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Monotone waves for non-monotone and non-local monostable reaction–diffusion equations

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Abstract

We propose a new approach for proving existence of monotone wavefronts in non-monotone and non-local monostable diffusive equations. This allows to extend recent results established for the particular case of equations with local delayed reaction. In addition, we demonstrate the uniqueness (modulo translations) of obtained monotone wavefront within the class of all monotone wavefronts (such a kind of conditional uniqueness was recently established for the non-local KPP-Fisher equation by Fang and Zhao). Moreover, we show that if delayed reaction is local then each monotone wavefront is unique (modulo translations) within the class of all non-constant traveling waves. Our approach is based on the construction of suitable fundamental solutions for linear integral-differential equations. We consider two alternative scenarios: in the first one, the fundamental solution is negative (typically holds for the Mackey–Glass diffusive equations) while in the second one, the fundamental solution is non-negative (typically holds for the KPP-Fisher diffusive equations).

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1. Main results and discussion

Introduction In this work, we study the existence and uniqueness of *monotone* wavefronts $u(x,t) = \phi(x+ct)$ for the monostable delayed non-local reaction–diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(x-y)g(u(t-h,y))dy, \ u \geq 0, \tag{1}$$

when the reaction term $g : \mathbb{R}_+ \to \mathbb{R}_+$ neither is *monotone* nor defines a *quasi-monotone* functional in the sense of Wu–Zou [46] or Martin–Smith [32] and when the non-negative kernel $K(s)$ is Lebesgue integrable on $\mathbb{R}$. Equation (1) is an important object of studies in the population dynamics, see [2,3,6,15,19,22,29,30,33,34,40,41,44,47–49]. Taking formally $K(s) = \delta(s)$, the Dirac delta function, we obtain the diffusive Mackey–Glass type equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \ u \geq 0, \tag{2}$$

another popular focus of investigation, see [2,21,26,43] for more details and references.

In the classical case, when $h = 0$, all wavefronts to the monostable equation (2) are monotone and, given a fixed admissible wave velocity c, all of them are generated by a unique front by means of translations. The same monotonicity-uniqueness principle is valid for certain subclasses of equations (2) with $h > 0$ (e.g. when g is monotone [43]) and even for equations (1) (e.g. when g is a monotone and globally Lipschitzian function, with the Lipschitz constant $g'(0)$, and when additionally $K(s) = K(-s)$, $s \in \mathbb{R}$ [30,40]). However, if the reaction term is non-local and g is non-monotone, monotonicity and uniqueness are not longer obligatory front's characteristics. For example, [41] provides conditions sufficient for non-monotonicity of wavefronts' profiles for non-local equation (1) with compactly supported kernel K. Co-existence of multiple wavefronts for non-local models is also known from [23,36]. All this explains our interest in establishing effective criteria for the existence and uniqueness of *monotone* wavefronts for the monostable *non-monotone* non-local (or delayed) reaction–diffusion equations. Remarkably, this problem has recently attracted attention of several researchers. In this regard, the most studied model was the non-local KPP-Fisher equation [5,7,16,23,35,36]

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)(1 - \int_{\mathbb{R}} K(x-y)u(t,y)dy), \tag{3}$$

and its local delayed version [5,14,27,24,20,21,46] (called the diffusive Hutchinson's equation)

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)(1 - u(t-\tau,x)). \tag{4}$$

The above cited papers elaborated a complete characterization of models (3) and (4) possessing monotone wavefronts. Moreover, the absolute uniqueness (i.e. uniqueness within the class of all wavefronts) of monotone wavefronts to (4) and the conditional uniqueness (i.e. uniqueness within the subclass of monotone wavefronts) of monotone wavefronts to (3) were also proved in these works. As we have mentioned, in general, monotone and non-monotone wavefronts can coexist in (3) [23,36].

In the case of model (1) having non-monotone function g, the existence of *monotone* wavefronts was analyzed only for the particular case of equation (2) in [21], with the help of the

Hale–Lin functional-analytic approach and a continuation argument. This method required a detailed analysis of a family of linear differential Fredholm operators associated with (2). The discrete Lyapunov functionals of Mallet-Paret and Sell for delayed differential equations were also used in an essential way. Therefore the task of extension of the approach developed in [21] to non-local equations (1) seems to be quite difficult (if anyhow possible). Consequently, the main goal of this paper is to provide an alternative technique allowing to analyze monotonicity of wavefronts for non-monotone and non-local equation (1). A key feature of this technique consists in reduction of the wave profile equation for (1) to new non-obvious convolution equations (see Sections 2 and 6). The obtained nonlinear equations are then studied by means of various already established methods. In particular, we use the Berestycki–Nirenberg sliding method as well as an extension of the Diekmann–Kaper theory developed in [2,19].

In the subsequent parts of this section, we state the key hypotheses used in the paper and briefly discuss our main theorems together with a key auxiliary assertion.

Main assumptions

(M) $g \in C(\mathbb{R}_+),\ g(s) > 0$ for $s > 0$, and the equation $g(s) = s$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Moreover, g is differentiable at the equilibria with $g'(0) > 1$ and $g'(\kappa) < 0$.
(ST) $g(s) - g'(\kappa)s$ is non-decreasing on $[0, \kappa]$. Observe that the last assumption implies the sub-tangency property of g at κ: $g(s) \le g(\kappa) + g'(\kappa)(s - \kappa),\ s \in [0, \kappa]$.
(K) $K \ge 0$ and $\int_{\mathbb{R}} K(s)ds = 1$. Moreover, $\int_{\mathbb{R}} K(s)e^{-\lambda s}ds < \infty$ for each $\lambda \in \mathbb{R}$.

Example 1. If g is differentiable on $[0, \kappa]$ then the hypothesis **(ST)** amounts to the inequality **(ST′)**: $g'(s) \ge g'(\kappa)$ satisfied for all $s \in [0, \kappa]$. For the following non-local version of the popular Nicholson's blowflies diffusive equation

$$u_t(t,x) = u_{xx}(t,x) - du(t,x) + p\int_{\mathbb{R}} K(x-y)u(t-h,y)e^{-u(t-h,y)}dy, \quad p > d > 0,$$

the assumptions **(M)** and **(ST′)** are equivalent to the inequalities $e < p/d \le e^2$. We note that the bulk of information concerning the Nicholson's diffusive equation is obtained for a simpler (monotone) case when $1 < p/d \le e$ (cf. the recent works [47] and [39]).

Remark 2. In this paper, we are concerned with the classical wavefront solutions of equations (1), (2), so that rather weak smoothness conditions mentioned in **(M)** are sufficient for our purposes. On the other hand, in order to ensure the existence and uniqueness of classical solution of the initial value problem

$$u(s,x) = w_0(s,x),\ s \in [-h, 0],\ x \in \mathbb{R}, \tag{5}$$

for equation (1) (or (2)), we have to impose some additional restrictions on g, cf. [17]. For instance, suppose that $h > 0$, that $w_0(s,x)$ is continuous, bounded and uniformly Hölder continuous in $x \in \mathbb{R}$, and that $|g(u) - g(v)| \le L_g|u - v|,\ u, v \in [0, \kappa]$. Then the Cauchy problem (1), (5) (or (2), (5)) can be solved by the method of steps, where in the first step we have to look for the solution of the inhomogeneous linear equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(y)g(w_0(t-h, x-y))dy,\ t \in [0,h],\ x \in \mathbb{R},$$

satisfying the initial condition $u(0,x) = w_0(0,x)$.

Main results: existence Clearly, $u(t,x) = \phi(x+ct)$ is a front solution of equation (1) if and only if the profile $y = \phi(t)$ solves the boundary value problem

$$y''(t) - cy'(t) - y(t) + \int_{\mathbb{R}} K(t-s)g(y(s-ch))ds = 0,\ y(-\infty) = 0,\ y(+\infty) = \kappa,\ y(t) \geq 0. \tag{6}$$

For kernel K satisfying **(K)**, we will consider the characteristic functions

$$\chi_0(z,c,h) = z^2 - cz - 1 + g'(0)e^{-chz}\int_{\mathbb{R}} e^{-zs}K(s)ds, \quad z \in \mathbb{C},$$

$$\chi_\kappa(z,c,h) = z^2 - cz - 1 + g'(\kappa)e^{-chz}\int_{\mathbb{R}} e^{-zs}K(s)ds, \quad z \in \mathbb{C},$$

associated with the linearizations of (6) at the equilibria 0 and κ, respectively. We will need the following three subsets $\mathcal{D}_0, \mathcal{D}_\kappa, \mathcal{D}_{\mathfrak{L}} := \overline{\mathcal{D}_0} \cap \mathcal{D}_\kappa$ of the half-plane $(h,c) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\mathcal{D}_\kappa = \{(h,c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_\kappa(z,c,h) \text{ has at least one positive and one negative simple zeros}\};$$

$$\mathcal{D}_0 = \{(h,c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_0(z,c,h) \text{ has exactly two positive zeros } \mu_0 < \mu_1\}.$$

The geometric description of the open domain $\mathcal{D}_0$ is well known and it is summarized in the following assertion:

Proposition 3. *Assume that $g'(0) > 1$. Then for each $h \geq 0$ there exists a unique $c = c_\#(h) \in \mathbb{R}$ such that $\chi_0(z,c,h)$ with this c has a unique positive double zero. The function $c_\# : \mathbb{R}_+ \to \mathbb{R}$ is C^∞-continuous and strictly decreasing. Furthermore, $\mathcal{D}_0$ coincides with the set $\{(h,c) \in \mathbb{R}_+ \times \mathbb{R} : c > c_\#(h)\}$ and $\overline{\mathcal{D}_0} = \{(h,c) \in \mathbb{R}_+ \times \mathbb{R} : c \geq c_\#(h)\}$.*

Proof. For example, see [19, Lemma 22] and [3, Theorem 1.1]. By [3] if, in addition, $K(s) = K(-s)$, $s \in \mathbb{R}$, then $0 < c_\#(h) = O(1/h)$ at $+\infty$. In general, however, $c_\#(h)$ can take negative values, cf. [19]. □

Remark 4. To simplify the notation, we suppose in **(K)** that the bilateral Laplace transform of K exists for all $z \in \mathbb{C}$. Under this assumption, $\chi''(x,c,h) > 0$ for all $x \in \mathbb{R}$ and $\chi(0,c,h) = g'(0) - 1 > 0$, so that either $\chi_0(z,c,h)$ has exactly two real zeros (counted with multiplicity) or it does not have any real zero. It is also known (e.g. see [2,44]) that **(K)** can be weakened till

(K′) $K \geq 0$, $\int_{\mathbb{R}} K(s)ds = 1$ and $\int_{\mathbb{R}} K(s)e^{-\lambda s}ds < \infty$ for each $\lambda \in (\alpha_-, \alpha_+)$, where $\alpha_- < 0 < \alpha_+$ and $\lim_{\lambda \to \alpha_+^-} \int_{\mathbb{R}} K(s)e^{-\lambda s}ds = +\infty$.

In such a case, the above definition of $\mathcal{D}_0$ should be replaced with

$$\mathcal{D}_0' = \{(h, c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_0(z, c, h) \text{ has at least one positive zero}\}.$$

It is easy to see that the boundary of $\mathcal{D}_{\mathfrak{L}}$ in $\mathbb{R}_+ \times \mathbb{R}$ consists from the curves determined either from the system $\chi_0(z, c, h) = 0$, $\chi_0'(z, c, h) = 0$ or from the system $\chi_\kappa(z, c, h) = 0$, $\chi_\kappa'(z, c, h) = 0$. Thus, in each particular case, the shape of the domains $\mathcal{D}_\kappa$, $\mathcal{D}_{\mathfrak{L}}$ can be identified after some work. For instance, if $K(s)$ is the Dirac's delta and $g'(\kappa) < 0$, then $\mathcal{D}_{\mathfrak{L}}$ is a simply connected domain whose boundary contains a non-empty segment of the half-line $\{h = 0, c \geq 0\}$ [21]:

$$\mathcal{D}_{\mathfrak{L}} = \{(h, c) : h \in [0, h_*],\ h_* \leq +\infty,\ 0 < c_\#(h) \leq c < c^*(h)\}.$$

Here the smooth decreasing function $c^* : [0, +\infty) \to (0, +\infty]$ is defined as follows [21]: i) $c^*(h') = +\infty$, if $\chi_\kappa(z, c, h') = z^2 - cz - 1 + g'(\kappa)e^{-ch'z}$ has a simple negative zero for each $c \geq 0$; ii) if $\chi_\kappa(z, c, h')$ has a double negative zero for some $c = c' > 0$, then such a value of c' is unique and we set $c^*(h') = c'$. It is easy to prove that $c^*(h) = +\infty$ for all $h \in [0, h_a]$, where h_a is the positive root of the equation $eh|g'(\kappa)|e^h = 1$, and that $c^*(h)$ is finite for $h > h_a$. Next, it was shown in [21, Lemma 1.3] that equation $c_\#(h) = c^*(h)$ has at most one solution, denoted by h_* (if exists). If $c_\#(h) < c^*(h)$ for all positive h, we set $h_* = +\infty$. For the particular case of the diffusive Nicholson's equation, two possible forms of $\mathcal{D}_{\mathfrak{L}}$, one with $h_* = +\infty$ and another with $h_* < +\infty$, are presented on Figure 2 in [21]. The aforementioned topological characteristics of $\mathcal{D}_{\mathfrak{L}}$ are essential for the use of a continuation argument in [21]. For general kernels K, however, the set $\mathcal{D}_{\mathfrak{L}}$ eventually might be more complicated (for instance, not connected). One of advantages of our present approach is that it does not require any connectedness property from $\mathcal{D}_{\mathfrak{L}}$:

Theorem 5. *Assume* **(M)**, **(K)**, **(ST)** *and that g is sub-tangential at the equilibrium* 0*: $g(s) \leq g'(0)s$, for all $s \in [0, \kappa]$. Then for each point (h, c) in the closure $\overline{\mathcal{D}_{\mathfrak{L}}}$ of the set $\mathcal{D}_{\mathfrak{L}}$, equation (1) has at least one wavefront $u(t, x) = \phi_c(x + ct)$ with strictly increasing profile $\phi_c(s)$, $s \in \mathbb{R}$.*

As we have mentioned, the conclusion and the proof of Theorem 5 are also valid when $K(s)$ is the Dirac delta function, i.e. for the local equation (2). Thus it is enlightening to compare criterion of front's monotonicity for (2) established in [21, Theorem 2.2] and Theorem 5. These two results almost coincide except for two important details: g in [21] must be more smooth ($C^{1,\gamma}$-continuous on $[0, \kappa]$) and must have a unique critical point on $(0, \kappa)$. That is, the unimodal form of g is assumed in [21] instead of the condition **(ST)**. Clearly, even if these both requirements are fulfilled for the classical population models (Nicholson's blowflies model, hematopoiesis model), they are independent: so that Theorem 5 and [21, Theorem 2.2] complement each other in the case of local delayed equations. This comparison also shows that for some classical 'delayed' models (such as equation (2) with $h > 0$ and $g(x) = pxe^{-x}$, $e < p \leq e^2$) condition $(h, c) \in \overline{\mathcal{D}_{\mathfrak{L}}}$ of Theorem 5 is necessary and sufficient for the existence of monotone wavefronts. However, as our 'dual' existence result, Theorem 11, shows, this condition fails to be necessary for the 'advanced' models (such as equation (2) with $h < 0$ and the same nonlinearity $g(x) = pxe^{-x}$, $e < p \leq e^2$).

