Computing the Value of Information of Quadratic Decision Problems and its Non-Negativity Conditions*

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Abstract

There are many examples in the literature of non-cooperative games in which players prefer not to have additional information in order to improve their payoff. We present a general game in which, if one of the players improves his payoff upon obtaining more information, the other player's payoff worsens in such a way that there is a net social loss due to having more information. How can we ensure this does not occur? The results of this paper are (1) the mathematical expression of the (social) value of information in a quadratic non-cooperative game, and (2) the conditions that ensure the social value of information is non-negative.

Key words. Information structure; Nash equilibrium; value of information.

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1 Introduction

The information value problem poses the question of the amount a decision-maker is willing to pay for increasing the *quantity of available information* and thus improve his decision. The answer will depend on which of two basic contexts are under consideration: a decision-maker who has no interaction with other players, or one who does have strategic interaction (typically a non-cooperative game). In general terms, the value

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of information is always non-negative for a decision-maker without interaction as in the classic work of Blackwell is shown (Blackwell 1951, [5]), but in a game situation it can be negative. This article presents the non-negativity conditions for the value of information in the case of a game with quadratic cost functions.

There are many examples in the literature of non-cooperative games in which players prefer not to have additional information in order to improve their payoff (see for example Kamien 1990, [14]; Neyman 1991, [18]; Gossner 2000, [9]) for the general Bayesian games case; and Bean 1997, [3]; Korilis 1999, [15] for a non-cooperative transportation network).

We present a general quadratic bayesian game in which, if one of the players improves his payoff upon obtaining more information, the other player's payoff worsens in such a way that there is a net social loss due to having more information. How can we ensure this does not occur? The results of this paper are (1) the mathematical expression of the (social) value of information in a quadratic non-cooperative game, and (2) the conditions that ensure the social value of information is non-negative.

Because the class of games presented here have resource constraints, our model is similar to the classic *Problem of the Commons* popularized and extended by Garrett Hardin in his 1968 Science essay *The Tragedy of the Commons*, [10]. Given the structure of this class non-cooperative games, the Nash equilibrium solution causes an over utilization of the common resources with a consequent loss of society. Although in our case, it is not possible to talk about an over-utilization of the information resource, we can however deal with the fundamental problem, how to ensure that the use of information has a social positive value.

Because the quadratic nature of the loss functions we consider, a large class of games of this type may be found in the literature. For example, the equilibrium solution to the classic Cournot duopoly (Fudenberg and Tirole 1991, [8], p. 215) can be obtained as a Nash equilibrium solution of the this game in which each player must approximate the demand of a single homogeneous good. In Hinich and Enelow's spatial voting theory (1984, [12]) the authors model voters' objective function as an approximation of their ideal policy. Pursuit-evasion (Isaacs 1965, [13], p. 67) can also be modeled as an approximation game.

The article is structured as follows: in the following section, we model the problem of decision-making in the absence of strategic interaction and showed that, in this case, the value of information is always non-negative. The model of the information that is developed in this section is extended in the following section (Section 3) to the case where the players must decide in the presence of strategic interaction. This section demonstrates the existence and uniqueness of Nash equilibrium, and identifies conditions under which the social value of the information is non-negative.

2 The Case Without Strategic Interaction

2.1 Some Examples

In this section we introduce the model of the problem of Bayesian decision without strategic interaction that we extend in the next section to the Bayesian game and, moreover, we show that the value of information in the case of an agent without strategic interaction is always non-negative.

Example 2.1 (Unconstrained Discrete Case)**.** In a decision-maker context without strategic interaction, suppose that he must estimate the demand of a product which is represented by a random variable taking two different values: ω_H for *high* demand or ω_L for *l*ow demand. However, the agent is faced to two situations depending on the availability of information. In the first situation, he has only an *a priori* information about de demand ω; we say the the agent is *uninformed.* In the second situation, he has, in addition to the *a priori* information, a signal ξ of ω, such that it reports two values ξ_H and ξ_L, indicating an high and low demand respectively; we say that the agent is *informed.*

The *a priori* mass function of ω is given by a function g defined as $g(\omega_H) \doteq \theta$, where $\theta \in \;]0,1[$, and $g(\omega_L) \doteq 1-\theta$. The signal is distributed according to the marginal mass function h as follows: $h(\xi_H|\omega_H) = \alpha$, $h(\xi_L|\omega_H) = 1-\alpha$, $h(\xi_L|\omega_L) = \beta$ and $h(\xi_H|\omega_L) = 1-\beta$. As with the θ parameter, α, β must be in the open interval $]0,1[$. The marginal mass function h can be represented in the following matrix form

$$\begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

Now, we try to answer the question about the value of the signals for the agent. How much is the agent willing to pay for this *information service*? To answer this question, the agent needs to define a *loss function* to measure how close is his estimate demand to the *actual* demand, and so to measure the information value. In the bayesian theory, is customary to define this loss function as an approximation problem (see Berger 1985, [4], p. 136) as follows: let $\overline{x}$ and $\overline{y}$ be the best demand estimation in the uninformed situation and in the informed situation respectively, the loss function is given by $\mathbf{E}(\overline{x}-\omega)^2$ and $\mathbf{E}(\overline{y}-\omega)^2$ for each case, and the value of the information is given by the difference $\mathbf{E}(\overline{x}-\omega)^2 - \mathbf{E}(\overline{y}-\omega)^2$.

Since the information value depend on the best approximation in each situation, they are calculated by the agent as follows. In the uninformed situation, the best approximation is given by

$$\overline{x} = \mathbf{E}(\omega) = g(\omega_H)\omega_H + g(\omega_L)\omega_L = \theta\omega_H + (1-\theta)\omega_L,$$

while in the informed situation, the agent calculate first an *a posteriori* mass function of ω, i.e. an actualized mass function g of ω, given the signal ξ. Next, with the actualized mass function of ω, he calculates the best approximation. In bayesian theory is customary to denote this *a posteriori* (actualized) mass function as $g(\omega|\xi)$. The best approximation, in the informed situation, is given by

$$\overline{y}_H = \mathbf{E}(\omega|\xi_H) = g(\omega_H|\xi_H)\omega_H + g(\omega_L|\xi_H)\omega_L,$$

in case the agent observes signal ξ_H, and

$$\overline{y}_L = \mathbf{E}(\omega|\xi_L) = g(\omega_H|\xi_L)\omega_H + g(\omega_L|\xi_L)\omega_L,$$

in case the agent observes signal ξ_L. We write the best solution as $\overline{y} \doteq (\overline{y}_H, \overline{y}_L)$. Now, the *a posteriori* mass function of ω is given by

$$g(\omega_i|\xi_j) = \frac{h(\xi_j|\omega_i)g(\omega_i)}{\sum_{i,j\in\{H,L\}} h(\xi_j|\omega_i)g(\omega_i)} \quad \text{for all } i,j \in \{H,L\},$$

which corresponds to the Bayes Theorem (see Berger 1985, [4], p. 126). As example, we calculate $g(\omega_H|\xi_H)$

$$g(\omega_H|\xi_H) = \frac{h(\xi_H|\omega_H)g(\omega_H)}{h(\xi_H|\omega_H)g(\omega_H) + h(\xi_H|\omega_L)g(\omega_L)} = \frac{\alpha\theta}{\alpha\theta + (1-\beta)(1-\theta)}.$$

To simplify the algebraic manipulation, we assign to ω_H the value 12, the high demand, and to ω_L the value 6. Furthermore, we define $\alpha \doteq \beta$ and write β at all. With these assumptions, the best approximation in each situation is obtained by

$$\overline{x} = \frac{1}{6}(1+\theta),$$

in the uninformed case, and

$$\overline{y}_H = 6\frac{3\,\theta\,\beta - \theta - \beta + 1}{2\,\theta\,\beta - \theta - \beta + 1} \quad \text{and} \quad \overline{y}_L = 6\,\frac{3\,\theta\,\beta - 2\,\theta - \beta}{2\,\theta\,\beta - \theta - \beta},$$

in the informed situation. Whereas the information value can be obtained in terms of θ and β, as the following picture shows

By observing Figure 1, we note that the information value is higher when $\theta = 1/2$ (and β is near of 0 or 1), because it means that the agent has a *a priori* information that is indifferent between the high and the low demand. Likewise, the information value is lower if $\beta = 1/2$ (and $\theta \in\ ]0,1[$), because, in this case, the signals report no additional knowledge about the demand. ■

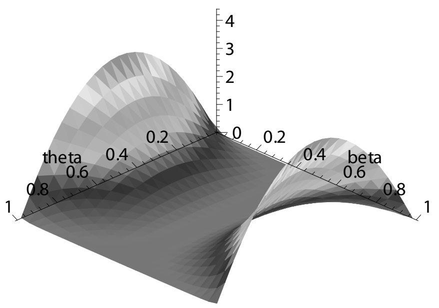

Figure 1: Information Value of the Unconstrained Discrete Case

Example 2.2 (Unconstrained Gaussian Case)**.** Let us consider the approximation problem of a random demand like in Example 2.1, but in the continuous context with gaussian probability densities. The agent wants to approximate the random variable ω representing, the future demand for a particular product, who has a *a priori* knowledge. Suppose that the agent is faced with two situations. In the first situation, the agent does not have any additional information to *a priori* knowledge of ω (the agent is *uninformed*). In the second situation, the agent has a signal ξ of the random variable ω (i.e. the agent is *informed*). Let ω be the random demand distributed as a normal with *a priori* density g given by

$$g(\omega) \doteq \frac{1}{\sqrt{2\pi}\sigma_\omega} \exp\left(-\frac{(\omega - \overline{\omega})^2}{2\sigma_\omega^2}\right),$$

i.e. ω is distributed as normal with mean $\overline{\omega}$ and standard deviation equal to σ_ω. Besides the signal ξ is distributed as normal with mean ω and standard deviation σ_ξ, then it has a density

$$h(\xi|\omega) \doteq \frac{1}{\sqrt{2\pi}\sigma_\xi} \exp\left(-\frac{(\xi - \omega)^2}{2\sigma_\xi^2}\right).$$

