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Existence of a competitive equilibrium when all goods are indivisible[☆]

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ABSTRACT

This paper investigates an economy where all consumption goods are indivisible at the individual level, but perfectly divisible at the overall level of the economy. In order to facilitate trading of goods, we introduce a perfectly divisible parameter that does not enter into consumer preferences — fiat money. When consumption goods are indivisible, a Walras equilibrium does not necessarily exist. We introduce the rationing equilibrium concept and prove its existence. Unlike the standard Arrow–Debreu model, fiat money can always have a strictly positive price at the rationing equilibrium. In our set up, if the initial endowment of fiat money is dispersed, then a rationing equilibrium is a Walras equilibrium. This result implies the existence of a dividend equilibrium or a Walras equilibrium with slack.

1. Introduction

Most economic models assume that goods are perfectly divisible. The rationale behind this assumption might be that the minimal unit of a good is sufficiently insignificant so that its indivisibility can be neglected. Then, one should be able to approximate an economy with a sufficiently small level of indivisibility of the goods, by some idealized economy where all goods are perfectly divisible. Consequently, it would be reasonable to expect that a competitive equilibrium in this idealized economy should be an approximation of some competitive outcome of the economy with indivisible goods. In the case of a finite set of consumers, Henry (1970) shows that indivisibility of goods may lead to non-existence of a Walras equilibrium. Shapley and Scarf (1974) show that even the core may be empty. So the question arises, what type of competitive outcome of an economy with indivisible goods would be approximated by a Walras equilibrium of an economy with perfectly divisible goods?

In order to enable us to address this question, we present a model where indivisibility is negligible at the level of the overall economy, but relevant at the individual level. This is achieved by assuming discrete consumption sets, but a continuum of agents. In order to facilitate the exchange in such a setting, similar to Drèze and Müller (1980), we add a parameter to the economy providing wealth, in addition to the value of the initial endowment in consumption goods. This parameter might be interpreted as fiat money. It has no intrinsic value whatsoever, since it does not enter into consumers' preferences.[1]

Indivisibility of consumption goods implies that the Walras demand may fail to be upper hemi-continuous. This leads us to introduce a regularized notion of demand, called weak demand. Under the assumptions we consider, the weak demand will always be an upper hemi-continuous correspondence. In the standard convex framework with continuous preferences, it would coincide with the Walras demand at points where the budget set has a non-empty interior. Based on the weak demand, we will then define a rationing equilibrium, where the consumers' demand is a refinement of the auxiliary weak demand. At a rationing equilibrium, in order to formulate their demand, consumers will need in addition to the prices an aggregate knowledge on the demand supply imbalance in the market summarized by an endogenously

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1 In the convex case, when the non-satiation assumption does not necessarily hold for consumers, or when some prices are fixed, one may establish the existence of a competitive equilibrium by allowing for the possibility that some agents spend more than the value of their initial endowment. The corresponding generalization of the Walras equilibrium is called dividend equilibrium or equilibrium with slack (see also Aumann and Drèze (1986), Balasko (1982), Makarov (1981) and Mas-Colell (1992) among others). Kajii (1996) shows that the dividend approach is equivalent to considering a Walras equilibrium with an additional commodity called fiat money. If local non-satiation holds and prices are flexible, fiat money is worthless and we are back in the standard Arrow–Debreu setting. However, if satiation occurs, fiat money may have a positive price at equilibrium. If a consumer does not want to spend his entire income on goods, he can spend the remainder on fiat money, provided that fiat money has a positive price. In our set up, since all goods are indivisible, local non-satiation cannot hold at any point.

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determined salient cone. This cone indicates the net trade directions for which rationing can occur. We will then be interested in the situations where agents only prefer points in their budget set which would require the execution of a net trade in the cone, for which rationing can occur. The cone is salient, implying that, if there exists a somehow excessive demand for a certain net trade direction, then there occurs no rationing in the exact inverse net trade direction.

