

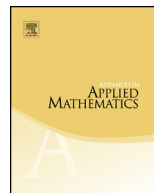


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A probabilistic analysis of a discrete-time evolution in recombination



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ABSTRACT

We study the discrete-time evolution of a recombination transformation in population genetics. The transformation acts on a product probability space, and its evolution can be described by a Markov chain on a set of partitions that converges to the finest partition. We describe the geometric decay rate to this limit and the quasi-stationary behavior of the Markov chain when conditioned on the event that the chain does not hit the limit.

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1. Introduction

Here we study the evolution of the following transformation Ξ acting on the set of probability measures μ on a product measurable space $\prod_{i \in I} \mathcal{A}_i$,

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$$\Xi[\mu] = \sum_{J \subseteq I} \rho_J \mu_J \otimes \mu_{J^c}.$$

The vector $\rho = (\rho_J : J \subseteq I)$ is a probability vector, μ_J and μ_{J^c} are the marginals of μ on $\prod_{i \in J} \mathcal{A}_i$ and $\prod_{i \in J^c} \mathcal{A}_i$ respectively, and \otimes means that these marginals are combined in an independent way.

The analysis of Ξ should give an insight in the study of the genetic composition of population under recombination. Genetic information is encoded in terms of sequences of symbols indexed by a finite set of sites. In the process of recombination the children sequences are derived from two parents, a subset of sites (J) is encoded with the maternal symbols and the complementary set (J^c) is encoded with the paternal symbols. The above equation expresses that the pair of sets (J, J^c) constitute a probabilistic object distributed according to ρ . By taking $\rho_J + \rho_{J^c}$ as the weight of the binary partition $\{J, J^c\}$ we can always consider binary partitions instead of sets.

The evolution ($\Xi^n[\mu]$) has been mainly studied in the context of single cross-overs, that is where $I = \{1, \dots, K\}$ and the pairs of sets (J, J^c) are of the form $J = \{i : i < j\}$, $J^c = \{i : i \geq j\}$. This evolution was introduced by H. Geiringer [11], and firstly solved in the continuous-time case by E. Baake and M. Baake [2], where it is also supplied an important corpus of ideas and techniques to study the discrete-time evolution. More detailed discussions on some of the pioneering works, comments on other significant results, including [6,9,10], as well as the interpretation of the above equation in a broader perspective of recombination in population genetics, can be found in the introductory sections of references [2,4,5] and [15,16].

When studying single cross-over recombination, one the main objectives in [15] and [4] is to express the iterated $\Xi^n[\mu]$ in a simple form which allows its dynamics to be understood. The main tools are Möbius inversion formulae, and commutation relations between Ξ and recombinators, which are idempotent operators that commute, so act as projectors. In my view, some of the main results in this body of works are:

- Theorem 1 in [4] and Proposition 3.3 in [15], that supply a one step recursive decomposition for Ξ^n in terms of the recombinators and give an expression of $\Xi^n[\mu]$ serving to the analysis of the convergence of $\Xi^n[\mu]$ to the distribution $\bigotimes_{J \in \mathcal{D}^*} \mu_J$, where \mathcal{D}^* is the partition whose atoms are the nonempty intersections of the sets J, J^c with $\rho(J) > 0$;
- the construction of a Markov chain by following the ancestry of the genetic material of a selected individual from a population; and Theorem 3 in [4], which states a relation between (Ξ^n) and the Markov chain.

Recently, in [5], the continuous-time evolution was studied in a framework of general partitions other than the binary partitions $\{J, J^c\}$ considered in [2,4] and [15]. It corresponds to study the evolution of the following transformation Ξ acting on the set of probability measures μ on a product measurable space $\prod_{i \in I} \mathcal{A}_i$,

$$\Xi[\mu] = \sum_{\delta \in \mathcal{G}} \rho_\delta \bigotimes_{J \in \delta} \mu_J.$$

Here \mathcal{G} is a set of partitions of the finite set I , $\rho = (\rho_\delta : \delta \in \mathcal{G})$ is a strictly positive probability vector, μ_J is the marginal of μ on $\prod_{i \in J} \mathcal{A}_i$, and $\bigotimes_{J \in \delta} \mu_J$ is the product measure. Its evolution was recently studied in [3].

We state our results in this general setting. The analysis is done by successive refinements of the partitions in \mathcal{G} , and a key element turns out to be the partition $\mathcal{D}^\mathcal{G}$ which is the common refinement of all the partitions in \mathcal{G} .

In Proposition 3.2 in Section 3 we associate to the evolution (Ξ^n) a Markov chain (Y_n) , whose transition probabilities starting from the coarsest partition $\{I\}$ give the decomposition $\Xi^n[\mu] = \sum_{\delta} \mathbb{P}(Y_n = \delta) \otimes_{K \in \delta} \mu_K$. The Markov chain can be seen as a random walk on the set of partitions with respect to operation refinement, and with the one-step transition law ρ . This Markov chain converges to $\mathcal{D}^\mathcal{G}$, then $\otimes_{K \in \mathcal{D}^\mathcal{G}} \mu_K$ should be the genetic composition that one observes after a long period of time (when considering only recombination). One of our aims is to understand the geometric decay of the chain to this limit.

The main result of this work is Theorem 4.1 shown in Section 4. There, we study in detail the geometric decay of the Markov chain (Y_n) , in particular, we show that the geometric decay rate is $\eta = \max\{P_{\delta,\delta} : \delta \neq \mathcal{D}^\mathcal{G}\}$. In this theorem the quasi-stationary behavior of the chain (Y_n) conditioned on not hitting the limit point $\otimes_{J \in \mathcal{D}(\mathcal{G})} \mu_J$ is also studied. We recall that quasi-stationarity gives very precise information on the deviations of the behavior from the limit measure $\otimes_{J \in \mathcal{D}^\mathcal{G}} \mu_J$. We describe the limiting conditional behavior of the chain and we state a ratio limit of the probabilities of not hitting the limit point. This ratio limit allows us to construct the Q -chain, which is the chain that never hits $\mathcal{D}^\mathcal{G}$, this is done in Corollary 4.4. We also show that the partitions δ^* satisfying $P_{\delta^*,\delta^*} = \eta$, are the limit points of the Q -chain. Hence, the laws $\otimes_{K \in \delta^*} \mu_K$ are the candidates to be the genetic composition of a population after a long period of time but when the limit $\otimes_{K \in \mathcal{D}^\mathcal{G}} \mu_K$ has not been attained.

We emphasize that our results on quasi-stationarity are not a consequence of any already published result in the literature of quasi-stationarity because the Markov chain (Y_n) is not irreducible on the class of non-absorbing states, so we are not able to use the Perron–Frobenius theory. All these results presented here require entirely new computations. Quasi-stationary distributions for finite Markov chains were studied in [8]. In population dynamics quasi-stationarity have been studied mostly in relation to population extinction, see for instance Section 2.6 in [12], and [14,7] for a wide ranging bibliography on the subject. We note that in this work, the absorbing state is not the empty population as happens when studying extinction phenomena.

In Section 2 we fix notation on partitions. In Section 3 we introduce the Markov chain on partitions and in Section 4 we state our main results. Examples containing explicit computations are developed in these sections.

2. The recombination transformation

2.1. Partitions

Let us fix some notation on partitions on finite sets. Let I be a nonempty finite set. A *partition* δ of I is a collection of nonempty and pairwise disjoint sets that cover I . Any of the sets L belonging to the partition $\delta = \{L : L \in \delta\}$ is called an atom of δ . We note by $\mathbb{S}(I)$ the family of partitions of I .

For $\delta, \delta' \in \mathbb{S}(I)$, δ' is said to be finer than δ or δ is coarser than δ' , we note $\delta \preceq \delta'$, if every atom of δ' is contained in an atom of δ , this is an order relation. The finest partition is $\{\{i\} : i \in I\}$, and the coarsest one is the trivial partition $\{I\}$ having a single atom. The common refinement between two partitions $\delta, \delta' \in \mathbb{S}(I)$ is noted by $\delta \vee \delta'$ and its atoms are the nonempty elements of the family of sets $\{L \cap L' : L \in \delta, L' \in \delta'\}$. The operation \vee is commutative, associative and $\{I\}$ is its unit element because $\{I\} \vee \delta = \delta$ for all $\delta \in \mathbb{S}(I)$. One has $\delta \preceq \delta'$ if and only if $\delta \vee \delta' = \delta'$.