From the bayesian decision theory, we know that the best approximation of ω is given by $\overline{x} = \overline{\omega}$ in the case of ignorance and $\overline{y}(\xi) = \mathbf{E}(u|\xi)$, in the case where the agent has a signal ξ. In the latter case, the best approximation $\overline{y}$ can be calculated as follows (see Berger 1985, pp. 127 and 161, [4]):

$$\begin{aligned} \overline{y}(\xi) = \mathbf{E}(\omega|\xi) &= \frac{\int_{\mathbb{R}} \omega h(\xi|\omega) g(\omega) d\omega}{\int_{\mathbb{R}} h(\xi|\omega) g(\omega) d\omega} \\ &= \frac{\sigma_\xi^2}{\sigma_\omega^2 + \sigma_\xi^2}\overline{\omega} + \frac{\sigma_\omega^2}{\sigma_\omega^2 + \sigma_\xi^2}\xi. \end{aligned}$$

Let us observe that the best approximation $\overline{y}$ is a linear function of the signal ξ. With these calculations, we obtain the information value of the signal like in Example 2.1 given by

$$\frac{1}{2}(\mathbf{E}(\overline{x}-\omega)^2 - \mathbf{E}(\overline{y}-\omega)^2) = \frac{1}{2}\frac{\sigma_\omega^4}{\sigma_\omega^2+\sigma_\xi^2}.$$

Note, like in Example 2.1, the information value is always non-negative, and it is higher if the standard deviation of the *a priori* density function, σ_ω, of ω is high or the standard deviation of the signal, σ_ξ, is low. ■

The following example develops the approximation problem in a broader context of probabilities spaces.

Example 2.3 (Unconstrained Generalized Incomplete Case)**.** Let us suppose Ω to be a (non empty) set of states of the nature and a structure of information $\mathcal{B}$ defined as a σ-algebra in Ω. Moreover, we define the probability space in the usual manner $(\Omega, \mathcal{B}, \mathbf{P})$ and the events space as $\mathcal{L}^2(\Omega, \mathcal{B}, \mathbf{P})$, that is, the set of $\mathcal{B}$-measurable random variables defined in Ω with finite variance. Let us consider two sub-σ-algebras of $\mathcal{B}$, $\mathcal{E}$ and $\mathcal{F}$, such that $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{B}$. This can be interpreted as the information structures less informative ($\mathcal{E}$) and more informative ($\mathcal{F}$). The strategies space is defined as $\mathcal{L}^2(\Omega, \mathcal{E}, \mathbf{P})$, a subspace of $\mathcal{L}^2(\Omega, \mathcal{B}, \mathbf{P})$ in the case less informative and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ in the case more informative. Notice that the strategies space can be also identified with the space of available information, that is, all the random variables with finite variance depend on the signal are also possible strategies of available to the agent.

We now obtain the formulations of Examples 2.1 and 2.2 from this general model in probabilities spaces. In Example 2.1, based on discrete random variables, Ω is the set $\{\omega_H, \omega_L\} \times \{\xi_H, \xi_L\}$ and the σ-algebra $\mathcal{B}$ is the power set of $\{\omega_H, \omega_L\} \times \{\xi_H, \xi_L\}$. The sub-$\sigma$-algebra $\mathcal{E}$ (in the uninformed situation) is given by the σ-algebra trivial, i.e. $\{\emptyset, \Omega\}$, whereas the sub-σ-algebra $\mathcal{F}$ (in the informed situation) is generated by the set[1] $\{(\omega_H, \xi_H), (\omega_L, \xi_H)\}$, i.e. the agent can differentiate between the signal ξ_H or ξ_L, but not the true state of the nature ω_H or ω_L. Let us observe that $\mathcal{E} \subsetneq \mathcal{F} \subsetneq \mathcal{B}$. Furthermore, the probability measure is given by

$$\mathbf{P}(\{(\omega_i, \xi_j)\}) \doteq h(\xi_j|\omega_i)g(\omega_i) \quad \text{for all} \quad (\omega_i, \xi_j) \in \Omega,$$

where g and h are defined as before.

In Example 2.2 of continuous gaussian random variables, Ω is $\mathbb{R}^2$, with elements $(\omega, \xi) \in \Omega$; the σ-algebra $\mathcal{B}$ is $\mathcal{B}(\mathbb{R}^2)$, the set of Borel subset of $\mathbb{R}^2$; the sub-σ-algebra $\mathcal{E}$ is given by $\{\emptyset, \Omega\}$; and the sub-σ-algebra $\mathcal{F}$ by $\{\{\emptyset, \mathbb{R}\} \otimes \mathcal{B}(\mathbb{R})\}$. The probability measure is given by and

[1] A σ-algebra in Ω generated by a subset A of Ω, written by $\sigma(A)$, is defined as the intersection of all the σ-algebra in Ω that contain A. See Dudley 1989, [7], p. 64.

$$\mathbf{P}(B) \doteq \int_B h(\xi|\omega)g(\omega)d\omega d\xi \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^2),$$

where g and h are the probability densities defined in the respective Example unconstrained gaussian case.

Finally, in both cases, the loss function to be minimized is given by

$$f(v) \doteq \frac{1}{2}\mathbf{E}(v-\omega)^2,$$

where ω is a random variable of the probabilities space $(\Omega, \mathcal{B}, \mathbf{P})$. In the uninformed case, ω is being approximated by the strategy $\overline{x}$, that is an element of the strategies subspace $\mathcal{L}^2(\Omega, \mathcal{E}, \mathbf{P})$; whereas in the informed case, the sub-σ-algebra $\mathcal{F}$ rather than $\mathcal{E}$ is considered. From probability theory, we know that the optimal solution of the problem on the σ-algebra $\mathcal{E}$ (resp. $\mathcal{F}$) is given by $\overline{x} = \mathbf{E}(\omega|\mathcal{E})$, where $\mathbf{E}(\cdot|\mathcal{E})$ is the conditional expectation given the information structure $\mathcal{E}$ (see Dudley 1989, [7], Theorem 10.2.9). ■

2.2 Non-Negativity of the Information Value

We now consider the general case and show that the Information Value is always non-negative. Let V be a Hilbert space (an **events space**, modeled as $\mathcal{L}^2(\Omega, \mathcal{B}, \mathbf{P})$ in the previous example), endowed with an inner product $\langle v, w\rangle$ for all $v, w \in V$ (modeled as $\mathbf{E}(vw)$ in the example), which gives rise to the norm $|\cdot| \doteq \langle \cdot, \cdot\rangle^{1/2}$ on V. Further, let us define the closed subspace E (resp. F) as the **strategies space** (modeled as $\mathcal{L}^2(\Omega, \mathcal{E}, \mathbf{P})$ and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ respectively in the preceding example).

Let K be the set of strategies imposed by the available resources, which is a convex, closed subset of V, and consider $\mathcal{V} = \mathcal{V}_K$ the family of closed subspaces E of V, satisfying $K_E \doteq K \cap E \neq \emptyset$. Here K_E is the **set of feasible strategies**. Furthermore, let the **loss function** $f : V \to \mathbb{R} \cup \{+\infty\}$ being a convex, lower semicontinuous (lsc) function such that $\text{dom} f \doteq \{x \in V : f(x) < +\infty\} \neq \emptyset$. Let us consider a function $h : \mathcal{V} \times V \to \mathbb{R} \cup \{+\infty\}$ such that for all $x \in V$, for all $E, F \in \mathcal{V}$, $E \subseteq F$, it satisfies

$$h(F, x) \leq h(E, x). \tag{2.1}$$

A very interesting case occurs when $h(E, x) = i_{K_E}(x)$ is the indicator function defined on V, by $i_{K_E}(x) = 0$ if $x \in K_E$, and $+\infty$ elsewhere. Given $E \in \mathcal{V}_K$ and setting

$$f_E(x) \doteq f(x) + h(E, x),$$

we are concern with the following optimization problem without strategic interaction

$$\min\{f(x) + h(E, x) : \ x \in V\}, \tag{$P_1(E)$}$$

which is an unconstrained minimization problem. Its optimal value is denoted by[2] $\min P_1(E)$. For any closed subspace F of V containing $E \in \mathcal{V}_K$ (thus $F \in \mathcal{V}_K$), it is defined the information value of F with respect to E as

$$I_1(E,F) \doteq \min P_1(E) - \min P_1(F).$$

This value expresses the maximum amount the agent is willing to pay for obtaining additional information, and the next theorem asserts that the agent always prefers to have more information, since additional information has no cost, as expected.

Theorem 2.4 (Non-negativity of I_1)**.** *Let $E, F \in \mathcal{V}_K$ and f, h as above satisfying (2.1) with $E \subseteq F$ and* $\min P_1(F) > -\infty$*. Then, the information value is non-negative, that is, $I_1(E,F) \geq 0$.*

Proof. It follows from
$I_1(E,F) = \min P_1(E) - \min P_1(F) \geq \min P_1(F) - \min P_1(F) = 0.$ □

Some illustrative examples will be exhibited in Subsection 2.3.