The main result of this paper is the existence of a rationing equilibrium, with a strictly positive price for fiat money (Theorem 4.1). However, when the consumers' initial endowment in fiat money is dispersed, rationing occurs at most for a null set of the consumers, and then the rationing equilibrium reduces to a Walras equilibrium. Hence, the present paper also establishes a Walras equilibrium existence result for the case where all consumption goods are indivisible. As one can always construct a dispersed initial endowment in fiat money, our result also implies the existence of a dividend equilibrium or Walras equilibrium with slack.

The efficiency and core equivalence properties of a rationing equilibrium are studied in Florig and Rivera (2010). There it is proven that rationing equilibria satisfy the First and Second Welfare theorems for weak Pareto optimality, and that they coincide with the rejective core proposed by Konovalov (2005). The asymptotic behavior of rationing equilibria when discrete consumption sets converge to convex sets is studied in Florig and Rivera (2015).

So far, we have not mentioned the vast literature on indivisible goods, which can be roughly divided into two different approaches. The first approach follows Shapley and Scarf (1974), who model a market without perfectly divisible goods, assuming only one commodity per agent. Under suitable conditions, they prove that the core of the economy is non-empty and that a competitive equilibrium exists. Subsequent extensions of their results can be found in Inoue (2008, 2014), Konishi et al. (2001), Sönmez (1996) and Wako (1984). For these models, the existence of a competitive equilibrium depends strongly on the number of agents and/or the number of indivisible goods existing in the economy.

The second approach follows Henry (1970), and considers an economy with indivisible commodities, but at least one perfectly divisible commodity, which might be interpreted as commodity money (see Bikhchandani and Mamer (1997), Broome (1972), Khan and Yamazaki (1981), Mas-Colell (1977), Quinzii (1984), van der Laan et al. (2002); see Bobzin (1998) for a survey). All these contributions suppose that money satisfies overriding desirability, i.e. it is so desirable by the agents that an adequate amount of money could replace the consumption of any bundle of indivisible goods. Under such an assumption they can prove the non-emptiness of the core and the existence of a Walras equilibrium.

The approach we follow is similar to the one developed by Dierker (1971), who established the existence of a quasi-equilibrium for exchange economies without perfectly divisible consumption goods. However in that approach, agents do not necessarily receive an individually rational commodity bundle at a quasi-equilibrium, a drawback that a rationing equilibrium overcomes.

This paper is organized as follows. Mathematical notations used throughout this paper are presented in Section 2. The economic model, as well as the equilibrium notions, are introduced in Section 3. Section 4 is devoted to present the equilibrium existence results. Most of the proofs are established in Appendix A.

2. Mathematical notations

In the following, x^t denotes the transpose of $x \in \mathbb{R}^m$, $x \cdot y = x^t y$ the inner product between $x,\ y \in \mathbb{R}^m$, $\|x\|$ the Euclidean norm of x, and $x^\perp = \{p \in \mathbb{R}^m : p \cdot x = 0\}$ is the hyperplane in $\mathbb{R}^m$ orthogonal to x. The origin of $\mathbb{R}^m$ is 0_m, and the closed ball of radius ε centered at $x \in \mathbb{R}^m$ is denoted by $\mathbb{B}(x, \varepsilon)$. Additionally, $\operatorname{cl} K$, $\operatorname{int} K$ and $\operatorname{conv} K$ denote, respectively, the closure, interior and the convex hull of subset $K \subseteq \mathbb{R}^m$, and the positive hull of K is

$$\operatorname{pos} K = \left\{ \sum_{i=1}^{s} \mu_i x_i : \mu_i \geq 0,\ x_i \in K,\ i = 1, \ldots, s,\ s \in \mathbb{N} \right\}.$$

For a couple of sets $K_1, K_2 \subseteq \mathbb{R}^m$, $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^m$, we denote $\xi K_1 = \{\xi x : x \in K_1\}$, $p \cdot K_1 = \{p \cdot x : x \in K_1\}$ and $K_1 \pm K_2 = \{x_1 \pm x_2 : x_1 \in K_1,\ x_2 \in K_2\}$, while the set-difference between K_1 and K_2 is denoted as $K_1 \setminus K_2$.

