}.$$

Observe that

$$Q(\tau) := \int_{\mathbb{R}} a\,(v - u)(\cdot, \tau)\, dx = \int_{\mathbb{R}} v^p(\cdot, \tau)\, dx - \int_{\mathbb{R}} u^p(\cdot, \tau)\, dx.$$

In particular, this implies for every fixed $\tau \ll -1$ there exists a point (x, τ), such that $x \in [x(\tau) - 1, x(\tau)]$ and

$$0 \le v(x, \tau) - u(x, \tau) \le D_{\lambda, \lambda', h, h'}\, e^{d\tau}.$$

Recalling that $v(x, \tau) \approx 1 - C_\lambda e^{\lambda \gamma_\lambda \tau} e^{-\gamma_\lambda (x+h)}$, whenever $x - \lambda\tau \gg 1$ and $x \in [x(\tau) - 1, x(\tau)]$, we conclude that

$$u(x, \tau) \ge 1 - \bar{D}_{\lambda, \lambda', h, h'}\, e^{d\tau}.$$

This implies that

$$m(\tau) := \max_{\mathbb{R}} u(\cdot, \tau) \ge 1 - \bar{D}_{\lambda, \lambda', h, h'}\, e^{d\tau}. \tag{2.16}$$

On the other hand, using (2.4) we have

$$u(x, \tau) \le v(x, \tau) \le v(x(\tau), \tau) = 1 - C_{\lambda, \lambda', h, h'}\, e^{d\tau} + o(e^{d\tau}).$$

Hence,

$$1 - \bar{D}_{\lambda, \lambda', h, h'}\, e^{d\tau} \le m(\tau) \le 1 - \bar{C}_{\lambda, \lambda', h, h'}\, e^{d\tau} \tag{2.17}$$

for $\bar{C}_{\lambda, \lambda', h, h'} := C_{\lambda, \lambda', h, h'} + 1$. This in particular implies that $m(\tau) > 0$ for all $\tau \le \tau_0$ if $\tau_0 \ll 0$. Hence, there exists a number $T = T_{\lambda, \lambda', h, h'}$ such that $m(\tau) > 0$ for all $t \le T_{\lambda, \lambda', h, h'}$ and we may assume that $T_{\lambda, \lambda', h, h'}$ is the maximal such time (note that $T_{\lambda, \lambda', h, h'}$ may be equal to $+\infty$). Standard estimates then imply that $u(x, \tau) > 0$ for all $(x, \tau) \in \mathbb{R} \times (-\infty, T_{\lambda, \lambda', h, h'})$. We also have that $u(\cdot, \tau)$ satisfies conditions (2.12). $\square$

Next we show how to distinguish between solutions that we have constructed using different parameters. More precisely, we have the following result.

Proposition 2.3 (*Distinguishing Between Solutions*). *Let* $\lambda, \lambda', \bar{\lambda}, \bar{\lambda}' > 1$ *and* $(\lambda, \lambda', h, h') \ne (\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}')$, *then* $u_{\lambda, \lambda', h, h'} \ne u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'}$.

Proof. We will prove the proposition in two steps.

Step 1. *Fix* $h, h', \bar{h}, \bar{h}'$. *If* $\lambda, \lambda', \bar{\lambda}, \bar{\lambda}' > 1$ *and* $(\lambda, \lambda') \ne (\bar{\lambda}, \bar{\lambda}')$, *then* $u_{\lambda, \lambda', h, h'} \ne u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'}$.

To prove the claim we argue by contradiction. Assume that $(\lambda, \lambda') \ne (\bar{\lambda}, \bar{\lambda}')$ and $u_{\lambda, \lambda', h, h'} \equiv u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'}$. For simplicity we call this solution u. Without loss of generality we may assume that $\lambda < \bar{\lambda}$. By (2.5) we have

$$d = \frac{\gamma_\lambda \gamma_{\lambda'} + (p-1)}{p} = \frac{\gamma_{\bar{\lambda}} \gamma_{\bar{\lambda}'} + (p-1)}{p} = \bar{d}$$