2.3 Computing the Information Value

We now compute the information value when the loss function has a particular quadratic form in the case without strategic interaction. To that end some notations are needed. Let V be the events space as before, and R be the **resources space**, with V and R being Hilbert spaces. Let $A : V \to R$ be the **technology operator** which is linear and bounded; it transforms strategies into resources. Given $b \in \mathcal{R}(A) = A(V)$ a resource, the convex closed set of resource constrains is given by $K = \{x \in V : Ax = b\}$. We define $\mathcal{V}_K$, as before, to be the family of closed strategies subspaces E of V, so that (the feasible strategies) $K_E \doteq K \cap E \neq \emptyset$. In order to define the loss function, we consider a linear bounded operator $M : V \to V$ and an **objective** $u \in V$ that the agent wants to approximate by using the feasible strategies.
We are concern with the problem

$$\min\{\frac{1}{2}|Mx - u|^2 : x \in K_E\} \qquad (Q_1(E))$$

We impose the following hypothesis on $Q_1(E)$:

Hypothesis (H1): For each $E \in \mathcal{V}_K$, the problem $Q_1(E)$ has a solution (Theorem 1.1. of Aubin 1993 in [1]).

[2]Sufficient conditions ensuring existence of solutions to problem $(P_1(E))$ can be found in Aubin 1993, Theorem 1.1, [1].

Lemma 2.5. *If $\overline{x} \in K_E$ is a solution to $Q_1(E)$, then*

$$\langle M^*M\overline{x} - M^*u, v\rangle = 0 \quad \textit{for all } v \in \mathcal{N}(A) \cap E.$$

Proof. By setting $f(x) = \frac{1}{2}|Mx - u|^2$ we get $\nabla f(\overline{x}) = M^*M\overline{x} - M^*u$, and since $x - \overline{x} \in \mathcal{N}(A) \cap E$, we apply Lemma A.4 with $C = K_E$ to obtain the desired result. □

Before going further, we need to introduce the following concepts. For a given $E \in \mathcal{V}_K$, we say that $A : V \to R$, as above, is **limiting observable** through E if $\overline{A^*(R)} \subseteq E$, or equivalently, if $\mathcal{N}(A)^\perp \subseteq E$ since $\overline{A^*(R)} = \mathcal{N}(A)^\perp$ by Theorem 5.22.6 of Naylor 1982, [16]. Similarly, we say that A is **complementary limiting observable** through E if $\overline{A^*(R)}^\perp \subseteq E$, or equivalently, if $\mathcal{N}(A) \subseteq E$.

Moreover, we say that E **reduces** a linear bounded operator $T : V \to V$ if $T(E) \subseteq E$ and $T(E^\perp) \subseteq E^\perp$. It means that T is completely characterized by its restrictions to E and $E^\perp$ (see de paragraph before de Theorem 5.22.4 of Naylor 1982, [16]). It often happens that these restrictions of T are simpler than T itself. Notice that, $T^*(E) \subseteq E \iff T(E^\perp) \subseteq E^\perp$, by definition of T^* and the equality $E = E^{\perp\perp}$ since E is a closed subspace. Consequently, when T is selfadjoint, that is, $T^* = T$, the reduction of T through E is equivalent to the invariance of E under T, that is, $T(E) \subseteq E$. The next example shows an instance of A which clarifies our notion of observability.

Example 2.6 (Constrained Generalized Incomplete Case)**.** Take $A : V \to \mathbb{R}$ defined by

$$A(x) \doteq \mathbf{E}(ax) \quad (a \in V)$$

as the technology operator. It is not difficult to prove that,

$$\mathcal{N}(A)^\perp \subseteq E \iff a \in E.$$

This justifies our notion of observability, which says roughly speaking that a is *observable* if $a \in E$, i.e., A is observable through E. Here, $A^* : \mathbb{R} \to V$, is defined by $A^*\lambda = \lambda a$, and thus $A^*(\mathbb{R}) = \mathcal{N}(A)^\perp$.

Proposition 2.7. *Let $E \in \mathcal{V}_K$, A be the technology operator such that either A is limiting observable through E or complementary limiting observable through E. If $\overline{x}$ is a solution to problem $Q_1(E)$, then*

$$P_E M^*M\overline{x} - P_E M^*u \in \mathcal{N}(A)^\perp$$

Thus, if $M^*M(E) \subseteq E$, we have

$$M^*M\overline{x} - P_E M^*u \in \mathcal{N}(A)^\perp.$$

Proof. Because of Proposition 2.5, we have

$$\langle M^*M\overline{x} - M^*u, P_{E\cap\mathcal{N}(A)}v\rangle = 0 \quad \text{for all } v \in V.$$

But, if $\mathcal{N}(A) \subseteq E$ or $\mathcal{N}(A)^\perp \subseteq E$ then $P_{E\cap\mathcal{N}(A)} = P_E P_{\mathcal{N}(A)}$ by Lemma A.1. Thus

$$\langle P_E M^*M\overline{x} - P_E M^*u, P_{\mathcal{N}(A)}v\rangle = 0 \quad \text{for all } v \in V,$$

which implies that

$$P_E M^*M\overline{x} - P_E M^*u \in \mathcal{N}(A)^\perp,$$

proving the result. □

The following theorem is important by itself. Given two closed subspaces, one including the other, such a theorem expresses under some mild assumptions that a solution associated to one subspace can be obtained from a solution corresponding to the other.

Theorem 2.8. *Let $E, F \in \mathcal{V}_K$, A be the technology operator such that A^* is limiting observable through E with $E \subseteq F$. Assume that the closed subspaces E, F and $\mathcal{N}(A)$ reduce M^*M. If $\overline{x}$ and $\overline{y}$ are solutions to $Q_1(E)$ and $Q_1(F)$ respectively, then*

$$M^*M\overline{y} = M^*M\overline{x} + P_F M^*u - P_E M^*u.$$

Consequently, if $M^*M : V \to V$ is an isomorphism, then

$$\overline{y} = \overline{x} + (M^*M)^{-1}(P_F M^*u - P_E M^*u).$$

Proof. From the previous proposition and the reduction property of E and F by M^*M, it follows that

$$M^*M\overline{x} - P_E M^*u \in \mathcal{N}(A)^\perp \quad \text{and} \quad M^*M\overline{y} - P_F M^*u \in \mathcal{N}(A)^\perp.$$

Then

$$\langle M^*M\overline{x} - M^*M\overline{y} + P_F M^*u - P_E M^*u, z\rangle = 0 \quad \text{for all} \quad z \in \mathcal{N}(A).$$

Since $P_F M^*u - P_E M^*u \in E^\perp \subseteq \mathcal{N}(A)$ and

$$M^*M(\overline{x} - \overline{y}) \in M^*M(\mathcal{N}(A) \cap F) \subseteq M^*M(\mathcal{N}(A)) \subseteq \mathcal{N}(A),$$

a preceding equality implies

$$M^*M\overline{x} - M^*M\overline{y} + P_F M^*u - P_E M^*u \in \mathcal{N}(A),$$

and therefore

$$M^*M\overline{x} - M^*M\overline{y} + P_F M^*u - P_E M^*u = 0.$$

Obviously, if $\mathcal{N}(A) = \{0\}$ there is nothing to prove. □

We are now in a position to give a formula for the information value of F with respect to E.

Corollary 2.9. *Assume that E, F, A, M satisfy the assumptions of the previous theorem with $E \subseteq F$. If $\overline{x}$, $\overline{y}$ are solutions to $Q_1(E)$ and $Q_1(F)$ respectively, and $M^*M : V \to V$ is an isomorphism, the value of information is expressed by*

$$\begin{aligned} I_1(E,F) =& \langle P_{F^\perp} M^*u, (M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle \\ &+ \frac{1}{2}|M(M^*M)^{-1}P_F M^*u|^2 - \frac{1}{2}|M(M^*M)^{-1}P_E M^*u|^2. \end{aligned}$$

Proof. We apply the previous theorem to have

$$\begin{aligned} I_1(E,F) &= \frac{1}{2}|M\overline{x} - u|^2 - \frac{1}{2}|M\overline{y} - u|^2 \\ &= -\langle M\overline{x} - u, M(M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle \\ &\quad - \frac{1}{2}|M(M^*M)^{-1}(P_F M^*u - P_E M^*u)|^2. \end{aligned}$$

Let us denote by α the first term of the right hand side and by β the second one. Then, since $P_F M^*u - P_E M^*u \in E^\perp$ and $\overline{x} \in E$, one obtain

$$\begin{aligned} \alpha &= -\langle \overline{x}, P_F M^*u - P_E M^*u\rangle + \langle u, M(M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle \\ &= \langle P_F M^*u + P_{F^\perp} M^*u, (M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle \\ &= \langle P_F M^*u, (M^*M)^{-1}P_F M^*u\rangle - \langle P_F M^*u, (M^*M)^{-1}P_E M^*u\rangle \\ &\quad + \langle P_{F^\perp} M^*u, (M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle \\ &= |M(M^*M)^{-1}P_F M^*u|^2 - \langle P_F M^*u, (M^*M)^{-1}P_E M^*u\rangle \\ &\quad + \langle P_{F^\perp} M^*u, (M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \beta &= -\frac{1}{2}|M(M^*M)^{-1}(P_F M^*u - P_E M^*u)|^2 \\ &= -\frac{1}{2}|M(M^*M)^{-1}P_F M^*u|^2 + \langle M(M^*M)^{-1}P_F M^*u, M(M^*M)^{-1}P_E M^*u\rangle \\ &\quad - \frac{1}{2}|M(M^*M)^{-1}P_E M^*u|^2 \\ &= -\frac{1}{2}|M(M^*M)^{-1}P_F M^*u|^2 + \langle P_F M^*u, (M^*M)^{-1}P_E M^*u\rangle \\ &\quad - \frac{1}{2}|M(M^*M)^{-1}P_E M^*u|^2. \end{aligned}$$