A convex set K of $\mathbb{R}^m$ is called a convex cone if $0_m \in K$ and $\xi K \subset K$ for all $\xi > 0$. A convex cone K is said to be *salient* if $K \cap -K = \{0_m\}$. In what follows, $\mathcal{C}_m$ stands for the set of salient cones of $\mathbb{R}^m$.

We follow Rockafellar and Wets (1998) to denote

$$\mathbb{N}_\infty = \{\mathrm{N} \subseteq \mathbb{N} : \mathbb{N} \setminus \mathrm{N} \text{ is finite}\} \quad \text{and}$$
$$\mathbb{N}^*_\infty = \{\mathrm{N} \subseteq \mathbb{N} : \mathrm{N} \text{ is infinite}\},$$

and given $\{x_n\}_{n\in\mathbb{N}}$ a sequence of elements in $\mathbb{R}^m$, for $\mathrm{N} \in \mathbb{N}^*_\infty$ or $\mathrm{N} \in \mathbb{N}_\infty$, we write $x_n \to_{\mathrm{N}} x$ to say that the subsequence $\{x_n\}_{n\in\mathrm{N}}$ converges to $x \in \mathbb{R}^m$; the subset of accumulation points of this subsequence is denoted by

$$\operatorname{acc}\{x_n\}_{n\in\mathrm{N}} = \left\{ x \in \mathbb{R}^m : \exists \mathrm{N}' \subseteq \mathrm{N},\ \mathrm{N}' \in \mathbb{N}^*_\infty,\ x_n \to_{\mathrm{N}'} x \right\}.$$

The outer limit of a sequence of subsets $\{K_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^m$, also known as topological limes superior, is the set

$$\limsup_{n\to\infty} K_n = \left\{ x \in \mathbb{R}^m : \exists \mathrm{N} \in \mathbb{N}^*_\infty,\ \exists x_n \in K_n,\ n \in \mathrm{N}, \text{ with } x_n \to_{\mathrm{N}} x \right\}.$$

The outer limit of a correspondence $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^m$ is

$$\limsup_{x\to\bar{x}} F(x) = \bigcup_{\{x_n\to\bar{x}\}} \limsup_{n\to\infty} F(x_n). \tag{1}$$

The fact that $\limsup_{x\to\bar{x}} F(x) = F(\bar{x})$ is not equivalent to the upper hemi-continuity of F at $\bar{x}$.[2] However, when F is closed valued, by Theorem 5.19 in Rockafellar and Wets (1998) it follows that the equivalence holds under the condition that F is locally bounded at $\bar{x}$, i.e., for some neighborhood V of $\bar{x}$ the set $F(V) \subset \mathbb{R}^n$ is bounded.

We now remind the integral of a correspondence $F : K_1 \rightrightarrows K_2$, where $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subseteq \mathbb{R}^\ell$. For the aim of this paper, it suffices to assume that K_1 is a compact set, and that K_2 is a closed set. Provided they are non-empty sets, using the standard Lebesgue measure, the set of Lebesgue integrable functions from K_1 to K_2 is $L^1(K_1, K_2)$, and the Lebesgue integral of $f \in L^1(K_1, K_2)$ is denoted by $\int_{K_1} f(t)dt$. Following Aubin and Frankowska (1990), § 8.6, we define

$$\int_{K_1} F(t)dt = \left\{ \int_{K_1} f(t)dt : f \in L^1(K_1, K_2), f(t) \in F(t) \text{ for a.e. } t \in K_1 \right\}.$$

3. The model

3.1. The economy and assumptions

By abuse of notation, we denote by $L = \{1, \ldots, L\}$ the finite sets of consumption goods, and let I and J be the finite subset of consumers and firms, respectively. We assume that each type of

