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A DIRECT PROOF OF THE EXISTENCE OF PURE STRATEGY EQUILIBRIA IN LARGE GENERALIZED GAMES WITH ATOMIC PLAYERS

ALVARO J. RIASCOS VILLEGAS AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. Consider a game with a continuum of players where only a finite number of them are atomic. Objective functions and admissible strategies may depend on the actions chosen by atomic players and on aggregate information about the actions chosen by non-atomic players. Only atomic players are required to have convex sets of admissible strategies and quasi-concave objective functions. In this context, we prove the existence of pure strategy Nash equilibria, a result that extends Rath (1992, Theorem 2) to generalized games and gives a direct proof of a special case of Balder (1999, Theorem 2.1). Our proof has the merit of being simple, based only on standard fixed point arguments and finite dimensional real analysis.

Keywords. Generalized games, Non-convexities, Pure-strategy Nash equilibrium.
JEL Classification numbers. C72, C62.

1. Introduction

In a seminal paper, Schmeidler (1973) proved that in non-convex games with a continuum of players the set of pure strategy equilibria is non-empty provided that (i) all agents are non-atomic, and (ii) objective functions depends only on their own strategy and on the average of the actions chosen by the other players. Essentially, this last assumption convexify the game, because the integral of any correspondence is a convex set (Aumann (1964)).

In this paper, we extend Schmeidler's result to large generalized games with a finite number of atomic players. In our framework, both objective functions and admissible strategies may depend

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on the strategies of atomic players and on messages which aggregate information about strategies chosen by non-atomic players (i.e., not necessarily on the average of these actions). By extending the proof given by Rath (1992, Theorem 2) of Schmeidler (1973) classical result, we provide a short and direct proof of the existence of a pure Nash equilibria in our generalized game, without need to purify a mixed strategy equilibrium. Our theorem is a special case of Balder (1999, Theorem 2.1) but his prove use extensively functional analysis. Thus, one of the merits of our proof is its simplicity, because it is based only on standard fixed point arguments and finite dimensional real analysis. Furthermore, our theorem is general enough to cope with interesting applications.

A natural application of our result is to general equilibrium theory. Essentially, to prove equilibrium existence it is usual to find bounds on endogenous variables and search an equilibrium allocation as an equilibrium in an abstract generalized game. In this type of generalized game, consumers and firms maximize their objective functions taking prices as given, and there are atomic players that determine prices, asset returns, taxes or any other endogenous variable that are taken as given by consumers or firms. Thus, for equilibrium models where agents have non-convex choice sets or their objective function are not necessarily quasi-concave, our main result may help researcher to find an equilibrium.

The rest of the paper is organized as follows: in Section 2 we present our non-convex large generalized game and we prove the existence of a pure strategy Nash equilibrium. In Section 3, we discuss the relation to the existing literature.

2. Pure strategy equilibria in a non-convex generalized game

Let $\mathcal{G}(T,(K_t,\Gamma_t,u_t)_{t\in T},h)$ be a generalized game with an infinite set of players $T = T_1 \cup T_2$, where $T_1 \subset \mathbb{R}$ is a compact finite measure continuum set of non-atomic players with respect to Lebesgue measure λ,[1] and T_2 is a finite set of atomic players. Each player $t \in T_1$ has a compact and non-empty action space $K_t \subset \widehat{K}$, where $\widehat{K} \subset \mathbb{R}^n$ is a compact set and $\bigcap_{t\in T_1} K_t \neq \emptyset$. On the other hand, each player $t \in T_2$ has a compact, convex and non-empty action space $K_t \subset \mathbb{R}^{n_t}$, with $n_t \in \mathbb{N}$.[2]

A *profile of actions* for players in T_1 is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Since T_2 is finite, a *profile of actions* for the players in T_2 is a vector $a := (a_i; i \in T_2) \in \Pi_{t\in T_2}\mathbb{R}^{n_t}$ such that $a_t \in K_t$, for any $t \in T_2$. Let $\mathcal{F}(T_i)$ be the space of all profiles of actions of agents in T_i, with $i \in \{0,1\}$. Also, given $t \in T_2$, let $\mathcal{F}_{-t}(T_2)$ be the set of profiles $a_{-t} := (a_j; j \in T_2 \setminus \{t\})$ of actions take by players $j \in T_2 \setminus \{t\}$.