Main results: uniqueness The Diekmann–Kaper theory [2,13,19,47] is instrumental in establishing the absolute uniqueness of semi-wavefronts to equation (1). In order to apply this theory, it is necessary to transform the profile equation (6) into a suitable nonlinear convolution equation [13,46]

$$\phi(t) = \int_{\mathbb{R}} N_j(t-s)g_j(\phi(s-ch))ds, \tag{7}$$

with an appropriate kernel $N_j \in L_1(\mathbb{R}, \mathbb{R}_+)$ and continuous monostable nonlinearity $g_j : \mathbb{R}_+ \to \mathbb{R}_+$. Certainly, N_j, g_j in (7) depend on c, h, K, g and the choice of specific N_j, g_j depends on the goals of investigation. So far, the usual way was to take $g_j = g$: in such a case, assuming that $g \in C^{1,\alpha}[0,\kappa]$, K satisfy **(M)**, **(K)** and $|g'(s)| \le g'(0)$, $s \in [0,\kappa]$, the absolute uniqueness of each semi-wavefront profile $\phi : \mathbb{R} \to [0,\kappa]$ for (6) was proved in [2, Theorem 7]. In this work, we take other $g_j \neq g, N_j$ in equation (7), with these functions, the Diekmann–Kaper method leads to slightly different conclusions. For instance, it can be proved that each semi-wavefront $\phi : \mathbb{R} \to [0,\kappa]$ is absolutely unique if $(h,c) \in \mathfrak{D}_{\mathfrak{L}}$ and $K, g \in C^{1,\alpha}[0,\kappa]$ satisfy **(K)**, **(M)** and $g'(\kappa) \le g'(s) \le g'(0)$, $s \in [0,\kappa]$.

However, even monotone wavefronts might be not absolutely unique [23,36]. In order to prove the conditional uniqueness of a monotone wavefront, here we will use the sliding method developed by Berestycki and Nirenberg [8]. This technique was successfully applied in [10–12,31,43] to prove the uniqueness of *monotone* wavefronts without imposing any Lipschitz condition on g. The main idea of the sliding technique consists in moving (either horizontally or vertically) one wavefront profile (say, $\psi(t)$) up to having a common point of tangency with another (fixed) wavefront profile (say, $\phi(t)$). Technically, this includes consideration of one-dimensional family of differences $\psi(t+a) - \phi(t)$, $a \in \mathbb{R}$ (or $\psi(t) + a - \phi(t)$, $a \in \mathbb{R}$) and analysis of its behavior as $t \to \pm\infty$. Then a suitable form of the maximum principle is invoked in order to establish that $\psi(t) \equiv \phi(t+a_0)$ for some a_0, cf. [8]. In the present paper, we consider sliding solutions to prove the following:

Theorem 6. *Assume* **(M)**, **(K)** *and* **(ST)**. *In addition, let g be C^1-smooth in some neighborhood of κ and there exist $C > 0$, $\theta \in (0,1]$, $\Delta > 0$ such that*

$$\left|g(u)/u - g'(0)\right| \le Cu^{\theta}, \quad u \in (0,\Delta]. \tag{8}$$

Fix some $(h,c) \in \mathfrak{D}_{\mathfrak{L}}$, and suppose that $u_1(t,x) = \phi(x+ct)$, $u_2(t,x) = \psi(x+ct)$ are two monotone traveling fronts of equation (1). Then $\phi(s) = \psi(s+s_0)$, $s \in \mathbb{R}$, for some s_0.

Remark 7. As a by-product of the proofs of Theorems 5, 6, we obtain the following: Assume **(K)** and **(M)** where $g'(\kappa) \ge 0$ is considered instead of $g'(\kappa) < 0$. If, in addition, $g(s) \le g'(0)s$, $s \in [0,\kappa]$, and g is monotone and satisfies the smoothness conditions of Theorem 6, then for each point (h,c) in the closure of the set $\mathfrak{D}_0$, equation (1) has a unique (up to a translation) monotone wavefront $u(t,x) = \phi_c(x+ct)$.

We will say that some c is an admissible speed of propagation for (1) (or for (2)) if there exists a positive wave solution $u = \phi(x+ct)$ to (1) (to (2), respectively) such that $\phi(-\infty) = 0$ and $\liminf_{t\to+\infty} \phi(t) > 0$. We call such a wave solution semi-wavefront. As [23,36] reveals, proper semi-wavefronts and monotone fronts can co-exist in non-local monostable equations.

Nevertheless, as the next result shows, the statement of Theorem 6 can be strengthened for the case of local delayed reaction:

Theorem 8. *Assume* **(M)**, **(K)** *and* **(ST)**. *Then each semi-wavefront $u = \phi(x + ct)$, $(h, c) \in \mathfrak{D}_{\mathfrak{L}}$, for equation (2) is actually a monotone front. Therefore, if g also satisfies the smoothness conditions of Theorem 6, then $u = \phi(x + ct)$ is the unique (up to translation) wavefront solution of (2) propagating with the speed c.*

An auxiliary result A correct choice of N_j and g_j in (7) may indicate a shortest way for establishing various properties of profiles (including their existence and uniqueness). For instance, all above mentioned front's monotonicity criteria for the KPP-Fisher equations (3) and (4) were obtained after discovering a satisfactory form of the associated convolution equation, see [16,20,27]. Similarly, an important part of this paper is focused on reducing equation (6) to the 'optimal' convolution equation:

Theorem 9. *Assume* **(M)** *and* **(K)**. *Then for each point $(h, c) \in \mathfrak{D}_{\mathfrak{L}}$, there exist g_1, positive ϵ and kernels $N_1, -v > 0$ given by*

$$N_1 = -(1+\xi)K * v := -(1+\xi)\int_{\mathbb{R}} K(s)v(t-s)ds, \ g_1(s) = \frac{g(s)+\xi s}{1+\xi}, \ \xi := |g'(\kappa)| + \epsilon,$$

such that the boundary value problem (6) has a solution if and only if equation (7) has a non-negative solution satisfying the boundary conditions of (6). Furthermore, $\int_{\mathbb{R}} N_1(s)ds = 1$ and $\int_{\mathbb{R}} N_1(s)e^{-\lambda s}ds < \infty$ for all λ from some maximal finite interval $(\gamma_l, \gamma_r) \ni \{0\}$. Continuous function v is C^∞-smooth on $\mathbb{R}_-$ and $\mathbb{R}_+$ and has a unique minimum point at $t = 0$. In fact, v is strictly monotone on $\mathbb{R}_-$ and $\mathbb{R}_+$ and it is strictly convex on $\mathbb{R}_-$.

The function v in Theorem 9 is called the fundamental solution, actually it is a distributional solution of the non-local equation

$$y''(t) - cy'(t) - y(t) + (g'(\kappa) - \epsilon)\int_{\mathbb{R}} K(t-s)y(s-ch)ds = \delta(t),$$

where $\delta(t)$ is the Dirac delta function. The inequality $v(t) < 0, t \in \mathbb{R}$, is an important part of the statement of Theorem 9 and Section 2 is completely devoted to proving this and other properties of $v(t)$.

Remark 10. For equation (2), a more explicit form of $N_1(t)$ can be obtained:

$$N_1(t) = -(1+\xi)v(t,\xi), \quad \text{where } v(t,\xi) = -\frac{1}{\chi'(\lambda_0(\xi))}\begin{cases} \tilde{u}(t), & t \geq 0, \\ e^{\lambda_0(\xi)t}, & t < 0, \end{cases}$$

$\chi(z) = z^2 - cz - 1 - \xi e^{-chz}$, $\lambda_0(\xi)$ is the unique positive zero of $\chi(z)$, and $\tilde{u}(t)$ is the solution of the following initial value problem:

$$u''(t) - cu'(t) - u(t) - \xi u(t - ch) = 0, \tag{9}$$
$$u(s) = e^{\lambda_0(\xi)s},\ s \in [-ch, 0], \quad u'(0) = -(\lambda_0(\xi) - c + \xi che^{-\lambda_0(\xi)ch}).$$

When $\xi = 0$, this formula for the fundamental solution $v(t, \xi)$ for (9) is well known, cf. (19). We note that the explicit exponential form of $v(t)$ for negative t allowed us to prove the monotonicity of all wavefronts under conditions of Theorem 8. On the other hand, the one-sided Laplace transform $\hat{v}(z) = \int_0^{+\infty} e^{-zt}v(t)dt$ of $v(t)$ can also be easily found:

$$\hat{v}(z) = \frac{1}{\chi(z)} - \frac{1}{\chi'(\lambda_0)}\frac{1}{z - \lambda_0(\xi)}.$$

This function is analytic in the half-plane $\{\Re z > \lambda_1(\xi)\}$ where $\lambda_1(\xi)$ is the biggest negative zero of $\chi(z)$. As the Laplace transform of the negative function, $-\hat{v}(x)$, $x \in (\lambda_1(\xi), +\infty)$, provides a new example of completely monotone function, cf. [4,45].

Once the hypotheses **(ST)**, **(M)** and **(K)** are assumed, the optimality of our main existence/uniqueness results has to be explained in terms of optimality of the set $\mathfrak{D}_{\mathfrak{L}} = \overline{\mathfrak{D}_0} \cap \mathfrak{D}_\kappa$. In the ideal case, the closure of $\mathfrak{D}_{\mathfrak{L}}$ must contain the set $\mathfrak{M}$ of all pairs (h, c) for which (6) has a unique (modulo translation) monotone solution. Since it is well known (cf. [19]) that $\mathfrak{M} \subseteq \overline{\mathfrak{D}_0}$, we need only to justify the choice of $\mathfrak{D}_\kappa$. The necessity of the presence of at least one negative zero of $\chi_\kappa(z, c, h)$ in the definition of $\mathfrak{D}_\kappa$ for the existence of monotone fronts was proved both for the delayed equation (2) (e.g. see [21]) and the non-local equation (1) (at least when K has compact support, cf. [41, Theorem 6]). However, the necessity of the presence of at least one positive zero of $\chi_\kappa(z, c, h)$ in the definition of $\mathfrak{D}_\kappa$ is not so clear. Certainly, in some situations (e.g. for equation (2) with $h > 0$), at least one positive zero of $\chi_\kappa(z, c, h)$ exists automatically for all parameters in $\overline{\mathfrak{D}_0}$[1]; on the other hand, in other cases (e.g. if we take equation (2) with large $h < 0$), $\chi_\kappa(z, c, h)$ may have only negative real roots. At this point, it is enlightening to recall the Fang–Zhao monotonicity criterion [16] for equation (3) where it is required from the characteristic equation

$$char_\kappa(z) := z^2 - cz - \int_{\mathbb{R}} K(s)e^{-zs}ds = 0,$$

related to the linearization at the steady state $u = 1$ of the wave profile equation

$$\phi''(t) - c\phi'(t) + \phi(t)(1 - \int_{\mathbb{R}} K(s)\phi(t - s)) = 0, \quad t \in \mathbb{R}, \tag{10}$$

only to have a negative root. However, more weak restrictions on the real zeros of $char_\kappa(z)$ assumed in [16] go hand in hand with more simple structure of the linearization $\phi''(t) - c\phi'(t) + \phi(t) = 0$ of the profile equation (10) at $u = 0$. In fact, this linear equation possesses a *non-negative* fundamental solution. Surprisingly, the same combination of conditions (i.e. existence

[1] In fact, a unique positive zero of $\chi_\kappa(z, c, h)$ plays an essential role in the proof of monotonicity criterion in [21] (more precisely, in the proof of surjectivity of associated Fredholm operators, see Proposition 3.2 and Lemma 3.3 in [21]).

of non-negative fundamental solution for the linearization of the profile equation (6) at 0 together with existence of at least one negative zero of $\chi_\kappa(z,c,h)$) can also be sufficient for the presence of monotone wavefronts in (1). Such an alternative existence result will show, however, that one has to pay some price for each simplification in the definition of $\mathcal{D}_\kappa$. In our case, the non-local term in equation (1) will be supposed to have trivial left interaction.

Non-negative fundamental solutions and a 'dual' existence result In various applied models, the derivative $g'(s)$, $s \in [0,\kappa]$, attains its maximal value at 0. This suggests the consideration of the following hypothesis (which is in some sense *'dual'* to **(ST)**):

(ST$_*$) $g'(0)s - g(s)$ is non-decreasing on $[0,\kappa]$. In particular, $g(s) \le g'(0)s$, $s \in [0,\kappa]$.