Hence,

$$\begin{aligned} I_1(E,F) &= \alpha + \beta \\ &= \frac{1}{2}|M(M^*M)^{-1}P_F M^*u|^2 - \frac{1}{2}|M(M^*M)^{-1}P_E M^*u|^2 \\ &\quad + \langle P_{F^\perp} M^*u, (M^*M)^{-1}(P_F M^*u - P_E M^*u)\rangle, \end{aligned}$$

which is the desired result. □

Corollary 2.10. *Assume that E, F, A satisfy the assumptions of the previous theorem with $E \subseteq F$. If M is the identity operator in V, the value of information is expressed by*

$$I_1(E,F) = \frac{1}{2}|P_F u|^2 - \frac{1}{2}|P_E u|^2. \tag{2.2}$$

Example 2.11 (Unconstrained Generalized Incomplete Case)**.** In this Example, we calculate the information value in the context of probability spaces by using Formula 2.2 of Corollary 2.10. In this context, the agent minimizes on $E = \mathcal{L}^2(\Omega, \mathcal{E}, \mathbf{P})$ (respectively on $F = \mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ the loss function

$$f(v) \doteq \frac{1}{2}\mathbf{E}(v-u)^2,$$

where $u \in V = \mathcal{L}^2(\Omega, \mathcal{B}, \mathbf{P})$. From Example 2.3, we already know that the optimal solution to this problem is $\overline{x} = \mathbf{E}(u|\mathcal{E})$ in the uninformed case (resp. $\overline{y} = \mathbf{E}(u|\mathcal{F})$ in the informed case). The approximation problem can be expressed in abstract form as minimizing

$$f(v) \doteq \frac{1}{2}|v-u|^2,$$

where $u \in V$ and $v \in E$ or F depending on the uninformed or informed situation respectively. From functional analysis, we know that the optimal solution to this problem is $\overline{x} = P_E u$ (resp. $\overline{y} = P_F u$), where P_E (resp. P_F) is the orthogonal projector from V into E (resp. F). This linear bounded projector is idempotent ($P_E \circ P_E = P_E$) and self-adjoint ($\langle P_E v, w\rangle = \langle v, P_E w\rangle$ for all $v, w \in V$). In the context of the probabilities space, this projector takes the form $\mathbf{E}(\cdot|\mathcal{E})$ to approximate onto E, or $\mathbf{E}(\cdot|\mathcal{F})$ to approximate onto F respectively (see Dudley 1989, [7], Theorem 10.2.9). Hence,

$$0 \leq I_1(E,F) = \frac{1}{2}\mathbf{E}(\mathbf{E}(u|\mathcal{F})^2) - \frac{1}{2}\mathbf{E}(\mathbf{E}(u|\mathcal{E})^2).$$

■

Example 2.12 (Unconstrained Gaussian Case)**.** We use Formula 2.2 of Corollary 2.10 to calculate the information value in the case of continuous gaussian variables. Indeed,

$$
\begin{aligned}
0 \leq I_1(E,F) &= \frac{1}{2}\mathbf{E}(\mathbf{E}(u|\mathcal{F})^2) - \frac{1}{2}\mathbf{E}(\mathbf{E}(u|\mathcal{E})^2) \\
&= \frac{1}{2}\mathbf{E}\left(\frac{\sigma_\xi^2}{\sigma_\omega^2+\sigma_\xi^2}\overline{\omega} + \frac{\sigma_\omega^2}{\sigma_\omega^2+\sigma_\xi^2}\xi\right)^2 - \frac{1}{2}\mathbf{E}(\overline{\omega}^2) \\
&= \frac{1}{2}\frac{\sigma_\omega^4}{\sigma_\omega^2+\sigma_\xi^2},
\end{aligned}
$$

which is the same to the one obtained in Example 2.2. ■

Example 2.13 (Unconstrained Discrete Case – revisited)**.** In this Example, we calculate the information value obtained in Example 2.1 by using Formula 2.2 from Corollary 2.10 in the Hilbert spaces context. Nevertheless, we first determinate the matrix form of the operators P_E and P_F, likewise we find the matrix form of the norm included in it. We do not apply the formula directly in the discrete probability space context, but we develop the same model in an equivalent formulation in vectorial spaces with finite dimension. The aim of this Example is to illustrate and understand the abstract form of our formulation in Hilbert spaces, and furthermore to show how it solves a wide family of practical situations, where the variables are discrete and mostly finite.

First, we express in matrix form the norm $|\cdot|$ of Formula 2.2 and then we obtain the expression of the orthogonal projectors P_E and P_F. Recall that in Example 2.11, the space V is given by $\mathcal{L}^2(\Omega, \mathcal{B}, \mathbf{P})$, where Ω is given by the product $\{\omega_H, \omega_L\} \times \{\xi_H, \xi_L\}$; $\mathcal{B}$ is the power set in Ω and the probability measure defined by

$$\mathbf{P}(\{(\omega_i, \xi_j)\}) \doteq h(\xi_j|\omega_i)g(\omega_i) \quad \text{for all} \quad (\omega_i, \xi_j) \in \Omega.$$

Let us define following events on this space: the event *high demand signal* by $\{(\omega, \xi) : \xi = \xi_H\}$, that is equal to $\{(\omega_H, \xi_H), (\omega_L, \xi_H)\}$; likewise we define the event *low demand signal* by $\{(\omega, \xi) : \xi = \xi_L\}$. Furthermore, we define the following random variables as the characteristic function of the respective sets by

$$\mathbf{1}_{\bullet j}(\omega, \xi) \doteq \begin{cases} 1, & \text{if } \xi = \xi_j; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we define the random variables as the characteristic function of the *atoms* sets of Ω as follows

$$\mathbf{1}_{ij}(\omega, \xi) \doteq \begin{cases} 1, & \text{if } \omega \doteq \omega_i \text{ and } \xi = \xi_j; \\ 0, & \text{otherwise.} \end{cases}$$

With these definitions, we can express V as a space generated by the linear combination of members of the finite set $\{\mathbf{1}_{HH}, \mathbf{1}_{HL}, \mathbf{1}_{LH}, \mathbf{1}_{LL}\}$, so that we can write each random variable of V as $v(\omega_i, \xi_j) = \sum_{i,j} \mathbf{1}_{ij}(\omega_i, \xi_j) v_{ij}$, where we use the notation v_{ij} for the value $v(\omega_i, \xi_j)$ and $i, j \in \{H, L\}$. Likewise, the sub-space F is generated by the set $\{\mathbf{1}_{\bullet H}, \mathbf{1}_{\bullet L}\}$. Now, we can write the inner product of V as

$$\begin{aligned}
\langle v, w\rangle &\doteq \mathbf{E}(vw) \\
&= \int_\Omega v(\omega,\xi) w(\omega,\xi) h(\xi|\omega) g(\omega) d\omega d\xi \\
&= \sum_{i,j\in\{H,L\}} \mathbf{1}_{ij}(\omega_i,\xi_j) v_{ij} w_{ij} h_{ji} g_i \\
&= \sum_{i,j\in\{H,L\}} v_{ij} w_{ij} h_{ji} g_i,
\end{aligned}$$

which we can write (with notation abuse) as $v^T B w$, where B is a diagonal matrix given by

$$B \doteq \operatorname{diag}(\theta\alpha, \theta(1-\alpha), (1-\theta)(1-\beta), (1-\theta)\beta).$$

Then, we can say that the space V is the equivalent to the space (denoted by $V \equiv \mathbb{R}^4$), while

$$E \equiv \operatorname{span}\{(1,1,1,1)\} \quad \text{and} \quad F \equiv \operatorname{span}\{(1,0,1,0),(0,1,0,1)\}.$$

Furthermore, the loss function can be expressed as

$$f(v) \doteq \frac{1}{2}|v-u|^2 \equiv \frac{1}{2}(v-u)^T B (v-u).$$

We now determine the orthogonal projector P_F. This is done by using the inequality of Lemma A.4, which becomes an equality on subspace E. Let u be any element from V. Then the projection of $u \doteq (u_{HH}, u_{HL}, u_{LH}, u_{LL})$ on F, which we call $\overline{y}$, must satisfy

$$\langle y, u-\overline{y}\rangle \equiv y^T B(u-\overline{y}) = 0 \quad \text{for all } y \in F.$$

This equality is valid particularly for the elements e_i of the base F, from which we obtain the following system of two equations

$$\langle e_i, u-\overline{y}\rangle \equiv e_i^T B(u-\overline{y}) = 0 \quad \text{for all } i \in \{H, L\},$$

and in turn obtain the following result:

$$\overline{y}_H = \frac{\alpha\theta}{\theta\alpha+(1-\theta)(1-\beta)} u_{HH} + \frac{(1-\theta)(1-\beta)}{\theta\alpha+(1-\theta)(1-\beta)} u_{LH}$$

and

$$\overline{y}_L = \frac{(1-\alpha)\theta}{\theta(1-\alpha)+(1-\theta)\beta}u_{HL} + \frac{(1-\theta)\beta}{\theta(1-\alpha)+(1-\theta)\beta}u_{LL}.$$