implying $\gamma_\lambda \gamma_{\lambda'} = \gamma_{\bar{\lambda}} \gamma_{\bar{\lambda}'}$. If $m(\tau) := \max_{\mathbb{R}} u(\cdot, \tau)$ then it satisfies (2.17). Let $x_{\max}(\tau)$ be a point such that $m(\tau) = u(x_{\max}(\tau), \tau)$. If $v, \bar{v}$ are the approximating solutions corresponding to $u_{\lambda, \lambda', h, h'}, u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'}$ respectively, then we have $u = u_{\lambda, \lambda', h, h'} \le v$ and $u = u_{\bar{\lambda}, \bar{\lambda}', \bar{h}, \bar{h}'} \le \bar{v}$, which combined with (2.17) gives that

$$v(x_{\max}(\tau), \tau) \ge 1 - C_1\, e^{d\tau} \quad \text{and} \quad \bar{v}(x_{\max}(\tau), \tau) \ge 1 - C_1\, e^{d\tau}. \tag{2.18}$$

In addition, if $x(\tau)$, $\bar{x}(\tau)$ denote the maximum points of $v(\cdot,\tau)$, $\bar{v}(\cdot,\tau)$ respectively, we have

$$v(x_{\max}(\tau),\tau) \leq v(x(\tau),\tau) \leq 1 - C_2\, e^{d\tau} \quad \text{and} \quad \bar{v}(x_{\max}(\tau),\tau) \leq \bar{v}(\bar{x}(\tau),\tau) \leq 1 - C_2\, e^{d\tau}. \qquad (2.19)$$

Using the asymptotics (1.14) and the estimates (2.18), (2.19) we conclude that

$$e^{-\gamma_\lambda(x_{\max}(\tau)-\lambda\tau)} \approx e^{-\gamma_{\bar{\lambda}}(-x_{\max}(\tau)-\bar{\lambda}\tau)},$$

yielding

$$x_{\max}(\tau) = \frac{\gamma_\lambda - \gamma_{\lambda'} + (p-1)}{p}\,\tau + O(1) = \frac{\gamma_{\bar{\lambda}} - \gamma_{\bar{\lambda}'} + (p-1)}{p}\,\tau + O(1). \qquad (2.20)$$

This in particular implies that $\gamma_\lambda - \gamma_{\lambda'} = \gamma_{\bar{\lambda}} - \gamma_{\bar{\lambda}'}$. In addition, by (2.20) and (2.3) we have

$$x_{\max}(\tau) \approx x(\tau) + O(1) = \bar{x}(\tau) + O(1).$$

On the other hand, by (2.13) we have

$$\int_{\mathbb{R}} (v^p - u^p)\, dx \leq C_1 e^{d\tau}$$

and

$$\int_{\mathbb{R}} (\bar{v}^p - u^p)\, dx \leq C_2 e^{d\tau}.$$

Recalling that $u \leq v$ and $u \leq \bar{v}$ we conclude that

$$\int_{\mathbb{R}} |v^p - \bar{v}^p|\, dx \leq C\, e^{d\tau}.$$

Since $x(\tau)$ and $\bar{x}(\tau)$ are comparable for $\tau \ll -1$, without a loss of any generality we may assume $x(\tau) \leq \bar{x}(\tau)$. Then we have

$$\int_{-\infty}^{x(\tau)} |v_\lambda^p(x + h - \lambda\tau) - v_{\bar{\lambda}}^p(x + \bar{h} - \bar{\lambda}\tau)|\, dx \leq C e^{d\tau}$$

implying

$$\int_{-\infty}^{x(\tau)+h-\lambda\tau} |v_\lambda^p(y) - v_{\bar{\lambda}}^p(y + (\lambda - \bar{\lambda})\tau + \bar{h} - h)|\, dy \leq C\, e^{d\tau}.$$

Using the asymptotics (1.14) and that $(\lambda - \bar{\lambda})\,\tau \gg 1$ (since $\lambda < \bar{\lambda}$), the previous inequality gives

$$\int_{M}^{2M} |e^{-\gamma_\lambda y} - e^{-\gamma_{\bar{\lambda}}(y+\bar{h}-h+(\lambda-\bar{\lambda})\tau)}|\, dy \leq C e^{d\tau}, \qquad (2.21)$$

for a big constant $M \gg 1$. Estimate (2.21) holding for any $\tau \ll -1$ forces $\lambda = \bar{\lambda}$, which concludes the proof of Step 1.