In this subsection, we discuss how our previous arguments can be modified when **(ST$_*$)** is considered instead of **(ST)**. We also state here an analog of Theorem 5 for such a case while an analog of Theorem 9 will be stated and proved later, in Section 6. All this requires the redefinition of the set $\mathcal{D}_\mathfrak{L}$ (its analog will be denoted as $\mathcal{D}^*_\mathfrak{L}$) and of the fundamental solution $v(t)$ (its analog will be denoted as $w(t)$), as well as the use of appropriate functions N_j, g_j, $j>1$, in equation (7). Revisiting our proofs for the case when **(ST)** is assumed, we see that new fundamental solution $w(t)$ should be non-negative for $(h,c) \in \mathcal{D}^*_\mathfrak{L}$ and that it should satisfy, as a distribution, the equation

$$y''(t) - cy'(t) - y(t) + g'(0)\int_{\mathbb{R}} K(t-s)y(s-ch)ds = \delta(t)$$

(the formal definition of $w(t)$ is given in Section 6). This means that conditions assuring the non-negativity of $w(t)$ should be expressed in terms of the characteristic function $\chi_0(z,c,h)$. But as we have already mentioned, this function must have two positive roots (counted with multiplicity) each time when equation (1) has a semi-wavefront. Since $\chi_0''(x,c,h) > 0$, $x \in \mathbb{R}$, this means that $\chi_0(z,c,h)$ cannot have negative zeros when equation (1) has a semi-wavefront. On the other hand, if $\chi_0(z,c,h)$ has complex zeros with negative real parts, the fundamental solution $w(t)$ oscillates at $+\infty$ (cf. Lemma 19 and Remark 20). This shows that $w(t)$ can be non-negative only when all zeros of $\chi_0(z,c,h)$ have positive real parts. In such a case, however, $w(t)$ decays super-exponentially as $t \to +\infty$ that makes the analysis of its positivity at $+\infty$ highly non-trivial. Nonetheless, for each fixed pair $(h,c) \in \mathcal{D}_0$, the analysis of non-negativity of $w(t)$ can be successfully realized when the support $\operatorname{supp} K$ of $K(s)$ belongs to the interval $(-\infty, -ch]$. This leads us to the following definition:

$$\mathcal{D}^*_0 = \{(h,c) \in \mathbb{R}_+ \times \mathbb{R} : \chi_0(z,c,h) \text{ (a) has exactly two positive zeros } \mu_0 < \mu_1; \\ \text{and (b) } \operatorname{supp} K \subset (-\infty, -ch]\}.$$

In Section 6, we prove that if $(h,c) \in \mathcal{D}^*_0$ then $\chi_0(z,c,h)$ does not have zeros with non-positive real parts. It is clear also that if $\operatorname{supp} K \subset (-\infty,-ch]$ then $\chi_\kappa(z,c,h)$ has exactly one negative simple zero (say, $\lambda_1(g'(\kappa))$). Thus we can set $\mathcal{D}^*_\mathfrak{L} := \overline{\mathcal{D}^*_0}$:

Theorem 11. *Assume* **(M)**, **(ST$_*$)**, **(K)** *and* $g(s) \le g(\kappa) + g'(\kappa)(s-\kappa)$, $s \in [0,\kappa]$. *Then for each point* (h,c) *in the closure* $\overline{\mathcal{D}^*_0}$ *of the set* $\mathcal{D}^*_0$, *equation (1) has at least one wavefront* $u(t,x) = \phi_c(x+ct)$ *with non-decreasing profile* $\phi_c(s)$, $s \in \mathbb{R}$.

Finally, the organization of the paper is as follows. In Sections 2, 6, we study properties of the fundamental solutions $v(t,\xi)$ and $w(t)$. These studies are resumed in Theorems 9 and 27 which are formally proved in Step I of Section 3 and at the beginning of Section 6, respectively. The convolution equation (7) is then used to prove Theorems 5, 6, 11 in Sections 3, 4 and 6, respectively. The proof of Theorem 8 is given in Section 5.

2. Negativity of the fundamental solution

2.1. The fundamental solution: definitions and properties

Fix $c, d, \xi \in \mathbb{R}$, kernel $K(s)$ satisfying **(K)** and consider the linear integral-differential inhomogeneous equation

$$y''(t) - cy'(t) - dy(t) - \xi \int_{\mathbb{R}} K(t-s)y(s-ch)ds + f(t) = 0, \tag{11}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and the characteristic function

$$\chi(z,\xi) = z^2 - cz - d - \xi e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s)ds, \quad z \in \mathbb{C},$$

does not have zeros on the imaginary axis (in such a case, we will say that equation (11) is hyperbolic). Suppose, for a moment, that f is compactly supported and that, for this inhomogeneity, equation (11) has a solution $y : \mathbb{R} \to \mathbb{R}$ exponentially decaying, together with its first derivative $y'(t)$, at $\pm\infty$. Then, applying the bilateral Laplace transformation to (11), we find easily that this equation has a solution $y(t) = -v * f(s)$, which is the convolution of f with the bilateral Laplace inverse $v(t,\xi)$ of $1/\chi(\lambda,\xi)$. Since $y(t)$ is a bounded function, the formula $y(t) = -v * f(s)$ shows that the inverse Laplace transform should be applied to $1/\chi(\lambda,\xi)$ considered on the maximal vertical analyticity strip $\Pi(\lambda_l.\lambda_r) := \{z : \lambda_l < \Re z < \lambda_r\}$ that includes the imaginary axis (observe that $\lambda_l < 0 < \lambda_r$ since the imaginary axis does not contain any singular point of $1/\chi(\lambda,\xi)$). The function $v(\cdot,\xi) : \mathbb{R} \to \mathbb{C}$ is called the fundamental solution for equation (11). The above said and the inversion theorem imply that

$$v(t,\xi) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut}du}{u^2 + ciu + d + \xi e^{-iuch} \int_{\mathbb{R}} K(s)e^{-ius}ds}, \; t \in \mathbb{R}. \tag{12}$$

We view this formula as a formal definition of the fundamental solution for equation (11).

Lemma 12. *Suppose that $\chi(z,\xi)$ does not have pure imaginary zeros. Then $v(\cdot,\xi) : \mathbb{R} \to \mathbb{R}$ is a real valued function which is infinitely differentiable with respect to t on the set $\mathbb{R} \setminus \{0\}$ where it also satisfies equation (11) with $f(t) \equiv 0$. Moreover, the limits $v'(0^-,\xi)$, $v'(0^+,\xi)$ exist and $v'(0^+,\xi) - v'(0^-,\xi) = 1$ (thus the limits $v''(0^-,\xi)$, $v''(0^+,\xi)$ exist and $v''(0^+,\xi) - v''(0^-,\xi) = c$).*

Proof. Indeed, v is a real valued function because of the presentation

$$v(t,\xi) = -\frac{1}{\pi}\int_0^{+\infty} \frac{p(u)\cos(tu) + q(u)\sin(tu)}{p^2(u)+q^2(u)}du, \ t \in \mathbb{R}, \tag{13}$$

where p, q satisfying $p(u) = p(-u), q(u) = -q(-u), u \in \mathbb{R}$, are defined by

$$p(u) := u^2 + d + \xi C(u), \ q(u) := cu - \xi S(u),$$

where, due to the Lebesgue–Riemann lemma, the functions

$$C(u) := \int_{\mathbb{R}} K(s)\cos(u(ch+s))ds, \quad S(u) := \int_{\mathbb{R}} K(s)\sin(u(ch+s))ds,$$

are vanishing, together with their derivatives of all orders, at ∞. In particular, this implies that

$$P(+\infty) = 0, \ \text{ where } \ P(u) := \frac{up(u)}{p^2(u)+q^2(u)},$$

while derivatives of all orders $k = 1, 2, 3, \dots,$

$$P^{(k)}(u) = (u^{-1})^{(k)}(1+o(1)) = (-1)^k k! u^{-k-1}(1+o(1)), \ u \to +\infty,$$

are monotone at $+\infty$. Therefore, by the Dirichlet test of the uniform convergence of improper integrals [50, p. 421], the integral

$$\frac{1}{\pi}\int_0^{+\infty} \frac{up(u)\sin(tu)}{p^2(u)+q^2(u)}du$$

converges uniformly for t on each compact subset of $\mathbb{R} \setminus \{0\}$. In consequence (cf. [50, p. 426]),

$$v'(t,\xi) = \frac{1}{\pi}\int_0^{+\infty} \frac{up(u)\sin(tu) - uq(u)\cos(tu)}{p^2(u)+q^2(u)}du, \ t \neq 0, \tag{14}$$

exists for all $t \neq 0$. Note that the term $uq(u)/(p^2(u)+q^2(u))$ is Lebesgue integrable on $\mathbb{R}_+$ so that the function

$$I_2(t) = \int_0^{+\infty} \frac{uq(u)\cos(tu)}{p^2(u)+q^2(u)}du$$

is continuous on $\mathbb{R}$. Hence, in order to prove the existence of $v'(0^+,\xi), v'(0^-,\xi)$, we only need to take into account the integral

$$I_1(t) := \int_0^{+\infty} P_0(u)\frac{\sin(tu)}{u}du = \int_0^{+\infty}(1-P_1(u))\frac{\sin(tu)}{u}du = \frac{\pi\,\mathrm{sign}\,t}{2} - \int_0^{+\infty} P_1(u)\frac{\sin(tu)}{u}du$$

where $t \neq 0$, $|u^{-1}\sin(tu)| \leq |t|,\ u > 0$, and

$$P_0(u) := \frac{u^2 p(u)}{p^2(u)+q^2(u)}, \quad P_1(u) = \frac{p(u)(1+\xi C(u)) + q^2(u)}{p^2(u)+q^2(u)} \in L_1(\mathbb{R}_+).$$

Thus $I_1(0^+) = \pi/2,\ I_1(0^-) = -\pi/2$, so that $v'(0^+,\xi) - v'(0^-,\xi) = 1$.

Finally, in view of formulas (13), (14), a direct computation gives, for $t \neq 0$,

$$v'(t,\xi) - cv(t,\xi) - \int_0^t v(s,\xi)ds - \xi\int_0^t du\int_{\mathbb{R}} K(s)v(u-s-ch,\xi)ds$$

$$= \frac{1}{\pi}\int_0^{+\infty}\frac{\left(p^2(u)+q^2(u)\right)\sin(tu) + (q(u)p(u)-p(u)q(u))\cos(tu)}{u(p^2(u)+q^2(u))}du$$

$$+ \frac{1}{\pi}\int_0^{+\infty}\frac{\xi p(u)S(u) + \xi q(u)C(u) + dq(u)}{u(p^2(u)+q^2(u))}du$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\frac{\xi p(u)S(u) + \xi q(u)C(u) + dq(u)}{u(p^2(u)+q^2(u))}du.$$

Consequently, $v''(t,\xi)$ exists for $t \neq 0$ and $v(t,\xi)$ satisfies equation (11) with $f(t) \equiv 0$ for all $t \neq 0$. In fact, if $t > 0$ (the case $t < 0$ is similar) then

$$v'(t,\xi) = \frac{1}{t\pi}\int_0^{+\infty}\left[P\left(\frac{v}{t}\right)\sin v + Q\left(\frac{v}{t}\right)\cos v\right]dv, \quad \text{where } Q(u) := \frac{-uq(u)}{p^2(u)+q^2(u)}.$$

This shows that all derivatives $v^{(j)}(t,\xi),\ t > 0$, exist. □

It follows from (12) that $v(\pm\infty,\xi) = 0$ as the Fourier transform of a function from $L_1(\mathbb{R})$. In fact, some additional work shows that actually $v(t)$ is exponentially decaying at $\pm\infty$:

Lemma 13. *If equation (11) is hyperbolic then $v \in W^{1,1}(\mathbb{R})$. In addition, $|v(t)| \leq Ce^{-\gamma|t|}$, $t \in \mathbb{R}$, for some positive C,γ and $v'' \in L_1(\mathbb{R}_\pm)$ (so that $v'(\pm\infty,\xi) = 0$).*

Proof. A simple inspection of the characteristic equation

$$z^2 - cz - d = \xi e^{-chz}\int_{\mathbb{R}} e^{-zs}K(s)ds, \tag{15}$$

shows that, in view of the hyperbolicity of (11), there exists $\gamma > 0$ such that the vertical strip $\{|\Re z| < 2\gamma\}$ does not contain roots of (15). But then we can shift the path of integration in the inversion formula for the Laplace transform (e.g. see [2, p. 88]) to obtain

$$v(t,\xi) = \frac{e^{\pm\gamma t}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut}du}{(\pm\gamma + iu)^2 - c(\pm\gamma + iu) - d - \xi \int_{\mathbb{R}} K(s)e^{-(\pm\gamma+iu)s}ds} = e^{\pm\gamma t}\sigma_{\pm}(t),$$

where $\sigma_{\pm}(\infty) = 0$. Next, by (14),

$$v'(t,\xi) = -\frac{1}{2\pi} \lim_{T\to+\infty} \int_{-T}^{T} \frac{iue^{iut}du}{u^2 + ciu + d + \xi e^{-iuch} \int_{\mathbb{R}} K(s)e^{-ius}ds}, \quad t \neq 0.$$

Therefore similarly, for $t \neq 0$,

$$v'(t,\xi) = \frac{e^{\pm\gamma t}}{2\pi} \lim_{T\to+\infty} \int_{-T}^{T} \frac{(\pm\gamma + iu)e^{iut}du}{(\pm\gamma + iu)^2 - c(\pm\gamma + iu) + d - \xi \int_{\mathbb{R}} K(s)e^{-(\pm\gamma+iu)s}ds},$$

where the latter limit also exists in $L_2(\mathbb{R})$ and represent the Fourier transform of an element of $L_2(\mathbb{R})$. Thus $v'(t,\xi) = e^{\pm\gamma t}\rho_{\pm}(t),\ t \neq 0$, where $\rho_{\pm} \in L_2(\mathbb{R})$. By the Hölder inequality, $v' \in L_1(\mathbb{R}_{\pm})$ so that $v' \in L_1(\mathbb{R})$. Finally, since v satisfies the equation (11) on $\mathbb{R}_{\pm}$, we conclude that $v''(t) = cv'(t) + v(t) + \xi K * v(t - ch)$ also belongs to $L_1(\mathbb{R}_{\pm})$. □

In the sequel, we will denote by $C_b(\mathbb{R})$ the Banach space of all real valued bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ endowed with the norm $|f| = \sup_{t\in\mathbb{R}} |f(t)|$. Below, we also use the notation $C_b^1(\mathbb{R}) = \{f \in C_b(\mathbb{R}) : f'(t) \in C_b(\mathbb{R})\}$.