Then, P_F can then be written in matrix form as

$$P_F = \begin{pmatrix} \frac{\alpha\theta}{\theta\alpha+(1-\theta)(1-\beta)} & 0 & \frac{(1-\theta)(1-\beta)}{\theta\alpha+(1-\theta)(1-\beta)} & 0 \\ 0 & \frac{(1-\alpha)\theta}{\theta(1-\alpha)+(1-\theta)\beta} & 0 & \frac{(1-\theta)\beta}{\theta(1-\alpha)+(1-\theta)\beta} \\ \frac{\alpha\theta}{\theta\alpha+(1-\theta)(1-\beta)} & 0 & \frac{(1-\theta)(1-\beta)}{\theta\alpha+(1-\theta)(1-\beta)} & 0 \\ 0 & \frac{(1-\alpha)\theta}{\theta(1-\alpha)+(1-\theta)\beta} & 0 & \frac{(1-\theta)\beta}{\theta(1-\alpha)+(1-\theta)\beta} \end{pmatrix}.$$

which is the orthogonal projector into the subspace F. It is easy to prove that this matrix satisfies the definition of an orthogonal projector, namely it is idempotent ($P_F \circ P_F = P_F$) and self-adjoint ($\langle P_F v, w\rangle = \langle v, P_F w\rangle$ for all $v, w \in V$). Likewise, it can be the orthogonal projector P_E, which is given by

$$P_E = \begin{pmatrix} \alpha\theta & \theta(1-\alpha) & (1-\theta)(1-\beta) & (1-\theta)\beta \\ \alpha\theta & \theta(1-\alpha) & (1-\theta)(1-\beta) & (1-\theta)\beta \\ \alpha\theta & \theta(1-\alpha) & (1-\theta)(1-\beta) & (1-\theta)\beta \\ \alpha\theta & \theta(1-\alpha) & (1-\theta)(1-\beta) & (1-\theta)\beta \end{pmatrix}.$$

Thus, it can be computed $P_E u$ and $P_F u$, and the information value $I_1(E, F)$, which are here omitted, because they are the same as in Example 2.1. ■

2.4 Extension to a Convex Quadratic Problem

In what follows, we show that our model considered in the previous section is not much restricted. To that end, we first give the following definition.

Definition 2.14. Given a bounded linear operator Q defined on the real Hilbert space V. We say that Q is positive if

$$\langle Qv, v\rangle \geq 0 \quad \text{for all } v \in V.$$

The following theorem collects (Theorems 12.32 and 12.33 of Dixit 1979, [6]) the main properties of positive operators. We recall that Q^* stands for the adjoint operator of Q.

Theorem 2.15. *Let V be a Hilbert space and $Q : V \to V$ be a bounded linear operator. The following assertions holds:*

(a) *If Q is positive then $Q = Q^*$ and the spectrum $\sigma(Q) \subseteq [0, +\infty)$;*

(b) *if Q is positive then there exists an unique bounded linear operator M which is positive satisfying $Q = M^*M$. Moreover, M is invertible provided Q is so.*

Let us consider the cost function

$$\tilde{f}(v) \doteq \frac{1}{2}\langle Qv, v\rangle - \langle b, v\rangle \quad (v \in V),$$

where Q is as above and $b \in V$. By applying the previous theorem we show that any function of the above form can be reduced to our model.

Proposition 2.16. *Let f as above with Q having inverse, then there exists an invertible positive linear bounded operator M, such that*

$$\tilde{f}(v) = \frac{1}{2}|Mv - u|^2 - \frac{1}{2}|u|^2,$$

with $u = M^{-1}b$. As a consequence, the value of information associated to $\tilde{f}$ and to f coincides, where

$$f(v) = \frac{1}{2}|Mv - u|^2.$$

Proof. We simply apply the previous result to get M satisfying

$$\langle Qv, v\rangle = |Mv|^2 \quad \text{and} \quad \langle b, v\rangle = \langle Mv, u\rangle, \tag{2.3}$$

from which the conclusion follows. □

3 The Case with Strategic Interaction

3.1 Some Examples

Example 3.1 (Constrained Gaussian Case)**.** Consider two players who are attempting to approximate a common objective denoted by ω. Afterwards, we consider different objectives. The loss function of player i ($i \in \{1, 2\}$) is given by the distance between this objective and $\overline{x}_i$, his best approximation. However, each player's approximation can be affected by the action $m_{ij}x_j$ of the other player. The effect $m_{ij}x_j$ of player j on player i may be interpreted as the action taken by player j to disturb player i's approximation. If ω is a random variable, the cost function of player i is expressed as

$$\frac{1}{2}\mathbf{E}(x_i + m_{ij}x_j - \omega)^2 \quad \text{with } i \neq j.$$

Further, we consider that the players share a limited common resource, typically human or materials resources. This restriction of the resource is modeled as $\mathbf{E}(a_1x_1 + a_2x_2) = b$, where a_i represents the ability of player i to transform resources into *outputs* and b the quantity of resources available to both players.

To illustrate the conditions that ensure a non-negative information value, we consider three cases of static bayesian games. In the first case (Case A), both players have no information about and this knowledge is common to both players. In Case B, only one of the players has a signal ξ containing information on ω. Finally, in Case C both players have the same signal ξ. Case B is thus an asymmetric information bayesian game, whereas cases A and C are bayesian games with symmetric information.

To determine the three games' respective equilibria we begin by specifying more precisely certain items introduced above. Like in Example 2.2, the objective ω is a real random variable that has an *a priori* normal density function with mean 0 and standard deviation 1. This information is common to both players in all cases.

In case A, player i knows that the opponent j plays an optimal strategy $\overline{x}_j$ and also both know that they have no more information than the common knowledge. Therefore, if player 2's best strategy is $\overline{x}_2$, then the reaction of player 1 ($\overline{x}_1$) to player 2's strategy satisfies

$$f_1(\overline{x}_1, \overline{x}_2) = \min\{\frac{1}{2}\mathbf{E}(x_1 + m_{12}\overline{x}_2 - \omega)^2 : \mathbf{E}(a_1 x_1 + a_2\overline{x}_2) = b\}.$$

Given the strategy $\overline{x}_2$, player 1 tries to find the optimal solution to the previous minimization problem. Thanks to the conditions of the first order, then there is a *price of resource* $\overline{p}$, so that $\overline{x}_1 + m_{12}\overline{x}_2 - \overline{p}a_1 = 0$ and further $\mathbf{E}(a_1\overline{x}_1 + a_2\overline{x}_2) = b$, that in this Case is $a_1\overline{x}_1 + a_2\overline{x}_2 = b$. Similarly, because player 2 optimizes his loss function, he obtains strategies satisfying $\overline{x}_2 + m_{21}\overline{x}_1 - \overline{p}a_2 = 0$. These three linear equations together uniquely determine equilibrium strategies $\overline{x}_1$ and $\overline{x}_2$.

To make the effect of symmetry/asymmetry of information available to both players clear, we assume without loss of generality, that a_1 and a_2 are constant and equal to a with $a \neq 0$; $b = 1$ and $m_{21} = m_{12} = m$ with $m \neq 1$. In this way we can isolate the effect of the asymmetry of information from the other parameters. The loss functions of both players are obtained from the solution of the equilibrium conditions and take the following value[3]

$$f_1(\overline{x}_1, \overline{x}_2) = f_2(\overline{x}_1, \overline{x}_2) = \frac{1}{8}\left(\frac{1+m}{a}\right)^2 + \frac{1}{2}.$$

In Case B, player 1 observes the signal ξ. We assume that the marginal density of signal ξ is also normal, with mean ω and standard deviation 1. Both players, in this instance, know that player 1 has the signal ξ. If we suppose that the strategy of player 1 is linear affine and takes the expression $\overline{x}_1 = \alpha_1 + \beta_1\xi_1$, then the equilibrium solution satisfies the condition $\overline{x}_1 + m\overline{x}_2 - \overline{p}a = 0$ and the resource constrain $a\overline{x}_1 + a\overline{x}_2 = b$.

[3]Obtaining these loss functions involves certain manipulations that were performed using the software program Maple V. The optimal equilibrium solutions ($\overline{x}_i$) are not presented here as they do not contribute significantly to our analysis.

This equations with the optimality condition of player 2, i.e. $\overline{x}_2 + m\overline{x}_1 - \overline{p}a = 0$, gives:

$$f_1(\overline{x}_1, \overline{x}_2) = \frac{1}{8}\left(\frac{1+m}{a}\right)^2 + \frac{1}{4} \quad \text{and} \quad f_2(\overline{x}_1, \overline{x}_2) = \frac{1}{8}\left(\frac{1+m}{a}\right)^2 + \frac{1}{4}((m-1)^2 + 1).$$

Case C, in which both players observe ξ, is solved in analogous fashion with the following result:

$$f_1(\overline{x}_1, \overline{x}_2) = f_2(\overline{x}_1, \overline{x}_2) = \frac{1}{8}\left(\frac{1+m}{a}\right)^2 + \frac{1}{4}.$$

Having set out the foregoing results we now propose the following game. Assume that each player has the option of using the additional observation ξ and knows that the opponent can use the same information or not. As before, both players play simultaneously. The gain from using the information obviously depends on whether the other player is also using it or not. Thus, player 1 can opt to use not the information (Decision I) or to use it (Decision II), and player 2 must also decide whether use it (Decision 2) or not (Decision 1). The gain or loss to player i from using the information is obtained as the difference between the respective values of the cost function for the equilibrium strategies with and without the additional information. If, for example[4], $m = \sqrt{3} + 1$ and we define then the payoff matrix of the game as shown in Table 1.