Step 2. *Fix now $\lambda, \lambda' > 1$. If $h, h', \bar{h}, \bar{h}'$ satisfy $(h, h') \neq (\bar{h}, \bar{h}')$, then $u_{\lambda,\lambda',h,h'} \neq u_{\lambda,\lambda',\bar{h},\bar{h}'}$.*

To prove Step 2 we argue by contradiction. Assume that $(h, h') \neq (\bar{h}, \bar{h}')$ and $u := u_{\lambda,\lambda',h,h'} = u_{\lambda,\lambda',\bar{h},\bar{h}'}$. By translating $v_\lambda, v_{\lambda'}$ by $\bar{h}, \bar{h}'$ respectively, we may assume that $\bar{h} = \bar{h}' = 0$ (our proof is not using the exact choice of $v_\lambda, v_{\lambda'}$ so that $v_\lambda(0) = v_{\lambda'}(0) = 1/2$). Let v be the approximation of $u := u_{\lambda,\lambda',h,h'}$ given by $v := \min(v_\lambda(x + h - \lambda\tau), v_{\lambda'}(-x + h' - \lambda'\tau))$. We observe that

$$Q(\tau) := \int_{\mathbb{R}} a\,(v - u)(\cdot,\tau)\, dx = \int_{\mathbb{R}} v^p(\cdot,\tau)\, dx - \int_{\mathbb{R}} u^p(\cdot,\tau)\, dx.$$

Hence, the bounds $u \leq v$ and (2.15) yield

$$\left| \int_{\mathbb{R}} v^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} u^p(\cdot, \tau) \, dx \right| \leq C_{\lambda, \lambda', h, h'} \, e^{d\tau},$$

where d is given by (2.5) and depends only on λ, λ'. Similarly, if $\bar{v} := \min\left(v_\lambda(x - \lambda\tau), v_{\lambda'}(-x - \lambda'\tau) \right)$ is the approximation of $\bar{u} := u_{\lambda, \lambda', \bar{h}, \bar{h}'}$ with $\bar{h} = \bar{h}' = 0$, then

$$\left| \int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} u^p(\cdot, \tau) \, dx \right| \leq C_{\lambda, \lambda', \bar{h}, \bar{h}'} \, e^{d\tau}.$$

We conclude that

$$\left| \int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx \right| \leq C \, e^{d\tau}. \tag{2.22}$$

We will now show that if $(h, h') \neq (0, 0)$, then (2.22) cannot hold leading to a contradiction. To this end, denote by $x(\tau)$ the intersection point between $v_\lambda(x + h - \lambda\tau)$ and $v_{\lambda'}(-x + h - \lambda'\tau)$ and by $\bar{x}(\tau)$ the intersection point between $v_\lambda(x - \lambda\tau)$ and $v_{\lambda'}(-x - \lambda'\tau)$. We have

$$\int_{\mathbb{R}} v^p(\cdot, \tau) \, dx = \int_{-\infty}^{x(\tau)+h} v_\lambda^p(x - \lambda\tau) \, dx + \int_{x(\tau)-h'}^{+\infty} v_{\lambda'}^p(-x - \lambda'\tau) \, dy$$

and similarly

$$\int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx = \int_{-\infty}^{\bar{x}(\tau)} v_\lambda^p(x - \lambda\tau) \, dx + \int_{\bar{x}(\tau)}^{+\infty} v_{\lambda'}^p(-x - \lambda'\tau) \, dx.$$

Hence,

$$\int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} v^p(\cdot, \tau) \, dx = \int_{x(\tau)+h}^{\bar{x}(\tau)} v_\lambda^p(x - \lambda\tau) \, dx + \int_{\bar{x}(\tau)}^{x(\tau)-h'} v_{\lambda'}^p(-x - \lambda'\tau) \, dx.$$

By (2.3) we have

$$\bar{x}(\tau) - x(\tau) = \frac{h\gamma_\lambda - h'\gamma_\lambda'}{\gamma_\lambda + \gamma_\lambda'} + o(1)$$

which implies

$$\bar{x}(\tau) - x(\tau) = h - \frac{\gamma_{\lambda'}(h' + h)}{\gamma_\lambda + \gamma_\lambda'} + o(1) = -h' + \frac{\gamma_\lambda(h' + h)}{\gamma_\lambda + \gamma_\lambda'} + o(1). \tag{2.23}$$