Lemma 14. *If $v \in W^{1,1}(\mathbb{R})$ and function f is continuous and bounded then $v * f \in C_b^1(\mathbb{R})$ and $(v * f)' = v' * f$.*

Proof. Clearly, $|v * f(t)| \leq \sup_{t\in\mathbb{R}} |f(t)||v|_1,\ t \in \mathbb{R}$, and

$$v * f(t+\delta) - v * f(t) = \int_{\mathbb{R}} f(t-s)(v(s+\delta) - v(s))ds = \int_{\mathbb{R}} f(t-s) \int_{s}^{s+\delta} v'(u)duds$$

$$= \delta \int_{\mathbb{R}} v'(u)du \frac{1}{\delta} \int_{u-\delta}^{u} f(t-s)ds \ \text{ because of } \int_{\mathbb{R}} \int_{s}^{s+\delta} |v'(u)|duds = \delta|v|_1 < \infty.$$

$$\text{Since} \quad \left| \frac{1}{\delta} \int_{u-\delta}^{u} f(t-s)ds \right| \leq \sup_{t\in\mathbb{R}} |f(t)|, \quad \lim_{\delta\to 0} \frac{1}{\delta} \int_{u-\delta}^{u} f(t-s)ds = f(t-u),$$

we conclude that $v * f$ is differentiable on $\mathbb{R}$ and $(v * f)' = v' * f$. Note also that $|v' * f(t)| \le \sup_{t\in\mathbb{R}} |f(t)||v'|_1$, $t \in \mathbb{R}$. $\square$

Our next goal is, given a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, to prove the uniqueness of bounded solution $y(t)$ for the hyperbolic equation (11) and to justify the representation $y(t) = -v * f(t)$:

Corollary 15. *Suppose that $f \in C_b(\mathbb{R})$. If equation (11) is hyperbolic and v is the associated fundamental solution then the formula $u = -v * f$ gives the unique C^2-smooth bounded solution of (11).*

Proof. Invoking Lemmas 12, 13, 14 and using the formula

$$u(t) = -\int_{-\infty}^{t} v(t-s,\xi)f(s)ds - \int_{t}^{+\infty} v(t-s,\xi)f(s)ds,$$

we find easily that

$$u'(t) = -\int_{\mathbb{R}} v'(t-s,\xi)f(s)ds, \; u''(t) = -\int_{\mathbb{R}} v''(t-s,\xi)f(s)ds - f(t)(v'(0^+,\xi) - v'(0^-,\xi))$$

are continuous, bounded and satisfy equation (11).

On the other hand, assume that $u(t)$ is some classical bounded solution of equation (11). Then it is easy to see from (11) that $u'(t), u''(t)$ are also bounded on $\mathbb{R}$ and

$$u'' * v(t) = \int_{\mathbb{R}} u''(s)v(t-s)ds = \int_{\mathbb{R}} v'(t-s)u'(s)ds = u(t) + \int_{\mathbb{R}} v''(t-s)u(s)ds,$$

$$u' * v(t) = \int_{\mathbb{R}} u'(s)v(t-s)ds = \int_{\mathbb{R}} u(s)v'(t-s)ds.$$

In consequence, considering the convolution of equation (11) with the fundamental solution, we find that $u(t) + v * f(t) = 0$. $\square$

2.2. *Continuous function $v(t,\xi)$ as a distributional solution of a non-local equation*

It is worthwhile to analyze the fundamental solution $v(t,\xi)$ and some of its properties from the point of view of the theory of distributions. The distributions will be regarded in the standard way, as elements of the dual space $\mathcal{D}'(\mathbb{R})$ (recall that the space $\mathcal{D}(\mathbb{R})$ of test functions consists of compactly supported smooth functions). This perspective is quite useful since it helps to cope with more general non-local delayed differential operators

$$\mathcal{L}\phi(t) = \phi^{(n)}(t) + a_{n-1}\phi^{(n-1)}(t) + \cdots + a_1\phi'(t) + a_0\phi(t) + b_0 \int_{\mathbb{R}} K(t-s)\phi(s)ds + \sum_{j=1}^{m} b_j\phi(t-h_j).$$

We assume here that $\phi \in \mathcal{D}(\mathbb{R})$, $n \geq 2$, $a_i, b_j, h_j \in \mathbb{R}$ and that the operator $\mathcal{L}$ is hyperbolic in the sense that the characteristic function $\eta(z, \mathcal{L})$ determined from $(\mathcal{L})(e^{zt}) = \eta(z, \mathcal{L})e^{zt}$ does not have zeros on the imaginary axis. Obviously, this form of $\mathcal{L}$ includes the particular case of the operator defined by the left-hand side of equation (11).

Consider the formally adjoint operator $\mathcal{L}^*$ defined by

$$\mathcal{L}^*\psi(t) = (-1)^n\psi^{(n)}(t) + (-1)^{n-1}a_{n-1}\psi^{(n-1)}(t) + \cdots - a_1\psi'(t) + a_0\psi(t) + b_0 \int_{\mathbb{R}} K(s-t)\psi(s)ds + \sum_{j=1}^{m} b_j\psi(t+h_j), \quad \psi \in \mathcal{D}(\mathbb{R}).$$

Clearly, for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$, it holds that $\mathcal{L}\phi, \mathcal{L}^*\psi \in L_1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \psi(t)\mathcal{L}\phi(t)dt = \int_{\mathbb{R}} \phi(t)\mathcal{L}^*\psi(t)dt.$$

We have the following

Lemma 16. *Suppose that $\mathcal{L}$ is hyperbolic. Then function*

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut}du}{\eta(iu, \mathcal{L})}, \quad t \in \mathbb{R}, \tag{16}$$

is continuous, bounded and Lebesgue integrable on $\mathbb{R}$. Moreover, it is a distributional solution of the equation $\mathcal{L}v(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function:

$$\int_{\mathbb{R}} v(t)\mathcal{L}^*\phi(t)dt = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}). \tag{17}$$

In consequence, $v(t)$ is C^{n-2}-smooth on $\mathbb{R}$, C^n-smooth on $(-\infty, 0]$ and on $[0, +\infty)$, $\mathcal{L}v(t) = 0$ for all $t \neq 0$ and $v^{(n-1)}(0^+) - v^{(n-1)}(0^-) = 1$.

Proof. From $1/\eta \in L_1(\mathbb{R})$ we infer that $v \in C(\mathbb{R})$, $v(\pm\infty) = 0$. In fact, repeating the argument given in the first lines of the proof of Lemma 13, we obtain that $|v(t)| \leq Ce^{-\gamma|t|}$, $t \in \mathbb{R}$, for some positive C, γ. Now, using the Fubini theorem and integrating by parts, we find that

$$\int_{\mathbb{R}} v(t)\mathcal{L}^*\phi(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{L}^*\phi(t)dt \int_{\mathbb{R}} \frac{e^{iut}du}{\eta(iu,\mathcal{L})} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{du}{\eta(iu,\mathcal{L})}\int_{\mathbb{R}} e^{iut}\mathcal{L}^*\phi(t)dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{du}{\eta(iu,\mathcal{L})}\int_{\mathbb{R}} \phi(t)\mathcal{L}e^{iut}dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{du}{\eta(iu,\mathcal{L})}\int_{\mathbb{R}} \phi(t)\eta(iu,\mathcal{L})e^{iut}dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} du \int_{\mathbb{R}} \phi(t)e^{iut}dt = \phi(0).$$

In the last line, we are using the inversion formula for the Fourier transform [50].

Next, we observe that, on each interval (α, β) disjoint with $\{0\}$, continuous $v(t)$ is a distributional solution of the following inhomogeneous linear ordinary differential equation with constant coefficients and right-hand side $F \in C_b(\mathbb{R})$:

$$v^{(n)}(t) + a_{n-1}v^{(n-1)}(t) + \cdots + a_1 v'(t) + a_0 v(t) = F(t), \quad t \in (\alpha, \beta), \tag{18}$$
$$F(t) := -b_0 \int_{\mathbb{R}} K(t-s)v(s)ds - \sum_{j=1}^{m} b_j v(t-h_j).$$

It is well known [38] that, in such a case, $v(t)$ is also a classical solution on (α, β) of equation (18). Since $F \in C_b(\mathbb{R})$, we find that $v(t)$ has continuous derivatives up to order n on the interval $(-\infty, 0]$ and on the interval $[0, +\infty)$.

Finally, for $n \geq 2$ consider C^{n-2}-smooth function $T_n : \mathbb{R} \to [0, +\infty)$ defined as follows: $T_n(s) = 0$ for $s \leq 0$ and $T_n(s) = s^{n-1}/(n-1)!$ Clearly, T_n is a distributional solution of

$$T^{(n)}(t) + a_{n-1}T^{(n-1)}(t) + \cdots + a_1 T'(t) + a_0 T(t) = \delta(t) + F_1(t), \quad t \in \mathbb{R},$$

where piecewise continuous function $F_1 : \mathbb{R} \to \mathbb{R}$ (having a unique jump discontinuity at $t = 0$) is defined by $F_1(0) = 0$ and

$$F_1(t) := a_{n-1}T_n^{(n-1)}(t) + \cdots + a_1 T_n'(t) + a_0 T_n(t), \ t \neq 0.$$

But then the difference $W(t) = v(t) - T_n(t)$, being is a distributional solution of the ordinary differential equation

$$W^{(n)}(t) + a_{n-1}W^{(n-1)}(t) + \cdots + a_1 W'(t) + a_0 W(t) = F(t) - F_1(t), \quad t \in \mathbb{R},$$

is C^{n-1}-smooth function on $\mathbb{R}$. In consequence, $v(t) = W(t) + T_n(t)$ is a C^{n-2}-smooth function on $\mathbb{R}$ and $v^{(n-1)}(0^+) - v^{(n-1)}(0^-) = T_n^{(n-1)}(0^+) - T_n^{(n-1)}(0^-) = 1$. $\square$

Now, since for $f \in C_b(\mathbb{R})$ and continuous $v \in L_1(\mathbb{R})$

$$-\int_{\mathbb{R}} v * f(t)\mathcal{L}^*\phi(t)dt = \int_{\mathbb{R}} f(s)ds \int_{\mathbb{R}} v(t-s)\mathcal{L}^*\phi(t+s)dt = \int_{\mathbb{R}} f(s)\phi(s)ds,$$

we obtain the following

Corollary 17. *For each fixed $f \in C_b(\mathbb{R})$, the function $u = -v * f$ is a distributional solution of the equation $\mathcal{L}u(t) + f(t) = 0$. In consequence [38], $u(t)$ is also a bounded C^n-smooth classical solution on $\mathbb{R}$ of this equation.*

2.3. A criterion of negativity of the fundamental solution $v(t, \xi)$

In this subsection, assuming the hyperbolicity of equation (11), we establish a criterion of negativity of its fundamental solution $v(t, \xi)$. It is well known (and it is straightforward to check) that $v(t, \xi) < 0,\ t \in \mathbb{R}$, in the local and non-delayed case when $\xi = 0, d > 0$ and

$$v(s, 0) = \min\{e^{\lambda_1(0)s}, e^{\lambda_0(0)s}\}/(\lambda_1(0) - \lambda_0(0)), \tag{19}$$

with $\lambda_1(0) < 0 < \lambda_0(0)$ being the roots of the characteristic equation $\lambda^2 - c\lambda - d = 0$. Suppose now that $K(s)$ satisfies **(K)** and $\xi \geq 0,\ d + \xi > 0$. Then it is easy to see that there exists a unique positive number ξ^* such that $\chi(z, \xi)$ has both positive and negative finite zeros if and only if $\xi \in [0, \xi^*]$. In fact, since $\chi^{(4)}(s, \xi) < 0$ for all $s \in \mathbb{R}$, function $\chi(s, \xi),\ s \in \mathbb{R}$, for $\xi \in (0, \xi^*]$ can have at most four real zeros $\lambda_j(\xi)$, all of them being finite if

$$\int_{-\infty}^{0} K(s)ds \int_{0}^{+\infty} K(s)ds \neq 0.$$

In such a case, we will order them as $\lambda_2(\xi) \leq \lambda_1(\xi) < 0 < \lambda_0(\xi) \leq \lambda_{-1}(\xi)$. If $\int_{-\infty}^{0} K(s)ds = 0$ and $\xi \in [0, \xi^*]$, then there are exactly two negative and one positive finite roots $\lambda_2(\xi) \leq \lambda_1(\xi) < 0 < \lambda_0(\xi)$; by definition, we set $\lambda_{-1}(\xi) = +\infty$. A similar situation occurs when $\int_0^{+\infty} K(s)ds = 0,\ \xi \in [0, \xi^*]$, where it is convenient to set $\lambda_2(\xi) = -\infty$. Finally, we set $\lambda_2(0) = -\infty,\ \lambda_{-1}(0) = +\infty$. Observe that in either case the biggest negative root $\lambda_1(\xi)$ and the smallest positive root $\lambda_0(\xi)$ are finite numbers.

Lemma 18. *Suppose that $\xi \in [0, \xi^*]$, $c^2 + 4d > 0$. Then in the closed strip*

$$\overline{\Pi}(\lambda_2, \lambda_{-1}) := \{z : \lambda_2(\xi) \leq \Re z \leq \lambda_{-1}(\xi)\}$$

the function $\chi(z, \xi)$ does not have zeros different from $\lambda_j(\xi),\ j = -1, 0, 1, 2$.