Let \mathcal{G} be a nonempty family of partitions of I . We will associate to it the following sequence of families of partitions, which are the consecutive refinements with \mathcal{G} ,

$$\forall n \geq 1 : \quad \mathcal{G}^{n+1} = \mathcal{G}^n \vee \mathcal{G} = \{\mathcal{D} \vee \delta : \delta \in \mathcal{G}^n, \mathcal{D} \in \mathcal{G}\}.$$

Since every $\delta \in \mathcal{G}^n$ satisfies $\mathcal{D} \vee \delta = \delta$ for some element $\mathcal{D} \in \mathcal{G}$, we have $\mathcal{G}^n \subseteq \mathcal{G}^{n+1}$ for all $n \geq 1$. This sequence stabilizes in a finite number of steps, that is there exists $n_0 \geq 1$ such that $\mathcal{G}^{n_0+k} = \mathcal{G}^{n_0}$ for all $k \geq 0$. Let

$$\mathcal{G}^+ = \bigcup_{n \geq 0} \mathcal{G}^n = \mathcal{G}^{n_0}.$$

Denote by $\mathcal{D}^{\mathcal{G}}$ the partition which is the common refinement of all the partitions in \mathcal{G} , this is written

$$\mathcal{D}^{\mathcal{G}} = \bigvee_{\mathcal{D} \in \mathcal{G}} \mathcal{D}.$$

It is the finest partition in \mathcal{G}^+ , so $\delta \preceq \mathcal{D}^{\mathcal{G}}$ for all $\delta \in \mathcal{G}^+$. The atoms of $\mathcal{D}^{\mathcal{G}}$ are the nonempty intersections $\bigcap_{\mathcal{D} \in \mathcal{G}} L_{\mathcal{D}}$, where $(L_{\mathcal{D}} : \mathcal{D} \in \mathcal{G})$ varies over all the sequences of atoms of the partitions in \mathcal{G} .

Remark 2.1. $\mathcal{D}^{\mathcal{G}}$ is the unique element in \mathcal{G}^+ that satisfies $\mathcal{D}^{\mathcal{G}} \vee \mathcal{D} = \mathcal{D}^{\mathcal{G}}$ for all $\mathcal{D} \in \mathcal{G}$. Moreover, it also holds $\mathcal{D}^{\mathcal{G}} \vee \delta = \mathcal{D}^{\mathcal{G}}$ for all $\delta \in \mathcal{G}^+$. That is, $\mathcal{D}^{\mathcal{G}}$ is an absorbing element in (\mathcal{G}^+, \vee) .

Example. Let $I = \{1, 2, 3, 4\}$ and take $\mathcal{G} = \{\delta^1, \delta^2, \delta^3\}$ with $\delta^1 = \{\{I\}\}$, $\delta^2 = \{\{1, 2\}, \{3, 4\}\}$, $\delta^3 = \{\{1, 3\}, \{2, 4\}\}$. Then, $\mathcal{G}^2 = \mathcal{G} \vee \mathcal{G} = \{\delta^k \vee \delta^l : k, l = 1, 2, 3\}$. We

have $\delta^k \vee \delta^k = \delta^k$ for $k = 1, 2, 3$; $\delta^1 \vee \delta^k = \delta^k$ for $k = 2, 3$; and $\delta^2 \vee \delta^3 = \{\{i\} : i \in I\} = \mathcal{D}^{\mathcal{G}}$ (for instance $\{3\} = \{3, 4\} \cap \{1, 3\}$). Then, $\mathcal{G}^+ = \mathcal{G}^2 = \mathcal{G} \cup \{\mathcal{D}^{\mathcal{G}}\}$, so the sequence (\mathcal{G}^n) stabilize at $n_0 = 2$. \square

Note that the trivial partition $\{I\}$ belongs to \mathcal{G}^+ only when $\{I\} \in \mathcal{G}$. Since we will require to consider $\{I\}$, we introduce

$$\mathcal{G}_0^+ = \mathcal{G}^+ \cup \{\{I\}\},$$

and so $\mathcal{G}_0^+ = \mathcal{G}^+$ only when $\{I\} \in \mathcal{G}$.

Let us define the following relation,

$$\forall \delta, \delta' \in \mathcal{G}_0^+ : \delta \rightarrow \delta' \Leftrightarrow [\exists \mathcal{D} \in \mathcal{G} : \delta' = \delta \vee \mathcal{D}], \tag{1}$$

when this happens we say that δ is connected to δ' . By definition, $\delta \rightarrow \delta'$ implies $\delta \preceq \delta'$. Since for every $\delta \in \mathcal{G}^+$ there exists $\mathcal{D} \in \mathcal{G}$ such that $\delta \vee \mathcal{D} = \delta$, we get

$$\forall \delta \in \mathcal{G}^+ : \delta \rightarrow \delta. \tag{2}$$

On the other hand for $\delta = \{I\}$ we have

$$\{I\} \rightarrow \delta' \Leftrightarrow \delta' \in \mathcal{G}.$$

In particular, $\{I\} \rightarrow \{I\}$ if and only if $\{I\} \in \mathcal{G}$.

A path between $\delta \in \mathcal{G}_0^+$ and $\delta' \in \mathcal{G}^+$ is a sequence $(\delta_k : k = 1, \dots, r)$ in \mathcal{G}_0^+ with $r \geq 2$, such that $\delta_1 = \delta$, $\delta_r = \delta'$ and $\delta_k \rightarrow \delta_{k+1}$ for $k = 1, \dots, r - 1$. When there exists a path between δ and δ' one says that δ attains δ' . Obviously, if δ attains δ' then $\delta \preceq \delta'$. Notice that $\{I\}$ attains all $\delta \in \mathcal{G}^+$. We have that,

$$[(\delta_k : k = 1, \dots, r) \text{ is a path, } \delta_1 = \delta_r] \Rightarrow [\delta_k = \delta_1, k = 1, \dots, r]. \tag{3}$$

In other words, \mathcal{G}_0^+ endowed with the relation $[\delta \rightarrow \delta', \delta \neq \delta']$ has no cycles.

2.2. Product probability spaces

Let us introduce a product measurable space and the set of probability measures on it. Let $(\mathcal{A}_i, \mathcal{B}_i)$, $i \in I$, be a finite collection of measurable spaces and let $\prod_{i \in I} \mathcal{A}_i$ be a product space endowed with the product σ -field $\otimes_{i \in I} \mathcal{B}_i$. When all the \mathcal{A}_i are equal we note $\mathcal{A}_i = \mathcal{A}$, so $\prod_{i \in I} \mathcal{A}_i = \mathcal{A}^I$.

Denote by \mathcal{P}_I the set of probability measures on $(\prod_{i \in I} \mathcal{A}_i, \otimes_{i \in I} \mathcal{B}_i)$. Let $J \subseteq I$ and \mathcal{P}_J be the set of probability measures on $(\prod_{i \in J} \mathcal{A}_i, \otimes_{i \in J} \mathcal{B}_i)$. The marginal $\mu_J \in \mathcal{P}_J$ of $\mu \in \mathcal{P}_I$ on J is given by,

$$\forall C \in \otimes_{i \in J} \mathcal{B}_i : \mu_J(C) = \mu(C \times \prod_{i \in J^c} \mathcal{A}_i).$$

For $J = I$ we have $\mu_I = \mu$, and we put $\mu_\emptyset \equiv 1$ to get consistency in all the relations where it will appear, in particular in product measures.

Let $J, K \subseteq I, J \cap K = \emptyset$. For $\mu_J \in \mathcal{P}_J, \mu_K \in \mathcal{P}_K$, we denote by $\mu_J \otimes \mu_K$ its product measure. We have that \otimes is commutative and associative, $\mu_\emptyset = 1$ is the unit element, and \otimes is stable under restriction, that is, for all $J, K, M \subseteq I$ with $J \cap K = \emptyset$ and $M \subseteq J \cup K$,

$$(\mu_J \otimes \mu_K)_M = \mu_{J \cap M} \otimes \mu_{K \cap M}. \tag{4}$$

Remark 2.2. All the results of this work will be proven for product probability spaces $(\prod_{i \in I} X_i, \otimes_{i \in I} \mathcal{B}_i, \mu)$ with I finite, but with no restriction on the finite collection $((X_i, \mathcal{B}_i) : i \in I)$ of measurable spaces.

Remark 2.3. In our results it is not required that the operation \otimes is the product between probability measures. As it can be checked, the results can be extended to any operation \otimes that satisfies commutativity, associativity, $\mu_\emptyset = 1$ and stability under restriction (4).

2.3. The transformation

In the sequel we fix $\rho = (\rho_\delta : \delta \in \mathbb{S}(I))$ a probability vector on the set of partitions, so $\rho_\delta \geq 0$ for $\delta \in \mathbb{S}(I)$ and $\sum_{\delta \in \mathbb{S}(I)} \rho_\delta = 1$. From now on, we note by $\mathcal{G} = \{\delta \in \mathbb{S}(I) : \rho_\delta > 0\}$ the support of ρ .