Setting $\mu := \frac{\gamma_\lambda}{\gamma_\lambda + \gamma_\lambda'}$, $\mu' := \frac{\gamma_{\lambda'}}{\gamma_\lambda + \gamma_\lambda'}$ and combining the above yields

$$\int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} v^p(\cdot, \tau) \, dx = \int_{x(\tau)+h}^{x(\tau)+h-\mu'(h+h')} v_\lambda^p(x - \lambda\tau) \, dx$$

$$+ \int_{x(\tau)-h'+\mu(h+h')}^{x(\tau)-h'} v_{\lambda'}^p(-x - \lambda'\tau) \, dx + o(1). \tag{2.24}$$

For h, h' and $\tau \ll 0$ (depending on h, h') we have $v_\lambda(x - \lambda\tau) \geq 1/2$ and $v_{\lambda'}(-x - \lambda'\tau) \geq 1/2$ on the intervals over which those functions are integrated in (2.24). In addition, both integrals on the right hand side of (2.24) have the same sign. Hence,

$$\left| \int_{\mathbb{R}} \bar{v}^p(\cdot, \tau) \, dx - \int_{\mathbb{R}} v^p(\cdot, \tau) \, dx \right| \geq \frac{1}{2} (\mu + \mu') (h + h') + o(1) = \frac{1}{2} |h + h'| + o(1).$$

If $h + h' \neq 0$ this contradicts (2.22) and concludes the proof of the lemma. If $h' = -h$ then $v(x, \tau) = \bar{v}(x + h, \tau)$ for all τ, which means that the solutions $u_m, \bar{u}_m$ of (2.10) with initial data $v(\cdot, -m), \bar{v}(\cdot, -m)$ respectively satisfy $u_m(x, \tau) = \bar{u}_m(x + h, \tau)$ for all $\tau > -m$, hence the same will hold for the limits $u, \bar{u}$. Since $u = \bar{u}$,

this means that $u(x, \tau) = u(x + h, \tau)$ for any $x \in \mathbb{R}$. On the other hand, the fact that the solution u defines a metric that can be lifted to a smooth metric on S^n implies that $u(x, \tau) = C(\tau) e^x (1 + o(1))$, as $x \to -\infty$ with $C(\tau) > 0$, hence $u(x, \tau) = u(x + h, \tau)$ must imply that $h = 0$ which means that $(h, h') = (0, 0)$ and contradicts our assumption. The proof of Step 2 is now complete. $\square$

Proof of Theorem 2.1. The proof of the theorem is a direct consequence of Propositions 2.2 and 2.3. $\square$

3. The geometry of merging traveling waves

In this last section we derive the geometric properties of the ancient solution $u_{\lambda, \lambda', h, h'}$ of Eq. (1.8) on $\mathbb{R} \times (-\infty, T_{\lambda, \lambda', h, h'})$, as constructed in Theorem 2.1. The one parameter family of metrics $g_{\lambda, \lambda', h, h'}(\tau) := u_{\lambda, \lambda', h, h'}^{\frac{4}{n-2}}(\cdot, \tau) g_{cyl}$ can be lifted to a smooth one parameter family of metrics on $S^n \times (-\infty, T_{\lambda, \lambda', h, h'})$ which defines an ancient rotationally symmetric solution of the *rescaled Yamabe flow* on S^n, equation

$$\frac{\partial}{\partial \tau} g = -(R - 1) g. \tag{3.1}$$

We next prove the following result concerning the behavior of the Riemannian curvature of the metric $g_{\lambda, \lambda', h, h'}(\tau)$ near $\tau = -\infty$.