Proof. First, note that only zero of $\chi(z, \xi)$ on the line $\Re z = \lambda_j(\xi)$, where $j \in \{-1, 0, 1, 2\}$, is $\lambda_j(\xi)$. Indeed, if $z_j,\ \Re z_j = \lambda_j(\xi)$, denotes another root of equation (15) then, using the factorization $z^2 - cz - d = (z - A)(z - B)$ with real A, B, we get the following contradiction:

$$|\lambda_j^2 - c\lambda_j - d| < |z_j - A||z_j - B| = |z_j^2 - cz_j - d| \leq \xi e^{-ch\lambda_j} \int_{\mathbb{R}} e^{-\lambda_j s} K(s)ds = |\lambda_j^2 - c\lambda_j - d|.$$

Next, the right hand side of (15) is uniformly bounded in each closed strip $\overline{\Pi}(a, b)$ by

$$\xi^* \int_{\mathbb{R}} (e^{-a(ch+s)} + e^{-b(ch+s)})K(s)ds,$$

while $z^2 - cz - 1 \to \infty$ as $z \to \infty$. In consequence, there exists positive C which does not depend on ξ such that each zero z_k of $\chi(z,\xi)$, $\xi \in [0,\xi^*]$, in $\overline{\Pi}(\lambda_1(\xi), \lambda_0(\xi))$ satisfies $|\Im z_k| \le C$. Since $\lambda_j(\xi)$, $j = 0, 1$, are continuous functions of ξ and $\overline{\Pi}(\lambda_1(0), \lambda_0(0))$ does not contain non-real roots of $\chi(z,0)$, we find that either $\overline{\Pi}(\lambda_1(\xi), \lambda_0(\xi))$ contains only two zeros of $\chi(z,\xi)$ for all $\xi \in [0,\xi^*]$ or there exist $\xi_1 \in (0,\xi^*]$ and complex zero z_1 of $\chi(z,\xi)$ such that $\Re z_1 \in \{\lambda_1(\xi_1), \lambda_0(\xi_1)\}$. However, as we have just proved, the latter cannot happen.

Finally, suppose that $\chi(z_0,\xi) = 0$, $\lambda_2(\xi) < x_0 := \Re z_0 < \lambda_1(\xi)$ (the case when $\lambda_0(\xi) < \Re z_0 < \lambda_{-1}(\xi)$ can be treated analogously). Then we get the following contradiction

$$|z_0^2 - cz_0 - d| \le \xi e^{-chx_0} \int_{\mathbb{R}} e^{-x_0 s} K(s)ds < |x_0^2 - cx_0 - d| \le |z_0 - A||z_0 - B| = |z_0^2 - cz_0 - d|.$$

This completes the proof of the lemma. $\square$

Lemma 19. *Suppose that $\xi \in [0,\xi^*]$, $d > 0$, $h \ge 0$. Then $v(t,\xi) < 0$ for all $t \in \mathbb{R}$, $\xi \in [0,\xi^*]$. Moreover, $v(t,\xi)$ is sign-changing on $\mathbb{R}$ for each $\xi > \xi^*$ close to ξ^*.*

Proof. Due to Lemma 18, equation (11) is hyperbolic and therefore the fundamental solution exists. The proof of its negativity is divided in several steps. Recall that if $\xi \in [0,\xi^*)$ then $\chi'(\lambda_0(\xi)) > 0$, $\chi'(\lambda_1(\xi)) < 0$ and $\lambda_j(\xi)$, $j = 0, 1$, are simple zeros of $\chi(z,\xi)$.

Claim I. For each non-negative $\xi_0 < \xi^*$ there exist real numbers ν_0, ν_1, a neighborhood $\mathcal{O} \ni \xi_0$ and positive constants K, L such that, for all $\xi \in \mathcal{O}$,

$$\lambda_2(\xi) + L < \nu_1 < \lambda_1(\xi) - L, \; \lambda_0(\xi) + L < \nu_0 < \lambda_{-1}(\xi) - L, \tag{20}$$

$$v(t,\xi) = \rho_j(\xi)e^{\lambda_j(\xi)t} + r_j(t,\xi), \quad |r_j(t,\xi)| \le Ke^{\nu_j t}, \quad (-1)^{j+1}t \ge 0, \; j = 0, 1, \tag{21}$$

where $\rho_j(\xi) = (-1)^{j+1}/\chi'(\lambda_j(\xi)) < 0$, $j = 0, 1$, depend continuously on ξ.

We prove this claim for $j = 1$, the other case being similar. Fix some $\xi_0 < \xi^*$ and $\nu_1 \in (\lambda_2(\xi_0), \lambda_1(\xi_0))$. Then we can choose a neighborhood $\mathcal{O} \ni \xi_0$ and $L > 0$ sufficiently small to meet the condition (20) for all $\xi \in \mathcal{O}$. Next, after moving the integration path in the inversion formula (12) from $\Re z = 0$ to $\Re z = \nu_1$, we obtain that, for $t \ge 0$, $v(\xi,t) =$

$$\frac{e^{\lambda_1 t}}{\chi'(\lambda_1,\xi)} + \frac{1}{2\pi i}\int_{\nu_1 - i\cdot\infty}^{\nu_1 + i\cdot\infty} \frac{e^{tz}dz}{\chi(z,\xi)} = -\frac{e^{\lambda_1 t}}{|\chi'(\lambda_1,\xi)|} + \frac{e^{\nu_1 t}}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ist}ds}{\chi(\nu_1 + is,\xi)} =: e_1(t) + e^{\nu_1 t}q(t),$$

where $q(\pm\infty) = 0$ and

$$|q(t)| \le K = \sup_{\xi \in \mathcal{O}} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{ds}{|\chi(\nu_1 + is,\xi)|}.$$

Claim I implies that exponentially decaying function $v(t,\xi)$, $\xi \in \mathcal{O}$, is negative at $\pm\infty$. In particular, there exists a leftmost point $T_+(\xi) \ge 0$ such that $v(t,\xi) < 0$ for all $t > T_+(\xi)$. Analogously, $T_-(\xi) \le 0$ denotes the rightmost point such that $v(t,\xi) < 0$ for all $t < T_-(\xi)$. By (19), $T_\pm(0) = 0$.

Claim II. $v(t,\xi)$ is a continuous function of $t \in \mathbb{R}$, $\xi \in [0,\xi^*]$. Furthermore, $v(t,\xi)$ is bounded on $\mathbb{R}$, uniformly with respect to $\xi \in [0,\xi^*]$.

Indeed, observe that the function $g(u,\xi) := 1/\chi(iu,\xi)$ is continuous on $\mathbb{R} \times [0,\xi^*]$ and $|g(u,\xi)| \le G(u)$, $u \in \mathbb{R}$, $\xi \in [0,\xi^*]$, with $G \in L_1(\mathbb{R})$ defined by

$$G(u) = \begin{cases} \max\{1/|\chi(is,\xi)|,\ |s| \le 2+\xi^*, \xi \in [0,\xi^*]\}, & |u| \le 2+\xi^*, \\ 1/(u^2-\xi^*), & |u| > 2+\xi^*, \end{cases}$$

(recall that $\chi(\lambda,\xi) \neq 0$ when $\Re\lambda = 0$, $\xi \in [0,\xi^*]$). In particular, to prove the continuity statement, it suffices to apply the Lebesgue dominated convergence theorem to (12).

Claim III. On each closed interval $[0,\zeta] \subset [0,\xi^*)$, functions $T_\pm(\xi)$ are bounded.

Due to the compactness of $[0,\zeta]$, it is enough to prove that $T_\pm(\xi)$ are locally bounded. For example, consider $T_+(\xi)$ for $\xi \in \mathcal{O}$. We have that either $T_+(\xi) = 0$ or $T_+(\xi) > 0$ and

$$0 = v(T_+(\xi)) \le \rho_1(\xi)e^{\lambda_1(\xi)T_+(\xi)} + Ke^{\nu_1 T_+(\xi)},\ \xi \in \mathcal{O}.$$

In the latter case, for all $\xi \in \mathcal{O}$,

$$T_+(\xi) \le \frac{1}{\lambda_1(\xi)-\nu_1}\ln\frac{K}{|\rho_1(\xi)|} \le \frac{1}{L}\ln(K\sup_{\xi\in\mathcal{O}}|\chi'(\lambda_1(\xi),\xi)|).$$

Claim IV. $v(t,\xi) < 0$ for all $t \in \mathbb{R}$, $\xi \in [0,\xi^*]$. Furthermore, $v(t,\xi)$ is sign-changing on $\mathbb{R}$ for $\xi > \xi^*$ close to ξ^*.

Let $\xi_c \in [0,\xi^*]$ be the maximal number such that $v(t,\xi) < 0$, $t \in \mathbb{R}$, for all $\xi \in [0,\xi_c)$. For a moment, suppose that $\xi_c < \xi^*$. Since $v(t,0) < 0$ for all $t \in \mathbb{R}$, due to Claims II and III, $\xi_c > 0$ and $v(t_c,\xi_c) = 0$ for some $T_-(\xi_c) \le t_c \le T_+(\xi_c)$. Since $v(\pm\infty,\xi_c) = 0$, there exist two numbers $a_c < b_c$ where $v(t,\xi_c)$ reaches its absolute minima on the half-lines $(-\infty,t_c]$ and $[t_c,+\infty)$, respectively. Clearly, either a_c or b_c is different from zero. For instance, suppose that $a_c \neq 0$. Then $v(t,\xi_c)$ is differentiable at this point where $v(a_c,\xi_c) < 0$, $v'(a_c,\xi_c) = 0$, $v''(a_c,\xi_c) \ge 0$ and $K * v(a_c - ch) \le 0$. Obviously, this contradicts equation (11) with $f(t) \equiv 0$ at $t = a_c \neq 0$. This shows that $\xi_c = \xi^*$ and also implies that $v(t,\xi^*) \le 0$ for all $t \in \mathbb{R}$. Now, $\xi = \xi^*$ is a bifurcation point for some real zero of $\chi(z,\xi)$: for instance, suppose that for $\lambda_2(\xi^*) = \lambda_1(\xi^*) < 0$. Clearly, $\chi'(\lambda_1(\xi^*),\xi^*) = 0$ while $\chi''(\lambda_1(\xi^*),\xi^*) \neq 0$ for otherwise $\lambda_1(\xi^*)$ would be a triple negative zero of $\chi(z,\xi^*)$. Then, using the inversion formula again, we find that $v(t,\xi^*)$ is negative at $+\infty$ because of the relation $\lim_{t\to+\infty} v(t,\xi^*)t^{-1}e^{-\lambda_1(\xi^*)t} = 1/\chi''(\lambda_1(\xi^*),\xi^*) < 0$. Thus the above proof of negativity also works for $v(t,\xi^*)$. Next, for all $\xi > \xi^*$ close to ξ^*, function $\chi(z,\xi)$ has two simple complex conjugated zeros $\lambda_{1\pm}(\xi) := p(\xi) \pm iq(\xi)$, $\lambda_{1\pm}(\xi^*) := \lambda_1(\xi^*)$, such that the strip $\Pi(p(\xi),0]$ does not contain any zero of $\chi(z,\xi)$. Now, assuming that $v(t,\xi) \ge 0$, $t \in \mathbb{R}$, we infer from [45, Theorem 5b, p. 58] that the singularity of the Laplace transform $1/\chi(z,\xi)$ of $v(t,\xi)$, which is rightmost on the half-plane $\Re z \le 0$, should be a real number and not complex as $\lambda_{1\pm}(\xi)$. This contradiction proves the second part of Claim IV. □

Remark 20. It can be proved in Lemma 19 that actually $v(t,\xi)$ is oscillating (either at $+\infty$ or $-\infty$) for $\xi > \xi^*$ close to ξ^*. Here the assumption of smallness of $\xi - \xi^*$ is used only in order to assure the existence of zeros of $\chi(z,\xi)$ in the both half-planes $\{\Re z > 0\}$ and $\{\Re z < 0\}$. Observe that, due to a result by Iliev, [25, Theorem 3.2.46], the function $\int_{\mathbb{R}} e^{-zs}K(s)ds$ can have all its zero only in the half-plane $\{\Re z < 0\}$ even if non-negative continuous K has a compact support and $K(0) > 0$. In general, there is a lack of detailed knowledge regarding the distribution of zeros of $\chi(z, \xi)$ (e.g., see the discussion concerning the Riemann hypothesis in [25]). If K has a compact support, then the entire function $\chi(iz, \xi)$ is of class A and is of completely regular growth: in such a case, some general information about the distribution of zeros of $\chi(z, \xi)$ can be found in [28, Chapter V, Theorem 11].

Remark 21. The negativity (or positivity) of the fundamental solution (or of the Green function) for solving initial/boundary value problems for delayed differential equations is an important topic of the theory of functional differential equations. See the recent monographs [1,18] for more references concerning this problem.