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[1]In other words, $(T_1, \mathbb{B}(T_1), \lambda)$ is a measure space, where $\mathbb{B}(T_1)$ is the $\sigma-$ algebra of Borel sets of T_1.

[2]Through the article, we assume that Euclidean spaces are endowed with the topology induced by the Euclidean norm and, therefore, a set is compact if and only it is bounded and closed in this topology.

In $\mathcal{G}(T,(K_t,\Gamma_t,u_t)_{t\in T},h)$ players do not necessarily advance the actions chosen by players in T_1 or, alternatively, players do not necessarily take into account the actions of each non-atomic player to determine an optimal response. However, players will considerate, at the moment in which they make a decision, aggregated information of some characteristics of these actions. Formally, if agents $t \in T_1$ choose a profile of actions $f \in \mathcal{F}(T_1)$, then we assume that the relevant characteristics of this actions are coded by a continuous function $h : T_1 \times \widehat{K} \to \mathbb{R}^l$. Furthermore, each player in T will only take into account, for strategic purposes, aggregated information about these available characteristics through the message $m(f) = \int_{T_1} h(t,f(t))d\lambda$.

Since we want to concentrate in actions profiles for which messages are well defined, we say that f is a *strategic profile* of players in T_1 if both $f \in \mathcal{F}(T_1)$ and $h(\cdot,f(\cdot))$ is a measurable function from T_1 to $\mathbb{R}^l$.[3] Measurability restrictions are not necessary over the behavior of atomic players. For this reason, the set of strategic profiles of players in T_2 coincides with the space of profiles of actions $\mathcal{F}(T_2)$.

The set of messages associated to strategic profiles of non-atomic players is given by

$$M = \left\{ \int_{T_1} h(t,f(t))d\lambda : \quad f \in \mathcal{F}(T_1) \ \wedge \ h(\cdot,f(\cdot)) \text{ is measurable} \right\} \subset \mathbb{R}^l,$$

which is non-empty since $\bigcap_{t\in T_1} K_t$ is a non-empty set and h is a continuous function. Also, since the sets $\widehat{K}$ and T_1 are compact, for any profile of actions $f : T_1 \to \widehat{K}$, the function $h(\cdot,f(\cdot)) : T_1 \to \mathbb{R}^l$ is bounded. Thus, if it is measurable then it is integrable. For these reason, in the definition of the set of messages M we only require measurability of $h(\cdot,f(\cdot))$.

In our game, the messages about the strategic profiles of players in T_1 jointly with the strategic profiles of players in T_2 may restrict the set of admissible strategies available for a player $t \in T$. That is, given a vector $(m,a) \in M \times \mathcal{F}(T_2)$ the strategies available for a player $t \in T_1$ are given by a set $\Gamma_t(m,a) \subset K_t$, where $\Gamma_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty and compact values. Analogously, given $(m,a_{-t}) \in M \times \mathcal{F}_{-t}(T_2)$, the set of strategies available for a player $t \in T_2$ is $\Gamma_t(m,a_{-t}) \subset K_t$, where $\Gamma_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty, compact and convex values. We refer to correspondences $(\Gamma_t; t \in T)$ as correspondences of admissible strategies.