Definition 2.4. Define the following transformation $\Xi : \mathcal{P}_I \rightarrow \mathcal{P}_I$,

$$\Xi[\mu] = \sum_{\mathcal{D} \in \mathcal{G}} \rho_{\mathcal{D}} \bigotimes_{J \in \mathcal{D}} \mu_J. \quad \square$$

The common refinement of partitions in \mathcal{G} is $\mathcal{D}^{\mathcal{G}} = \bigvee_{\delta \in \mathcal{G}} \delta$. We claim that

$$\mu = \bigotimes_{L \in \mathcal{D}^{\mathcal{G}}} \mu_L \text{ is a fixed point for } \Xi : \Xi[\mu] = \mu.$$

In fact, we have $\mathcal{D}^{\mathcal{G}} = \mathcal{D}^{\mathcal{G}} \vee \mathcal{D}$ for all $\mathcal{D} \in \mathcal{G}$, so $\mu_J = \otimes_{L \in \mathcal{D}^{\mathcal{G}} : L \subseteq J} \mu_L$ for all $J \in \mathcal{D}$. Then, $\mu = \bigotimes_{J \in \mathcal{D}} \mu_J$ for all $\mathcal{D} \in \mathcal{G}$ and the claim $\Xi[\mu] = \mu$ is shown.

Remark 2.5. If one redefines I (as the set of atoms of the partition $\mathcal{D}^{\mathcal{G}}$) one can always assume that the atoms of $\mathcal{D}^{\mathcal{G}}$ are singletons, that is $\mathcal{D}^{\mathcal{G}} = \{\{i\} : i \in I\}$. We will not do it because there is no substantial gain in notation.

Remark 2.6. For the meaning of $\Xi[\mu]$ in population genetics, assume ρ only gives positive probability to binary partitions $\delta = \{J, J^c\}$. Suppose the genetic information is encoded

in terms of sequences of \mathcal{A}^I , with A finite, and let μ be the distribution of this genetic information in certain population. The maternal and paternal genetic information, are two random elements of this population that are noted U and V , it is assumed they are independent. The genetic information of the children is Z with $Z_J = U_J$, $Z_{J^c} = V_{J^c}$, where $\{J, J^c\}$ is a random binary partition, distributed as ρ and independent of the genetic information of the parents U and V . It is straightforward to show that the distribution of the children genetic information Z , is $\Xi[\mu]$.

2.4. Example

Let $I = \{0, \dots, 4r - 1\}$ be a set of $4r$ elements, and let $\mathcal{A} = \{A, C, G, T\}$ be the set of four nucleotides. Assume $\mathcal{G} = \{\delta_1, \delta_2\}$, where

$$\begin{aligned} \delta_1 &= \{J, J^c\} \text{ with } J = \{0, \dots, 2r - 1\}, J^c = \{2r, \dots, 4r - 1\}; \\ \delta_2 &= \{K, K^c\} \text{ with } K = \{0, 2, \dots, 4r - 2\}, K^c = \{1, 3, \dots, 4r - 1\}. \end{aligned}$$

That is, in δ_1 the set I is divided into two consecutive half segments, and in δ_2 the set I is divided into even and odd elements. Each atom in δ_1 and δ_2 has $2r$ elements. We have $\mathcal{D}^{\mathcal{G}} = \delta_1 \vee \delta_2 = \{J \cap K, J \cap K^c, J^c \cap K, J^c \cap K^c\}$ and each of these atoms has r elements. The atoms of $\mathcal{D}^{\mathcal{G}}$ are easily described, for instance $J^c \cap K$ is constituted by the even elements between $2r$ and $4r - 1$ inclusive.

Let $\mathbb{N} = \{0, 1, \dots\}$, and ν^1 and ν^2 be two stationary Markov probability measure on $\mathcal{A}^{\mathbb{N}}$: ν^1 with irreducible transition matrix $\Theta = (\Theta_{a,b} : a, b \in \mathcal{A})$ starting from its stationary vector $\pi = (\pi_a : a \in \mathcal{A})$, and ν^2 with transition matrix Θ^2 starting from π . Assume μ is the probability measure induced by ν^1 on \mathcal{A}^{4r} . Then, the marginals μ_J and μ_{J^c} are the probability measure induced by ν^1 on \mathcal{A}^{2r} . But the marginals μ_K and μ_{K^c} are the probability measure induced by ν^2 on \mathcal{A}^{2r} . Thus, for all $L \in \mathcal{D}^{\mathcal{G}}$, μ_L is the induced probability measure by ν^2 on \mathcal{A}^r . In the case $r = 1$, $I = \{0, 1, 2, 3\}$ has four sites, $\mathcal{D}^{\mathcal{G}} = \{\{i\} : i \in I\}$ and $\mu_{\{i\}} = \pi$ for $i \in I$, so $\otimes_{L \in \mathcal{D}^{\mathcal{G}}} = \pi^{\otimes 4}$, which is a Bernoulli measure.

3. The Markov chain

Let us introduce some probabilistic elements allowing to get a better insight of the sequence of transformations (Ξ^n). Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability measure and $(\Delta_j : \Omega \rightarrow \mathcal{G} : j \geq 1)$ be a sequence of independent and identically distributed random variables with common law ρ , so

$$\forall r \geq 1, \mathcal{D}_1, \dots, \mathcal{D}_r \in \mathcal{G} : \mathbb{P}(\Delta_j = \mathcal{D}_j, j = 1, \dots, r) = \prod_{j=1}^r \rho_{\mathcal{D}_j}.$$

Let us define the following sequence of random variables $(Y_n : n \geq 0)$. We take Y_0 a random variable with values in \mathcal{G}_0^+ and independent of $(\Delta_j : j \geq 1)$ (for instance $Y_0 = \delta$ a fixed values in \mathcal{G}_0^+ is allowed), and

$$Y_n = Y_0 \vee \left(\bigvee_{j=0}^n \Delta_j \right) \text{ for } n \geq 1.$$

The sequence $(Y_n : n \geq 0)$ takes values in \mathcal{G}_0^+ , but Y_n takes values in \mathcal{G}^+ for $n \geq 1$. Thus, in the case $\{I\} \notin \mathcal{G}$ we can start from $Y_0 = \{I\}$, but $Y_n \neq \{I\}$ for $n \geq 1$.

From $Y_{n+1} = Y_n \vee \Delta_{n+1}$ with Y_n and Δ_{n+1} independent, it is straightforward to show that $(Y_n : n \geq 0)$ is a Markov chain. By definition its transition matrix $P = (P_{\delta, \delta'} : \delta, \delta' \in \mathcal{G}_0^+)$ is given by

$$P_{\delta, \delta'} = \sum_{\mathcal{D} \in \mathcal{G} : \delta \vee \mathcal{D} = \delta'} \rho_{\mathcal{D}}. \tag{5}$$

(Notice that $\sum_{\delta' \in \mathcal{G}^+} P_{\delta, \delta'} = \sum_{\mathcal{D} \in \mathcal{G}} \rho_{\mathcal{D}} = 1$.) The matrix P can be seen as the one of a random walk in the set \mathcal{G}_0^+ with respect to operation \vee , and with one-step transition law ρ (it is a nearest neighbor random walk with respect to the oriented neighbor relation \rightarrow).

From definition and (1) we get

$$\forall \delta, \delta' \in \mathcal{G}_0^+ : P_{\delta, \delta'} > 0 \Leftrightarrow \delta \rightarrow \delta'. \tag{6}$$

For all $\delta \in \mathcal{G}^+$ we have $\delta \rightarrow \delta$ (see (2)), and so $P_{\delta, \delta} > 0$. Also note that $P_{\delta, \delta'} > 0$ implies $\delta \preceq \delta'$. From (6) and (3), we get that when the chain (Y_n) leaves an state δ it does never return to it. From Remark 2.1 we have $\mathcal{D}^{\mathcal{G}} \vee \mathcal{D} = \mathcal{D}^{\mathcal{G}}$ for all $\mathcal{D} \in \mathcal{G}$ and so

$$P_{\mathcal{D}^{\mathcal{G}}, \mathcal{D}^{\mathcal{G}}} = 1.$$

Hence, $\mathcal{D}^{\mathcal{G}}$ is an absorbing state for the chain (Y_n) . Also, Remark 2.1 implies that it is the unique absorbing point for the chain. Then

$$\forall \delta \in \mathcal{G}^+, \delta \neq \mathcal{D}^{\mathcal{G}} : 0 < P_{\delta, \delta} < 1. \tag{7}$$

Remark 3.1. Since, for all $\delta \in \mathcal{G}^+$, there exists a path $\delta_1 = \{I\} \rightarrow \dots \rightarrow \delta_r = \delta$, this path has strictly positive probability for the Markov chain.

