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Ambiguity and Long-Run Cooperation in Strategic Games

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Ambiguity and Long-Run Cooperation in Strategic Games

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Abstract

This paper studies the effects of ambiguity on long-run cooperation in infinitely repeated games with strategic players. Using a neo-additive capacities framework, which allows us to work with a utility function that parametrically captures the degree of ambiguity, we determine a critical condition under which players can cooperate in equilibrium. Then, this result is applied to canonical problems of strategic interaction and potential cooperation: the Prisoner's Dilemma and the Cournot and Bertrand duopoly models. The application leads to two main conclusions. First, ambiguity may alter the game structure to schemes where seeking conditions to sustain long-run cooperative agreements stops being desirable. In these cases, non-cooperation is more profitable in expected terms and is achievable as a short-run Nash equilibrium. This happens for parametric combinations usually characterized by large levels of ambiguity. Second, in cases where cooperation between individuals is still desirable, the critical discount factor needed to sustain the equilibrium can vary in very non-trivial ways with the ambiguity parameters. In some cases, games may not accept a feasible discount factor consistent with a cooperative equilibrium, even when the expected payoff of cooperating is larger.

Keywords: Ambiguity, Strategic Games, Long-Run Cooperation, Infinitely Repeated Prisoner's Dilemma, Cournot Duopoly, Bertrand Duopoly.

1. Introduction

Why individuals cooperate is a question that has been widely studied by several disciplines (Nowak and Highfield, 2011). In economics, a particular interest has been put on determining conditions under which cooperative behaviors may arise as a result of the interaction between strategic individuals. Depending on the game structure, cooperation could or could not be a potential equilibrium and, when being, conditions required for implementation may vary depending on the setting considered.

In this literature, repeated games are particularly interesting, as the possibility of generating long-run relationships between individuals may induce the implementation of sustained cooperative strategies even though non-cooperative behaviors are optimal from a static point of view (see Mailath and Samuelson, 2006). The canonical example in this regard is the Prisoner's Dilemma

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(henceforth, PD). In its static version, cooperation is dominated by defection. Nevertheless, in the infinitely repeated PD cooperation may arise as an equilibrium if individuals are patient enough, as long-run payoffs for sustained cooperation may offset short term gains for defecting. This line of reasoning has also being used to study the conditions under which, in markets characterized by imperfect competition, firms decide whether or not to sustain long-run collusive agreements.

This paper revisits the standard analysis of long-run cooperation in strategic games in a simple ambiguity setting.[2] We alter assumptions about individuals' decision-making by allowing strategic players to have ambiguous beliefs about their counterparts' actions. Given that, individuals partially distrust their predictions derived from the game's equilibrium and assign some degree of probability to potential deviations of their counterparts from the expected behavior. These potential deviations from expected rational behavior can be motivated in several ways. Players could doubt about their counterpart's rationality, could suspect about the other player's intentions, or could simply internalize that they could eventually make mistakes when implementing the optimal strategy. Generally speaking, individuals distrust their counterparts' behavior just because they do not feel they know enough about each other in order to fully trust in rational strategic behaviors.

Some developments in the ambiguity literature offer intuitive and simple frameworks to include these concerns in a standard model. In particular, we follow Eichberger, Kelsey, and Schipper (2009) strategy to model ambiguity in strategic games built on Chateauneuf, Eichberger, and Grant (2007) framework based on neo-additive capacities. In this model, individuals play a game that has well-behaved predictions about the counterpart's behavior in the absence of ambiguity, i.e game's equilibrium derives a probability distribution defined over the other player actions. Nevertheless, individuals may partially distrust this probability distribution. The degree of distrust an individual has over the probability distribution is understood as the *degree of ambiguity* the individual faces: the more ambiguous the situation is, the less weight the individual puts on the original probability distribution to make decisions. The remaining weight is mapped to the best or worst possible outcomes depending on the player's attitude towards ambiguity. If the individual is *optimistic* (*pessimistic*), the unassigned probability is assigned to the best (worst) possible outcome, i.e. to the other player's action that is more beneficial (harmful) for the individual.[3]

For our purpose, we extend Eichberger, Kelsey, and Schipper (2009) framework to infinitely repeated strategic games accounting for beliefs' updating concerns. Then, we look for conditions under which cooperative equilibria may arise in the long-run in cases where there exists short-run incentives to deviation. In particular, we characterize a condition for the critical discount factor needed for cooperation in the infinitely repeated game. This condition is a function of the game's expected payoffs and the individual's beliefs about the counterpart behavior, which in turn depend on the ambiguity level and the individual's attitude towards ambiguity. After formalizing our framework, we illustrate our results by revisiting two canonical examples in our ambiguity setting: the possibility of cooperation in the infinitely repeated PD, and the possibility of collusion

[2]In simple words, ambiguity extends the notion of risk by stating that not only the realization of different states of nature is unknown, but also the probabilities assigned to their realization. For a survey, see Etner, Jeleva, and Tallon (2012). Ambiguity emerged as an important topic in decision theory as it has been able to explain some facts that the standard theory has failed to (see, for example, Ellsberg, 1961, and Chen and Epstein, 2002).

[3]Throughout the paper, *pessimism* refers to ambiguity aversion.

in infinitely repeated duopoly models, specifically under Cournot and Bertrand competition. By analyzing these examples, we find that this simple and intuitive ambiguity perturbation may have strong implications on the possibility of sustaining cooperative agreements.

Applications display two main results. First, ambiguity may alter the game's structure to schemes where seeking for conditions to sustain long-run cooperative agreements stops being desirable. This may happen because of two different reasons. Given games with effective payoffs coherent with long-run cooperation desirability, expected payoffs induced by ambiguity can change in such a way that the non-cooperative equilibrium becomes Pareto-superior. In other words, it may no longer be interesting to ask for conditions for cooperating in the long-run since non-cooperation is more profitable in expected terms and is achievable as a short-run Nash equilibrium. As it is explained later with detail, this happens in some parametrizations where marginal increasings in ambiguity lower cooperation expected payoffs (since individuals anticipate potential harmful deviations) and increase non-cooperation expected payoff (since individuals anticipate potential benefitial deviations). As it is illustrated in the PD analysis, for some plausible exogenous payoffs, this feature may characterize a sizeable portion of the ambiguity and optimism parameter spaces. The other explanation comes from the fact that when payoffs are endogenous (i.e. depend on players' actions), as it is the case of the duopoly models, ambiguity may affect players' decisions in ways that lead to game structures different from the standard PD payoff ordering. As it is debated later, the analysis regarding cooperation in these new games becomes far more complex than the standard exercise.[4]

The second main result of the paper is that in cases where looking to implement a cooperative equilibrium is still relevant (i.e. in cases where the game structure is unchanged), the discount factor needed to sustain the equilibrium can vary in very non-trivial ways with both the ambiguity and optimism parameters. In most of the analyzed cases, there exists a positive relationship between ambiguity and the critical discount factor, being ambiguity detrimental for long-run cooperation (more patient players are needed for sustaining cooperative agreements). While this seems intuitive, a striking issue arises in this regard. For some parametric combinations, under the (standard) assumptions considered, games may not accept a feasible discount factor that ensures the possibility of agreeing on a cooperative equilibrium (i.e. the discount factor is required to be larger than one). That is, even when the expected payoff of cooperating is larger than the expected payoff of the non-cooperative equilibrium, ambiguity may erode the possibility of achieving mutually beneficial agreements (in effective, not expected, terms).

Although our results are not general, since some structure is imposed in order to derive our results, their implications are still of high relevance. We illustrate how ruling out ambiguity in the analyisis of strategic games will not generally be without loss of generality. Recall that the degree of ambiguity an individual faces can be understood as a situation-specific parameter (i.e. it can vary due to external reasons).[5] Then, in cases where cooperation between individuals is desirable, our

[4]For pessimistic firms facing *high* levels of ambiguity, short term deviation incentives are replaced for low-payoff multiple equilibria games. In limit cases with positive marginal costs, the declining of payoffs may lead firms to optimally not participate in the market.

[5]This is not necessary true regarding the attitude towards ambiguity, which usually is understood as an individual-specific parameter.

analysis stresses the importance of lowering the degree of ambiguity between individuals in order to increase the likelihood of cooperation (or, in more extreme cases, to make cooperation at least feasible). The converse is also true: in the duopoly models, when cooperation is not desirable, our analysis sheds light about firms' behavior and potential antitrust policies. In general, increasing the degree of ambiguity firms face (for example, through information disclosure regulations or restrictions in communication between firms) will lower the likelihood of achieving tacit collusions. This is consistent with the findings in Kandori and Matsushima (1998), Athey and Bagwell (2001) and Fonseca and Normann (2012) findings.

This paper contributes to three branches of the literature. First, the analysis of the potential achievement of cooperative equilibria in repeated games have been developed in incomplete information settings, for example, when the counterpart's *type* is uncertain or when there exists imperfect monitoring.[6] In this context, we contribute to this literature by proposing a framework to analyze the likelihood of cooperative equilibria when players have ambiguous beliefs on counterpart's actions. Second, this paper contributes to the understanding of the effect of ambiguity in strategic games. This strand of the literature has formalized some notions of ambiguity in strategic settings and has carried out different applications to analyze whether ambiguity may alter standard results.[7] We broaden the analysis by studying the likelihood of cooperation in an ambiguity environment. Third, as our applications to duopoly models suggest, our framework contributes to the understanding of antitrust behaviors in stochastic environments.[8] Regardless its simplicity, the proposed framework could fit well to analyze other related problems.