Theorem 3.1. *The solution $g_{\lambda, \lambda', h, h'}(\tau) := u_{\lambda, \lambda', h, h'}^{\frac{4}{n-2}}(\cdot, \tau) g_{cyl}$ defines a type I ancient solution to the Yamabe flow in the sense that the norm of its curvature operator is uniformly bounded, that is for any $\tau_0 < T_{\lambda, \lambda', h, h'}$, we have $\| Rm (g_{\lambda, \lambda', h, h'}) \| \leq C$ for all $\tau \in (-\infty, \tau_0)$.*

Remark 3.1. The statement of Theorem 3.1 exactly means that the unrescaled flow (1.1), whose scaling by $|t|$ yields to Eq. (3.1), is a type I ancient solution according to Definition 1.1.

Proof. Since our metric is conformally flat, the norm of its curvature operator $\| Rm \|$ can be expressed in terms of the powers (positive or negative) of the conformal factor, its first and second order derivatives. On the other hand, the conformal factor satisfies the equation of type (1.8) in the considered parametrization. Therefore, we see that if we have uniform upper and lower bounds on the conformal factor, by standard parabolic estimates we get uniform bounds on all its derivatives and therefore the uniform bound on $\| Rm \|$.

Estimate (2.13) will be crucial in proving this theorem, that is we have

$$\int_{\mathbb{R}} (v_{\lambda, \lambda', h, h'}^p - u_{\lambda, \lambda', h, h'}^p) \, dx \leq C e^{d\tau}.$$

Denote shortly by $v := v_{\lambda, \lambda', h, h'}$ and by $u := u_{\lambda, \lambda', h, h'}$. Since $u \leq v$, the previous estimate and the definition of $v := \min(u_{\lambda, h}, u_{\lambda', h'})$ imply the bound

$$\int_{-\infty}^{x(\tau)} (u_{\lambda, h}^p - u^p) \, dx \leq C \, e^{d\tau}$$

where $u_{\lambda, h} = v_\lambda(x - \lambda \tau + h)$ is a traveling wave coming in from the left and $x(\tau)$ given by (2.3) denotes the point where the two traveling waves $u_{\lambda, h}$ and $u_{\lambda', h'}$ intersect. Let $z = x - \lambda \tau + h$ and denote by $U(z, \tau) := u(z + \lambda \tau - h)$. If we perform this change of variables in the previous integral estimate we obtain,

$$\int_{-\infty}^{\bar{x}(\tau)} (v_\lambda(z)^p - U(z, \tau)^p) \, dz \leq C \, e^{d\tau} \tag{3.2}$$

where $\bar{x}(\tau) := x(\tau) - \lambda \tau + h$ and $U(z, \tau)$ satisfies the equation

$$U_\tau = U_{zz} + \lambda (U^p)_z - U + U^p. \tag{3.3}$$

We will obtain derivative estimates which hold for

$$-\infty < z < \bar{x}(\tau) + \frac{1}{2}, \tag{3.4}$$

since similar estimates may be obtained in the region $\bar{x}(\tau) - \frac{1}{2} < z < +\infty$ from the symmetry of our problem.

We go now from cylindrical to polar coordinates via the following coordinate change,

$$U(z,\tau) = \hat{u}(y,\tau)\,|y|^{\frac{2}{p-1}}, \qquad r = |y| = e^{\frac{p-1}{2}\,z} \tag{3.5}$$

where $\hat{u}(y,\tau)$ satisfies the equation

$$(\hat{u}^p)_\tau = \alpha\,\Delta\hat{u} + \beta\,r\,\hat{u}_r + \gamma\,\hat{u}, \tag{3.6}$$

for some constants $\alpha > 0$ and β, γ. Furthermore, the ancient solution $g_{\lambda,\lambda',h,h'} = \hat{u}^{\frac{4}{n-2}}\,g_{\mathbb{R}^n}$ has positive scalar curvature $R > 0$, which is equivalent to

$$\Delta_{\mathbb{R}^n}\hat{u} \le 0.$$

By the mean value theorem we have

$$\hat{u}(y_0,\tau) \ge C_n \int_{B(y_0,1)} \hat{u}(y,\tau)\,dy \tag{3.7}$$

for all $y_0 \in \mathbb{R}^n$. Assume first that $|y| \le 2M$ for a fixed number M. Then, $u \le v$ implies

$$\hat{u} \le \hat{v} = v_\lambda(z)\,|y|^{-\frac{2}{p-1}} \le C\,\frac{\min\{1, e^z\}}{|y|^{\frac{2}{p-1}}} = C\,\frac{\min\left\{1, |y|^{\frac{2}{p-1}}\right\}}{|y|^{\frac{2}{p-1}}} \le C. \tag{3.8}$$