We claim that $P_{\delta, \delta}$ is strictly increasing with \rightarrow , that is for $\delta, \delta' \in \mathcal{G}_0^+$ we have

$$\left[\delta \rightarrow \delta', \delta \neq \delta' \right] \Rightarrow P_{\delta, \delta} < P_{\delta', \delta'}. \tag{8}$$

In fact, if $\mathcal{D} \in \mathcal{G}$ satisfies $\delta = \delta \vee \mathcal{D}$, then it also satisfies $\delta' = \delta' \vee \mathcal{D}$, and so $P_{\delta, \delta} \leq P_{\delta', \delta'}$. Furthermore, there exists $\mathcal{D}_0 \in \mathcal{G}$ such that $\delta' = \delta \vee \mathcal{D}_0$, and so $\delta' = \delta' \vee \mathcal{D}_0$. Since $\delta \neq \delta'$ we get $P_{\delta', \delta'} \geq P_{\delta, \delta} + \rho_{\mathcal{D}_0}$, then (8) follows.

By abuse of notation, \mathbb{P} denotes below the law starting from $Y_0 = \{I\}$ and \mathbb{E} denotes its mean expected value.

Proposition 3.2. *Let $Y_0 = \{I\}$. For $\mu \in \mathcal{P}_I$ define the sequence of (random) probability measures*

$$\forall n \geq 0: \quad \mu^{(n)} = \bigotimes_{L \in Y_n} \mu_L.$$

Then,

$$\Xi^n[\mu] = \sum_{\delta \in \mathcal{G}^n} \mathbb{P}(Y_n = \delta) \bigotimes_{L \in \delta} \mu_L = \mathbb{E}(\mu^{(n)}). \tag{9}$$

Proof. The equality $\mathbb{E}(\mu^{(n)}) = \sum_{\delta \in \mathcal{G}^n} \mathbb{P}(Y_n = \delta) \otimes_{L \in \delta_n} \mu_L$ is straightforward.

Let us prove the first equality in (9), we will make it by an induction argument. We have $\Xi^0[\mu] = \mu$, $\mathbb{P}(Y_0 = \{I\}) = 1$ and $\mathbb{P}(Y_0 = \delta)$ for $\delta \neq \{I\}$. Then, (9) holds for $n = 0$. From $\mathcal{G}^1 = \mathcal{G}$ and $\mathbb{P}(Y_1 = \delta) = \rho_\delta$ for $\delta \in \mathcal{G}$, we get that (9) holds for $n = 1$.

Assume the statement is satisfied for $n \geq 1$, let us show it for $n + 1$. We have

$$\begin{aligned} \Xi^{n+1}[\mu] &= \Xi^n[\Xi[\mu]] = \sum_{\delta \in \mathcal{G}^n} \mathbb{P}(Y_n = \delta) \bigotimes_{K \in \delta} \Xi[\mu]_K \\ &= \sum_{\delta \in \mathcal{G}^n} \mathbb{P}(Y_n = \delta) \bigotimes_{K \in \delta} \left(\sum_{\mathcal{D} \in \mathcal{G}} \rho_{\mathcal{D}} \bigotimes_{L \in \mathcal{D}} \mu_L \right)_K \\ &= \sum_{\delta \in \mathcal{G}^n} \mathbb{P}(Y_n = \delta) \bigotimes_{K \in \delta} \left(\sum_{\mathcal{D} \in \mathcal{G}} \rho_{\mathcal{D}} \bigotimes_{L \in \mathcal{D}} \mu_{L \cap K} \right) \\ &= \sum_{\delta \in \mathcal{G}^n} \sum_{\mathcal{D} \in \mathcal{G}} \mathbb{P}(Y_n = \delta) \rho_{\mathcal{D}} \bigotimes_{K \in \delta} \bigotimes_{L \in \mathcal{D}} \mu_{L \cap K} \\ &= \sum_{\delta \in \mathcal{G}^n} \sum_{\mathcal{D} \in \mathcal{G}} \mathbb{P}(Y_n = \delta) \rho_{\mathcal{D}} \bigotimes_{L \cap K \in \mathcal{D} \vee \delta} \mu_{L \cap K}. \end{aligned} \tag{10}$$

We used $\mu_\emptyset = 1$ when the atoms are empty, and for stating (10) we used (4). Since $\mathcal{D} \vee \delta \in \mathcal{G}^{n+1}$ when $\delta \in \mathcal{G}^n$ and $\mathcal{D} \in \mathcal{G}$, we get the decomposition,

$$\begin{aligned} \Xi^{n+1}[\mu] &= \sum_{\delta' \in \mathcal{G}^{n+1}} \left(\sum_{\delta \in \mathcal{G}^n} \sum_{\mathcal{D} \in \mathcal{G}: \mathcal{D} \vee \delta = \delta'} \mathbb{P}(Y_n = \delta) \rho_{\mathcal{D}} \right) \bigotimes_{L \in \delta'} \mu_L \\ &= \sum_{\delta' \in \mathcal{G}^{n+1}} \mathbb{P}(Y_{n+1} = \delta') \bigotimes_{L \in \delta'} \mu_L. \end{aligned}$$

Hence, the result is shown. \square

4. Quasi-stationary behavior

Let us denote by \mathbb{P}_δ the law of (Y_n) starting from $Y_0 = \delta$ and by \mathbb{E}_δ the mean expected value associated to \mathbb{P}_δ . As before, $\mathbb{P} = \mathbb{P}_{\{I\}}$ and $\mathbb{E} = \mathbb{E}_{\{I\}}$.

4.1. Hitting times

Let us define the hitting times,

$$\forall B \subseteq \mathcal{G}_0^+ : \zeta_B = \inf\{n \geq 0 : Y_n \in B\}.$$

For a partition $\delta \in \mathcal{G}_0^+$ we put $\zeta_\delta = \zeta_{\{\delta\}}$ the first time the chain hits δ . For $\delta = \{I\}$ we have $\mathbb{P}(\zeta_{\{I\}} = 0) = 1$. The random time for hitting $\mathcal{D}^\mathcal{G}$ is noted,

$$\zeta = \zeta_{\mathcal{D}^\mathcal{G}} = \inf\{n \geq 0 : Y_n = \mathcal{D}^\mathcal{G}\}.$$

Since $\mathcal{D}^\mathcal{G}$ is an absorbing point, then $Y_{\zeta+n} = \mathcal{D}^\mathcal{G}$ for all $n \geq 0$. So, recalling the notation $\mu^{(n)} = \bigotimes_{K \in Y_n} \mu_K$, we have $\mu^{(\zeta+n)} = \bigotimes_{L \in \mathcal{D}^\mathcal{G}} \mu_L$ for $n \geq 0$.

When $\mathcal{G} = \{\delta^*\}$ is a singleton we get

$$\forall n \geq 1 : \Xi^n[\mu] = \Xi[\mu] = \bigotimes_{L \in \delta^*} \mu_L.$$

Then, the evolution is trivial. Hence, in the sequel we assume

$$|\mathcal{G}| \geq 2. \tag{11}$$

Our main result, which is stated and proven in next section, only requires (11) as unique hypothesis. This result examine ratio limits between quantities of the type $\mathbb{P}(\zeta > n, Y_n \in B)$. In the case $\mathcal{G} = \{\{I\}, \delta\}$ with $\delta \neq \{I\}$ the results and computations turn out to be trivial because $\mathcal{D}^\mathcal{G} = \delta$, $\mathbb{P}(\zeta > n, Y_n = \{I\}) = \mathbb{P}(\zeta > n) = \rho_{\{I\}}^n$. The case $\mathcal{G} = \{\{I\}, \delta, \delta'\}$ with $\delta, \delta' \neq \{I\}$ will be developed in Section 4.5.

4.2. The main result

Before stating the Theorem, let us introduce the set of points connected to $\mathcal{D}^\mathcal{G}$ which are different from it,

$$\Gamma = \{\delta \in \mathcal{G}_0^+ : \delta \rightarrow \mathcal{D}^\mathcal{G}, \delta \neq \mathcal{D}^\mathcal{G}\}. \tag{12}$$

This set is nonempty and its shapes can vary drastically with \mathcal{G} , for instance if $\mathcal{D}^\mathcal{G} \in \mathcal{G}$ then $\Gamma = \mathcal{G}_0^+ \setminus \{\mathcal{D}^\mathcal{G}\}$.

Theorem 4.1. Assume $|\mathcal{G}| \geq 2$. Then,

$$\mathbb{P}(\zeta < \infty) = 1. \tag{13}$$

Define

$$\eta = \max\{P_{\delta,\delta} : \delta \in \mathcal{G}_0^+, \delta \neq \mathcal{D}^{\mathcal{G}}\} \text{ and } \mathcal{F} = \{\delta \in \mathcal{G}^+ : P_{\delta,\delta} = \eta\}.$$

Then, $\eta \in (0, 1)$, $\emptyset \neq \mathcal{F} \subseteq \Gamma$ and $\mathbb{P}(\zeta_{\mathcal{F}} < \infty) > 0$.