The rest of the paper is organized as follows. Section 2 describes the theoretical setting proposed to analyze strategic repeated games under ambiguity. Section 3 restricts the attention to a specific subset of games and derives our main result for assessing the likelihood of long-run cooperation. Section 4 applies this framework to the previously mentioned canonical examples. Finally, Section 5 concludes.

2. Theoretical setting

2.1. Ambiguity and strategic games

We follow Chateauneuf, Eichberger, and Grant (2007) framework based on neo-additive capacities to model ambiguity.[9] A neo-additive capacity is a particular capacity that can be interpreted as a convex combination between an additive probability distribution and a capacity that only

[6]See, for example, Kreps, Milgrom, Roberts, and Wilson (1982) and Conlon (2003) for analysis of finitely repeated games with incomplete information on counterpart's type; Ellison (1994), Watson (1994), Aoyagi (1996) and Chan (2000) for a similar analysis in infinitely repeated games; and Kandori (1992), Sekiguchi (1997), Compte (1998), Compte (2002), Ely and Valimaki (2002), Piccione (2002), Cripps, Mailath, and Samuelson (2004) and Cripps, Mailath, and Samuelson (2007) for analysis of infinitely repeated games with imperfect monitoring.

[7]For ambiguity applications to strategic games, see Dow and Werlang (1994), Marinacci (2000), Haller (2000), Dimitri (2005) and Rothe (2011). For the analysis of oligopoly models under ambiguity, see Fontini (2005) and Eichberger, Kelsey, and Schipper (2009).

[8]See, for example, Green and Porter (1984), Rotemberg and Saloner (1986), Bagwell and Staiger (1997), Fonseca and Normann (2012), Kandori and Matsushima (1998), Athey and Bagwell (2001), Athey, Bagwell, and Sanchirico (2004) and Rojas (2012).

[9]For technical details about capacities and neo-additive capacities, see Appendix A.

distinguishes if a state is possible, impossible or certain. Given a space of actions X, the Choquet integral of the function $f : X \to \mathbb{R}$ with respect to the neo-additive capacity $v : 2^X \to \mathbb{R}_+$ is defined by

$$\int f dv \quad := \quad \delta(\alpha M + (1-\alpha)m) + (1-\delta)E_\pi f, \tag{1}$$

where δ is the degree of ambiguity, α is the degree of optimism,[10] E_π is the expectation induced by the probability distribution π (defined over X), $M = \max_{x\in X} f(x)$ and $m = \min_{x\in X} f(x)$. Chateauneuf, Eichberger, and Grant (2007) axiomatized (1) as a choice criterion under ambiguity.[11]

Eichberger, Kelsey, and Schipper (2009) follow this strategy to model ambiguity in strategic games. The authors propose a game in the form $G = \langle (S_i, u_i, \delta_i, \alpha_i, \pi_i)_{i=1,2} \rangle$, where S_i and u_i are the space of strategies and the utility function of player i, respectively. In this context, π_i (and, consequently, uncertainty) is defined over S_{-i}. Then, the expected utility under ambiguity of player i when choosing strategy s_i, is defined by

$$v(s_i; \delta_i, \alpha_i, \pi_i) := \delta_i(\alpha_i M_i(s_i) + (1-\alpha_i)m_i(s_i)) + (1-\delta_i)E_{\pi_i} u_i(s_i, s_{-i}), \tag{2}$$

where $s_i \in S_i$, $s_{-i} \in S_{-i}$, $M_i(s_i) = \max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$ and $m_i(s_i) = \min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$. In this context, ambiguity is understood as the uncertainty an individual faces regarding the other player's decisions. For simplicity, we will focus on symmetric games, i.e. $S_i = S$, $u_i = u$, $\delta_i = \delta$, $\alpha_i = \alpha$ and $\pi_i = \pi$, for $i = 1, 2$.

This framework for modeling ambiguity has three good properties. First, it has a clear interpretation. The individual faces a subjective additive probability measure, π, but does not trust it fully. The ambiguity parameter, δ, measures the degree of distrust on π. Then, the *unassigned* probability is mapped to the best and worst possible outcomes depending on the degree of optimism of the individual. Second, no assumptions are imposed on player's attitude towards ambiguity: the model flexibly represents both optimistic and pessimistic individuals. Third, the setting fits well on the strategic games modelling. Concretely, it makes sense to assume the existence of π as it can be derived endogenously from the game's equilibrium.[12]

2.2. Static (short-run) equilibrium definition

Following Eichberger, Kelsey, and Schipper (2009), define the best-response correspondence of player i, given a neo-additive capacity, as $R_i(\delta, \alpha, \pi) := \arg\max_{s_i\in S} v(s_i; \delta, \alpha, \pi)$. Then, we define a static (short-run) equilibrium as a pair $(s_1^*, s_2^*) \in S \times S$ such that $s_i^* \in R_i(\delta, \alpha, \pi)$, for $i = 1, 2$.[13]

[10] If $\alpha = 1$ ($\alpha = 0$), we say that the individual is completely optimistic (pessimistic).

[11] In a context of no ambiguity (i.e. $\delta = 0$), (1) is reduced to the standard Expected Utility model (Savage, 1954). For a completely ambiguous and pessimistic individual (i.e. $\delta = 1$ and $\alpha = 0$), (1) mimics the maxmin expected utility model (Gilboa and Schmeidler, 1989).

[12] There are other frameworks for modelling ambiguity which are built from the idea that it is not possible to assume the existence of subjective probability distributions (see Etner, Jeleva, and Tallon, 2012).

[13] Note that the equilibrium could also be defined as an equilibrium in beliefs. See Eichberger, Kelsey, and Schipper (2009) for details.

2.3. Dynamic setting and parameters' updating

The framework can be extended to a dynamic setting by modelling an infinitely repeated strategic game. In this context, in $t=0$ player i chooses a sequence of strategies, $\tilde{s}_i := \{s_{it}\}_{t=0}^{\infty} \in S^{\omega} := \prod_{i=1}^{\infty} S$, which, given a subjective discount factor, $\beta \in (0,1)$, induces an expected utility described by[14]

$$V\left(\tilde{s}_i; \delta_0, \alpha_0, \pi_0 \middle| s_{-i,t-1}\right) := \sum_{t=0}^{\infty} \beta^t \sum_{s \in S} \phi_{st} v(s_{i,t}; \delta_t, \alpha_t, \pi_t | s_{-i,t-1}), \tag{3}$$

where $s_{-i,t-1}$ is the strategy the other player played on the period $t-1$, and ϕ_{st} is the probability the individual assigns in $t=0$, to the other player playing strategy $s \in S$ in period $t-1$ (in other words, it is the ex-ante probability of arriving to the state induced by $s_{-i,t-1}$ in period t).[15] Note that the exogenous parameters (δ and α) and the additive probability distribution (π) now have a time subscript. This is due to the fact that, given the existence of ambiguity, stage realizations may contain information that could induce updating player's beliefs. We model the updating rule following Eichberger, Grant, and Kelsey (2010). The authors recommend using the Generalized Bayesian Updating Rule for capacities as (i) it preserves the utility index, and (ii) it keeps unchanged the ambiguity aversion parameter.[16]

The rule works as follows. Fix a conditioning event $E \subseteq S$ (in our case, E is a counterpart's action) and an unconditional neo-additive capacity v defined by (δ, α, π). The authors show that the Generalized Bayesian Updating rule for neo-additive capacities implies that v_E, the conditioned capacity, is also a neo-additive capacity defined by $(\delta_E, \alpha_E, \pi_E)$, where

$$\delta_E = \frac{\delta}{(1-\delta)\pi(E)+\delta}, \tag{4}$$

$$\alpha_E = \alpha, \tag{5}$$

$$\pi_E(A) = \frac{\pi(A \cap E)}{\pi(E)}, \quad \forall A \in S. \tag{6}$$

Note that for games with a unique equilibrium in pure strategies, π is always degenerated, i.e. always assigns probability 1 to the strategy $s_{-i}^* \in S$ associated with the equilibrium and 0 to every other strategy $s_{-i} \in S$ different from s_{-i}^*. This issue, which will be relevant for the cases analyzed in the following sections, implies that when the other individual plays the expected action (i.e. plays $E = s_{-i}^* \in S$ such that $\pi(E)=1$), then $\delta_E = \delta$ and $\pi_E(A) = \pi(A)$, $\forall A \in S$. Therefore, all parameters remain unchanged. On the other hand, when the other individual deviates from the expected action (i.e. plays $E = s_{-i} \in S$ different from s_{-i}^* such that $\pi(E)=0$), then $\delta_E = 1$ and the probability measure becomes irrelevant. This condition is stationary, as when $\delta = 1$, $\delta_E = 1$, $\forall E \in S$. This is intuitive, as the additive distribution stops being relevant for the individual when an *impossible* event (from a subjective point of view) is actually realized.

[14]For simplicity, it is assumed that the expected utility given by the sequence of strategies chosen in $t=0$ only depends on the strategies played on the last period, which are assumed to contain all relevant information available to that date. As is described later, this affects the parameters' updating dynamics.

[15]If S is infinite, ϕ_{st} in (3) is replaced by a probability distribution $f_t(s)$, with $\int_{s \in S} f_t(x)dx = 1$.

[16]The latter property is important as the ambiguity aversion parameter, α, is usually seen as an individual's intrinsic parameter and, therefore, it is reasonable to assume it fixed over time.