		Player 2	
		1	2
Player 1	*I*	$(0,0)$	$(-\frac{1}{2}, \frac{1}{4})$
	II	$(\frac{1}{4}, -\frac{1}{2})$	$(\frac{1}{4}, \frac{1}{4})$

Table 1: Payoff matrix of the information game

Observe that this game has a pure strategy Nash equilibrium that consists in both players preferring to use the additional information. In other words, the Nash equilibrium is $(\text{II}, 2)$.

In this paper we are interested in the problem represented in Table 1. Assume, then, that player 1 cannot use the available information or that the observation is too expensive for him to obtain it. In this situation, player 2 will use the available information to achieve a decrease $\frac{1}{4}$ in his costs while bringing about an increase $\frac{1}{2}$ in player 2's costs. In other words, while player 2 gains unilaterally, player 1 loses twice what player 1 gains. But of particular significance is that player 1's unilateral decision has a social cost equal to $\frac{1}{4}$. How can we ensure that this does not occur? What conditions must be imposed on the information available to the players and/or

[4]This value of m does not limit the generality of the analysis.

the interaction between them to ensure the social benefits of the information are non-negative? ■

Example 3.2 (Constrained Generalized Incomplete Case)**.** Let us first show how to represent the information structure available to the players in the context of probabilities spaces. The measure space $(\Omega, \mathcal{B})$ of the previous Example 3.1 is given by $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ where $\mathcal{B}(\mathbb{R}^2)$ are the Borel subset of $\mathbb{R}^2$. Depending of the case A, B or C, each player has a sub-σ-algebra of $\mathcal{B}(\mathbb{R}^2)$ representing the information structure available to him.

In Case A, both players are uninformed, that is, the information structure is the sub-σ-algebra $\{\emptyset, \Omega\}$ for each player. In Case B, player 1 has an signal represented by the information structure $\{\{\emptyset, \mathbb{R}\} \otimes \mathcal{B}(\mathbb{R})\}$, while the player 2 is uninformed. In Case C, both players receive the same signal, so that they have the information structure $\{\{\emptyset, \mathbb{R}\} \otimes \mathcal{B}(\mathbb{R})\}$. With these information structures, the probability measure is defined as in Example 2.3. ■

3.2 Definition of the Social Information Value

For $i = 1, 2$, let V_i be the event space for player i, which is Hilbert, both endowed with the inner product denoted by the same symbol $\langle \cdot, \cdot \rangle$. Let R be the **resources space** as Hilbert space too. For each $i = 1, 2$, let $A_i : V_i \to R$ be the **technology operator** of player i, being bounded and linear that transforms the strategies into resources. Similarly, for $i \neq j$, let the (i, j)**-interaction operator** $M_{ij} : V_j \to V_i$ which is bounded and linear. The element $u = (u_1, u_2) \in V_1 \times V_2$, where u_i is the individual **objective** of player i, is given. Let $E_1 \subseteq V_1$, $E_2 \subseteq V_2$, be the **strategies spaces** of player 1 and 2, which are closed subspaces; the **available resource** $b \in \mathcal{R}(A) = A(V_1 \times V_2)$ is given, where $A : V_1 \times V_2 \to R$ is defined by $A(x_1, x_2) = A_1x_1 + A_2x_2$.

Set $V \doteq V_1 \times V_2$. This is a Hilbert space too endowed with the inner product $\langle v, w \rangle \doteq \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle$ if $v_1 = (v_1, v_2) \in V_1 \times V_2$, $w = (w_1, w_2) \in V_1 \times V_2$. The bounded linear operator $M : V \to V$ defined by $M(v_1, v_2) = (v_1 + M_{12}v_2, v_2 + M_{21}v_1)$, is the **interaction operator** between both players. Let us denote by $E \doteq E_1 \times E_2$ and $K_E \doteq \{x \in E : Ax = b\}$.

The Nash equilibrium problem on E, $Q_2(E)$, consists in finding $\overline{x} = (\overline{x}_1, \overline{x}_2) \in K_E$ such that

$$\left.\begin{array}{ll} f_1(\overline{x}_1, \overline{x}_2, u_1) \leq f_1(x_1, \overline{x}_2, u_1) & \text{for all } x_1 \in E_1,\ A_1x_1 + A_2\overline{x}_2 = b \\ f_2(\overline{x}_1, \overline{x}_2, u_2) \leq f_2(\overline{x}_1, x_2, u_2) & \text{for all } x_2 \in E_2,\ A_1\overline{x}_1 + A_2x_2 = b \end{array}\right\} \qquad (Q_2(E))$$

Such an $\overline{x}$ is called a Nash equilibrium solution. The remainder of this section considers the following **loss functions**

$$f_i(x_1, x_2, u_i) \doteq \frac{1}{2}|x_i + M_{ij}x_j - u_i|^2 \ \ (i \neq j).$$

Obviously, the function f_i is convex in x_i. For a given $(\overline{x}_1, \overline{x}_2) \in K_E$, we set $K_{E_1} \doteq \{x_1 \in E_1 : A_1x_1 + A_2\overline{x}_2 = b\}$, $K_{E_2} \doteq \{x_2 \in E_2 : A_1\overline{x}_1 + A_2x_2 = b\}$, which are convex and closed sets. Obvioulsy $K_{E_1} \times K_{E_2} \subseteq K_E$.
The next proposition characterizes the Nash equilibrium solutions as solutions to a variational inequality problem, and it will be used to find a explicit formula for the information value.

Proposition 3.3. *Let $\overline{x} \in K_E$. Then, $\overline{x}$ is a Nash equilibrium solution to problem $Q_2(E)$ with the quadratic loss functions previously given if and only if*

$$\langle M\overline{x} - u, x - \overline{x}\rangle \geq 0 \quad \textit{for all } x = (x_1, x_2) \in K_{E_1} \times K_{E_2}.$$

Proof. If $\overline{x} = (\overline{x}_1, \overline{x}_2) \in K_E$ is a Nash equilibrium solution to $Q_2(E)$, then $\overline{x}_1$ (resp. $\overline{x}_2$) minimizes $f_1(\cdot, \overline{x}_2, u_1)$ (resp. $f_2(\overline{x}_1, \cdot, u_2)$) on K_{E_1} (resp. K_{E_2}). Then, by using Lemma A.4, we obtain

$$\langle \overline{x}_1 + M_{12}\overline{x}_2 - u_1, x_1 - \overline{x}_1\rangle \geq 0 \quad \text{for all } x_1 \in K_{E_1}$$

and

$$\langle \overline{x}_2 + M_{21}\overline{x}_1 - u_2, x_2 - \overline{x}_2\rangle \geq 0 \quad \text{for all } x_2 \in K_{E_2}.$$

Summing up the two expressions, it gives

$$\langle M\overline{x} - u, x - \overline{x}\rangle \geq 0 \quad \text{for all } x \in K_{E_1} \times K_{E_2}.$$

Conversely, assume that $\overline{x} \in K_E$ satisfies the previous variational inequality. In particular, if $x = (x_1, \overline{x}_2)$ with $x_1 \in K_{E_1}$ then

$$\langle \overline{x}_1 + M_{12}\overline{x}_2 - u_1, x_1 - \overline{x}_1\rangle \geq 0 \quad \text{for all } x_1 \in K_{E_1} \tag{3.1}$$

and analogously,

$$\langle \overline{x}_2 + M_{21}\overline{x}_1 - u_2, x_2 - \overline{x}_2\rangle \geq 0 \quad \text{for all } x_2 \in K_{E_2}. \tag{3.2}$$

By virtue of Lemma A.4, (3.1) and (3.2) assert that $\overline{x}$ is a Nash equilibrium solution to $Q_2(E)$. □

3.3 Computing de Social Information Value

We now define the (Social) information value for the problem $Q_2(E)$. Let F_1, F_2 be two closed subspaces of V such that $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$. Let $\overline{x} = (\overline{x}_1, \overline{x}_2) \in K_E$,

$\overline{y} = (\overline{y}_1, \overline{y}_2) \in K_F$ (with $F = F_1 \times F_2$) be Nash equilibrium solutions to problem $Q_2(E)$ and $Q_2(F)$ respectively. We define the **(social) information value of problem Q_2 of F with respect to E** by

$$I_2(E,F) \doteq \underbrace{f_1(\overline{x}_1,\overline{x}_2) + f_2(\overline{x}_1,\overline{x}_2)}_{\text{Solution in } E} - \underbrace{(f_1(\overline{y}_1,\overline{y}_2) + f_2(\overline{y}_1,\overline{y}_2))}_{\text{Solution in } F}.$$

The next result expresses the information value in terms of the interaction operator of the game.