The geometric rate of decay of $\mathbb{P}(\zeta > n)$ is η , and satisfies,

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) = \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) = \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty). \tag{14}$$

Let

$$\mathcal{G}_0^* = \mathcal{G}_0^+ \setminus \{\mathcal{D}^{\mathcal{G}}\} \text{ and } P^* = (P_{\delta,\delta'} : \delta, \delta' \in \mathcal{G}_0^*).$$

The quasi-limiting distribution on \mathcal{G}_0^* is given by,

$$\begin{aligned} \forall \delta \in \mathcal{F} : \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = \delta \mid \zeta > n) &= \frac{\mathbb{E}(\eta^{-\zeta_{\delta}}, \zeta_{\delta} < \infty)}{\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}, \\ \forall \delta \in \mathcal{G}_0^* \setminus \mathcal{F} : \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = \delta \mid \zeta > n) &= 0. \end{aligned} \tag{15}$$

The following ratio limit relation is satisfied,

$$\forall \delta \in \mathcal{G}_0^* : \lim_{n \rightarrow \infty} \frac{\mathbb{P}_{\delta}(\zeta > n)}{\mathbb{P}(\zeta > n)} = \frac{\mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}. \tag{16}$$

Both ratios vanish only when $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$. The vector

$$\varphi = (\varphi_{\delta} : \delta \in \mathcal{G}_0^*) \text{ given by } \varphi_{\delta} = \mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty), \tag{17}$$

is a right eigenvector of P^* with eigenvalue η (note that $\varphi_{\{I\}} = 1$).

Proof. From hypothesis (11) we have $\mathcal{G}^+ \setminus \{\mathcal{D}^{\mathcal{G}}\} \neq \emptyset$, then $\mathcal{F} \neq \emptyset$ and (7) gives $\eta \in (0, 1)$. We have

$$\forall \delta \in \mathcal{F} : P_{\delta,\delta} + P_{\delta,\mathcal{D}^{\mathcal{G}}} = 1. \tag{18}$$

In fact, let $\delta \in \mathcal{F}$ and $P_{\delta,\delta'} > 0$ for δ' different from δ and $\mathcal{D}^{\mathcal{G}}$. Then, (7) implies $\eta = P_{\delta,\delta} < P_{\delta',\delta'}$ which contradicts the definition of η , so $P_{\delta,\delta'} = 0$ for all $\delta' \notin \mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}$.

Let us now prove $\mathcal{F} \subseteq \Gamma$. By hypothesis, \mathcal{G} is not a singleton so $\mathcal{G} \neq \{\mathcal{D}^{\mathcal{G}}\}$, which implies $\mathcal{F} \neq \emptyset$. Since every $\delta \in \mathcal{F}$ satisfies $P_{\delta,\delta} < 1$, from (18) we deduce $P_{\delta,\mathcal{D}^{\mathcal{G}}} > 0$, so $\delta \in \Gamma$.

From (6), relation (18) can be written,

$$\forall \delta \in \mathcal{F} : \quad \delta \rightarrow \delta' \Leftrightarrow [\delta' = \delta \text{ or } \delta' = \mathcal{D}^{\mathcal{G}}].$$

Define,

$$\beta_0 = \max\{P_{\delta,\delta} : \delta \in \mathcal{G}_0^+, \delta \neq \mathcal{D}^{\mathcal{G}}, \delta \notin \mathcal{F}\}, \tag{19}$$

where we put $\beta_0 = 0$ if $\mathcal{G}_0^+ = \mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}$. Let us prove

$$\beta_0 < \eta. \tag{20}$$

This is trivial if $\mathcal{G}_0^+ = \mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}$. When $\mathcal{G}_0^+ \neq \mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}$, by definition of \mathcal{F} we have $P_{\delta,\delta} < \eta$ for all $\delta \in \mathcal{G}_0^+ \setminus (\mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\})$, so (20) holds.

Let us show (13). As already noted, when (Y_n) exits from some state it does never return to it. This fact together with inequality $P_{\delta,\delta} < 1$ for $\delta \neq \mathcal{D}^{\mathcal{G}}$, give

$$\forall \delta \in \mathcal{G}_0^+, \delta \neq \mathcal{D}^{\mathcal{G}} : \quad \mathbb{P}(\#\{n : Y_n = \delta\} < \infty) = 1.$$

So, since $\mathcal{D}^{\mathcal{G}}$ is an absorbing state for (Y_n) , we get (13): $\mathbb{P}(\zeta < \infty) = \mathbb{P}(\exists n : Y_n = \mathcal{D}^{\mathcal{G}}) = 1$.

On the other hand, the existence of paths from $\{I\}$ to \mathcal{F} with strictly positive probability gives $\mathbb{P}(\zeta_{\mathcal{F}} < \infty) > 0$.

Let us now turn to the proof of relations (14), (15) and (16). From (18) we get,

$$\forall \delta^* \in \mathcal{F}, n \geq 0 : \quad \mathbb{P}_{\delta^*}(Y_n = \delta^*) = \mathbb{P}_{\delta^*}(\forall j \leq n, Y_j = \delta^*) = \eta^n.$$

We have

$$\mathbb{P}(\zeta > n) = \mathbb{P}(\zeta > n, Y_n \notin \mathcal{F}) + \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}). \tag{21}$$

Since every $\delta \in \mathcal{G}^+$ is attained from $\{I\}$, we obtain the existence of $k_0 \geq 1$ such that

$$\forall \delta^* \in \mathcal{F} : \quad \mathbb{P}(\zeta_{\delta^*} \leq k_0) > 0.$$

Define $\alpha(\mathcal{F}) := \min\{\mathbb{P}(\zeta_{\delta^*} \leq k_0) : \delta^* \in \mathcal{F}\}$, so $\alpha(\mathcal{F}) > 0$. From the Markov property we get for all $\delta^* \in \mathcal{F}$,

$$\begin{aligned} \mathbb{P}(\zeta > n) &\geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j, \zeta > n) \geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \mathbb{P}_{\delta^*}(\zeta > n - j) \\ &\geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \mathbb{P}_{\delta^*}(Y_{n-j} = \delta^*) \geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \eta^{n-j} \geq \alpha(\mathcal{F}) \eta^n. \end{aligned} \tag{22}$$

To analyze the first term at the right hand side of equality (21), it will be useful to prove the following result involving the quantity β_0 defined in (19).

Lemma 4.2. *We have,*

$$\forall \theta > 0 \exists C' = C'(\theta) : \forall n \geq 0, \mathbb{P}(\forall j \leq n : Y_j \notin \mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}) \leq C'(\beta_0 + \theta)^n. \tag{23}$$

Proof of Lemma 4.2. Let $U = \mathcal{G}_0^+ \setminus (\mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\})$. Since $Y_j \in \mathcal{G}^+$ when $j \geq 1$, the result is trivial if $G^+ \setminus (\mathcal{F} \cup \{\mathcal{D}^{\mathcal{G}}\}) = \emptyset$. So, assume this last set is nonempty, in particular $U \neq \emptyset$. Fix $\delta_1 = \{I\}$. For every $s \geq 2$ consider the following family of paths,

$$\mathcal{C}(U, s) = \{(\delta_1, \dots, \delta_s) \in U^s : \forall r \leq s - 1, \delta_r \rightarrow \delta_{r+1} \text{ and } \delta_r \neq \delta_{r+1}\}.$$

So, $P_{\delta_r, \delta_{r+1}} > 0$ for all $r = 1, \dots, s - 1$, see (6). We have

$$\begin{aligned} &\mathbb{P}(\forall j \leq n : Y_j \in U) \\ &= \sum_{s \geq 2} \sum_{(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)} \prod_{r=1}^{s-1} P_{\delta_r, \delta_{r+1}} \left(\sum_{k_1, \dots, k_s \geq 0 : \sum_{r=1}^s k_r = n - s} P_{\delta_r, \delta_r}^{k_r} \right). \end{aligned}$$

When $(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)$ we have that every δ_k with $k \leq s$ satisfies $P_{\delta_k, \delta_k} \leq \beta_0$. On the other hand,

$$\#\{(k_1, \dots, k_s) : \forall r \leq s, k_r \geq 0; \sum_{r=1}^s k_r = n - s\} = \binom{n - 1}{s}.$$

Then,

$$\mathbb{P}(\forall j \leq n : Y_j \in U) \leq \sum_{s \geq 2} \binom{n - 1}{s} \beta_0^{n-s} \left(\sum_{(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)} \prod_{r=0}^{s-1} P_{\delta_r, \delta_{r+1}} \right).$$