2.4. Dynamic (long-run) equilibrium definition

We can extend the notion of the static equilibrium to the dynamic setting. Define the best-response correspondence of player i, given a neo-additive capacity, as $R_i(\delta, \alpha, \pi) := \arg\max_{\tilde{s}_i \in S^\omega} V(\tilde{s}_i; \delta, \alpha, \pi)$. Then, we define a dynamic (long-run) equilibrium as a pair $(\tilde{s}_1^*, \tilde{s}_2^*) \in S^\omega \times S^\omega$ such that $\tilde{s}_i^* \in R_i(\delta, \alpha, \pi)$, for $i = 1, 2$.

3. Looking for cooperative equilibrium in competitive games

Following our general theoretical setting, we restrict our attention to a specific type of games. Suppose a non-cooperative (strategic) game with a unique symmetric static (short-run) equilibrium in pure strategies. Denote by $n \in S$ the symmetric equilibrium strategy. Suppose the existence of some other strategy $c \in S$ such that $u(c, c) > u(n, n)$ but that cannot be implemented as a short-run equilibrium given incentives for deviation (i.e. $c \notin R(\delta, \alpha, \pi)$, when $\pi(c) = 1$). Denote by $d \in S$ the optimal response when the other individual plays c, with $u(d, c) > u(c, c)$. Nevertheless, in the infinitely repeated game, playing $\tilde{c}_i := \{s_{it} = c\}_{t=0}^{\infty}$, for $i = 1, 2$, could be an equilibrium if the (expected) present value of playing that sequence is greater or equal than the (expected) present value of playing any other feasible sequence of strategies.[17] We call this potential equilibrium a *cooperative equilibrium*.

We investigate the effects ambiguity may have on this long-run analysis. Assume that $u(n, d) > u(c, d)$. Therefore, deviation from the static cooperative equilibrium is not only profitable for the one that deviates from it, but it is also costly for the one that does not deviate (with respect to playing non-cooperative strategies). Therefore, a punishment scheme has to be defined in order to induce commitment on players to play c. We follow the standard *grim trigger* punishment scheme: an individual plays c until the other player deviates, punishing her by playing n forever.

This simple analysis is similar in structure to the standard case (i.e. non-ambiguity setting). What is interesting is that the discounted benefits of the different strategies are affected by ambiguity, as individuals internalize the possibility that the other player might choose an action different from the expected. Concretely speaking, uncertainty about the other player's behavior affects the computation of discounted benefits through three channels. First, ambiguity induces changes on the expected payoffs of the different strategies, as payoffs depend on the other player's action which is (by definition) partially unpredictable given uncertainty. Second, potential future deviations from the cooperative equilibrium may induce potential future punishments. Third, whenever an individual plays an action different from the expected, counterparts update their beliefs. Potential future belief updating is considered in the computation of expected payoffs.

Now we proceed to derive conditions for sustaining a cooperative equilibrium.

Proposition I: $\tilde{c}_i := \{s_{it} = c\}_{t=0}^{\infty}$, for $i = 1, 2$, is an equilibrium of the infinitely repeated game if the subjective discount factor, β, meets the following condition:

$$\frac{v_c^* - \beta\phi_c v_n^u}{1 - \beta\phi_c} \quad \geq \quad v_d^* + \frac{\beta\phi_c(v_n^* - v_n^u)}{1 - \beta\phi_n}, \tag{7}$$

[17] As it was argued before, this can be equivalently stated as an equilibrium in beliefs, as the potential equilibrium proposed implies $\pi(c) = 1$.

where

$$
\begin{aligned}
v_c^* &:= \delta(\alpha M(c) + (1-\alpha)m(c)) + (1-\delta)u(c,c), \\
v_n^* &:= \delta(\alpha M(n) + (1-\alpha)m(n)) + (1-\delta)u(n,n), \\
v_d^* &:= \delta(\alpha M(d) + (1-\alpha)m(d)) + (1-\delta)u(d,c), \\
v_n^u &:= \alpha M(n) + (1-\alpha)m(n), \quad \textit{and}
\end{aligned}
$$

ϕ_s *is the probability the individual assigns to the other individual playing the expected action in the equilibrium in strategy s (i.e. playing s), for $s = c, n$.*

Define as β^* the value of β that meets (7) with equality. When $\phi_c = \phi_n = \phi$, (7) is reduced to

$$\frac{v_c^* - \beta\phi v_n^*}{1-\beta\phi} \quad \geq \quad v_d^*, \tag{8}$$

which implies that,

$$\beta^* = \phi^{-1}\frac{(v_d^* - v_c^*)}{(v_d^* - v_n^*)}. \tag{9}$$

Proof: See Appendix B.

Define the critical discount factor, β^*, as the discount factor that meets equation (7) with equality. Cooperation will be always feasible if players face a discount factor greater or equal than β^*. Note that β^* cannot always be analytically determined, given the non-linear nature of the problem. Particular cases in which $\phi_c = \phi_n = \phi$ allow to work with a more tractable expression.

Before proceeding to the applications, we want to make a final remark. Note that the existence of a feasible $\beta \in (0,1)$ meeting equation (7) is not guaranteed. As it is shown below, there may be cases where no $\beta \in (0,1)$ meets equation (7), thus being cooperation impossible to achieve in those cases, regardless of players' discount factor.[18] This is the main result of this paper: under standard conditions, ambiguity may erode cooperation possibilities, i.e. an intuitive perturbation in the assumptions of the analysis of long-run cooperation in strategic games can have important implications on the main conclusions.

4. Applications: revisiting canonical examples under ambiguity

This section applies the proposed framework to canonical standard problems in order to analyze the implications that ambiguity may have on the likelihood of sustaining long-run cooperative equilibria. We look at the long-run analysis of the infinitely repeated PD, and the Cournot and Bertrand duopolies. For each case, we first assess how their structure is affected by ambiguity, as it could change to the extent that our results above might not longer apply. Then, we evaluate the conditions for sustaining a cooperative equilibrium.

[18] Note that this is not equivalent to claim that the Folk Theorem fails in an ambiguity setting, as here we work both with a specific ambiguity model and with a specific punishment scheme. The analysis of the Folk Theorem in a more general setting is beyond the scope of this paper.

4.1. Infinitely repeated Prisoner's Dilemma

Consider the normal form representation of the PD

	C	N
C	(R,R)	(Q,T)
N	(T,Q)	(P,P)

where C and N stand for *Cooperate* and *Non Cooperate*, $T > R > P > Q$ and $\frac{Q+T}{2} < R$. In this case, the space of strategies $S = \{C, N\}$, and considering the notation above $c = C$ and $n = d = N$. Consequently, $M(c) = R$, $m(c) = Q$, $M(n) = M(d) = T$, $m(n) = m(d) = P$, $u(c,c) = R$, $u(d,c) = T$ and $u(n,n) = P$. Also, $v_n^u = \alpha T + (1-\alpha)P$, $\phi_c = (1-\delta)+\delta\alpha = 1-\delta(1-\alpha)$ and $\phi_n = (1-\delta) + \delta(1-\alpha) = 1 - \delta\alpha$. Therefore, for given values of T, R, P and Q, it is possible to compute all terms of equation (7) for each (δ, α) combination, in order to determine β^* and its relations with ambiguity.

For that purpose, we first look at the expected values of the cooperative and non-cooperative static equilibria, i.e.

$$v_c^* = \delta(\alpha R + (1-\alpha)Q) + (1-\delta)R, \quad \text{and} \tag{10}$$

$$v_n^* = \delta(\alpha T + (1-\alpha)P) + (1-\delta)P. \tag{11}$$

Notice that the analysis of the PD rises the question about the possibility of implementing a cooperative equilibrium in the long-run given that the cooperative equilibrium payoff, R, is larger than the non-cooperative equilibrium payoff, P. Note that ambiguity affects the equilibria's expected payoffs by decreasing the expected value of the cooperative equilibrium (as relatively pessimistic individuals internalize that the counterpart may deviate from the expected action to a non-cooperative behavior, thus losing $R - Q$) and increasing the expected value of the non-cooperative equilibrium (as relatively optimistic individuals internalize that the counterpart may deviate from the expected action to a cooperative behavior, thus obtaining $T - P$). In fact, while when $\delta \to 0$ (i.e. non-ambiguity case), $v_c^* > v_n^*$, when $\delta \to 1$ (i.e. highly ambiguous case), $v_c^* < v_n^*$. In the latter case, seeking to implement the cooperative equilibrium stops being desirable, as the static equilibrium is Pareto-superior.[19]

Concretely, given T, R, P and Q, we have that

$$v_n^* > v_c^* \quad \Leftrightarrow \quad \delta > (R-P)\left[\alpha(T-P) + (1-\alpha)(R-Q)\right]^{-1}. \tag{12}$$