The final result of this section shows that the geometric form of $v(t, \xi)$ for $\xi \in [0, \xi^*]$ is quite similar to the shape of $v(t, 0)$ given in (19):

Corollary 22. *If $\xi \in [0, \xi^*]$ then $v(t, \xi)$ has a unique minimum point at $t = 0$. Moreover, $v(t, \xi)$ is strictly monotone on $\mathbb{R}_-$ and $\mathbb{R}_+$. It is also strictly convex on $\mathbb{R}_-$.*

Proof. Indeed, as we have seen in the proof of Lemma 19, $v'(t, \xi)$ cannot change the sign on $(-\infty, 0)$ and $(0, +\infty)$ because otherwise $v(t, \xi)$ reaches a local minimum at some point of $\mathbb{R} \setminus \{0\}$. By the same reason, $v'(t, \xi)$ cannot vanish on an open interval. Finally, observe that all this implies that $v''(t, \xi) < 0$ for $t < 0$. □

3. Proof of Theorem 5

Case I: $(h, c) \in \mathcal{D}_{\mathfrak{L}}$ In this section, in view of the particular form of equation (1), we use $\chi(z, \xi)$ with $d = 1$:

$$\chi(z, \xi) = z^2 - cz - 1 - \xi e^{-chz} \int_{\mathbb{R}} e^{-zs} K(s) ds.$$

Since $g'(\kappa) < 0$, we have therefore that $\chi_\kappa(z, c, h) = \chi(z, |g'(\kappa)|)$. In this way, we are assuming that the equation $\chi(z, |g'(\kappa)|) = 0$ has at least one negative and one positive simple roots $\lambda_1(|g'(\kappa)|) < 0 < \lambda_0(|g'(\kappa)|)$. In consequence, for sufficiently small $\epsilon > 0$ we have that $\xi = |g'(\kappa)| + \epsilon \le \xi^*$ and the equation $\chi(z, \xi) = 0$ also has at least one negative and one positive simple roots $\lambda_1(\xi), \lambda_0(\xi)$:

$$\lambda_1(\xi) < \lambda_1(|g'(\kappa)|) < 0 < \mu_0 < \mu_1 < \lambda_0(|g'(\kappa)|) < \lambda_0(\xi).$$

With $g_1(s) = (g(s) + \xi s)/(1 + \xi)$, the profile equation (6) can be rewritten as

$$y''(t) - cy'(t) - y(t) - \xi \int_{\mathbb{R}} K(t-s) y(s-ch) ds + (1+\xi) \int_{\mathbb{R}} K(t-s) g_1(y(s-ch)) = 0. \tag{22}$$

By Corollary 15, this equation has at least one bounded solution $\phi(t)$ if and only if

$$\phi(t)=\mathcal{N}\phi(t),\ \text{where } \mathcal{N}\phi(t):=\int_{\mathbb{R}} N_1(t-s)g_1(\phi(s-ch))ds,\ N_1(s)=-(1+\xi)v*K(s). \tag{23}$$

In virtue of Lemma 19, the following properties of $N_1(s)$ are immediate: $N_1(s)>0,\ s\in\mathbb{R}$, $\int_R N_1(s)ds=1$. Furthermore, $\chi_1(0)=(1-g'(0))/(1+\xi)<0,\ \chi_1(\mu_0)=0$, where

$$\chi_1(z):=1-g_1'(0)\int_R e^{-zs}N_1(s)ds=\frac{\chi_0(z,c,h)}{\chi(z,\xi)}<\infty \text{ for all } z\in(\lambda_1(\xi),\lambda_0(\xi)). \tag{24}$$

On the other hand, $g_1(s)$ is strictly increasing on $[0,\kappa]$ where $g_1(\kappa)=\kappa,\ g_1(0)=0$ and

$$g_1'(\kappa)=\frac{\epsilon}{|g'(\kappa)|+\epsilon}\in(0,1),\ g_1(s)=\frac{g(s)+\xi s}{1+\xi}\le g_1'(0)s=\frac{g'(0)s+\xi s}{1+\xi}.$$

Therefore nonlinear convolution equation (23) can be analyzed within the framework of theory developed in [19]. Particularly, Theorem 7 in [19] guarantees the existence of a positive solution $y=\phi(t)$ to (22) satisfying the conditions $\phi(-\infty)=0,\ \phi(+\infty)=\kappa$. Moreover, it is easy to see that solution $\phi(t)$ provided by [19, Theorem 7] is a non-decreasing one if $g_1(s)$ is a non-decreasing function. For the sake of completeness, in Remark 23 below, we indicate the corresponding change in the proof of [19, Theorem 7]. Now, due to the positivity of $N_1(s)$, the profile $\phi(t)$ is actually a strictly increasing function: if $t_2>t_1$ then $\phi(t_2-s)\ge\phi(t_1-s),\ s\in\mathbb{R}$, $\phi(t_2-s)\not\equiv\phi(t_1-s)$, so that

$$\phi(t_2)=\int_{\mathbb{R}} N_1(s-ch)g_1(\phi(t_2-s))ds>\int_{\mathbb{R}} N_1(s-ch)g_1(\phi(t_1-s))ds=\phi(t_1).$$

Hence, the proof of Case I is completed if $g_1(s)$ is increasing on $\mathbb{R}_+$. Otherwise, consider some increasing continuous and bounded function $g_2(s)$ coinciding with $g_1(s)$ on $[0,\kappa]$ and such that $g_2'(\kappa)=g_1'(\kappa)$. But then, due to the first part of the proof, convolution equation (23) where g_1 is replaced with g_2 has a monotone solution $\phi:\mathbb{R}\to[0,\kappa]$. Since $g_1(s)\equiv g_2(s)$ on $[0,\kappa]$, the same function $\phi(t)$ solves (23).

Case II: (h,c) belongs to the boundary of the set $\mathcal{D}_{\mathfrak{L}}$ In such a case, there exists a sequence $\{(h_j,c_j)\}$ of points in $\mathcal{D}_{\mathfrak{L}}$ converging to (h,c). From Case I we conclude that for each point $\{(h_j,c_j)\}$ there exists a monotone positive solution $y=\phi_j(t),\ \phi_j(-\infty)=0,\ \phi_j(-\infty)=\kappa$, satisfying the profile equation

$$y''(t)-c_jy'(t)-y(t)+\int_{\mathbb{R}} K(t-s)g(y(s-c_jh_j))=0.$$

Since this equation is translation invariant, we can assume that $\phi_j(0)=\kappa/2$ for each j. Then it follows that $\phi_j(s)$ has a subsequence $\phi_{j_k}(t)$ converging (uniformly on compact subsets of $\mathbb{R}$) to a positive monotone solution $\phi(t),\ \phi(0)=\kappa/2$, of the limit equation (6) (e.g. see [19] or [41, Section 6] for more details). Now, the monotonicity of $\phi(t)$ implies that the boundary conditions in (6) are also satisfied (e.g. see Remark 23 below). This completes the proof of Theorem 5. □

Remark 23. In order to solve the following slightly modified version

$$\phi := \mathcal{A}\phi, \quad \text{where } \mathcal{A}\phi(t) := \int_{\mathbb{R}} N_1(s)\gamma_n(\phi(t-s-ch))ds, \tag{25}$$

$$\gamma_n(s) := \begin{cases} g_1'(0)s, & \text{for } s \in [0, 1/n], \\ \max\{g_1'(0)/n, g_1(s)\}, & \text{when } s \geq 1/n, \end{cases}$$

of equation $\mathcal{N}\phi = \phi$ in Section 4 of [19], we can use the iteration procedure $\phi_{j+1} = \mathcal{A}\phi_j$, $j = 0, 1, \dots$, $\phi_0(s) = n^{-1}\exp(\mu_0 s)$, $s \in \mathbb{R}$, instead of the Schauder fixed point theorem. For a small positive $\varepsilon > 0$, set $\phi^-(t) = \phi_0(t)(1 - e^{\varepsilon t})\chi_{\mathbb{R}_-}(t)$, where $\chi_{\mathbb{R}_-}(t)$ is the characteristic function of $\mathbb{R}_-$. Since $\gamma_n(s), \phi_0(t)$ are non-decreasing functions and $\phi_-(t) \leq \mathcal{A}\phi_-(t) \leq \phi_1(t) \leq \mathcal{A}\phi_0(t) \leq \phi_0(t)$, we conclude that each $\phi_j(t)$, $j \in \mathbb{N}$, is also a non-decreasing function and $\phi_0(t) \geq \phi_2(t) \geq \dots \geq \phi_j(t) \geq \dots \geq \phi_-(t)$. Then the limit $\phi(t) = \lim_{j\to+\infty}\phi_j(t)$ should be a positive non-decreasing and bounded solution of the equation $\phi = \mathcal{A}\phi$. Taking the limit in (25) as $t \to \pm\infty$, we obtain that $\phi(\pm\infty) = \gamma_n(\phi(\pm\infty))$ that immediately implies that $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$.

4. Proof of Theorem 6

In this section, we show how the use of convolution equation (22) helps to extend the front uniqueness result established for equation (2) with monotone birth function g (e.g. see [43, Theorem 1.2]) on the case of non-local and non-monotone model (1).

Lemma 24. *Fix some $(h, c) \in \mathcal{D}_{\mathfrak{L}}$ and suppose that $\phi, \psi : \mathbb{R} \to (0, \kappa]$ are two wavefront profiles satisfying equation (6) and such that ϕ is monotone and, for some finite T,*

$$\phi(t) < \psi(t), \quad t < T. \tag{26}$$

Then $\phi(t) < \psi(t)$ for all $t \in \mathbb{R}$.

Proof. Set $a_* = \inf \mathsf{A}$ where

$$\mathsf{A} := \{a \geq 0 : \psi(t) + a \geq \phi(t),\ t \in \mathbb{R}\}.$$

Note that $\mathsf{A} \neq \emptyset$ since $[\kappa, +\infty) \subset \mathsf{A}$. Clearly, $a_* \in \mathsf{A}$.

Now, if $a_* = 0$ then $\psi(t) \geq \phi(t)$, $t \in \mathbb{R}$. We claim that, in fact, $\psi(t) > \phi(t)$, $t \in \mathbb{R}$. Indeed, otherwise we can suppose that T is such that $\phi(T) = \psi(T)$. In this way, the difference $\psi(t) - \phi(t) \geq 0$ reaches its minimal value 0 at T. Then, recalling that $N_1(t) = -(1+\xi)v * K(t) > 0$ for all $t \in \mathbb{R}$, we get a contradiction:

$$0 = \psi(T) - \phi(T) = \int_{\mathbb{R}} N_1(s-ch)(g_1(\psi(T-s)) - g_1(\phi(T-s)))ds > 0. \tag{27}$$

In this way, Lemma 24 is proved when $a_* = 0$ and, consequently, we have to consider the case $a_* > 0$. Let $\sigma > 0$ be small enough to satisfy

$$\gamma_1 := \max_{s\in[\kappa-\sigma,\kappa+\sigma]} g_1'(s) < 1.$$

Case I. First, we take $Q > 0$ such that $\kappa \int_Q^{+\infty} N_1(s-ch)ds < a_*(1-\gamma_1)$ and suppose that T is large enough to have

$$\phi(t), \psi(t) \in (\kappa-\sigma, \kappa+\sigma), \; t \geq T - Q. \tag{28}$$

In such a case non-negative function

$$w(t) := \psi(t) + a_* - \phi(t), \quad w(\pm\infty) = a_* > 0,$$

reaches its minimal value 0 at some leftmost point t_m, where $\psi(t_m) - \phi(t_m) = -a_*$. Thus $\psi(t_m) < \phi(t_m)$ and therefore $t_m > T$ so that

$$\psi(t_m - s), \; \phi(t_m - s) \in (\kappa-\sigma, \kappa+\sigma), \quad s \leq Q.$$

In consequence, for some $\theta(s) \in (\kappa-\sigma, \kappa+\sigma)$, we obtain

$$\begin{aligned}
-a_* = \psi(t_m) - \phi(t_m) &= \int_{\mathbb{R}} N_1(s-ch)(g_1(\psi(t_m-s)) - g_1(\phi(t_m-s)))ds \\
&= \left(\int_{-\infty}^{Q} + \int_{Q}^{+\infty}\right) N_1(s-ch)(g_1(\psi(t_m-s)) - g_1(\phi(t_m-s)))ds \\
&> -a_*(1-\gamma_1) + \int_{-\infty}^{Q} N_1(s-ch)g_1'(\theta(s))(\psi(t_m-s) - \phi(t_m-s))ds \\
&\geq -a_* + a_*\gamma_1 - \gamma_1 a_* \int_{-\infty}^{Q} N_1(s-ch)ds \geq -a_*, \quad \text{a contradiction.}
\end{aligned}$$

Case II. If (28) does not hold, then, due to the convergence of profiles at $+\infty$ and the strict monotonicity of ϕ, we can find large $\tau > 0$ and $T_1 > T$ such that

$$\psi(t+\tau) > \phi(t), \; t < T_1, \quad \phi(t), \psi(t+\tau) \in (\kappa-\sigma, \kappa+\sigma), \; t \geq T_1 - Q.$$

Therefore, in view of the previous arguments, we obtain that

$$\psi(t+\tau) > \phi(t), \quad t \in \mathbb{R}. \tag{29}$$

Define now τ_* by

$$\tau_* := \inf\{\tau \geq 0 : \text{ inequality (29) holds}\}.$$

It is clear that $\psi(t+\tau_*) \geq \phi(t),\ t \in \mathbb{R}$. Now, using the same argument as in (27), we conclude that either $\psi(t+\tau_*) \equiv \phi(t)$ with $\tau_* = 0$ (a contradiction) or $\psi(t+\tau_*) > \phi(t),\ t \in \mathbb{R}$. In the latter case, if $\tau_* = 0$, then Lemma 24 is proved. Otherwise, $\tau_* > 0$ and for each $\varepsilon \in (0, \tau_*)$ there exists a unique $T_\varepsilon > T$ such that

$$\psi(t+\tau_*-\varepsilon) > \phi(t),\ t < T_\varepsilon,\ \psi(T_\varepsilon+\tau_*-\varepsilon) = \phi(T_\varepsilon).$$

It is immediate to see that $\lim T_\varepsilon = +\infty$ as $\varepsilon \to 0^+$. Indeed, if $T_{\varepsilon_j} \to T'$ for some finite T' and $\varepsilon_j \to 0^+$, then we get a contradiction: $\psi(T'+\tau_*) = \phi(T')$. Therefore, if ε is small, then

$$\psi(t+\tau_*-\varepsilon), \phi(t) \in (\kappa-\sigma, \kappa+\sigma),\ t \geq T_\varepsilon - Q,$$

that is $\psi(t+\tau_*-\varepsilon)$ and $\phi(t)$ satisfy condition (28). But then we get $\psi(t+\tau_*-\varepsilon) > \phi(t)$ for all $t \in \mathbb{R}$, in contradiction to the definition of τ_*. This means that $\tau_* = 0$ and the proof of Lemma 24 is completed. □