[2] In this paper we use the notion of upper hemi-continuity as stated in Hildenbrand (1974): for every open set O such that $F(\bar{x}) \subseteq O$ there is a neighborhood V of $\bar{x}$ such that $F(x) \subseteq O$ for every $x \in V$.

agent $i \in I$ and $j \in J$ corresponds to a continuum of identical individuals indexed by compact subsets $T_i \subset \mathbb{R}$, $i \in I$, and $T_j \subset \mathbb{R}$, $j \in J$, pairwise disjoint. Given that, the subsets of consumers and firms are respectively denoted by

$$\mathcal{I} = \bigcup_{i\in I} T_i \quad \text{and} \quad \mathcal{J} = \bigcup_{j\in J} T_j.$$

The type of producer $t \in \mathcal{J}$ is $j(t) \in J$, and each firm of type $j \in J$ is characterized by a production set $Y_j \subseteq \mathbb{R}^L$. The aggregate production set for firms of type $j \in J$ is the convex hull of $\lambda(T_j)Y_j$, where $\lambda(\cdot)$ denotes the standard Lebesgue measure in $\mathbb{R}$. A production plan for a firm $t \in \mathcal{J}$ is denoted by $y(t) \in Y_{j(t)}$, and the set of admissible production plans is

$$Y = \left\{ y \in L^1(\mathcal{J}, \cup_{j\in J} Y_j) : y(t) \in Y_{j(t)} \text{ a.e. } t \in \mathcal{J} \right\}.$$

The type of consumer $t \in \mathcal{I}$ is $i(t) \in I$, and each consumer of type $i \in I$ is characterized by a consumption set $X_i \subseteq \mathbb{R}^L$, an initial endowment of resources $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i : X_i \rightrightarrows X_i$. A consumption plan of an individual $t \in \mathcal{I}$ is denoted by $x(t) \in X_{i(t)}$, and the set of admissible consumption plans is

$$X = \left\{ x \in L^1(\mathcal{I}, \cup_{i\in I} X_i) : x(t) \in X_{i(t)} \text{ a.e. } t \in \mathcal{I} \right\}.$$

The total initial resources of the economy is $e = \sum_{i\in I} \lambda(T_i) e_i \in \mathbb{R}^L$, and for $(i, j) \in I \times J$, $\theta_{ij} \geq 0$ is the consumer of type i's share in firms of type j. For every $j \in J$, we assume that $\sum_{i\in I} \lambda(T_i)\theta_{ij} = 1$. In addition, we also assume that each consumer $t \in \mathcal{I}$ is initially endowed with an amount of fiat money $m(t) \in \mathbb{R}_+$, where $m \in L^1(\mathcal{I}, \mathbb{R}_+)$. Note that two consumers of the same type may be initially endowed with different amounts of fiat money.

An economy $\mathcal{E}$ is a collection

$$\mathcal{E} = \left(\{X_i, P_i, e_i\}_{i\in I}, \{Y_j\}_{j\in J}, \{\theta_{ij}\}_{(i,j)\in I\times J}, m, \{T_i\}_{i\in I}, \{T_j\}_{j\in J} \right),$$

and the feasible consumption–production plans of $\mathcal{E}$ are the elements of

$$A(\mathcal{E}) = \left\{ (x, y) \in X \times Y : \int_{\mathcal{I}} x(t)dt = \int_{\mathcal{J}} y(t)dt + e \right\}. \tag{2}$$

The following assumptions will be used at different parts in this paper. In our view, the strongest of the conditions below is the finiteness of consumption and production sets. Our existence results only need a weak survival condition (Assumption **S** below). Indeed, as we do not use a strong survival assumption, the interior of the convex hull of the consumption sets could be an empty set.