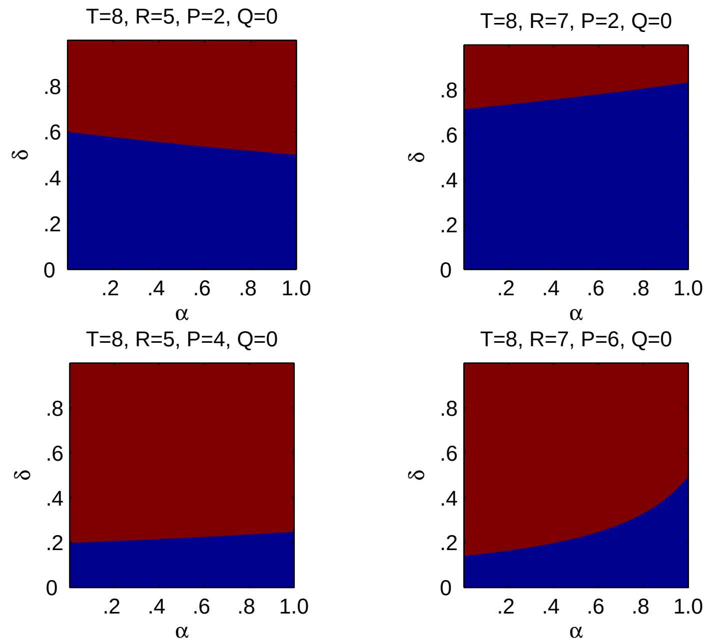
[19] Note that, consistent with Marinacci (2000), the static equilibrium is not affected by ambiguity. In fact, if π is a probability distribution that assigns probability p to the other individual playing C and probability $1-p$ to the other individual playing N, then

$$v(C;\delta,\alpha,\pi) = \delta\left(\alpha R + (1-\alpha)Q\right) + (1-\delta)(pR + (1-p)Q), \quad \text{and}$$

$$v(N;\delta,\alpha,\pi) = \delta\left(\alpha T + (1-\alpha)P\right) + (1-\delta)(pT + (1-p)P),$$

and, since $T > R$ and $P > Q$, endogenous π will imply $p = 0$ and, therefore, playing N will always be dominant strategy. Moreover, it remains being the only short-term equilibrium as $v(N;\delta,\alpha,\pi(p=1)) > v(C;\delta,\alpha,\pi(p=1))$ and, therefore, incentives for deviating from a cooperative equilibrium remain existing in an ambiguity context.

Figure 1: Infinitely repeated PD: Comparison of expected equilibria' payoffs

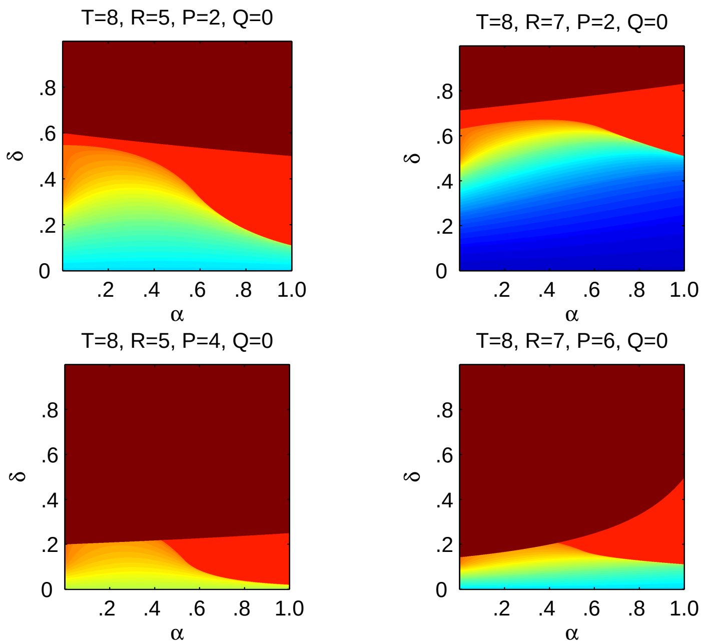


Note: Blue zones account for (δ, α) combinations such that $v_c^* > v_n^*$. Red zones account for (δ, α) combinations such that $v_n^* > v_c^*$.

Then, for a given α, there is a threshold level of ambiguity such that the game structure changes to scenarios in which the expected payoff associated with the cooperative equilibrium is strictly lower than the non-cooperative equilibrium expected payoff. This defines a critical function, $\delta^*(\alpha)$, which suggests that when $\delta > \delta^*(\alpha)$, asking for conditions to sustain a cooperative equilibrium stops being relevant, given that the static non-cooperative equilibrium is Pareto-superior (in expected terms) and is implementable as a short-run Nash equilibrium. This is illustrated in Figure 1 for different values of (T, R, P, Q). (δ, α) combinations marked with blue (red) represent cases in which $v_c^* > v_n^*$ ($v_c^* < v_n^*$). The line which separates the different areas correspond to $\delta^*(\alpha)$. It can be seen that for some payoffs, the possibility of changing the game structure is far from negligible since $\delta^*(\alpha)$ depends on the relationship between payoffs. The mechanism behind this is that higher levels of ambiguity give relatively more weight to the best and worst scenarios, which are larger when playing non-cooperative strategies. Namely, more ambiguous players are better off when playing N as potential deviations are associated with higher payoffs than when playing C. This is a first interesting insight: ambiguity may change the game's structure to scenarios in which asking for conditions to sustain cooperative equilibria is no longer relevant.

Despite this last finding, the result developed in the previous section is still valid for cases in which $v_c^* > v_n^*$. In the repeated game, given the punishment scheme previously described, cooperation is an equilibrium if the expected discounted benefits of cooperating are larger than the expected discounted benefits of deviating from the cooperative agreement and then being punished for that. Since in this case $\phi_c \neq \phi_n$ (except for $\alpha = 0.5$), equation (7) has not analytical solution and has

Figure 2: Infinitely repeated PD: Behavior of β^*



Note: Red zones account for (δ, α) combinations such that $v_n^* > v_c^*$. The remaining zones account for (δ, α) combinations such that $v_c^* > v_n^*$. In the latter, the bluer the color, the lower β^*; the more yellow the color, the higher β^*. The orange zones account for (δ, α) combinations for which no exists $\beta \in (0, 1)$ satisfying equation (7).

to be numerically addressed to see how the critical discount factor, β^*, behaves regarding ambiguity.

We proceed by considering the same cases exposed in Figure 1. Figure 2 illustrates what happens with the critical discount factor when the game structure remains unchanged. The former blue zone is replaced by two new zones: a color palette zone and an orange zone. The color palette zone is read as follows: for each pair (δ, α), the bluer the point, the smaller the critical discount factor. Conversely, the more yellow the point, the higher the critical discount factor required. Hence, numerical analysis of these examples suggest that for a given α, ambiguity decreases the possibility of cooperation in the infinitely repeated PD, as it demands more patient individuals for meeting the critical condition. While previous result seems intuitive, the orange zone accounts for a more striking insight. A certain (δ, α) combination marked in orange refers to the nonexistence of feasible discount factors (i.e. $\beta \in (0, 1)$) satisfying equation (7). In other words, orange zones represent cases in which although is profitable to implement cooperative agreements, ambiguity erodes the possibility by making infeasible the patience levels required. If we add the red and orange zones, we can see that ambiguity may turn cooperation infeasible for several portions of the (δ, α) parameter space.

To understand this, first note that the expected payoff of deviating falls with the ambiguity level as the outcome (T, Q) is less likely to happen. In principle, this should make cooperation more likely.

However, higher levels of ambiguity increase the expected value of non cooperating and lower the value of cooperating. Both effects combined more than compensate the less appealing deviation, generating that the possibility of cooperation decreases with ambiguity. The non-linear nature of the relationship yields on the fact that the ambiguity needed for compensating the effects varies with α.

The examples presented suggest that ambiguity may importantly affect the standard long-run analysis of this canonical game. First, it may affect the game structure making cooperation no longer a desirable scenario from a strategic point of view. Moreover, in cases when the structure remains unchanged, the necessary conditions for sustaining cooperative agreements may be more restrictive, even being infeasible in some cases regardless of players' patience.

4.2. Cournot Duopoly

Consider two firms competing in quantities, producing an homogeneous product with constant marginal cost k and facing an inverse demand function $P(Q) = \min\{A - bQ, 0\}$, where $A > k$, $b > 0$, and $Q = q_1 + q_2$, where q_i is the quantity produced by firm i. Also consider ambiguity on the competitor's action, which means firms internalize that the other firm may produce a quantity different from the expected, i.e. different from the standard competitive equilibrium. Suppose firms can produce any $q \in [0, \bar{q}]$, where $\bar{q}$ is the capacity constraint. For simplicity, we assume that $\bar{q}$ is such that $A - b\bar{q} = 0$. In this case, $S = [0, \bar{q}]$. Therefore, $M(s) = (A - bs)s - ks$, $\forall s \in S$, as the best scenario is always given by the other firm producing zero. Conversely, $m(s) = -ks$, $\forall s \in S$, as the worst scenario is always given by the other firm producing $\bar{q}$ and, therefore, taking the price to zero.

In this setting, n is the competitive quantity, q^n, which comes from the following maximization problem,

$$v_n^* = \max_{q^n \in [0,\bar{q}]} \quad \delta\alpha(A - bq^n)q^n + (1 - \delta)\left[A - b(q^n + q_j)\right] q^n - kq^n, \tag{13}$$

where q_j is the expected quantity produced by the other firm in the non-ambiguous scenario, i.e. what the firm expects the other firm to produce when things behave as they are supposed to. Taking first order condition leads to the reaction function,

$$q^n = R(q_j) = \frac{\delta\alpha A + (1 - \delta)(A - bq_j) - k}{2b(1 - \delta(1 - \alpha))}. \tag{14}$$

When solving for q^n, we make two different assumptions on q_j. The first is to assume that the other firm is expected to behave as a standard competitive firm and, therefore, $q_j = \frac{A-k}{3b}$ and $q^n = R(q_j)$. The second is to assume that the other firm is expected to be ambiguous, being the symmetric equilibrium given by $q^n = \frac{\delta\alpha A+(1-\delta)A-k}{2b(1-\delta(1-\alpha))+b(1-\delta)}$. Simulations for both cases are reported. Finally, v_n^* is given by

$$v_n^* = \delta\alpha(A - bq^n)q^n + (1 - \delta)\left(A - b\left(q^n + \frac{A - k}{3b}\right)\right) q^n - kq^n, \quad \text{or} \tag{15}$$

$$v_n^* = \delta\alpha(A - bq^n)q^n + (1 - \delta)\left(A - 2bq^n\right) q^n - kq^n, \tag{16}$$

depending on the assumption made over q_j.