Corollary 25. *Fix some (h, c) in the closure of $\mathfrak{D}_0$ and suppose that equation (6) possesses two monotone wavefronts ϕ and ψ. Then there exist $s_0, s_1 \in \mathbb{R}$ and $j \in \{0, 1\}$ such that either $\lim_{s\to-\infty} \phi(s+s_0)e^{-\mu_j s} = 1$, $\lim_{s\to-\infty} \psi(s+s_1)e^{-\mu_j s} = 1$, or $\lim_{s\to-\infty} \phi(s+s_0)e^{-\mu_j s}s^{-1} = -1$, $\lim_{s\to-\infty} \psi(s+s_1)e^{-\mu_j s}s^{-1} = -1$.*

Proof. First, we prove that every profile satisfies one of the given asymptotic formula, with j which might depend on the profile. For definiteness, we will take profile ϕ. We are going to apply some results of [2] to the convolution equation (23). It follows from (24) that the set $\{z : \sigma_K < \Re z < \gamma_K\}$, where $\sigma_K = \lambda_1(\xi), \gamma_K = \lambda_0(\xi)$, is the maximal open strip of convergence for the Laplace transform of N_1, cf. [45, Theorem 16b]. Moreover,

$$\lim_{x\to\gamma_K-} \int_{\mathbb{R}} N_1(s)e^{-sx}ds = +\infty \text{ and, in virtue of (21), } N_1(s) = O(e^{\lambda_0(\xi)s}),\ s \to -\infty.$$

Therefore, using condition (8) and a standard argument of the Diekmann–Kaper approach (cf. Step I of the proof of Theorem 3 in [2]), we find that, for some $j, k \in \{0, 1\}$ and $\rho > 0$, the Laplace transform $\int_{\mathbb{R}} \phi(s)e^{-zs}ds$ is analytic in the strip $0 < \Re z < \mu_j$, has a singularity at μ_j, and satisfies

$$\frac{\chi_0(z, c, h)}{\chi(z, \xi)} \int_{\mathbb{R}} \phi(s)e^{-zs}ds = D(z),$$

where $D(z)$ is analytic in a bigger strip $0 < \Re z < \mu_j + \rho$. Since clearly $\Phi_+(z) := \int_0^{+\infty} \phi(s) \times e^{-zs}ds$ is analytic in the half-plane $\{\Re z > 0\}$, we conclude that the function $Q(z) := D(z)\chi(z, \xi)/\chi_0(z, c, h) - \Phi_+(z)$ is meromorphic in $0 < \Re z < \mu_j + \rho$, where it has a unique singularity (a simple or double pole) at μ_j. Since $\Phi_-(z) := \int_{-\infty}^0 e^{-sz}\phi(s)ds = Q(z)$ for $\Re z \in (0, \mu_j)$ and $\phi(s)$ is positive and non-decreasing on $\mathbb{R}_-$, an application of the Ikehara theorem [9, Proposition 2.3] yields the required asymptotic formula.

Finally, we claim that ϕ and ψ have the same asymptotic behavior at $-\infty$. For example, suppose that $\phi(t) \sim e^{\mu_0 t}$ and $\psi(t) \sim e^{\mu_1 t}$ as $t \to -\infty$. Then for every fixed $\tau \in \mathbb{R}$ there exists $T(\tau)$ such that $\psi(t+\tau) < \phi(t)$ for all $t < T(\tau)$. Applying Lemma 24, we obtain that $\psi(s) < \phi(t)$ for every $s := t + \tau,\ t \in \mathbb{R}$, what obviously is false. □

Now we are in position to finalize the proof of Theorem 6. By Corollary 25, we can suppose that $\psi(t)$ and $\phi(t)$ have the same type of asymptotic behavior at $-\infty$. Consequently, $\psi(t+\tau), \phi(t)$ satisfy condition (26) of Lemma 24 for every small $\tau > 0$. But then $\psi(t+\tau) > \phi(t)$ for every small $\tau > 0$ that yields $\psi(t) \geq \phi(t),\ t \in \mathbb{R}$. By symmetry, we also find that $\phi(t) \geq \psi(t),\ t \in \mathbb{R}$, and Theorem 6 is proved. □

5. Proof of Theorem 8

We will show that conditions of Theorem 8 assure that each semi-wavefront $u = \phi(x+ct)$, $(h,c) \in \mathcal{D}_{\mathfrak{L}}$, of equation (2) is actually a monotone wavefront. Indeed, it is easy see that $0 < \phi(t) < \kappa,\ t \in \mathbb{R}$, since otherwise, without loss of generality, we can assume that $\phi(t_0) = \max_{s\in\mathbb{R}} \phi(s)$ for some t_0 that leads to the following contradiction:

$$\kappa \leq \phi(t_0) = \max_{s\in\mathbb{R}} \phi(s) = \int_{\mathbb{R}} N_1(t_0 - s) g_1(\phi(s-ch))ds < \int_{\mathbb{R}} N_1(t_0 - s) \max_{s\in\mathbb{R}} g_1(\phi(s))ds \leq \phi(t_0).$$

Next, we will need the following

Lemma 26. *Set $\Gamma(s) := g_1(\phi(s-ch))$. If the semi-wavefront $\phi(t)$ is increasing on $\mathbb{R}_-$ and satisfies $\phi'(0) = 0$ then, for $t \in [0, ch]$,*

$$\phi'(t) = \int_{-\infty}^{0} (N_1(t-s) - e^{\lambda_0(\xi)t} N_1(-s))d\Gamma(s) + \int_0^t (N_1(t-s) - e^{\lambda_0(\xi)(t-s)} N_1(0))d\Gamma(s).$$

Proof. Since $\Gamma(s)$ increases on $(-\infty, ch]$ and $\Gamma(-\infty) = 0$, all Riemann–Stieltjes integrals in the above formula are well defined and convergent. Next, note that $g(s) = g_1(s)(1+\xi) - \xi s$ is of bounded variation on $[0,\kappa]$. Thus, using [42, Remark 9(2)] together with Corollary 25, we conclude that $\phi(t)$ can have at most a finite number of critical points on each interval $(-\infty, \alpha]$. This implies that $\Gamma(s)$ has bounded variation on each $(-\infty, \alpha]$. Next, in view of Remark 10, after integrating by parts, we find that

$$\begin{aligned}
\phi'(t) &= \int_{-\infty}^{t} N_1'(t-s)\Gamma(s)ds + \int_t^{+\infty} \lambda_0(\xi) N_1(0) e^{\lambda_0(\xi)(t-s)}\Gamma(s)ds \\
&= \int_{-\infty}^{t} N_1(t-s)d\Gamma(s) + \int_t^{+\infty} N_1(0) e^{\lambda_0(\xi)(t-s)} d\Gamma(s) \\
&= \int_{-\infty}^{t} N_1(t-s)d\Gamma(s) + e^{\lambda_0(\xi)t} N_1(0) \left(\int_0^{+\infty} e^{-\lambda_0(\xi)s} d\Gamma(s) - \int_0^t e^{-\lambda_0(\xi)s} d\Gamma(s) \right).
\end{aligned}$$

Since $\phi'(0) = 0$, it holds that

$$N_1(0) \int_0^{+\infty} e^{-\lambda_0(\xi)s} d\Gamma(s) = -\int_{-\infty}^{0} N_1(-s) d\Gamma(s)$$

and therefore

$$\begin{aligned}\phi'(t) &= \int_{-\infty}^{t} N_1(t-s)d\Gamma(s) + e^{\lambda_0(\xi)t}\left(-\int_{-\infty}^{0} N_1(-s)d\Gamma(s) - N_1(0)\int_0^t e^{-\lambda_0(\xi)s} d\Gamma(s)\right) \\ &= \int_{-\infty}^{0} (N_1(t-s) - e^{\lambda_0(\xi)t} N_1(-s))d\Gamma(s) + \int_0^t (N_1(t-s) - N_1(0)e^{\lambda_0(\xi)(t-s)})d\Gamma(s). \quad \square\end{aligned}$$

Now we can complete the proof of Theorem 8. From [42, Lemma 6], we know that $\phi'(t) > 0$ on some maximal interval $(-\infty, \sigma)$. If $\sigma = +\infty$, the corollary is proved. If σ is finite, without loss of generality we may take $\sigma = 0$. Then $\Gamma(t) = g_1(\phi(t - ch))$ is strictly increasing on $(-\infty, ch)$. But then Lemma 26 implies that $\phi'(t) \le 0$ for all $t \in (0, ch]$. Here, we are using the inequalities

$$N_1(t-s) < N_1(-s) < e^{\lambda_0(\xi)t} N_1(-s),\ s \le 0 < t;\ N_1(t-s) < N_1(0) < N_1(0)e^{\lambda_0(\xi)(t-s)},\ s < t.$$

Thus $\phi'(t) \le 0$ on some maximal interval $(0, \sigma_1)$. Note that σ_1 must be a finite real number since otherwise $\phi'(t) \le 0$ on $(0, +\infty)$ implying $\phi(+\infty) = 0$. However, this contradicts the uniform persistence property of semi-wavefronts [19]. In consequence, $\sigma_1 > ch$ is finite so that $\phi'(\sigma_1) = 0$, $\phi''(\sigma_1) \ge 0$ and $\phi(\sigma_1) \le \phi(\sigma_1 - ch)$. On the other hand, we know that $\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - ch)) = 0$ for all $t \in \mathbb{R}$ so that

$$\phi''(\sigma_1) - \phi(\sigma_1) + g(\phi(\sigma_1 - ch)) = 0,$$

from which we obtain $\kappa > \phi(\sigma_1 - ch) \ge \phi(\sigma_1) \ge g(\phi(\sigma_1 - ch)) > 0$, a contradiction.

Finally, the uniqueness statement of Theorem 8 follows from Theorem 6. $\square$

6. Proof of Theorem 11

We will need the following analog of Theorem 9.

Theorem 27. *Assume* **(M)** *and* **(K)**. *Then for each point* $(h, c) \in \mathcal{D}_0^*$, *there exist* g_2 *and kernels* $N_2, w \ge 0$ *given by*

$$N_2(t) = (g'(0) - 1)\int_{\mathbb{R}} K(s)w(t-s)ds,\ g_2(s) = \frac{g'(0)s - g(s)}{g'(0) - 1},$$

such that the boundary value problem (6) has a solution if and only if equation (7) (where $j = 2$*) has a non-negative solution satisfying the boundary conditions of (6). Furthermore,*

$\int_{\mathbb{R}} N_2(s)ds = 1$ and $\int_{\mathbb{R}} N_2(s)e^{-\lambda s}ds < \infty$ for all λ from some infinite interval $(-\infty, \gamma_r^*) \ni \{0\}$. In fact, $N_2(t) = 0$ for all $t \geq -ch$. Continuous function w is C^2-smooth on $\mathbb{R}_-$ and $w(t) \equiv 0$ on $\mathbb{R}_+$.

Proof. With $\xi \in [0, 1]$, consider the linear equation

$$y''(t) - cy'(t) - y(t) + g'(0) \int_{\mathbb{R}} K(s)y(t - \xi(s + ch))ds = 0. \tag{30}$$

For $(h, c) \in \mathcal{D}_0^*$, the characteristic function for (30) has the form

$$\chi_0(z, \xi) := z^2 - cz - 1 + g'(0) \int_{-\infty}^{0} e^{-z\xi s} K(s - ch)ds.$$

Thus $\chi_0(x, \xi)$, $x > 0$, is strictly increasing with respect to $\xi \in [0, 1]$. At the same time, $\chi_0(0, \xi) \equiv g'(0) - 1 > 0$, $\chi_0''(x, \xi) > 0$, $x \in \mathbb{R}$. This implies that if $(h, c) \in \mathcal{D}_0^*$ then $\chi_0(z, \xi)$ has exactly two positive simple zeros $\mu_0(\xi) < \mu_1(\xi)$ for each $\xi \in [0, 1]$. In fact, $\mu_0(\xi), -\mu_1(\xi)$ are increasing continuous functions of $\xi \in [0, 1]$ and $\mu_0(\xi) \leq \mu_0 = \mu_0(1) < \mu_1(1) = \mu_1 \leq \mu_1(\xi) \leq \mu_1(0)$.

In the next five claims, we define the fundamental solution $w(t)$ and study its properties.

Claim I. If $(h, c) \in \mathcal{D}_0^*$ then $\chi_0(z, \xi)$, $\xi \in [0, 1]$, does not have non-real zeros z_j with the real part $\Re z_j \leq \mu_1(\xi)$.

Proof of the claim. Note that if $\chi_0(z_j, \xi) = 0$ and $\Re z_j \leq \mu_1(\xi)$ then

$$|z_j^2 - cz_j - 1| = g'(0)\left|\int_{-\infty}^{0} e^{-z_j\xi s} K(s - ch)ds\right| \leq g'(0) \int_{-\infty}^{0} e^{-\mu_1(0)s} K(s - ch)ds.$$

Therefore some compact disk in $\mathbb{C}$ (not depending on $\xi \in [0, 1]$) contains all such zeros z_j of $\chi_0(z, \xi)$. Moreover, the vertical line $\Re z = \mu_1(\xi_0)$ does not contain non-real zeros z_k of $\chi_0(z, \xi)$. Indeed, assuming that such a zero z_k exists, we immediately obtain the following contradiction:

$$|(\mu_1(\xi_0))^2 - c\mu_1(\xi_0) - 1| < |z_k^2 - cz_k - 1| = g'(0)\left|\int_{-\infty}^{0} e^{-z_k\xi_0 s} K(s - ch)ds\right|$$

$$\leq g'(0) \int_{-\infty}^{0} e^{-\mu_1(\xi_0)\xi_0 s} K(s - ch)ds = |(\mu_1(\xi_0))^2 - c\mu_1(\xi_0) - 1|.$$

In consequence, since the zeros $\mu_0(\xi), \mu_1(\xi)$ are simple, the number of zeros of $\chi_0(z, \xi)$ lying in the half-plane $\Re z \leq \mu_1(\xi)$ does not depend on ξ and is equal to 2. $\square$

In consequence, if $(h, c) \in \mathcal{D}_0^*$ then equation (30) is hyperbolic for all $\xi \in [0, 1]$ that allows us to apply Lemma 16 and define the fundamental solution $w(t, \xi)$ for (30);

$$w(t, \xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut} du}{\chi_0(iu, \xi)}, \quad t \in \mathbb{R}. \tag{31}$$

Arguing as in Claim II of Lemma 19, we obtain that the function $w(t, \xi)$ is continuous and bounded on $\mathbb{R} \times [0, 1]$. It is easy to see (cf. [20]) that

$$w(t, 0) = \frac{1}{\mu_1(0) - \mu_0(0)} \begin{cases} 0, & t \geq 0, \\ e^{\mu_0(0)t} - e^{\mu_1(0)t}, & t < 0. \end{cases}$$

In the next two claims, we extend the main properties of $w(t, 0)$ for $w(t, \xi)$ with $\xi \in [0, 1]$.