Since a path $(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)$ necessarily satisfies $s \leq |I|$ (because the elements δ_r are different and become finer when r increases), we get that $\mathcal{C}(U, s) \neq \emptyset$ implies $s \leq |I|$. Then, the index s in the sum can be restricted to be smaller than or equal to $|I|$. So,

$$C_1 = \sum_{s \geq 2} \sum_{(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)} \prod_{r=0}^{s-1} P_{\delta_r, \delta_{r+1}} = \sum_{s=2}^{|I|} \sum_{(\delta_1, \dots, \delta_s) \in \mathcal{C}(U, s)} \prod_{r=0}^{s-1} P_{\delta_r, \delta_{r+1}} < \infty.$$

On the other hand, for $\theta' \in (0, 1)$ we have

$$C_2(\theta') = \max_{s \leq |I|} \sup_{n \geq 1} \binom{n - 1}{s} (1 - \theta')^{n - |I|} < \infty.$$

Then,

$$\mathbb{P}(\forall j \leq n : Y_j \in U) \leq C_1 \cdot C_2(\theta')\beta_0^{n-|I|}/(1 - \theta')^{n-|I|}.$$

Hence, by taking $\theta' \in (0, 1)$ such that $\beta_0/(1 - \theta') < \beta_0 + \theta$ we get that the constant

$$C' = (\beta_0 + \theta)^{-|I|}C_1 \cdot C_2(\theta') < \infty$$

makes the job in (23). \square

Continuation with the proof of Theorem 4.1.

From (20) we can fix $\theta > 0$ such that $\beta_0 + \theta < \eta$. Hence, from (22) and (23) we find

$$\mathbb{P}(Y_n \notin \mathcal{F} | \zeta > n) \leq C'' ((\beta_0 + \theta)/\eta)^n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

with $C'' = C'/\alpha(\mathcal{F})$. Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \in \mathcal{F} | \zeta > n) = 1. \tag{24}$$

Let us examine the second term at the right hand side of equality (21). For every $\delta^* \in \mathcal{F}$ we have

$$\begin{aligned} \mathbb{P}(\zeta > n, Y_n = \delta^*) &= \sum_{j=1}^n \mathbb{P}(\zeta > n, \zeta_{\delta^*} = j) \\ &= \sum_{j=1}^n \mathbb{P}(\zeta_{\delta^*} = j)\mathbb{P}_{\delta^*}(\zeta > n - j) \\ &= \sum_{j=1}^n \mathbb{P}(\zeta_{\delta^*} = j)\eta^{n-j} = \eta^n \left(\sum_{j=1}^n \eta^{-j}\mathbb{P}(\zeta_{\delta^*} = j) \right). \end{aligned}$$

Since

$$\begin{aligned} \mathbb{P}(\zeta_{\delta^*} = j) &\leq \mathbb{P}(\zeta_{\mathcal{F}} = j) \\ &\leq \mathbb{P}(\forall n \leq j - 1 : Y_n \notin \mathcal{F} \cup \{\mathcal{D}^G\}) \leq C'(\beta_0 + \theta)^{j-1}, \end{aligned}$$

and $\beta_0 + \epsilon < \eta$, we get $\sum_{j=1}^{\infty} \eta^{-j}\mathbb{P}(\zeta_{\delta^*} = j) < \infty$. Hence,

$$\begin{aligned} \forall \delta^* \in \mathcal{F} : \lim_{n \rightarrow \infty} \eta^{-n}\mathbb{P}(\zeta > n, Y_n = \delta^*) &= \sum_{j=1}^{\infty} \eta^{-j}\mathbb{P}(\zeta_{\delta^*} = j) \\ &= \mathbb{E}(\eta^{-\zeta_{\delta^*}}, \zeta_{\delta^*} < \infty) < \infty. \end{aligned} \tag{25}$$

Now, for $\delta^* \in \mathcal{F}$ we have

$$\zeta_{\delta^*} < \infty \Rightarrow [\zeta_{\mathcal{F}} < \infty \text{ and } \forall \delta' \in \mathcal{F} \setminus \{\delta^*\}, \zeta_{\delta'} = \infty].$$

Then, for j finite,

$$\{\zeta_{\mathcal{F}} = j\} = \bigcup_{\delta^* \in \mathcal{F}} \{\zeta_{\delta^*} = j\}$$

and the union is disjoint. So, $\eta^{-\zeta_{\mathcal{F}}} \mathbf{1}_{\zeta_{\mathcal{F}} < \infty} = \sum_{\delta^* \in \mathcal{F}} \eta^{-\zeta_{\delta^*}} \mathbf{1}_{\zeta_{\delta^*} < \infty}$. Hence,

$$\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \sum_{\delta^* \in \mathcal{F}} \mathbb{E}(\eta^{-\zeta_{\delta^*}}, \zeta_{\delta^*} < \infty) < \infty. \tag{26}$$

Then, from (25) we deduce

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) = \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty). \tag{27}$$

Therefore, from relations (25) and (27), we get (15). Also (15) and (26) give

$$\sum_{\delta \in \mathcal{F}} \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = \delta \mid \zeta > n) = \frac{1}{\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)} \sum_{\delta \in \mathcal{F}} \mathbb{E}(\eta^{-\zeta_{\delta}}, \zeta_{\delta} < \infty) = 1,$$

and so the quasi-limiting distribution (15) is a distribution (there is no loss of mass).

Now, relation (14) is a consequence of relations (24) and (27) because they imply

$$\begin{aligned} \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) &= \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) \\ &= \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty). \end{aligned}$$

Let us show (16). First, assume δ is such that $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) > 0$, that is there exists a path with strictly positive probability from δ to some nonempty subset of \mathcal{F} . A similar proof as the one showing (14) gives

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}_{\delta}(\zeta > n) = \mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty), \tag{28}$$

so (16) is satisfied. Now, let $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$. Then, $\mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 0$ and in (16) we have $\mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) / \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 0$. We claim that in this case we also have $\lim_{n \rightarrow \infty} \mathbb{P}_{\delta}(\zeta > n) / \mathbb{P}(\zeta > n) = 0$. In fact $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$ implies

$$\begin{aligned} (\beta_0 + \theta)^{-n} \mathbb{P}_{\delta}(\zeta > n) &= (\beta_0 + \theta)^{-n} \mathbb{P}_{\delta}(\zeta > n, \zeta_{\mathcal{F}} > n) \\ &= (\beta_0 + \theta)^{-n} \mathbb{P}(\forall j \leq n : Y_j \notin (\mathcal{F} \cup \{\mathcal{D}^g\})) < \infty. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) > 0$ and $\beta_0 + \theta < \eta$, the claim follows and (16) is shown.

The last statement to be proven is that φ defined in (17) is a right eigenvector of P^* with eigenvalue η . First take $\delta \in \mathcal{F}$. We have $\mathbb{P}_\delta(\zeta_{\mathcal{F}} = 0) = 1$ and so $\varphi_\delta = \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 1$. From (18) and $P_{\delta,\delta} = \eta$ we get

$$(P^*\varphi)_\delta = \sum_{\delta':\delta' \neq \mathcal{D}^g, \delta \rightarrow \delta'} P_{\delta,\delta'} \mathbb{E}_{\delta'}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = P_{\delta,\delta} = \eta = \eta \varphi_\delta.$$

Now, let δ be such that $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) = 0$, so $\varphi_\delta = 0$. Then $P_{\delta,\delta'} > 0$ implies $\mathbb{P}_{\delta'}(\zeta_{\mathcal{F}} < \infty) = 0$ and so $(P^*\varphi)_\delta = 0 = \eta \varphi_\delta$.

Now take $\delta \notin \mathcal{F}$ with $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) > 0$. From the Markov property we get,

$$\begin{aligned} \varphi_\delta &= \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \sum_{\delta':\delta' \neq \mathcal{D}^g, \delta \rightarrow \delta'} \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty, Y_1 = \delta') \\ &= \sum_{\delta':\delta' \neq \mathcal{D}^g, \delta \rightarrow \delta'} P_{\delta,\delta'} \eta^{-1} \mathbb{E}_{\delta'}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \eta^{-1} (P^*\varphi)_\delta. \end{aligned}$$

Notice that (28) gives $\varphi_\delta \in (0, \infty)$. Then, the result is shown, and this finishes the proof of the theorem. \square

Remark 4.3. Let $R = (R_{k,l} : k, l \in \mathcal{K})$ be a stochastic matrix defined on a finite set \mathcal{K} . Define the relation

$$k \hookrightarrow l \Leftrightarrow [R_{k,l} > 0, k \neq l],$$

which is the graph of connections defined by R except by loops. Assume that $(\mathcal{K}, \hookrightarrow)$ has no cycles and $k \hookrightarrow l$ implies $R_{k,k} < R_{l,l}$. (Note that the matrix P defined on \mathcal{G}_0^+ by (5) satisfies these properties, this follows from (6), (3) and (8)). One of the referees has observed that Theorem 4.1 can be extended for the Markov chain defined by a matrix R that satisfy the above conditions on the graph of connections.