On the other hand, c is given by collusive quantity, q^m, which is the (equally splitted) standard optimal monopoly production,

$$q^m = \frac{A-k}{4b}, \tag{17}$$

which leads to

$$v_c^* = \delta\alpha(A - bq^m)q^m + (1-\delta)(A - 2bq^m)q^m - kq^m. \tag{18}$$

Finally, d is given by the optimal reaction to the collusive agreement, $q^d = R\,(q^m)$, which leads to,

$$v_d^* = \delta\alpha(A - bq^d)q^d + (1-\delta)\left[A - b(q^d + q^m)\right] q^d - kq^d. \tag{19}$$

Since $\phi_c = \phi_n = 1-\delta$, the critical discount factor can be analytically determined by equation (9). Before proceeding with the analysis of whether ambiguity affects the game structure and the effects of ambiguity on the discount factor, let's recall the similar PD structure that tacit collusions in oligopoly models have.

The canonical analysis of tacit collusions in oligopoly models following a repeated PD logic arises from the fact that payoffs (firms' benefits) have a similar structure to the PD, i.e. $\pi\left(q^d, q^m\right) > \pi\left(q^m, q^m\right) > \pi\left(q^n, q_j\right)$, where $q_j = \frac{A-k}{3b}$ or $q_j = q^n$, depending on the assumption made. And where $\pi(q_i, q_j) = (A - b(q_i + q_j) - k)q_i$ are the benefits when the firm produces q_i and the counterpart produces q_j. To simplify notation, we will refer to these benefits as π_d, π_c, and π_n, respectively. While the mentioned ordering is always met in the non-ambiguity setting, it is worth asking if it holds in an ambiguity environment, since optimal quantities now depend on the ambiguity parameters, which in turn determine benefits.[20] Therefore, it is important to check which (δ, α) combinations are characterized by alternative payments orderings, in order to rule them out in our subsequent analysis.

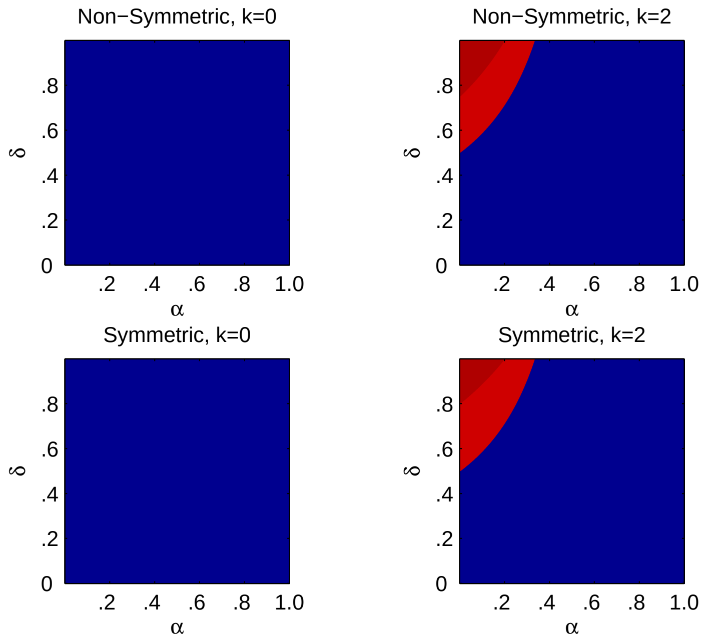
In order to address that concern, we proceed by computing the benefits for each (δ, α) pair, for given values of A, b and k. For k, we consider two cases: zero and positive marginal costs. Figure 3 illustrates the analysis for the two different assumptions made on counterpart benchmark behavior, unambiguous (or non-symmetric) and symmetric, and for both cases regarding the marginal cost.[21]

When marginal cost is zero, both with and without symmetry, payoffs ordering is unaltered with respect to the non-ambiguity case. Nevertheless, this is not always true when marginal costs are positive. When firms are relatively pessimist and face *high* levels of ambiguity, the problem stops behaving as a particular case of the PD (red zones). Here two cases arise. First, when ambiguity and pessimism are *moderately* high, deviation stops being profitable. Light-red zones account for this fact in the right panels in Figure 3. Concretely, payoffs ordering changes to $\pi_c > \pi_d > \pi_n$ and $\pi_c > \pi_n > \pi_d$, in the non-symmetric and symmetric cases, respectively. Therefore, with effective

[20] This was not the case in the PD, since payoffs were exogenous and not affected by ambiguity parameters.

[21] While results are shown for $A = 10$, $b = 1$ and $k = 2$ (when being positive), qualitative implications of the numerical analysis do not rely on these specific values. Altering parameter values may marginally alter the zones' magnitude, but not their shape and existence.

Figure 3: Cournot duopoly: Game's structure



Note: $A = 10$ and $b = 1$. Blue zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$). Red zones account for (δ, α) combinations such that PD structure no longer holds. The light red zones accounts for games defined by $\pi_c > \pi_d > \pi_n$ and $\pi_c > \pi_n > \pi_d$, in the non-symmetric and symmetric cases, respectively. The dark red zone accounts for (δ, α) combinations such that for firms is no longer profitable to enter the market.

payoffs considerable lower in magnitude with respect to the non-ambiguity case, the game transits to a multiple equilibria structure.[22] Moreover, when ambiguity and pessimism are *large enough*, a second change in the game structure happens. This is illustrated by the dark red zones of Figure 3. At this point, it is no longer profitable for firms to enter the market, as they put considerable weight to the scenario where their counterparts set the price equal to zero, thus only facing losses when production is positive. In this case, π_n and π_d are equal to zero. In brief, a first insight about this application is that the duopoly may no longer behave as a particular case of the PD when marginal costs are positive and, therefore, dynamics about their market performance work in a different way regarding the standard tacit collusion analysis. This is interesting, since zero marginal costs are usually assumed *without loss of generality* in some applications of these models.[23]

[22] Note that this not means that now collusion can be achieved as a short run-equilibrium, since firms make decisions looking at expected payoffs, and in these cases, expected payoff of non-cooperation is larger.

[23] To better understand the mechanism, note that in the zones that are not characterized by $\pi_d > \pi_c > \pi_n$ but firms do not leave the market (light red zones), the optimal quantity for a firm wishing to defect from a cooperative agreement is lower than the optimal collusive quantity, as opposed to the regular setup where the optimal deviation quantity is larger than the collusive agreement. This makes the potential deviation to generate a smaller total quantity and hence a higher price than the monopoly one. Consequently, this results in a lower profit (i.e. $\pi_c > \pi_d$)

Departing from the previous consideration, we can now replicate the analysis done in the previous section to the cases in which the payoff ordering is consistent with a PD structure. First, we address the comparison between expected payoffs derived from sustaining cooperative and non-cooperative equilibria. Figure 4 illustrates the analysis for the four cases considered. As in previous section, we mark with dark red (blue) the (δ, α) pairs in which $v_n^* > v_c^*$ ($v_n^* < v_c^*$). Note that again, a large area arises in which the cooperative equilibrium is Pareto-dominated (in expected terms) by the non-cooperative equilibrium in the static game. To understand this, recall that when firms are relatively optimistic and face high levels of ambiguity, they give more probability to the other firm not producing anything. Therefore, collusive agreements are less profitable for firms in expected terms (although not in effective terms) since potential benefitial deviations are more profitable when firms are not cooperating.[24]

Finally, Figure 5 pools the analysis done above and adds the critical discount factor that would sustain a cooperative equilibrium in the long-run in cases where the PD structure is maintained and still exist incentives to seek cooperative equilibria (when $v_c^* > v_n^*$). Figure 5 is read as Figure 2. It can be seen that when there are no marginal costs, the figure shows a similar pattern than the PD case: ambiguity increases the critical discount factor needed for cooperation (color palette zone, from blue to yellow) up to a point that cooperation is infeasible (orange zone). This suggests, as in the previous section, that ambiguity is detrimental for cooperation and can lead to infeasibility for a large fraction of the parameter space. Where marginal costs are positive, that pattern is seen for firms with a minimum level of optimism. When firms are relatively pessimist, the critical discount factor decreases up to the point the game is altered.[25]

4.3. Bertrand Duopoly

We finally address the case of the Bertrand duopoly. Consider two firms competing in prices, producing an heterogeneous product with constant marginal cost k. Imperfect substitution implies that demand for firm i is given by $D(p_i, p_{-i}) = \max\{0, a + bp_{-i} - cp_i\}$, where $c > b$, $a > k$ and $a + bk - cK > 0$, with $K = \frac{a}{c}$. Without loss of generality, we can restrict the strategy space to $S = [k, K]$.[26] Therefore, $M(s) = (a - bs + cK)(s - k)$, $\forall s \in S$, as the best scenario is always given by the other firm setting the price equal to K. Conversely, $m(s) = (a - bs + ck)(s - k)$, $\forall s \in S$, as the worst scenario is always given by the other firm setting the price equal to k.