Lemma 3.4. *The information value of problem $Q_2(E)$ can be written by*

$$I_2(E,F) = \frac{1}{2}|M\overline{x} - u|^2 - \frac{1}{2}|M\overline{y} - u|^2.$$

Proof. By taking into account the definition of M, we write

$$\begin{aligned}
I_2(E,F) &= \frac{1}{2}|\overline{x}_1 + M_{12}\overline{x}_2 - u_1|^2 + \frac{1}{2}|\overline{x}_2 + M_{21}\overline{x}_1 - u_2|^2 \\
&\quad - \left(\frac{1}{2}|\overline{y}_1 + M_{21}\overline{y}_1 - u_1|^2 + \frac{1}{2}|\overline{y}_2 + M_{21}\overline{y}_1 - u_2|^2\right) \\
&= \frac{1}{2}|\overline{x}_1 + M_{12}\overline{x}_2|^2 + \frac{1}{2}|\overline{x}_2 + M_{21}\overline{x}_1|^2 \\
&\quad - \langle \overline{x}_1 + M_{12}\overline{x}_2, u_1\rangle - \langle \overline{x}_2 + M_{21}\overline{x}_1, u_1\rangle + \frac{1}{2}|u_1|^2 + \frac{1}{2}|u_2|^2 \\
&\quad - \left(\frac{1}{2}|\overline{y}_1 + M_{12}\overline{y}_2|^2 + \frac{1}{2}|\overline{y}_2 + M_{21}\overline{y}_1|^2\right) \\
&\quad - \left(-\langle \overline{y}_1 + M_{12}\overline{y}_2, u_1\rangle - \langle \overline{y}_2 + M_{21}\overline{y}_1, u_2\rangle + \frac{1}{2}|u_1|^2 + \frac{1}{2}|u_2|^2\right) \\
&= \frac{1}{2}|M\overline{x} - u|^2 - \frac{1}{2}|M\overline{y} - u|^2.
\end{aligned}$$

This proves the desired equality. □

Now, our aim is to establish an explicit formula for the information value. To that end, we need an extension of the notion of observability introduced in Section 2. Given $E_1 \subseteq V_1$ and $E_2 \subseteq V_2$ as above, bounded and linear operators $A_i : V_i \to R$ $(i = 1,2)$ we say that (A_1, A_2) is **limiting observable** through $E_1 \times E_2$ if $(\mathcal{N}(A_1) \times \mathcal{N}(A_2))^\perp \subseteq E_1 \times E_2$. It is equivalent to say that each A_i is limiting observable in the sense introduced in Subsection 2.3, i.e., if $\mathcal{N}(A_i)^\perp \subseteq E_i$ for $i = 1,2$, since $\mathcal{N}(A_1)^\perp \times \mathcal{N}(A_2)^\perp = (\mathcal{N}(A_1) \times \mathcal{N}(A_2))^\perp$, Due to the inclusion $\mathcal{N}(A_1) \times \mathcal{N}(A_2) \subseteq \mathcal{N}(A)$ where $A(x_1,x_2) \doteq A_1(x_1) + A_2(x_2)$, we conclude that, if (A_1, A_2) is limiting observable through $E_1 \times E_2$ then A is limiting observable through $E = E_1 \times E_2$, i.e., $\mathcal{N}(A)^\perp \subseteq E$.

The next example shows an instance of A_1, A_2 justifying the notion of observability just introduced.

Example 3.5. Let us consider as before $A_1 x_1 = \mathbf{E}(a_1 x_1)$, $A_2 x_2 = \mathbf{E}(a_2 x_2)$, where $(a_1, a_2) \in V_1 \times V_2$, and $A(x_1, x_2) = \mathbf{E}(a_1 x_1 + a_2 x_2)$. It is not difficult to prove that (set $E = E_1 \times E_2$),

$$\mathcal{N}(A_1)^\perp \times \mathcal{N}(A_2)^\perp \times \subseteq E_1 \times E_2 \Longrightarrow (a_1, a_2) \in E_1 \times E_2 \Longleftrightarrow \mathcal{N}(A)^\perp \subseteq E.$$

The first implicancy become an equivalence if $\mathcal{N}(A_1) \times \mathcal{N}(A_2) = \mathcal{N}(A)$. Here, $A^* : \mathbb{R} \to V$, is defined by $A^*\lambda = \lambda(a_1, a_2)$, and thus $A^*(\mathbb{R}) = \mathcal{N}(A)^\perp$.

Lemma 3.6. *Assume that (A_1, A_2) is limiting observable through $E_1 \times E_2 = E$. If $\overline{x} \in K_E$ is a Nash equilibrium solution to problem $Q_2(E)$, then*

$$P_E M\overline{x} - P_E u \in (\mathcal{N}(A_1) \times \mathcal{N}(A_2))^\perp.$$

Moreover, if $M(E) \subseteq E$, then

$$M\overline{x} - P_E u \in (\mathcal{N}(A_1) \times \mathcal{N}(A_2))^\perp.$$

Proof. Thanks to Proposition 3.3, we have that

$$\langle M\overline{x} - u, x - \overline{x} \rangle \geq 0 \quad \text{for all } x = (x_1, x_2) \in K_{E_1} \times K_{E_2}.$$

Thus

$$\langle M\overline{x} - u, z \rangle = 0 \quad \text{for all } z \in (\mathcal{N}(A_1) \times \mathcal{N}(A_2)) \cap E.$$

It implies that

$$\langle M\overline{x} - u, P_{E \cap (\mathcal{N}(A_1) \times \mathcal{N}(A_2))} v \rangle = 0 \quad \text{for all } v \in V.$$

We apply Lemma A.1 to obtain $\langle P_E M\overline{x} - P_E u, P_{\mathcal{N}(A_1) \times \mathcal{N}(A_2)} v \rangle = 0$ for all $v \in V$, from which the conclusion follows. □

Remark 3.7. The invariance of E under M is implied by requiring that M_{ij}: $M_{ij}(E_j) \subseteq E_i$, $i \neq j$ as one can easily showed.
On the other hand, the condition $M(\mathcal{N}(A)) \subseteq \mathcal{N}(A_1) \times \mathcal{N}(A_2)$ imposed in the next theorem, is implied by the two assumptions

$$M_{12}(\mathcal{N}(A_2)) \subseteq \mathcal{N}(A_1), \; M_{21}(\mathcal{N}(A_1)) \subseteq \mathcal{N}(A_2),$$

provided $\mathcal{N}(A_1) \times \mathcal{N}(A_2) = \mathcal{N}(A)$.

Theorem 3.8. *Let, for $i = 1, 2$, $E_i \subseteq V_i$, $F_i \subseteq V_i$ be closed subspaces. Set $E = E_1 \times E_2$, $F = F_1 \times F_2$, and assume that: $E \subseteq F$, (A_1, A_2) is limiting observable through $E_1 \times E_2$, $M(E) \subseteq E$, $M(F) \subseteq F$ and $M(\mathcal{N}(A)) \subseteq \mathcal{N}(A_1) \times \mathcal{N}(A_2)$. If $\overline{x}$ and $\overline{y}$ are Nash equilibrium solutions to $Q_2(E)$ and $Q_2(F)$ respectively, then*

(a) $M\overline{y} = M\overline{x} + P_F u - P_E u$;

(b) *The social information value is given by the Formula*

$$I_2(E,F) = \frac{1}{2}|P_F u|^2 - \frac{1}{2}|P_E u|^2; \tag{3.3}$$

which is non-negative, i.e.

(c) $I_2(E,F) \geq 0$.

Proof. (a): By the previous Lemma,

$$M\overline{x} - P_E u \in (\mathcal{N}(A_1) \times \mathcal{N}(A_2))^{\perp} \quad \text{and} \quad M\overline{y} - P_F u \in \mathcal{N}(A_1) \times \mathcal{N}(A_2))^{\perp}.$$

Then

$$\langle z, M\overline{x} - P_E u - M\overline{y} + P_F u\rangle = 0 \quad \text{for all } z \in \mathcal{N}(A_1) \times \mathcal{N}(A_2).$$

Since $\overline{x} - \overline{y} \in \mathcal{N}(A)$,

$$P_F u - P_E u \in E^{\perp} \subseteq \mathcal{N}(A_1) \times \mathcal{N}(A_2), \quad M(\overline{x} - \overline{y}) \in \mathcal{N}(A_1) \times \mathcal{N}(A_2),$$

we immediately obtain

$$M\overline{y} = M\overline{x} + P_F u - P_E u.$$

(b): From Part (a), we have

$$\begin{aligned} 2I_2(E,F) &= |M\overline{x} - u|^2 - |M\overline{y} - u|^2 = -|P_F u - P_E u|^2 - 2\langle M\overline{x} - u, P_F u - P_E u\rangle \\ &= -|P_F u|^2 - |P_E u|^2 + 2\langle P_F u, P_E u\rangle + 2\langle u, P_F u - P_E u\rangle, \end{aligned}$$

since $P_F u - P_E u \in E^{\perp}$ and $M(E) \subseteq E$. Writing $u = P_F u + P_{F^{\perp}} u$, the previous equality reduces

$$I_2(E,F) = \frac{1}{2}|P_F u|^2 - \frac{1}{2}|P_E u|^2.$$

(c): It follows from the inequality (see (2) of Lemma A.1)

$$|P_E u| = |P_E P_F u| \leq |P_E||P_F u| \leq |P_F u|.$$

□

Example 3.9 (Unconstrained Generalized Incomplete Case)**.** The first condition to assure the non-negativity of the information value is about the invariance of the subspaces E and F under the interaction operator M.

Then, we introduce the concept of observability of the interaction operator. In game theory, a stochastic parameter of a game is defined as a *type of a player* (see Harsanyi 1967, [11]), i.e. a stochastic variable that is known by the player, while the *type* of the opponent musts be estimated, in the bayesian sense, by the own *types* of the player.