Here we have used the estimate $v_\lambda(z) \le \min(1, e^z)$ which follows from the bounds $v_\lambda \le 1$ and (1.14). Since $p > 1$, (3.7) and (3.8) imply

$$\hat{u}(y_0,\tau) \ge C \int_{B(y_0,1)} \hat{u}(y,\tau)^p\,dy. \tag{3.9}$$

On the other hand, after the coordinate change (3.5), estimate (3.2) becomes

$$\int_{B(0,e^{\frac{p-1}{2}\,\bar{x}(\tau)})} \frac{(\hat{v}_\lambda^p - \hat{u}^p)}{|y|^{\frac{n}{2}-1}}\,dy \le C e^{d\tau},$$

where $B\left(0, e^{\frac{p-1}{2}\,\bar{x}(\tau)}\right)$ is the euclidean ball in $\mathbb{R}^n$ of radius $e^{\frac{p-1}{2}\,\bar{x}(\tau)}$. Note that for $|y_0| \le 2M$, and $\tau << -1$ sufficiently small so that $e^{\frac{p-1}{2}\,\bar{x}(\tau)} \gg 1$, the previous estimate yields

$$c \int_{B(y_0,1)} (\hat{v}_\lambda^p - \hat{u}^p)\,dy \le \int_{B(0,e^{\frac{p-1}{2}\,\bar{x}(\tau)})} \frac{(\hat{v}_\lambda^p - \hat{u}^p)}{|y|^{\frac{n}{2}-1}}\,dy \le C\,e^{d\tau},$$

where $c = c(M)$ is a constant uniform in time. Hence,

$$\int_{B(y_0,1)} \hat{u}(y,\tau)^p\,dy \ge \int_{B(y_0,1)} \hat{v}_\lambda(y)^p\,dy - Ce^{d\tau}, \qquad |y_0| \le 2M \tag{3.10}$$

where $C = C(M)$. Combining (3.9) and (3.10) yields

$$\hat{u}(y_0,\tau) \ge C \int_{B(y_0,1)} \hat{v}_\lambda(y)^p\,dy - C\,e^{d\tau} \ge c > 0$$

for $\tau \le \tau_0$ sufficiently small and all $|y| \le 2M$. This together with (3.8) imply

$$c \le \hat{u}(y,\tau) \le C, \quad \tau \le \tau_0, \qquad |y| \le 2M. \tag{3.11}$$

Having (3.11), Eq. (3.6) is a uniformly parabolic equation for $(y, \tau) \in B(0, 2M) \times (-\infty, \tau_0)$, so standard parabolic estimates applied to Eq. (3.6) imply we have all uniform bounds on the derivatives of $\hat{u}$ in the region $B(0, \frac{3M}{2}) \times (-\infty, -2\tau_0)$. Since $\hat{u}^{\frac{4}{n-2}}$ is the conformal factor of our metric $g_{\lambda, \lambda', h, h'}$ in polar coordinates, by the discussion at the beginning of the proof we have

$$\| \operatorname{Rm}(y, \tau) \| \leq C, \quad \tau \leq \tau_0, \qquad |y| \leq M$$

for a uniform constant C. Equivalently, in z coordinates this means

$$\| \operatorname{Rm}(z, \tau) \| \leq C, \quad \tau \leq \tau_0, \ z \leq \frac{2}{p-1} \log M. \tag{3.12}$$

Observe this estimate implies that we have the curvature uniformly bounded in the tip region of our ancient solution.