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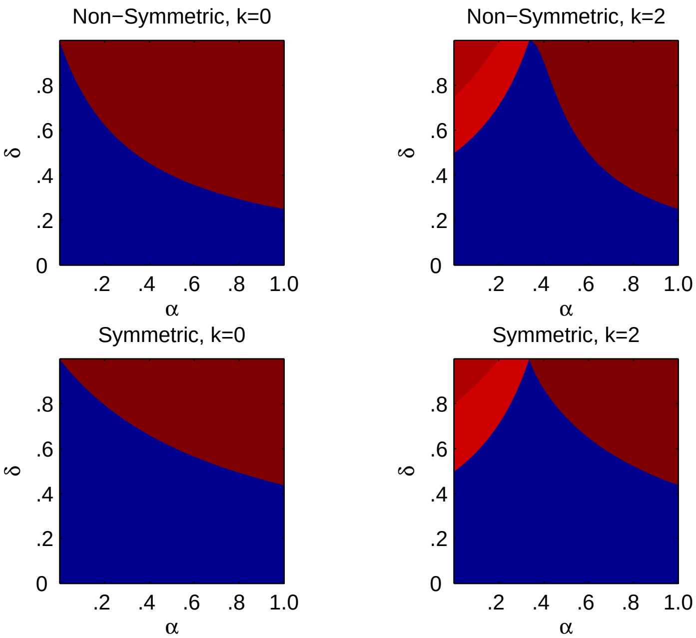
and in a deviation that never takes place. Recall that in these zones, firms assign large probabilities to counterparts deviating from the expected benchmark behavior to a case in which their production is high enough to set the price to zero.

[24]To better understand the mechanism, note that zones characterized by $v_n^* > v_c^*$ are associated with large quantities of the non-cooperative equilibrium. This occurs because higher levels of ambiguity lead optimistic firms to produce more (as optimal decision relies more on the best potential scenario). Then, collusive agreements are less profitable than competing and expecting beneficial deviations.

[25]Again, the explanation is in the non-linearity of the problem (regarding δ) that affects optimal quantities and, therefore, benefits. When firms are pessimistic, all expected payoffs decrease with ambiguity, but the short-term expected benefit from defecting does it faster. That increases the likelihood of sustaining a collusion, but in a context with extremely low payoffs, with respect to the non-ambiguity case.

[26]Choosing prices below k will always be dominated by not entering the market while choosing prices above K will always be dominated by choosing K (in both cases, strictly dominated if $k > 0$).

Figure 4: Cournot duopoly: Comparison of expected equilibria



Note: $A = 10$ and $b = 1$. Blue zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* > v_n^*$. Dark red zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* < v_n^*$. Light red zones account for (δ, α) combinations such that PD structure no longer holds. Among them, the lighter zones accounts for games defined by $\pi_c > \pi_d > \pi_n$ and $\pi_c > \pi_n > \pi_d$, in the non-symmetric and symmetric cases, respectively, while the darker zone accounts for (δ, α) combinations such that for firms is no longer profitable to enter the market.

In this case, n is given by p^n, which comes from the following optimization problem[27]

$$v_n^* = \max_{p^n \in [k,K]} (p^n - k)(a - cp^n) + (p^n - k)b\left((1-\delta)p_j + \delta\alpha K + \delta(1-\alpha)k\right), \qquad (20)$$

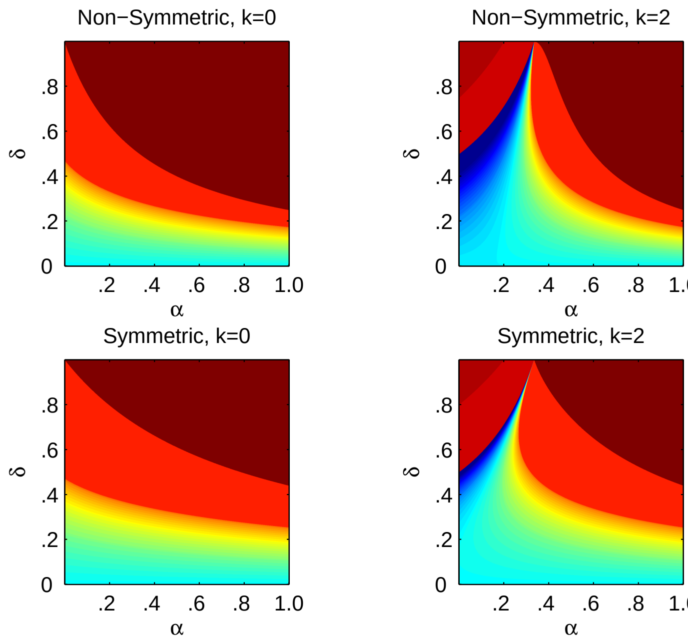
where p_j is the expected price chosen by the other firm in the non-ambiguous scenario, i.e. the price that the firm expects the other firm to set when things behave as they are supposed to. Taking first order condition leads to the reaction function,

$$p^n = R(p_j) = \frac{a + b\left[(1-\delta)p_i + \delta\alpha K + \delta(1-\alpha)k\right] + ck}{2c}. \qquad (21)$$

For solving for p^n, we make the two same assumptions as in the Cournot case. The first is to assume that the other firm is expected to behave as a standard competitive firm and, therefore, $p_j = \frac{a+ck}{2c-b}$ and $p^n = R(p_j)$. The second is to assume that the other firm is expected to be ambiguous, being the

[27]The optimization problem is given by $v_n^* = \max_{p^n \in [k,K]} \delta(\alpha(p^n - k)(a + bK - cp^n) + (1-\alpha)(p^n - k)(a + bk - cp^n)) + (1-\delta)(p^n - k)(a + bp_j - cp^n)$ and simplifies to (20).

Figure 5: Cournot duopoly: Behavior of β^*



Note: $A = 10$ and $b = 1$. Dark red zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* < v_n^*$. Light red zones account for (δ, α) combinations such that PD structure no longer holds. Among them, the lighter zones accounts for games defined by $\pi_c > \pi_d > \pi_n$ and $\pi_c > \pi_n > \pi_d$, in the non-symmetric and symmetric cases, respectively, while the darker zone accounts for (δ, α) combinations such that for firms is no longer profitable to enter the market. The remaining zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* > v_n^*$. In the latter, the bluer the color, the lower β^*; and the more yellow the color, the higher β^*. The orange zones account for (δ, α) combinations for which no exists $\beta \in (0, 1)$ satisfying equation (9).

symmetric equilibrium given by $p^n = \frac{a+\delta\alpha K+\delta(1-\alpha)k+ck}{2c-b(1-\delta)}$. Simulations for both cases are reported. Given that, v_n^* is given by

$$v_n^* = (p^n - k)(A - cp^n) + (p^n - k)b\left((1-\delta)\left(\frac{a+ck}{2c-b}\right) + \delta\alpha K + \delta(1-\alpha)k\right), \quad \text{or} \tag{22}$$

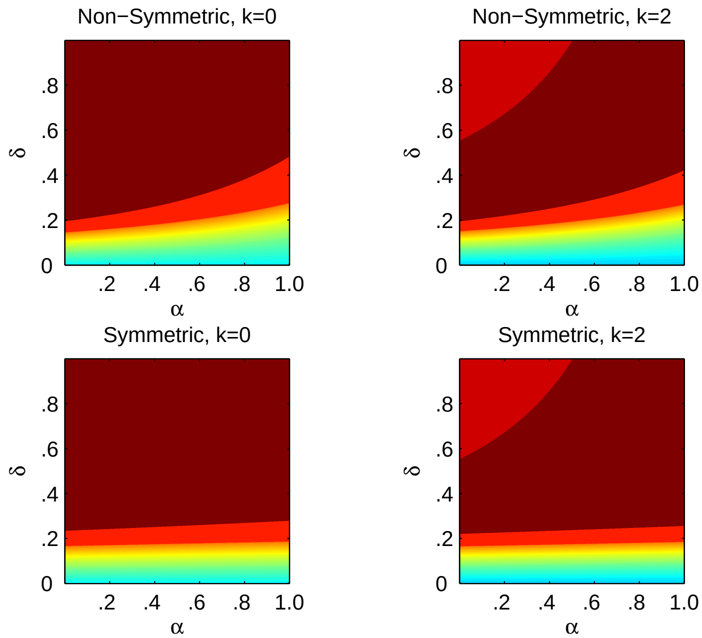
$$v_n^* = (p^n - k)(A - cp^n) + (p^n - k)b\left((1-\delta)p^n + \delta\alpha K + \delta(1-\alpha)k\right), \tag{23}$$

depending on the assumption made over p_j.

On the other hand, c is given by the collusive price, p^m, which comes from the maximization of the joint profits, $(p_1 - k)(a + bp_2 - cp_1) + (p_2 - k)(a + bp_1 - cp_2)$. Solution to this problem is given by $p^m = \frac{1}{2}\left(\frac{a}{c-b} + k\right)$. This leads to

$$v_c^* = (p^m - k)(a - cp^m) + (p^m - k)b\left((1-\delta)p^m + \delta\alpha K + \delta(1-\alpha)k\right). \tag{24}$$

Figure 6: Bertrand duopoly: Numerical results



Note: $a = 10$, $b = 0.5$, $c = 1$ (and, therefore, $K = 10$). Dark red zones account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* < v_n^*$. Light red zones account for (δ, α) combinations such that PD structure no longer holds ($\pi_c > \pi_d > \pi_n$). The remaining zones account account for (δ, α) combinations such that the PD structure holds ($\pi_d > \pi_c > \pi_n$) and $v_c^* > v_n^*$. In the latter, the bluer the color, the lower β^*; and the more yellow the color, the higher β^*. The orange zones account for (δ, α) combinations for which no exists $\beta \in (0, 1)$ satisfying equation (9).