Given a set $A \subset \mathbb{R}^k$, let $\mathcal{U}(A)$ be the collection of continuous functions $u : A \to \mathbb{R}$. Assume that $\mathcal{U}(A)$ is endowed with the sup norm topology. We suppose that each player $t \in T_1$ has an objective function $u_t \in \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$ and each player $t \in T_2$ has an objective function $u_t \in \mathcal{U}(M \times \mathcal{F}(T_2))$ which we assume is quasi-concave in its own strategy. Finally, we assume that

[3] In Schmeidler (1973) and Rath (1992), players are non-atomic and take into account only the average of actions chosen by the others players. Thus, following our notation, $l = n$ and $h(t,x) = x$. Therefore, they define strategic profiles as measurable functions from the set of players to the set of actions.

the mapping $U : T_1 \to \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$ defined by $U(t) = u_t$ is measurable.[4]

Definition. *A pure strategy Nash equilibrium of the generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t\in T}, h)$ is given by strategic profiles $(f^*, (a_t^*; t \in T_2))$ such that, for any non-atomic player $t \in T_1$,*

$$u_t(f^*(t), m(f^*), a^*) \geq u_t(f(t), m(f^*), a^*), \quad \forall f(t) \in \Gamma_t(m^*, a^*),$$

and for any atomic player $t \in T_2$, $u_t(m(f^*), a^*) \geq u_t(m(f^*), a_t, a_{-t}^*)$, $\forall a_t \in \Gamma_t(m^*, a_{-t}^*)$, where the message $m(f^*) := \int_{T_1} h(t, f^*(t))d\lambda$ belongs on the set M.

Note that, in our definition of Nash equilibrium, every agent maximize his objective function, while in Balder (1999) and Rath (1992) in equilibrium almost everyone maximizes. However, since objective functions are continuous and action spaces compact, given an equilibrium for any of the games studied in these articles, it is always possible to change the allocations associated to the set of non-atomic players that do not maximizes, give to each of them an optimal plan, without changing the integrability of the action profile or the value of messages. Thus, Theorem 2 in Rath (1992) and Theorem 2.1 in Balder (1999) assure the existence of Nash equilibria where each player maximize.

Theorem. *Consider a generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t\in T}, h)$ where,*

1. *The set of players is $T_1 \cup T_2$, where T_1 is a compact finite measure set of non-atomic players and T_2 is a finite set of atomic players.*
2. *For any $t \in T$, action spaces K_t are non-empty and compact, correspondences of admissible strategies Γ_t are continuous and have non-empty and compact values, and objective functions u_t are continuous.*
3. *Each atomic player has a convex set of actions, a convex-valued correspondence of admissible strategies, and an objective function which is quasi-concave on its own strategy.*
4. *There exists a compact set $\widehat{K}$ such that, for any $t \in T_1$, $K_t \subset \widehat{K}$ and $\bigcap_{t\in T_1} K_t$ is non-empty.*
5. *The function $h : T_1 \times \widehat{K} \to R^l$ is continuous.*
6. *The mapping $U : T_1 \to U(\widehat{K} \times M \times F(T_2))$, which associates with any $t \in T_1$ the objective function u_t, is measurable.*

Then, there exists a pure strategy Nash equilibrium.

Proof. We divide the proof in five steps.

[4]Suppose that there is a finite number of types on the set of non-atomic agents, T_1. That is, there is a finite partition of T_1 into Lebesgue measurable sets $\{I_1, \ldots, I_r\}$ such that, two players t and t' are identical if belongs into the same element of the partition. In this case, the restriction about measurability of U is trivially satisfied.

(1) *The space of messages $M \subset \mathbb{R}^l$ is non-empty, compact and convex.*

As we remark in the previous section, M is non-empty as a consequence of the fact that $\bigcap_{t\in T_1} K_t \neq \emptyset$. Essentially, if we fix $k \in \bigcap_{t\in T_1} K_t$, the function $g : T_1 \to \widehat{K}$ defined by $g(t) = k$ for any $t \in T_1$ belongs to $\mathcal{F}(T_1)$ and $h(\cdot, g(\cdot))$ is trivially measurable. Therefore, $\int_{T_1} h(t, g(t))d\lambda$ is well defined and M is a non-empty subset of $\mathbb{R}^l$.