Claim II. $w(t, \xi) = 0, \ t \geq 0$, for every $\xi \in [0, 1]$. Thus $w'(0-, \xi) = -1$ so that $w(t, \xi) > 0$ for all t from some maximal non-empty set $(T_+(\xi), 0)$.

Proof of the claim. Indeed, by Lemma 16 function $w_1(t, \xi) = w(-t, \xi)$ satisfies, for all $t \neq 0$, the linear functional differential equation with infinite delay

$$y''(t) + cy'(t) - y(t) + g'(0) \int_{-\infty}^{0} K(s - ch) y(t + \xi s) ds = 0. \tag{32}$$

Consider the initial value problem

$$y(s) = y_0(s), \ s \leq 0, \quad y'(0) = b \tag{33}$$

for (32), where $(b, y_0) \in \mathcal{B}$, and the vector space

$$\mathcal{B} := \mathbb{C} \times \{\varphi : \varphi \text{ is Lebesgue measurable complex function on } \mathbb{R}_-, \int_{-\infty}^{0} |\varphi(s)|^2 e^s ds < \infty\}$$

is provided with the complete norm $|(b, \varphi)|_{\mathcal{B}} = |\varphi(0)| + |b| + \left(\int_{-\infty}^{0} |\varphi(s)|^2 e^s ds\right)^{1/2}$. Then the linear operator

$$(b, \varphi) \in \mathcal{B} \longrightarrow (-cb + \varphi(0) - g'(0) \int_{-\infty}^{0} K(s - ch) \varphi(\xi s) ds, b) \in \mathbb{C}^2$$

is bounded in virtue of assumption **(K)** and the Cauchy–Schwarz inequality. Since $\chi_0(-z, \xi)$ is the characteristic function for (32), all its zeros have negative real parts once $(h, c) \in \mathcal{D}_0^*$. Therefore [37, Theorem 4.4] guarantees that, for some universal constants $\alpha > 0$, $K > 0$,

$$|(y'(t), y(t+\cdot))|_{\mathcal{B}} \le K e^{-\alpha t} |(b, y_0(\cdot))|_{\mathcal{B}}, \; t \ge 0, \tag{34}$$

for the unique solution $y(t)$ of the initial value problem (33).

Now, clearly, $(w_1'(0-, \xi), w_1(\cdot, \xi)) \in \mathcal{B}$. As a consequence, the initial value problem (33) where $y_0(s) = w_1(s, \xi), b = w_1'(0-, \xi)$ has a unique solution $\tilde{w}_1(t, \xi)$ on $[0, +\infty)$. In view of (34), $\tilde{w}_1$ is bounded on $[0, +\infty)$. Hence, the function

$$\hat{w}_1(t, \xi) = \begin{cases} w_1(t, \xi), & t \le 0, \\ \tilde{w}_1(t, \xi), & t > 0, \end{cases}$$

belongs to the space $C^2(\mathbb{R})$ and satisfies (32) for all $t \in \mathbb{R}$. Because of (34), this is possible only when $\hat{w}_1(t, \xi) \equiv 0$. $\square$

Claim III. For each non-negative $\xi_0 \in [0, 1]$ there exist real number ν, a neighborhood $\mathcal{O} \ni \xi_0$ and positive constant M such that, for all $\xi \in \mathcal{O}$,

$$\nu > \sup_{\xi \in \mathcal{O}} \mu_1(\xi), \; w(t, \xi) = \rho_0(\xi) e^{\mu_0(\xi)t} - \rho_1(\xi) e^{\mu_1(\xi)t} + r(t, \xi), \; |r(t, \xi)| \le M e^{\nu t}, \; t \le 0,$$

where $\rho_j(\xi) = (-1)^{j+1} / \chi_0'(\mu_j(\xi), \xi) > 0, \; j = 0, 1$, depend continuously on ξ.

Proof of the claim. Clearly, we can choose a small neighborhood $\mathcal{O} \ni \xi_0$ and $\nu > \sup_{\xi \in \mathcal{O}} \mu_1(\xi)$ sufficiently close to $\mu_1(\xi_0)$ and such that the vertical strip $\{z : \mu_1(\xi) < \Re z < \nu\}$ does not contain any zero of $\chi_0(z, \xi)$ when $\xi \in \mathcal{O}$. Then, after moving the integration path in the inversion formula (16) from $\Re z = 0$ to $\Re z = \nu$, we obtain that, for $t \le 0$,

$$\begin{aligned} w(\xi, t) &= -\frac{e^{\mu_0(\xi)t}}{\chi_0'(\mu_0(\xi), \xi)} - \frac{e^{\mu_1(\xi)t}}{\chi_0'(\mu_1(\xi), \xi)} + \frac{1}{2\pi i} \int_{\nu - i\cdot\infty}^{\nu + i\cdot\infty} \frac{e^{tz} dz}{\chi_0(z, \xi)} \\ &= \rho_0(\xi) e^{\mu_0(\xi)t} - \rho_1(\xi) e^{\mu_1(\xi)t} + \frac{e^{\nu t}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ist} ds}{\chi_0(\nu + is, \xi)} =: e_0(t) - e_1(t) + e^{\nu t} q(t, \xi), \end{aligned}$$

where $q(-\infty, \xi) = 0$ and

$$|q(t, \xi)| \le M = \sup_{\xi \in \mathcal{O}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{ds}{|\chi_0(\nu + is, \xi)|}. \quad \square$$

Claim III implies that exponentially decaying function $w(t, \xi)$, $\xi \in \mathcal{O}$, is positive at $-\infty$. In particular, there exists the rightmost point $T_-(\xi) \le 0$ such that $w(t, \xi) > 0$ for all $t < T_-(\xi)$. Recall that $T_+(\xi) < 0$ denotes the leftmost point such that $w(t, \xi) > 0$ for all $t \in (T_+(\xi), 0)$. In particular, $T_-(0) = 0$, $T_+(0) = -\infty$, $w(T_-(\xi), \xi) = 0$.

Claim IV. Function $T_-(\xi)$ is bounded on $[0, 1]$.

Proof of the claim. It is enough to prove that $T_-(\xi)$ is locally bounded, so that we will consider $T_-(\xi)$ for $\xi \in \mathcal{O}$ where $\mathcal{O}$ was defined in Claim III. We have that

$$0 = w(T_-(\xi), \xi) \geq \rho_0(\xi)e^{\mu_0(\xi)T_-(\xi)} - (\rho_1(\xi) + M)e^{\mu_1(\xi)T_-(\xi)}, \ \xi \in \mathcal{O},$$

and therefore, for all $\xi \in \mathcal{O}$,

$$T_-(\xi) \geq \frac{1}{\mu_1(\xi) - \mu_0(\xi)} \ln \frac{\rho_0(\xi)}{\rho_1(\xi) + M} \geq \frac{1}{\mu_1(1) - \mu_0(1)} \inf_{\xi\in\mathcal{O}} \ln \frac{\rho_0(\xi)}{\rho_1(\xi) + M}. \quad \square$$

Claim V. $w(t, \xi) > 0$ for all $t < 0$, $\xi \in [0, 1]$.

Proof of the claim. Indeed, otherwise there exists $\xi_c \in (0, 1]$ such that $w(t, \xi) > 0$, $t < 0$, for all $\xi \in (0, \xi_c)$ and $w(t_c, \xi_c) = 0$ at its rightmost zero t_c from $(-\infty, 0)$ (since $w(t, 0) > 0$ for $t < 0$, such ξ_c is well defined due to Claims III and IV). Clearly, $T_-(\xi_c) \leq t_c < 0$ and $w'(t_c, \xi_c) = 0$, $w''(t_c, \xi_c) \geq 0$. Thus, in view of equation (30),

$$0 \geq \int_{\mathbb{R}} K(s)w(t_c - \xi_c(s + ch), \xi_c)ds = \int_{t_c/\xi_c}^{0} K(s - ch)w(t_c - \xi_c s, \xi_c)ds \geq 0.$$

Since $w(t, \xi_c) > 0$ for all $t \in (t_c, 0)$, we obtain that $K(t - ch) = 0$ almost everywhere on $[t_c/\xi_c, 0]$. In this way,

$$\int_{\mathbb{R}} K(s)w(t - \xi_c(s + ch), \xi_c)ds = \int_{t/\xi_c}^{0} K(s - ch)w(t - \xi_c s, \xi_c)ds = 0 \text{ for } t \in [t_c, 0].$$

Therefore

$$w''(t, \xi_c) - cw'(t, \xi_c) - w(t, \xi_c) = 0 \text{ for all } t \in [t_c, 0].$$

Since $w(t_c, \xi_c) = w'(t_c, \xi_c) = 0$, in view of the uniqueness theorem we can conclude that $w(t, \xi) = 0$ for all $t \in [t_c, 0]$, a contradiction. $\square$

Hence, the function $w(t) := w(t, 1)$ has all properties mentioned in the statement of the theorem and it is the required fundamental solution for equation (30). In particular, $\int_{\mathbb{R}} w(t)dt = (\chi_0(0, c, h))^{-1} = (g'(0) - 1)^{-1}$. The properties of $N_2(t)$ are now obvious, note here that

$$N_2(t) = (g'(0) - 1) \int_{t}^{-ch} K(s)w(t - s)ds.$$

Finally, arguing as in Corollary 15, we find the boundary value problem (6) has a solution if and only if equation (7) has a non-negative solution satisfying the boundary conditions of (6). Theorem 27 is proved. □

In order to complete the proof of Theorem 11, one might try, as before, to apply the general existence theory developed for convolution equations in [19]. However, a straightforward application of [19, Theorem 7] to equation (7) with $j = 2$ (which was successfully realized for the case $j = 1$ in the proof of Theorem 5) is not possible now because it holds $g_2'(0) = 0$ instead of the required inequality $g_2'(0) > 1$. In particular, such a 'degeneracy' of g_2 at 0 implies that the well known sub- and super-solutions $\phi^-(t),\ \phi_0(t)$ indicated in Remark 23 cannot be used in the case when $j = 2$ in (7). Exactly the same situation happened in the studies of the KPP-Fisher equations (3), (4) (where also the non-negative fundamental solution was used). We recall that, for these equations, new sub- and super-solutions were constructed in [16,20] by gluing together pieces of non-oscillating eigenfunctions for linearizations of (3), (4) at the steady states 0 and 1. Since the approach of [16,20] is more technically involved than the rather standard method of [19], in this paper we propose the following simple alternative idea.

Fix some $(h, c) \in \mathcal{D}_0^*$ and realize the change of variables $y(t) = \kappa - \psi(-t)$ in (6). We will obtain the boundary value problem $\psi(-\infty) = 0,\ \psi(+\infty) = \kappa,\ \psi(t) \le \kappa$ for equation

$$\psi''(t) + c\psi'(t) - \psi(t) + \int_{\mathbb{R}} K(s - ch)(\kappa - g(\kappa - \psi(t + s)))ds = 0. \tag{35}$$

Let $w_-(t)$ be the fundamental solution for

$$\psi''(t) + c\psi'(t) - \psi(t) + g'(0) \int_{\mathbb{R}} K(s - ch)\psi(t + s)ds = 0,$$

then

$$w_-(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iut}du}{\chi_0(-iu, 1)} \equiv w(-t), \quad t \in \mathbb{R},$$

cf. (31). In consequence, arguing as in Corollary 15, we obtain that

$$\psi(t) = \int_{\mathbb{R}} N_3(s)g_3(\psi(t - s))ds,$$

where

$$N_3(t) = (g'(0) - 1) \int_{\mathbb{R}} w(-s)K(s - t - ch)ds, \quad g_3(u) = \frac{g'(0)u + g(\kappa - u) - \kappa}{g'(0) - 1}.$$

Observe that $N_3(t) \ge 0,\ t \in \mathbb{R},\ N_3(t) = 0,\ t \le 0,\ \int_{\mathbb{R}} N_3(s)ds = 1,\ g_3(0) = 0,\ g_3(\kappa) = \kappa,$

$$g_3'(0) = \frac{g'(0) - g'(\kappa)}{g'(0) - 1} > 1, \ g_3'(\kappa) = 0, \ g_3(u) > u, \ u \in (0, \kappa),$$

and $g_3(u)$ is non-decreasing on $[0, \kappa]$ in view of **(ST$_*$)**. Since $g(u)$ is sub-tangential at κ, we find that $g_3(u)$ is sub-tangential at 0 (i.e. $g_3(u) < g_3'(0)u, \ u \in (0, \kappa)$). Furthermore, the function

$$\chi_1^*(z) := 1 - g_3'(0) \int_{\mathbb{R}} N_3(s) e^{-sz} ds = 1 - (g'(0) - g'(\kappa)) \frac{\int_{\mathbb{R}} K(-s - ch) e^{-sz} ds}{\chi_0(-z, c, h)} = \frac{\chi_\kappa(-z, c, h)}{\chi_0(-z, c, h)}$$

is analytic in the half-plane $\{\Re z > -\mu_0\}$ and satisfies

$$\chi_1^*(0) = \frac{g'(\kappa) - 1}{g'(0) - 1} < 0, \quad \chi_1^*(-\lambda_1(g'(\kappa))) = 0.$$

In consequence, in order to complete the proof of Theorem 11, it suffices to apply Theorem 7 from [19] and argues as in the proof of Theorem 5. □

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