Finally, d is given by the optimal reaction to the collusive agreement, $p^d = R(p^m)$, which leads to,

$$v_d^* = (p^d - k)(a - cp^d) + (p^d - k)b\left((1-\delta)p^m + \delta\alpha K + \delta(1-\alpha)k\right).$$

Since $\phi_c = \phi_n = 1 - \delta$, critical discount factor is determined by equation (9). We perform the same numerical analysis than the one made in previous subsections. Results are summarized in Figure 6. The analysis suggest almost the same as in previous cases. First, ambiguity may alter the effective payoffs ordering. This happens to highly pessimistic and ambiguous firms with positive marginal costs, where again the new structure is characterized by $\pi_c > \pi_d > \pi_n$.[28] Second, in cases when the payoff ordering remains unchanged, an area arises in which cooperation is no longer desirable in expected terms. Finally, in cases when cooperation is desirable, ambiguity makes it less likely, since it demands more patient firms. In some cases, it even makes it infeasible. Important from this application is the fact that the zones in which cooperation fails to be feasible comprises a highly significant portion of the parameter.

[28] Unlike the Cournot analysis, firms always stays on the market, regardless the values of δ and α.

5. Conclusions

This paper studies the effects of ambiguity on long-run cooperation in repeated strategic games. We show how the inclusion of ambiguity in strategic games can change in nontrivial ways the structure of them, altering the conditions under which long-run cooperation can be sustained. By using neo-additive capacities, we assess the requirements on the discount factor for a cooperative long-run equilibrium in a two-player game, considering beliefs' updating in players' strategies. In this framework, players partially distrust their own beliefs about other player's behavior and place themselves in the best and worst cases depending on their attitude towards ambiguity. As the concepts of ambiguity and optimism are intuitively and flexibly parametrized, we can theoretically explore the effects ambiguity has on the likelihood of sustaining a cooperative equilibrium in the long-run.

Equipped with this, we proceed to analyze three applications to get insights about specific effects on different setups. We study the infinitely repeated Prisoner's Dilemma, and the Cournot and Bertrand duopoly models. These applications lead to the following conclusions. First, ambiguity may alter the game's structure to schemes where seeking conditions to sustain long-run cooperative agreements stops being desirable. In these cases, non-cooperation is more profitable in expected terms and is achievable as a short-run Nash equilibrium. Second, ambiguity is in general detrimental for cooperation. Whenever the question of a reachable long-run equilibrium is still relevant, all three applications studied result in critical discount factors which increase with the level of ambiguity. Furthermore, for some parametric combinations, games may not accept a feasible discount factor consistent with a cooperative equilibrium, even when the expected payoff of cooperating is larger.

The implications of these results are relevant for understanding cooperative equilibria in strategic games in contexts of uncertainty. Moreover, they can be relevant for the design of antitrust policies. The knowledge players have with respect to their counterparts' expected actions can have substantial effects on the set of equilibria that can arise. For instance, in more informed environments, i.e. settings with low levels of ambiguity, players should face increasing chances of attaining cooperative equilibria compared to less informed ones. This is connected to what policies can foster or discourage the existence of tacit collusion. Higher certainty about competitors' decisions may make more attractive for firms to agree on production or price decisions.

Appendix A. Capacities and Neo-Additive Capacities

Given a finite space X and its correspondent power set 2^X, a capacity $v : 2^X \to \mathbb{R}_+$ is a function that satisfies

$$\begin{aligned} v(\phi) &= 0, \\ v(A) &\leq v(B) \quad \text{if } A \subseteq B, \\ v(X) &= 1. \end{aligned}$$

A capacity is said to be convex if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ (concave if the relation holds with $\geq$). Hence, capacities not necessarily comply the additivity law of probabilities. In this setting, integrating a function $f : X \to \mathbb{R}$ with respect to a capacity v (the analogous of an expectation in the additive probability framework) is done by using Choquet integrals (Choquet, 1954). When the capacity is additive, the Choquet integral is equivalent to the Riemann integral.

Intuitively, capacities can capture ambiguous beliefs as, given their non-additivity, the sum of the likelihood assigned to the realization of the different states does not necessarily sum one. For example, weight assigned to the union of two excluding acts may be greater than the sum of the weights assigned to each act individually. In that case, associated with a convex capacity, it is said that the individual is ambiguity averse.

A neo-additive capacity, proposed by Chateauneuf, Eichberger, and Grant (2007), is a particular type of capacity defined by

$$v(A) \; := \; (1-\delta)\pi(A) + \delta\mu_\alpha^{\mathcal{N}}(A),$$

for all $A \subset X$, where $\delta \in [0,1]$, π is an additive probability distribution defined over X and $\mu_\alpha^{\mathcal{N}}$ is a Hurwicz capacity exactly congruent with $\mathcal{N} \subset X$ with an $\alpha \in [0,1]$ degree of optimism, defined by

$$\mu_\alpha^{\mathcal{N}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{N}, \\ \alpha & \text{if } A \notin \mathcal{N} \text{ and } S \setminus A \notin \mathcal{N}, \\ 1 & \text{if } S \setminus A \in \mathcal{N}, \end{cases} \tag{A.1}$$

where S is the set of all possible states and $\mathcal{N} \subset X$ is the set of *null* events, i.e. the set of states whose realization is impossible. Chateauneuf, Eichberger, and Grant (2007) show that the Choquet integral of a neo-additive capacity is given by (1) and axiomatize the functional form as a utility function under ambiguity.

Appendix B. Proof of Proposition I

For formal issues, let's assume that $s_{-i,-1} = c$ for $i = 1, 2$. Therefore, history allows to raise the question about the conditions for sustaining the cooperative sequence as a dynamic equilibrium.

In the described setting, $(\{s_{it} = c\})_{t=0}^{\infty}$ for $i = 1, 2$ is an equilibrium if the present value of always cooperating is larger than the present value of deviating from the cooperative agreement and then being punished by the scheme proposed.

We first look on the present value of always cooperating. In $t = 0$ the expected payoff of the cooperative agreement is given by

$$v_c^* \quad := \quad \delta(\alpha M(c) + (1-\alpha)m(c)) + (1-\delta)u(c,c).$$

In $t = 1$, the individual sees what the counterpart played in $t = 0$. If the counterpart played c in the previous period, then the individual keeps playing c and makes no update on the parameters. Therefore, the expected payoff is again v_c^*. The individual predicts that this will happen with probability ϕ_c, which is bounded from below by $(1-\delta)$.[29] However, if the counterpart deviated on the previous period (i.e. played any strategy $s_{-i} \in S$ different from c), then the individual punishes the counterpart by playing n and updates the ambiguity parameter, δ, to 1. This situation gives an expected payoff of

$$v_n^u \quad := \quad \alpha M(n) + (1-\alpha)m(n).$$

The later scenario is stationary, as the individual will play n forever and will not make any further update, regardless the other player's future actions. Adding up, the expected payoff of the cooperative agreement in $t = 1$, seen from $t = 0$, is given by $\phi_c v_c^* + (1-\phi_c)v_n^u$.

A similar argument is applied recursively for future periods. If the counterpart played c in $t = 0$ and, therefore, the individual keeps on playing c in $t = 1$, in $t = 2$ sees what the counterpart played in $t = 1$ and decides how to behave following the rule described in the previous paragraph. Hence, the expected payoff of the cooperative agreement in $t = 2$, seen from $t = 0$, is given by $\phi_c^2 v_c^* + \phi_c(1-\phi_c)v_n^u + (1-\phi_c)v_n^u = \phi_c^2 v_c^* + (1-\phi_c^2)v_n^u$. Straightforward calculations allow to conclude that the expected payoff of the cooperative equilibrium in $t = T$, seen from $t = 0$, is given by $\phi_c^T v_c^* + (1-\phi_c^T)v_n^u$.

Thereby, given the subjective discount factor, β, the present value at $t = 0$ of playing the cooperative strategy is given by

$$\begin{aligned} PV_c &= v_c^* + \beta\left[\phi_c v_c^* + (1-\phi_c)v_n^u\right] + \beta^2\left[\phi_c^2 v_c^* + (1-\phi_c^2)v_n^u\right] + \ldots \\ &= v_c^*\left[1 + \beta\phi_c + (\beta\phi_c)^2 + \ldots\right] + v_n^u\left[\beta(1-\phi_c) + \beta^2(1-\phi_c^2) + \ldots\right] \\ &= v_c^*\sum_{s\geq 0}(\beta\phi_c)^s + \beta v_n^u\sum_{s\geq 0}\beta^s - \beta\phi_c v_n^u\sum_{s\geq 0}(\beta\phi_c)^s \\ &= \frac{v_c^* - \beta\phi_c v_n^u}{1-\beta\phi_c} + \frac{\beta v_n^u}{1-\beta}. \end{aligned} \tag{B.1}$$

[29] It could be higher, for example, if $M(c) = u(c,c)$.