Now, a large class of linear bounded operators M can be expressed as $\lambda_E P_E + \lambda_E P_{E^\perp}$ (see Naylor 1982, [16], p. 398), where λ_E is an element of $\mathbb{R}$. For this reason, the invariance of E under M means that the operator M is *totally determined by* E. In other words, if the *types of players determine totally the operator* M, then the subspace E is invariant under the operator M. If this is valid for the subspace F too, we say that **interaction operator** M **is observable**.

For instance, in Example 3.1, the operator M takes the form

$$M(v_1, v_2) \doteq (v_1 + M_{12}v_2, v_2 + M_{21}v_1)$$

and M_{ij} is given by $M_{ij}v_j \doteq m_{ij}v_j$, where $m_{ij} \in \mathbb{R}$, i.e. M_{ij} corresponds to the multiplication for a real number. In this case, if there is symmetry of information (i.e. $E_1 = E_2$) between the players, then the interaction operator is observable.

The second condition is about the capacity of each player to observe their own technology matrix. To make this concept clear, we say that an event v of V is **observable** if player i has complete information about it, i.e. v is a member of E_i. Now we extend this concept to the observability of the technology matrix. The following Example clarifies this concept

Recall that the technology matrix A is given by

$$A(x_1, x_2) \doteq \mathbf{E}(a_1x_1 + a_2x_2).$$

It is not difficult to prove that, if the events a_i are observable to the player i, then $\mathcal{N}(A)^\perp \subseteq E$. In fact, if a_i is observable to player i, i.e. $a_i \in E_i$, we prove that $E^\perp \subseteq \mathcal{N}(A)$. For this, let consider an event (y_1, y_2) of V, so that $\mathbf{E}(y_1x_1 + y_2x_2) = 0$, i.e. $(y_1, y_2) \in E^\perp$. Then $\mathbf{E}(a_1y_1 + a_2y_2) = 0$, because each element a_i is observable. From here $(y_1, y_2) \in \mathcal{N}(A)$ is obtained. ■

Example 3.10 (Constrained Gaussian Case)**.** As we have seen in Example 2.12, we try to use Formula 3.3 of Theorem 3.8 to obtain the social information value between the Nash Equilibrium solutions $(I, 1)$ and $(II, 2)$ of Example 3.1. In this case, the operator M and the technology matrix A are both observable, then we can apply Formula 3.3 of Theorem 3.8.

$$\begin{aligned}
I_2(E, F) &= \frac{1}{2}|P_F u|^2 - \frac{1}{2}|P_E u|^2 \\
&= \frac{1}{2}|P_{F_1}u_1|^2 + \frac{1}{2}|P_{F_2}u_2|^2 - \frac{1}{2}|P_{E_1}u_1|^2 - \frac{1}{2}|P_{E_2}u_2|^2 \\
&= \frac{1}{2}\mathbf{E}(\mathbf{E}(\omega|\mathcal{F}_1)^2 + \mathbf{E}(\omega|\mathcal{F}_2)^2) - \frac{1}{2}\mathbf{E}(\mathbf{E}(\omega|\mathcal{E}_1)^2 + \mathbf{E}(\omega|\mathcal{E}_2)^2) \\
&= \frac{1}{2}(\mathbf{E}(\frac{1}{4}\xi^2) + \mathbf{E}(\frac{1}{4}\xi^2)) - 0 = \frac{1}{2}.
\end{aligned}$$

Let us note that Theorem 3.8 ensures the value of information is independent on the interaction operator M, on the restriction of resources determined by the technology matrix A, and on the available resource b. It only depends on the objectives u_i and the subspace information available E and F. ■

Example 3.11 (Duopoly with Differentiated Products)**.** Following Dixit 1979, [6] and Singh and Vives 1984, [17], consider a two-firm industry producing two differentiated products indexed by i. To simplify the exposition, we assume that production is costless. We assume the following (inverse) demand for product i is given by the price

$$\omega - m_i v_i - m v_j \quad \text{with} \quad m_i > 0, m_i^2 > m^2,$$

where $v_i \geq 0$ is the production level each firm and $i \neq j$. Thus, we assume that each product is produced by a different firm facing the given demands. The assumption $m_i^2 > m^2$ implies that the effect of increasing v_i on the price of product i is larger than the effect of the same increase on the price of product j. A common technic to describe this is to say that *own-price effect* dominates the *cross-price effect*.

We note that each firm i choose the production level to maximize their utility. In this example, this is posed as an equilibrium problem, i.e.

$$f_i(\overline{v}_i, \overline{v}_j) \doteq \min\{v_i(m_i v_i + m v_j - \omega) : v_i \geq 0\} \quad \text{with } i, j = 1, 2; i \neq j.$$

Let us observe that the social cost $\widetilde{f}$ can be obtained by the sum of each cost, i.e.

$$\widetilde{f}(v_i, v_j) \doteq f_1(v_1, v_2) + f_2(v_1, v_2),$$

that is expressed as $\widetilde{f}(v) = \langle Qv, v\rangle - (\omega v_1 + \omega v_2)$, where Q is given by

$$\begin{pmatrix} m_1 & m \\ m & m_2 \end{pmatrix},$$

and $v \doteq (v_1, v_2)$. We note that the matrix Q is invertible, since it is supposed that the *own-price effect* dominates the *cross-price effect*, i.e. $m_i^2 > m^2$. ■

A Auxiliary results

Lemma A.1. *Let be V a real Hilbert space, and E_1, E_2 closed subspaces of V. If either $E_2 \subseteq E_1$ or $E_2^\perp \subseteq E_1$, then the following assertions hold:*

(1) $E_1 = (E_1 \cap E_2) \oplus (E_1 \cap E_2^\perp)$ *and*

(2) $P_{E_1 \cap E_2} = P_{E_1} P_{E_2} = P_{E_2} P_{E_1}$.

Proof. (1) Let $x \in E_1$. Obviously $x = x' \oplus x''$, with $x' \in E_2$, $x'' \in E_2^\perp$.

- Assume first that $E_2 \subseteq E_1$. Then $x' \in E_1$ and thus $x'' = x - x' \in E_1 - E_1 = E_1$. Then $x = x' + x''$ with $x' \in E_1 \cap E_2$ and $x'' \in E_1 \cap E_2^\perp$. Morover, if $x' + x'' = y' + y''$ with $x', y' \in E_1 \cap E_2$ and $x'', y'' \in E_1 \cap E_2^\perp$, then $x' - y' \in E_1 \cap E_2$ and $x'' - y'' \in E_1 \cap E_2^\perp$. In particular, $x' - y' \in E_2 \cap E_2^\perp = \{0\}$, i.e., $x' = y'$, and then $x'' = y''$.
- In case $E_2^\perp \subseteq E_1$, by noticing that $E_2^\perp$ is also a closed subspace of V, we simply replace E_2 by $E_2^\perp$ in the previous case.

(2) Let be $u \in V$. By definition, we have

$$x = P_{E_1 \cap E_2} u \iff \begin{cases} x \in E_1 \cap E_2 \\ \langle u - x, v \rangle = 0 \quad \text{for all } v \in E_1 \cap E_2. \end{cases}$$

Since $x \in E_1 \cap E_2$, we obtain

$$\begin{aligned} \langle u - x, v \rangle = 0 \ \text{ for all } v \in E_1 \cap E_2 &\iff \langle P_{E_2} u - x, v \rangle = 0 \ \text{ for all } v \in E_1 \cap E_2 \\ &\iff \langle P_{E_2} u - x, v \rangle = 0 \ \text{ for all } v \in E_1. \end{aligned}$$

The last equivalence was obtained from Part (1). Hence, $x = P_{E_1} P_{E_2} u$, and then $P_{E_1 \cap E_2} u = P_{E_1} P_{E_2} u$, proving the first equality of Part (2). For the second equality we simply remind that every projection operator is selfadjoint and idempotent, and the fact $(P_{E_1} P_{E_2})^* = P_{E_2}^* P_{E_1}^*$. □

Remark A.2. Let E, F be closed subspaces of V such that $E \subseteq F$, then $F^\perp \subseteq E^\perp$. From the previous lemma, it follows that

$$E^\perp = (E^\perp \cap F) \oplus (E^\perp \cap F^\perp) \ \text{ and } P_{E^\perp \cap F} = P_{E^\perp} P_F.$$

Corollary A.3. *If $E_2^\perp \subseteq E_1$, then*

$$E_1 = (E_1 \cap E_2) \oplus (E_1 \cap E_2^\perp) \quad \textit{and} \quad E_2 = (E_2 \cap E_1) \oplus (E_2 \cap E_1^\perp).$$

Consequently,

$$P_{E_1 \cap E_2} = P_{E_1} P_{E_2} = P_{E_2} P_{E_1}.$$

Proof. Since $E_2^\perp \subseteq E_1$ implies $E_1^\perp \subseteq E_2$, we obtained the result after applying the preceding lemma twice. □

Finally, we formulate the necessary first order optimality condition, who prove can be found in any general textbook of mathematical programming (see for example Bazaraa et al. 1979, Theorem 3.4.3, [2]).

Lemma A.4. *Let be $C \subseteq V$ be a closed and convex set, and $f : V \to R$ a function continuously differentiable on an open set containing C. If $\overline{x}$ minimizes f on C, then*

$$\langle \nabla f(\overline{x}), x - \overline{x} \rangle \geq 0 \quad \textit{for all } x \in C.$$

If further f is convex, then any $\overline{x} \in C$ satisfying the previous inequality minimizes f on C.

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