Assumption F. *For all $i \in I$, and for all $j \in J$, the sets X_i and Y_j are finite.*

Assumption C. *For all $i \in I$, P_i is irreflexive and transitive.*

Assumption S. *For all $i \in I$, $e_i \in \operatorname{conv} X_i - \sum_{j\in J} \theta_{ij} \operatorname{conv}\left(\lambda(T_j)Y_j\right)$.*

Assumption M. *The function $m : \mathcal{I} \to \mathbb{R}_+$ is bounded and for a.e. $t \in \mathcal{I}$, $m(t) > 0$.*

Assumption D. *For all $M \in \mathbb{R}$, $\lambda(\{t \in \mathcal{I} : m(t) = M\}) = 0$.*

3.2. Supply, demand and equilibrium concepts

For $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$, $K \in \mathcal{C}_L$ and $j \in J$,

$$\pi_j(p) = \lambda(T_j) \sup_{z\in Y_j} p \cdot z, \quad S_j(p) = \operatorname{argmax}_{z\in Y_j} p \cdot z,$$
$$\sigma_j(p, K) = \left\{ z \in S_j(p) : p \neq 0_L \Rightarrow \left(Y_j - \{z\}\right) \cap K = \{0_L\} \right\},$$

are respectively, the profit, the Walras and the rationing supply of type $j \in J$ firms.[3] Observe that, by definition, $\sigma_j(p, K) \subseteq S_j(p)$. Moreover, when $p \neq 0_L$, for

$$K(p) = \{0_L\} \cup \{z \in \mathbb{R}^L : p \cdot z > 0\} \in \mathcal{C}_L, \tag{3}$$

we have $\sigma_j(p, K(p)) = S_j(p)$.

The income of consumer $t \in \mathcal{I}$ is denoted by

$$w_t(p, q) = p \cdot e_{i(t)} + qm(t) + \sum_{j\in J} \theta_{i(t)j}\pi_j(p),$$

and the resulting budget set is

$$B_t(p, q) = \left\{ \xi \in X_{i(t)} : p \cdot \xi \leq w_t(p, q) \right\}.$$

Given that,

$$d_t(p, q) = \left\{ \xi \in B_t(p, q) : B_t(p, q) \cap P_{i(t)}(\xi) = \emptyset \right\},$$
$$D_t(p, q) = \limsup_{(p',q')\to(p,q)} d_t(p', q'),$$

and

$$\delta_t(p, q, K) = \left\{ \xi \in D_t(p, q) : P_{i(t)}(\xi) - \{\xi\} \subseteq K \right\},$$

are, respectively, the Walras, weak and rationing demand for consumer $t \in \mathcal{I}$.

Remark 3.1. We observe that, by definition, $d_t(p, q) \subseteq D_t(p, q)$ and $\delta_t(p, q, K) \subseteq D_t(p, q)$. For $p \neq 0_L$ and the cone given by the relation in (3), $d_t(p, q) \subseteq \delta_t(p, q, K(p))$. Moreover, for $\xi \in \delta_t(p, q, K(p))$ such that $p \cdot \xi = w_t(p, q)$, we have $\xi \in d_t(p, q)$.

Definition 3.1. Given $(x, y, p, q) \in A(\mathcal{E}) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K \in \mathcal{C}_L$, we call

(a) (x, y, p, q) a Walras equilibrium with fiat money of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in d_t(p, q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,
(b) (x, y, p, q) a weak equilibrium of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in D_t(p, q)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in S_{j(t)}(p)$,
(c) (x, y, p, q, K) a rationing equilibrium of $\mathcal{E}$, if for a.e. $t \in \mathcal{I}$, $x(t) \in \delta_t(p, q, K)$ and for a.e. $t \in \mathcal{J}$, $y(t) \in \sigma_t(p, K)$.

Remark 3.2. Note that perfect divisibility of fiat money is not enough to guarantee that the Walras demand $d_t(p, q)$ is upper hemi-continuous when goods are indivisible.[4] However, when $d_t(p, q)$ is closed valued and locally bounded (which we will ensure), its outer regularization $D_t(p, q)$ is upper hemi-continuous (cf. Section 2).

The following proposition is proven in Appendix A. It has important implications for the relationships among the equilibrium concepts we have defined.

Proposition 3.1.