Now, lets refer to the deviating strategy. In $t = 0$, the expected payoff of deviating from the cooperative agreement is given by

$$v_d^* \ := \ \delta(\alpha M(d) + (1-\alpha)m(d)) + (1-\delta)u(d,c).$$

After deviating, the individual knows that the counterpart will punish her by playing n forever, so she will play n forever as well. Nevertheless, other player's actions may induce updating on the individual's parameters if they do not match the expected behavior. Other player's expected behavior in $t = 0$ is to play c (which the individual predicts it will happen with probability ϕ_c), and in $t \geq 1$ is to play n (which the individual predicts it will happen with probability ϕ_n, which is also downward bounded by $(1-\delta)$).[30]

In $t = 1$ the individual sees what the counterpart played on $t = 0$. If the counterpart played c in the previous period, then the individual makes no update and perceives an expected payoff of

$$v_n^* \ := \ \delta(\alpha M(n) + (1-\alpha)m(n)) + (1-\delta)u(n,n).$$

However, if the counterpart played any strategy $s_{-i} \in S$ different from c in the previous period, then the individual updates the ambiguity parameter to 1 and perceives an expected payoff of v_n^u. Again, the later situation is stationary. Adding up, the expected payoff of the deviating strategy in $t = 1$, seen from $t = 0$, is given by $\phi_c v_n^* + (1-\phi_c)v_n^u$.

Again, a recursive argument is followed. If the counterpart played c in $t = 0$ and, therefore, the individual made no update in $t = 1$, in $t = 2$ sees what the counterpart played in $t = 1$ and decides how to behave. If the counterpart played n, then the individual makes no update and perceives an expected payoff of v_n^*. Conversely, if the counterpart deviated from expected behavior, then the expected payoff is v_n^u. Hence, the expected payoff of the deviating strategy in $t = 2$, seen from $t = 0$, is given by $\phi_c\phi_n v_n^* + \phi_c(1-\phi_n)v_n^u + (1-\phi_c)v_n^u = \phi_c\phi_n v_n^* + (1-\phi_c\phi_n)v_n^u$. Straightforward calculations allow to conclude that the expected payoff of the deviating strategy in $t = T$, seen from $t = 0$, is given by $\phi_c\phi_n^{(T-1)}v_n^* + \left(1-\phi_c\phi_n^{(T-1)}\right)v_n^u$.

Thereby, given the subjective discount factor, β, the present value of playing the deviating strategy is given by

$$\begin{aligned}
PV_d &= v_d^* + \beta\left[\phi_c v_n^* + (1-\phi_c)v_n^u\right] + \beta^2\left[\phi_c\phi_n v_n^* + (1-\phi_c\phi_n)v_n^u\right] + ... \\
&= v_d^* + \beta\phi_c v_n^*\left[1 + \beta\phi_n + (\beta\phi_n)^2 + ...\right] + \beta v_n^u\left[1 + \beta + \beta^2 + ...\right] - \beta\phi_c v_n^u\left[1 + \beta\phi_n + (\beta\phi_n)^2 + ...\right] \\
&= v_d^* + \beta\phi_c v_n^* \sum_{s\geq 0}(\beta\phi_n)^s + \beta v_n^u \sum_{s\geq 0}\beta^s - \beta\phi_c v_n^u \sum_{s\geq 0}(\beta\phi_n)^s \\
&= v_d^* + \frac{\beta\phi_c(v_n^* - v_n^u)}{1-\beta\phi_n} + \frac{\beta v_n^u}{1-\beta}. \qquad \text{(B.2)}
\end{aligned}$$

Putting together (B.1) and (B.2) yields to (7).

Finally, (9) comes straightforward after algebra by replacing $\phi_c = \phi_n = \phi$ in (7). ■

[30]It could be higher, for example, if $m(n) = u(n,n)$.

References

Aoyagi, M. (1996): "Reputation and Dynamic Stackelberg Leadership in Infinitely Repeated Games," *Journal of Economic Theory*, 71(2), 378–393.

Athey, S., and K. Bagwell (2001): "Optimal Collusion with Private Information," *RAND Journal of Economics*, 32(3), 428–65.

Athey, S., K. Bagwell, and C. Sanchirico (2004): "Collusion and Price Rigidity," *Review of Economic Studies*, 71(2), 317–349.

Bagwell, K., and R. Staiger (1997): "Collusion Over the Business Cycle," *RAND Journal of Economics*, 28(1), 82–106.

Chan, J. (2000): "On the Non-Existence of Reputation Effects in Two-Person Infinitely-Repeated Games," Economics Working Paper Archive 441, The Johns Hopkins University,Department of Economics.

Chateauneuf, A., J. Eichberger, and S. Grant (2007): "Choice under uncertainty with the best and worst in mind: Neo-additive capacities," *Journal of Economic Theory*, 137(1), 538–567.

Chen, Z., and L. Epstein (2002): "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica*, 70(4), 1403–1443.

Choquet, G. (1954): "Theory of capacities," in *Annales de l'institut Fourier*, vol. 5, pp. 131–295. Institut Fourier.

Compte, O. (1998): "Communication in Repeated Games with Imperfect Private Monitoring," *Econometrica*, 66(3), 597–626.

——— (2002): "On Failing to Cooperate When Monitoring Is Private," *Journal of Economic Theory*, 102(1), 151–188.

Conlon, J. R. (2003): "Hope springs eternal: Learning and the Stability of Cooperation in Short Horizon Repeated Games," *Journal of Economic Theory*, 112(1), 35–65.

Cripps, M. W., G. J. Mailath, and L. Samuelson (2004): "Imperfect Monitoring and Impermanent Reputations," *Econometrica*, 72(2), 407–432.

——— (2007): "Disappearing private reputations in long-run relationships," *Journal of Economic Theory*, 134(1), 287–316.

Dimitri, N. (2005): "Cooperation with Strategy-Dependent Uncertainty Attitude," Quaerni 457, Universita degli Studi di Siena, Dipartimento di Economia Politica.

Dow, J., and S. R. D. C. Werlang (1994): "Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction," *Journal of Economic Theory*, 64(2), 305 – 324.

Eichberger, J., S. Grant, and D. Kelsey (2010): "Comparing Three Ways to Update Choquet Beliefs," *Economics Letters*, 107(2), 91–94.

Eichberger, J., D. Kelsey, and B. C. Schipper (2009): "Ambiguity and social interaction," *Oxford Economic Papers*, pp. 355–379.

Ellison, G. (1994): "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching," *Review of Economic Studies*, 61(3), 567–588.

Ellsberg, D. (1961): "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics*, 75, 643–669.

Ely, J. C., and J. Valimaki (2002): "A Robust Folk Theorem for the Prisoner's Dilemma," *Journal of Economic Theory*, 102(1), 84–105.

Etner, J., M. Jeleva, and J.-M. Tallon (2012): "Decision theory under ambiguity," *Journal of Economic Surveys*, 26(2), 234–270.

Fonseca, M. A., and H. T. Normann (2012): "Explicit vs. Tacit Collusion. The Impact of Communication in Oligopoly Experiments," *European Economic Review*, 56(8), 1759–1772.

Fontini, F. (2005): "Cournot Oligopoly Under Strategic Uncertainty With Optimistic And Pessimistic Firms," *Metroeconomica*, 56(3), 318–333.

Gilboa, I., and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18(2), 141–153.

Green, E. J., and R. H. Porter (1984): "Noncooperative Collusion under Imperfect Price Information," *Econometrica*, 52(1), 87–100.

Haller, H. (2000): "Non-Additive Beliefs in Solvable Games," *Theory and Decision*, 49, 313–338.

Kandori, M. (1992): "The Use of Information in Repeated Games with Imperfect Monitoring," *Review of Economic Studies*, 59(3), 581–593.

Kandori, M., and H. Matsushima (1998): "Private Observation, Communication and Collusion," *Econometrica*, 66(3), 627–652.

Kreps, D. M., P. Milgrom, J. Roberts, and R. Wilson (1982): "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," *Journal of Economic Theory*, 27(2), 245–252.

Mailath, G., and L. Samuelson (2006): *Repeated Games and Reputations: Long-Run Relationships*. Oxford University Press.

Marinacci, M. (2000): "Ambiguous Games," *Games and Economic Behavior*, 31(2), 191 – 219.

Nowak, M., and R. Highfield (2011): *Altruism, Evolution, and Why We Need Each Other to Succeed*. Free Press.

Piccione, M. (2002): "The Repeated Prisoner's Dilemma with Imperfect Private Monitoring," *Journal of Economic Theory*, 102(1), 70–83.

Rojas, C. (2012): "The Role of Demand Information and Monitoring in Tacit Collusion," *RAND Journal of Economics*, 43(1), 78–109.

Rotemberg, J., and G. Saloner (1986): "A Supergame-Theoretic Model of Price Wars during Booms," *American Economic Review*, 76(3), 390–407.

Rothe, J. (2011): "Uncertainty Aversion and Equilibrium in Extensive Games," in *Contributions to Game Theory and Management, Vol. IV: The Fourth International Conference Game Theory and Management*, ed. by L. Petrosyan, and N. Zenkevich, pp. 389–406. St. Petersburg University, Russia.

Savage, L. (1954): *The foundations of statistics*. New-York: John Wiley.

Sekiguchi, T. (1997): "Efficiency in Repeated Prisoner's Dilemma with Private Monitoring," *Journal of Economic Theory*, 76(2), 345–361.

Watson, J. (1994): "Cooperation in the Infinitely Repeated Prisoners' Dilemma with Perturbations," *Games and Economic Behavior*, 7(2), 260–285.