Let us now focus on the inner part of our solution that turns out to have the asymptotics of a cylindrical metric. More precisely, we will assume now that $|y| \geq M/2$ which according to (3.5) means

$$z \geq \frac{2}{p-1} \log \frac{M}{2}$$

and also that $z \leq \bar{x}(\tau) + 1$, since we are interested in deriving estimates in the region (3.4). Recall that the estimate (2.13) can be rewritten as

$$\int_{\mathbb{R}} \hat{a} \, (v - u) \, dx \leq C \, e^{d\tau},$$

where $\hat{a} = p \int_0^1 (sv + (1-s)u)^{p-1} \, ds$. This implies

$$\int_{\lambda\tau - h + \frac{2}{p-1} \log \frac{M}{2}}^{x(\tau)+1} \hat{a} \, (v_\lambda(x - \lambda\tau + h) - u(x, \tau)) \, dx \leq C \, e^{d\tau}.$$

Let $z = x - \lambda\tau + h$ be the coordinate change in the previous integral. Then,

$$\int_{\frac{2}{p-1} \log \frac{M}{2}}^{\bar{x}(\tau)+1} a(z, \tau) \, (v_\lambda(z) - U(z, \tau)) \, dz \leq C \, e^{d\tau} \tag{3.13}$$

where $a(z, \tau) := \hat{a}(z + \lambda\tau - h, \tau)$. Note that $a(z, \tau) \geq v_\lambda(z)^{p-1}$ and we may choose $M \geq 2$ so that $\log M/2 \geq 0$, hence

$$1 \geq a(z, \tau)^{\frac{1}{p-1}} \geq v_\lambda(z) \geq \frac{1}{2}, \quad z \in \left(\frac{2}{p-1} \log \frac{M}{2}, \bar{x}(\tau) \right) \tag{3.14}$$

since $v_\lambda(z)$ increases in z and $v_\lambda(0) = \frac{1}{2}$ by our normalization. Set $w(z, \tau) := v_\lambda(z) - U(z, \tau)$. Then $w \leq v_\lambda \leq 1$. Hence, (3.13) and (3.14) imply that for any $q > 1$,

$$\int_{\frac{2}{p-1} \log \frac{M}{2}}^{\bar{x}(\tau)+1} w(z, \tau) \, dz \leq C e^{d\tau}. \tag{3.15}$$

On the other hand, since both $U(z, \tau)$ and $v_\lambda(z)$ satisfy Eq. (3.3) we get that $w(z, \tau)$ satisfies

$$(a \, w)_\tau = w_{zz} + \lambda w_z - w + a \, w$$

and by (3.14) the equation is uniformly parabolic. Hence, standard parabolic estimates applied to it and estimate (3.15) yield the C^k bound

$$\| w \|_{C^k(U_\tau)} \leq C_k e^{d\tau}$$

holding on $U_\tau := \left(\frac{2}{p-1} \log \frac{3M}{4}, \bar{x}(\tau) \right) \times (-\infty, 2\tau_0)$ and $\tau_0 \ll -1$ is sufficiently small. In particular, we have

$$U(z, \tau) \geq v_\lambda(z) - C \, e^{d\tau} \geq c > 0, \quad \text{for } (z, \tau) \in U_\tau.$$

This implies the bound

$$\| \operatorname{Rm}(z, \tau) \| \leq C, \quad \tau \leq 2\tau_0, \qquad \frac{2}{p-1} \log M \leq z \leq \bar{x}(\tau). \tag{3.16}$$

Finally, estimates (3.12) and (3.16) yield a desired uniform bound on $\| \operatorname{Rm} \|$ for our ancient solution $g_{\lambda,\lambda',h,h'}$ for all $\tau \leq 2\tau_0$ and all $x \leq x(\tau)$. Recall that for $x \geq x(\tau)$ we get the uniform curvature bound using the same analysis as above (the only difference is that this time we need to consider the soliton that is coming in from the right). This finishes the proof of the theorem. $\square$

Remark 3.2. Let $u_{\lambda,\lambda',h,h'}$ be the ancient solution to (1.8) as in Theorem 2.1. Then, we will next observe that the metric $g_{\lambda,\lambda',h,h'} := (u_{\lambda,\lambda',h,h'})^{\frac{4}{n-2}} g_{cyl}$ has nonnegative Ricci curvature. Indeed, recall that $u_{\lambda,\lambda',h,h'} = \lim_{m \to +\infty} u_m$, where u_m is the solution of the initial value problem (2.10). It is sufficient to see that each u_m has nonnegative Ricci curvature, since then we can pass to the limit $m \to +\infty$. Indeed, we have seen that the convergence of $\{u_m\}$ to $u_{\lambda,\lambda',h,h'}$ is uniform on compact subsets of $\mathbb{R} \times (\infty, +\infty)$ and that $u_{\lambda,\lambda',h,h'} > 0$ on $\mathbb{R} \times (-\infty, T_{\lambda,\lambda',h,h'})$, where $T_{\lambda,\lambda',h,h'}$ is uniform in m. Standard regularity arguments on the quasilinear equation (1.8) imply that the convergence is C^∞ on compact subsets of $\mathbb{R} \times (\infty, T_{\lambda,\lambda',h,h'})$, from which our claim readily follows.