The extension works as follows. Fix some $k_0 \in \mathcal{K}$. Assume $R_{k_0,k_0} \neq 1$. Then, since there is no cycles, if $k_0 \hookrightarrow k_1 \dots \hookrightarrow k_r$ is a path then all the points are different, so $r \leq |\mathcal{K}|$. Let $U(k_0)$ be the set of maximal paths starting from k_0 (maximal means that they cannot be extended by a path $k_0 \hookrightarrow k_1 \dots \hookrightarrow k_r \hookrightarrow k_{r+1}$). If $k_0 \hookrightarrow k_1 \dots \hookrightarrow k_r$ is one of these maximal paths, we necessarily have $R_{k_r,k_r} = 1$ (in the contrary the path could be extended), so the terminal points of the maximal paths in $U(k_0)$ are absorbing points for R . Let $A(k_0)$ be the set of all the absorbing points that can be attained from k_0 and $\mathcal{K}(k_0)$ be the set of all the points that can be attained from k_0 and which are not in $A(k_0)$. Then, a similar proof as that of Theorem 4.1 allows to show that

$$\eta(k_0) = \max\{R_{k,k} : k \in \mathcal{K}(k_0)\}$$

is the geometric decay rate of the hitting time of $A(w_0)$ when the Markov chain starts from k_0 . Also, in this more general frame, the quasi-limiting behavior and the ratio limit result can be formulated and proved similarly as in Theorem 4.1,

4.3. The Q-chain

The results in [Theorem 4.1](#) allow us to describe the Q-chain, which in our case is the Markov chain that avoids the singleton $\{\otimes_{L \in \mathcal{D}^g} \mu_L\}$. The Q-chain was introduced in [\[1\]](#) for branching processes. Other developments on Q-chains, including finite Markov chains, can be found in [\[7\]](#). For the next result we recall that $\mathcal{G}_0^* = \mathcal{G}_0^+ \setminus \{\mathcal{D}^g\}$.

Corollary 4.4. *For all $\delta_i \in \mathcal{G}_0^*$, $i = 1, \dots, k$, the following limit exists*

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n),$$

and it vanishes if some δ_i satisfies $\mathbb{P}_{\delta_i}(\zeta_{\mathcal{F}} < \infty) = 0$.

Denote by $\partial_{\mathcal{F}}$ the class of partitions that can attain \mathcal{F} , that is

$$\partial_{\mathcal{F}} = \{\delta \in \mathcal{G}_0^* : \mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) > 0\}.$$

Then, the matrix $Q = (Q_{\delta, \delta'} : \delta, \delta' \in \partial_{\mathcal{F}})$ given by

$$Q_{\delta, \delta'} = \eta^{-1} P_{\delta, \delta'} \frac{\mathbb{E}_{\delta'}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}_{\delta}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}, \tag{29}$$

is an stochastic matrix on $\partial_{\mathcal{F}}$, and it is satisfied

$$\forall \delta_i \in \partial_{\mathcal{F}}, i = 0, \dots, j : \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) = \prod_{i=0}^{j-1} Q_{\delta_i, \delta_{i+1}}. \tag{30}$$

So, Q is the transition matrix of the Markov chain that never hits $\otimes_{L \in \mathcal{D}^g} \mu_L$ and

$$\forall \delta \in \mathcal{F} : Q_{\delta, \delta} = 1 \text{ and } \forall \delta \in \partial_{\mathcal{F}} \setminus \mathcal{F} : Q_{\delta, \delta} < 1. \tag{31}$$

That is, the elements in \mathcal{F} are the unique absorbing points of the chain Q .

Proof. Let us prove that Q is an stochastic matrix. Let φ be the right eigenvector of P^* with eigenvalue η given in [\(17\)](#). The component φ_{δ} vanishes when $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$. Let $\delta \in \partial_{\mathcal{F}}$, so $\varphi_{\delta} > 0$. We have $P_{\delta, \delta'} = 0$ if $\delta \not\rightarrow \delta'$ and

$$\mathbb{P}_{\delta'}(\zeta_{\mathcal{F}} < \infty) = 0 \text{ implies } \frac{\varphi_{\delta'}}{\varphi_{\delta}} = \frac{\mathbb{E}_{\delta'}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}_{\delta}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)} = 0.$$

Then, since φ is a right eigenvector with eigenvalue η we get

$$\sum_{\delta' \in \partial_{\mathcal{F}}} Q_{\delta, \delta'} = \eta^{-1} \left(\sum_{\delta' \in \partial_{\mathcal{F}}} P_{\delta, \delta'} \frac{\varphi_{\delta'}}{\varphi_{\delta}} \right) = \eta^{-1} \left(\frac{\eta \varphi_{\delta}}{\varphi_{\delta}} \right) = 1.$$

From the Markov property we obtain for $n > j$,

$$\mathbb{P}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) = \mathbb{P}(Y_i = \delta_i, i = 1, \dots, j) \frac{\mathbb{P}_{\delta_j}(\zeta > n - j)}{\mathbb{P}(\zeta > n)}.$$

Now we use the ratio limit result (16). This limit vanishes if $\mathbb{P}_{\delta_j}(\zeta_{\mathcal{F}} < \infty) = 0$ and it also vanishes when $\mathbb{P}_{\delta_i}(\zeta_{\mathcal{F}} < \infty) = 0$ for some $i < j$ because $P_{\delta_i, \delta_{i+1}} > 0$ implies $\mathbb{P}_{\delta_{i+1}}(\zeta_{\mathcal{F}} < \infty) = 0$. When $\delta_i \in \partial_{\mathcal{F}}$ for $i = 0, \dots, j$, relation (16) gives

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j) \frac{\mathbb{P}_{\delta_j}(\zeta > n - j)}{\mathbb{P}_{\delta_0}(\zeta > n)} \\ &= \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j) \frac{\varphi_{\delta_j}}{\varphi_{\delta_0}} \eta^{-j} \\ &= \prod_{l=0}^{j-1} \left(\eta^{-1} P_{\delta_l, \delta_{l+1}} \frac{\varphi_{\delta_{l+1}}}{\varphi_{\delta_l}} \right). \end{aligned} \tag{32}$$

In (32) we used $\lim_{n \rightarrow \infty} \mathbb{P}(\zeta > n - j) / \mathbb{P}(\zeta > n) = \eta^{-j}$, which is a consequence of (14). Then, relation (30) is proven.

The diagonal terms of Q satisfy

$$Q_{\delta, \delta'} = \eta^{-1} P_{\delta, \delta}. \tag{33}$$

By definition of η we get $Q_{\delta, \delta} = 1$ for all $\delta \in \mathcal{F}$ and $Q_{\delta, \delta} < 1$ when $\delta \in \partial_{\mathcal{F}} \setminus \mathcal{F}$. Then, the result follows. \square

Remark 4.5. Hence, once the Q -chain hits one of the states in \mathcal{F} it remains in it forever. So, the partitions in \mathcal{F} will be candidates for the points one observes if, after a long time has elapsed, the chain has not attained $\mathcal{D}^{\mathcal{G}}$. From (29), we get $Q_{\delta, \delta'} > 0$ implies $P_{\delta, \delta'} > 0$. Then, from (6) we find

$$Q_{\delta, \delta'} > 0 \Rightarrow \delta \rightarrow \delta'. \tag{34}$$

On the other hand, from (33) and (8) we get

$$[\delta \rightarrow \delta', \delta \neq \delta'] \Rightarrow Q_{\delta, \delta} < Q_{\delta', \delta'}. \tag{35}$$

Therefore, from relations (31), (34) and (35), we can apply the techniques developed in Theorem 4.1 for the Q -chain whose set of limit points is \mathcal{F} (which is not necessarily a singleton). From (33), the geometric decay rate of the chain Q to the limit points \mathcal{F} is given by

$$\eta' = \max\{Q_{\delta,\delta} : \delta \in \partial_{\mathcal{F}} \setminus \mathcal{F}\} = \eta^{-1} \max\{P_{\delta,\delta} : \delta \in \partial_{\mathcal{F}} \setminus \mathcal{F}\}.$$

The quasi-limiting behavior and ratio limit results can be stated similarly. Moreover, we can apply to the chain Q that avoids \mathcal{F} the same construction as that used for P and $\{D^{\mathcal{G}}\}$ in Corollary 4.4. Obviously we require that $\partial_{\mathcal{F}} \setminus \mathcal{F}$ has more than two points in order that this to be non trivial. This will give another (Q -)chain whose limit points will compose the set of partitions δ in $\partial_{\mathcal{F}} \setminus \mathcal{F}$ that maximizes $P_{\delta,\delta}$. This last construction can be also iterated similarly.