The set M is convex because the integral of a correspondence in a non-atomic measurable space is a convex set (see Aumann (1964)): consider the correspondence $Q : T_1 \twoheadrightarrow \mathbb{R}^l$ defined by $Q(t) = h(t, K_t)$, for any $t \in T_1$. Then $M = \int\limits_{T_1} Q(t)d\lambda$ and, therefore, M is convex.[5]

Let $\widetilde{Q} : T_1 \twoheadrightarrow \mathbb{R}^l$ be the correspondence defined by $\widetilde{Q}(t) = h(T_1, \widehat{K})$, for any $t \in T_1$. Then $M = \int\limits_{T_1} Q(t)d\lambda \subset \int\limits_{T_1} \widetilde{Q}(t)d\lambda = \text{convexhull}(h(T_1, \widehat{K}))$. Therefore, since h is continuous, M is a subset of a compact set. Thus, it remains to prove that M is closed. Let $\{m_k\}_{k\in\mathbb{N}} \subset M$ be a sequence that converges to a vector $m \in \mathbb{R}^l$. Since $m_k \in M$, $m_k = \int\limits_{T_1} h_k(t)d\lambda$, where $h_k : T_1 \to \mathbb{R}^l$ is a mesurable function and $h_k = h(\cdot, f_k(\cdot))$ for some $f_k \in \mathcal{F}(T_1)$. For each t, $\{h(t, f_k(t))\}_{k\in\mathbb{N}} \subset Q(t)$, which is a compact set. Thus, every limit point of $\{h_k(t)\}_{k\in\mathbb{N}}$ is contained in $Q(t)$. Also, since h is continuous, T_1 is compact and $\cup_{t\in T_1} K_t \subset \widehat{K}$, it is easy to see that $\{h_k\}_{k\in\mathbb{N}}$ is uniformly bounded by an integrable function. By Aumann (1976), it follows that the limit point of $\int\limits_{T_1} h_k(t)d\lambda$ belongs to $\int\limits_{T_1} Q(t)d\lambda$. Therefore, the space of messages is compact.

(2) *Best-reply correspondences are closed with non-empty and compact values.*

For any $t \in T_1$, define the best-reply correspondence $B_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ as

$$B_t(m, a) = \text{argmax}_{f(t)\in\Gamma_t(m,a)} u_t(f(t), m, a).$$

Analogously, for any atomic player $t \in T_2$ the best-reply correspondence $B_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is defined by $B_t(m, a_{-t}) = \text{argmax}_{a_t\in\Gamma_t(m,a_{-t})} u_t(m, a_t, a_{-t})$.

It follows that, under conditions of item (i) in the statement of the Theorem, and as a consequence of the Berge's Maximum Theorem, best-reply correspondences have closed graph and non-empty compact values.

(3) *For any atomic player $t \in T_2$, his best-reply correspondence has convex values.*

A direct consequence of item (ii) in the statement of the Theorem.

(4) *The correspondence $\Omega : M \times \mathcal{F}(T_2) \twoheadrightarrow M$ defined by $\Omega(m, a) = \int_{T_1} h(t, B_t(m, a))d\lambda$ is closed and has non-empty and convex values.*

[5]This follow immediately from the definition of integral of a correspondence and the fact that we do not require action profiles to be measurable.

Given $(m, a) \in M \times \mathcal{F}(T_2)$, and by an identical argument to those made by Rath (1992, pages 430-431), there exists a measurable function $f \in \mathcal{F}(T_1)$ such that $f(t) \in B_t(m, a)$ for any $t \in T_1$. Thus, since h is continuous, $h(\cdot, f(\cdot))$ is measurable and, therefore, Ω has non-empty values.

The correspondence Ω has convex values, since for any $(m, a) \in M \times \mathcal{F}(T_2)$, the set $\Omega(m, a)$ is the integral of the correspondence $t \twoheadrightarrow h(t, B_t(m, a))$.

