ON REALIZATIONS OF THE GELFAND CHARACTER OF A FINITE GROUP

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Abstract. We show that the Gel'fand character χ_G of a finite group G (i.e. the sum of all irreducible complex characters of G) may be realized as a " twisted trace" $g \mapsto Tr(\rho_g \circ T)$ for a suitable involutive linear automorphism of $L^2(G)$, where ρ stands for the right regular representation of G in $L^2(G)$. We prove further that, under certain hypotheses, T may be obtained as $T(f) = f \circ L$, where L is an involutive antiautomorphism of the group G so that $Tr(\rho_g \circ T) = |\{h \in G : hg = L(h)\}|$. We also give in the case of the group $G = PGL(2, \mathbb{F}_q)$ a positive answer to a question of K. W. Johnson asking whether it is possible to express the Gel'fand character χ_G as a polynomial in a single irreducible character η of G.

Keywords: Gelfand character, twisted trace, total character, Steinberg character, Gelfand Model.

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1. Introduction

The realization of the Gel'fand character χ_G of a finite group G, i.e. the sum of all ordinary irreducible characters of G is an old problem [9, 3, 12]. One approach to this problem is to try to obtain χ_G by twisting the trace of some very natural representation (V, π) of G, like the regular representation, by a suitable linear automorphism T of its underlying space V, so as to obtain $\chi_G(g) = Tr(\pi_g \circ T)$ for all $g \in G$. Recall that twisted traces appear in many contexts in mathematics [2, 3, 6].

For example it is a result in the character theory of finite groups that the central function $\theta_1 : G \to \mathbf{C}$, defined by $\theta_1(g) = |\{h \in G : h^2 = g\}|$ is a generalized character which satisfies $\theta_1(g) = \sum_{\pi \in \hat{G}} \nu(\pi)\chi_\pi(g)$ where

$\nu(\pi) = \frac{1}{|G|}\sum_{g\in G} \chi_\pi(g^2)$ is the number of Frobenius- Schur of the character χ_π of the irreducible representation π of G.

If (π, V) is an irreducible complex representations of a compact group G and π is self-contragradient then there exists a non degenerate bilinear form B of V, unique up to scalar multiple, such that $B(v, w) = \epsilon_(\pi)B(w, v)$ where $\epsilon_(\pi) = \pm 1$. Frobenius -Schur [4] proved that $\nu(\pi) = \epsilon(\pi)$ and if $\nu(\pi) = 1$ then $\Pi(G)$ is conjugated to a subgroup of the orthogonal group $O(n)$ and

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