4.4. A class of quasi-stationary distributions

Let us give an explicit class of quasi-stationary distributions, that must be compared with the irreducible case where there is a unique quasi-stationary distribution.

Let $\nu = (\nu_{\delta} : \delta \in \mathcal{G}_0^*)$ be a probability measure on \mathcal{G}_0^* . If necessary, ν will be identified with its extension on \mathcal{G}_0^+ with $\nu_{\mathcal{D}^{\mathcal{G}}} = 0$. We say that ν is supported by some subset $\tilde{\mathcal{G}} \subseteq \mathcal{G}_0^*$ if $\nu(\tilde{\mathcal{G}}) = 1$. We denote by ν' the row vector associated to ν .

Proposition 4.6. *Every probability measure ν on \mathcal{G}_0^* supported on \mathcal{F} satisfies $\nu'P^* = \eta\nu'$ and it is a quasi-stationary distribution, that is it satisfies*

$$\forall n \geq 1, \forall \delta \in \mathcal{G}_0^* : \mathbb{P}_{\nu}(Y_n = \delta \mid \zeta > n) = \nu_{\delta}. \tag{36}$$

Proof. With the above notation and by using (18) we get,

$$(\nu'P^*)_{\delta} = P_{\delta,\delta}\nu_{\delta} = \eta\nu_{\delta},$$

so $\nu'P^* = \eta\nu'$. By iteration we find $\nu'P^{*n} = \eta^n\nu'$. Note that this is equivalent to

$$(\nu'P^{*n})_{\delta} = \mathbb{P}_{\nu}(Y_n = \delta) = \mathbb{P}_{\nu}(\forall j \leq n Y_j = \delta) = \eta^n\nu'_{\delta}.$$

Now

$$\mathbb{P}_{\nu}(\zeta > n) = \sum_{\delta \in \mathcal{F}} (\nu'P^{*n})_{\delta} = \eta^n \left(\sum_{\delta \in \mathcal{F}} \nu_{\delta} \right) = \eta^n.$$

Hence, relation (36) is proven. \square

4.5. Example

Let $\delta^1 = \{I\}$ and δ^2, δ^3 be two different partitions and different from δ^1 . Let ρ be the probability vector on the set of partitions such that $\rho_{\delta^1} \geq 0, \rho_{\delta^2} > 0, \rho_{\delta^3} > 0$, and $\rho_{\delta^1} + \rho_{\delta^2} + \rho_{\delta^3} = 1$. Then $\mathcal{G} = \{\delta^2, \delta^3\}$ if $\rho_{\delta^1} = 0$ or $\mathcal{G} = \{\delta^1, \delta^2, \delta^3\}$ if $\rho_{\delta^1} > 0$. The

partition $\delta^4 = \delta^2 \vee \delta^3$ is strictly finer than δ^2 and δ^3 because these last partitions are different. We have $\mathcal{D}^{\mathcal{G}} = \delta^4$ and $\mathcal{G}^+ = \mathcal{G} \cup \{\delta^4\}$.

Denote $a = \rho_{\delta^1}$, $b = \rho_{\delta^2}$, $c = \rho_{\delta^3}$. Then, $a + b + c = 1$, $a \geq 0$ and $b, c > 0$. To write the matrix P we order the states in the obvious way, identifying k with δ^k for $k = 1, 2, 3, 4$. So, the matrix P is

$$P = \begin{pmatrix} a & b & c & 0 \\ 0 & a+b & 0 & c \\ 0 & 0 & a+c & b \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

because $P_{\delta^1, \delta^k} = \rho_{\delta^k}$ for $k = 1, 2, 3$; $P_{\delta^2, \delta^2} = \rho_{\delta^1} + \rho_{\delta^2}$, $P_{\delta^2, \delta^4} = \rho_{\delta^3}$; and $P_{\delta^3, \delta^3} = \rho_{\delta^1} + \rho_{\delta^3}$, $P_{\delta^3, \delta^4} = \rho_{\delta^2}$.

We have $\Gamma = \{\delta^2, \delta^3\}$ because these are the unique partitions δ such that $\delta \rightarrow \mathcal{D}^{\mathcal{G}}$ and $\delta \neq \mathcal{D}^{\mathcal{G}}$ (see (12)). The geometric decay rate of convergence to $\mathcal{D}^{\mathcal{G}}$ is $\eta = \max\{a+b, a+c\}$. If $b > c$ then $\mathcal{F} = \{\delta^2\}$, if $b < c$ then $\mathcal{F} = \{\delta^3\}$ and when $b = c$, then $\mathcal{F} = \{\delta^2, \delta^3\}$.

Let us assume $a > 0$ so $\mathcal{G} = \{\delta^1, \delta^2, \delta^3\}$ (the case $a = 0$ is easier to analyze). The restriction P^* of P to $\mathcal{G}_0^* = \{\delta^1, \delta^2, \delta^3\}$ is the substochastic matrix,

$$P^* = \begin{pmatrix} a & b & c \\ 0 & a+b & 0 \\ 0 & 0 & a+c \end{pmatrix}.$$

By induction, the iterates P^{*n} have the form

$$P^{*n} = \begin{pmatrix} a^n & \alpha_n & \beta_n \\ 0 & (a+b)^n & 0 \\ 0 & 0 & (a+c)^n \end{pmatrix},$$

where α_n and β_n satisfy:

$$\alpha_{n+1} = a\alpha_n + b(a+b)^n, \alpha_1 = b \text{ and } \beta_{n+1} = a\beta_n + c(a+c)^n, \beta_1 = c.$$

This gives $\alpha_n = \sum_{j=1}^n a^{j-1}b(a+b)^{n-j}$ and $\beta_n = \sum_{j=1}^n a^{j-1}c(a+c)^{n-j}$.

We have

$$\mathbb{P}(\zeta > n) = a^n + \alpha_n + \beta_n = a^n + \sum_{j=1}^n a^{j-1}(b(a+b)^{n-j} + c(a+c)^{n-j}).$$

Note that $a^{j-1}b = \mathbb{P}(\zeta_{\delta^2} = j)$ and $a^{j-1}c = \mathbb{P}(\zeta_{\delta^3} = j)$. Let us assume $b = c$, so $\eta = (a+b)$ and $\mathcal{F} = \{\delta^2, \delta^3\}$ (the cases $b > c$ or $b < c$ can be analyzed in a similar way). For $k = 2, 3$ we have

$$\mathbb{P}(Y_n = \delta^k) = \sum_{j=1}^n \mathbb{P}(\zeta_{\delta^k} = j, \zeta > n) = \sum_{j=1}^n a^{j-1} b (a + b)^{n-j},$$

and so,

$$\mathbb{P}(Y_n = \delta^k \mid \zeta > n) = \frac{\sum_{j=1}^n b a^{j-1} (a + b)^{n-j}}{a^n + 2 \sum_{j=1}^n a^{j-1} b (a + b)^{n-j}}.$$

This converges to 1/2 when $n \rightarrow \infty$ because $a < a + b$.

From (16), the right eigenvector φ is given by

$$\varphi = (\varphi_{\delta^1}, \varphi_{\delta^2}, \varphi_{\delta^3}) \text{ with } \varphi_{\delta^k} = \lim_{n \rightarrow \infty} \frac{\mathbb{P}_{\delta^k}(\zeta > n)}{\mathbb{P}_{\delta^1}(\zeta > n)}, k = 1, 2, 3.$$

So, $\varphi_{\delta^1} = 1$. Let $k = 2, 3$, from $\mathbb{P}_{\delta^k}(\zeta > n) = (a + b)^n$ we get

$$\varphi_{\delta^k} = \lim_{n \rightarrow \infty} \frac{(a + b)^n}{a^n + 2 \sum_{j=1}^n a^{j-1} b (a + b)^{n-j}} = \frac{1}{2 \sum_{j=1}^{\infty} a^{j-1} b (a + b)^{-j}} = \frac{1}{2},$$

because $\frac{b}{a+b} \sum_{j=0}^{\infty} \left(\frac{a}{a+b}\right)^j = 1$. Then the Q -matrix is

$$Q = \begin{pmatrix} a/(a + b) & b/(2(a + b)) & b/(2(a + b)) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, in agreement with Proposition 4.6, any vector of the form $(0, x, y)$ is a left eigenvector of P^* with eigenvalue $\eta = a + b$. It can be checked that the right eigenvectors associated to η are of the form $(x + y, x, y)$.

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the Examples serve to answer the suggestions and questions of Referee 2. The Referees also give valuable insights on the decomposition results works in [2,4] and [15]. This was complemented by helpful discussions the author had with Ellen Baake and Michael Baake during the School Information and Randomness 2016.

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