Since for any $t \in T_1$, the best-reply correspondence B_t has closed graph, for any $t \in T_1$ the correspondence that associate to each $(m, a) \in M \times \mathcal{F}(T_2)$ the set $h(t, B_t(m, a))$ has also closed graph (a direct consequence of the continuity of the function h and the fact that $B_t(m, a) \subset \widehat{K}$ for any $(t, m, a) \in T_1 \times M \times \mathcal{F}(T_2)$). On the other hand, since T_1 and $\widehat{K}$ are compact and h continuous, there exists a bounded function $v : T_1 \to \mathbb{R}^l$ such that $-v(t) \leq h(t, f(t)) \leq v(t)$, for any $t \in T_1$, $f \in \mathcal{F}(T_1)$ and $\int_{T_1} v(t) d\lambda$ is finite. Therefore, the correspondence that associate to each $(m, a) \in M \times \mathcal{F}(T_2)$ the integral on T_1 of the correspondence $t \twoheadrightarrow h(t, B_t(m, a))$ has closed graph (a consequence of the main result in Aumann (1976)). In other words, Ω is closed.

(5) *The generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ has a pure strategy Nash equilibrium.*

Define the correspondence $\Gamma : M \times \mathcal{F}(T_2) \twoheadrightarrow M \times \mathcal{F}(T_2)$ by $\Gamma(m, a) = (\Omega(m, a), (B_t(m, a_{-t}))_{t \in T_2})$. Then Γ is closed and has nonempty, convex and compact values. Therefore, applying Kakutani's Fixed Point Theorem, we conclude that Γ has a fixed point, i.e. there exists $(m^*, a^*) \in M \times \mathcal{F}(T_2)$ such that $(m^*, a^*) \in \Gamma(m^*, a^*)$.

Thus, for some $f^* \in \mathcal{F}(T_1)$, $m^* = \int_{T_1} h(t, f^*(t)) d\lambda$ and $f^*(t) \in B_t(m^*, a^*)$, for any $t \in T_1$. Also, for any $t \in T_2$, $a_t^* \in B_t(m^*, a_{-t}^*)$. These properties assure that (f^*, a^*) constitutes a pure strategy Nash equilibrium of the generalized game. □

3. Discussion of related literature

Rath (1992) main result on games with compact action spaces (Theorem 2, page 430) is an elementary prove of Schmeilder (1973) classical result. On the other hand, Balder (1999, Theorem 2.1, page 212) is a generalization of Rath (1992) to generalized games. Our theorem is a special case of Balder (1999) but it still extend Rath (1992) on some dimensions, because we consider generalized games, in with admissible strategies and objective functions may depend on players actions and, also, we allow for atomic players.

There are generalizations of our theorem that are quite straightforward but we think they would obscure the elementary nature of our prove. For example, we could avoid assuming actions spaces share a common action for every agent (i.e., $\bigcap_{t \in T_1} K_t \neq \emptyset$) and rather use an argument along Remark 8 in Rath (1992, page 432). Similar arguments to Remark 6 in Rath's article would allow us to avoid fixing a topology in $\mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$. On the other hand, we could also relax substantially the hypothesis of our coding function h. In particular, as in Balder (1999), we could assume that h is a vector valued function of Carathéodory functions.

Finally, we should stress that the most important difference of our theorem with Balder's Theorem 2.1 is the fact that we assume sets of strategic profiles to be integrable bounded codifications of actions profiles (i.e., are integrable functions—obtained by the codification of actions profiles—with respect to a finite measure space). Integrability of action profiles is something which is clearly an ungrounded hypothesis in many applications making necessary to bound actions, prove equilibrium existence and arguing somehow that bound are innocuous (for example, by constructing a sequence of equilibriums for less stringent bounds and then arguing that this sequence has a convergent subsequence that is also an equilibrium).

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