We will next observe how one may show that each solution $g_m(\cdot, \tau) := u_m(\cdot, \tau) g_{cyl}$ of (2.10) has nonnegative Ricci curvature. The initial data of u_m at $\tau = -m$ is $v_{\lambda,\lambda',h,h'}(\cdot, -m)$. We recall that for every τ, we have defined $v_{\lambda,\lambda',h,h'}$ by (2.2), namely $v_{\lambda,\lambda',h,h'}(\cdot, \tau) = \min\left(v_\lambda(x - \lambda\tau + h), v_{\lambda'}(-x - \lambda'\tau + h')\right)$ where v_λ, $v_{\lambda'}$ are traveling wave solutions of Eq. (1.11). It has been shown in [5] (Section 4, Proposition 4.5) that both metrics defined via conformal factors v_λ, $v_{\lambda'}$ respectively, have nonnegative sectional curvatures.

Moreover, it has been observed in [5] (Section 4) that for a given smooth and rotationally symmetric metric $g := v(x) g_{cyl}$ where $g_{cyl} := dx^2 + g_{S^{n-1}}$, nonnegative sectional curvatures are equivalent to having

$$v_x^2 - v \, v_{xx} \geq 0 \quad \text{and} \quad 4v^2 - v_x^2 \geq 0. \tag{3.17}$$

Since each for each $\tau \in \mathbb{R}$, the functions $v_\lambda(\cdot, \tau), v_{\lambda'}(\cdot, \tau)$ satisfy (3.17) (up to the dilation performed in (1.9)), the minimum $v_{\lambda,\lambda',h,h'}(\cdot, \tau)$ also satisfies (3.17) (up to the same dilation) in the distributional sense and it is smooth on $\mathbb{R} \setminus \{x(\tau)\}$, where $x(\tau)$ denotes the point at which $v_\lambda(\cdot, \tau)$ and $v_{\lambda'}(\cdot, \tau)$ intersect. One can then show that there is an approximation $\{v_{\lambda,\lambda',h,h'}^\delta(\cdot, \tau)\}$, $\delta \in (0, \delta_0)$ of $v_{\lambda,\lambda',h,h'}(\cdot, \tau)$ each satisfying (3.17) and such that $v_{\lambda,\lambda',h,h'}^\delta(\cdot, \tau) \to v_{\lambda,\lambda',h,h'}(\cdot, \tau)$, as $\delta \to 0$ uniformly on compact subsets in $\mathbb{R}$ and also in C^∞ on compact subsets of $\mathbb{R} \setminus \{x(\tau)\}$.

Let u_m^δ be the solution to (2.10) with initial data $v_{\lambda,\lambda',h,h'}^\delta(\cdot, -m)$ instead of $v_{\lambda,\lambda',h,h'}(\cdot, -m)$. Since $g_m^\delta(\cdot, -m) := (v_{\lambda,\lambda',h,h'}^\delta)^{\frac{4}{n-2}}(\cdot, -m) g_{cyl}$ has nonnegative sectional curvatures, it also has nonnegative Ricci curvature and this is preserved by the Yamabe flow. It follows that $g_m^\delta(\cdot, \tau) := (u_m^\delta)^{\frac{4}{n-2}}(\cdot, \tau) g_{cyl}$ has nonnegative curvature and passing to the limit $\delta \to 0$, the same holds for $g_m(\cdot, \tau) := (u_m)^{\frac{4}{n-2}}(\cdot, \tau) g_{cyl}$. This sketches the proof of the claim about our solutions having nonnegative Ricci curvature.

Proof of Theorem 1.1. Theorem 1.1 follows as a direct consequence of Theorems 2.1 and 3.1. $\square$

Acknowledgments

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