(i) *Suppose Assumption **F** holds and $q\, m(t) > 0$, then*

$$D_t(p, q) = \left\{ \xi \in B_t(p, q) : \inf\left\{ p \cdot P_{i(t)}(\xi) \right\} \geq w_t(p, q),\ \xi \notin \operatorname{conv} P_{i(t)}(\xi) \right\}.$$

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[3] Note that in Florig and Rivera (2010), we used the convexity of K, but missed to state this in the definition of rationing supply and demand. Moreover, here we define the rationing supply in a less restrictive way than in Florig and Rivera (2010). However, in the proofs given therein, it is only the property as defined here which is used.

[4] For example, assume $J = \emptyset$, and suppose that the preference relation is defined by the utility function $u(x, y) = 2x + y$, the initial endowment is $e = (0, 1)$, $m = 1$ and the consumption set is $X = \{0, 1\} \times \{0, 1\}$; for $(p^n, q^n) = ((1+1/n, 1), 1/n^2) \to (p, q) = ((1, 1), 0)$ we have that $\lim_{n\to\infty} d(p^n, q^n) = (0, 1)$, and $d(p, q) = (1, 0)$. This implies that the Walras demand correspondence is not upper hemi-continuous at $p = (1, 1),\ q = 0$.

(ii) *If we further assume that Assumption **C** holds, then*

$$\delta_t(p,q,K)=\left\{\xi\in B_t(p,q):\inf\{p\cdot P_{i(t)}(\xi)\}\right.\\ \left.\geq w_t(p,q),\ P_{i(t)}(\xi)-\{\xi\}\subseteq K\right\}.$$

We note now that in the standard case of convex consumption sets, provided the budget set has a non-empty interior and preferences are continuous, the weak and rationing demand would then be the standard Walras demand.[5]

The next example should motivate the introduction of our main equilibrium concept, and illustrates why the weak equilibrium might only be viewed as an auxiliary concept. Moreover, this example shows the important role that fiat money plays in our framework. We consider a finite set of consumers. Without altering the conclusions, we could replace each consumer by a continuum of identical consumers with a constant initial endowment of fiat money per type and the same Lebesgue measure for each type.

Example 3.1. Consider an exchange economy with consumers indexed by $I=\{1,2,3,4\}$, and three goods. For $i\in I$, the consumption sets are given by $X_i=\{0,1,2,3\}^3$ and $m_i\geq 0$ is the endowment of fiat money for this agent. Preferences and endowment of resources are given by:

$$\begin{cases} u_1(x,y,z)=2x+y+z & e_1=(0,1,1),\\ u_2(x,y,z)=x+2y+z & e_2=(1,0,0),\\ u_3(x,y,z)=x+y+2z & e_3=(1,0,0),\\ u_4(x,y,z)=x+y+2z & e_4=(1,0,0).\end{cases}$$

On the one hand, if for some $m\geq 0$ and all $i\in I$, $m_i=m$, then there does not exist a Walras equilibrium.[6] On the other hand, when $0<m_4<m_3$, there exists a unique Walras equilibrium (x,p,q) (in terms of the allocation), with

$$p=(m_1+m_2+m_3+m_4)(1,1,1)\\ +(0,m_2,m_3)\in\mathbb{R}^3,\qquad q=1,$$

and

$$x_1=(2,0,0),\ x_2=(0,1,0),\ x_3=(0,0,1),\ x_4=(1,0,0). \tag{4}$$

We note, however, when $m_2=m_3\geq m_4$ there also exists a weak equilibrium allocation x^*, supported by the same price, with $x_1^*=(2,0,0)$, $x_2^*=(0,0,1)$, $x_3^*=(0,1,0)$ and $x_4^*=(1,0,0)$.[7] A situation like this would be, in a certain sense, "unstable" with respect to the information available to the consumers. For instance, if consumers 2 and 3 knew each other's preferences and allocations, they could continue exchanging leading to allocation $x=(x_1,x_2,x_3,x_4)$.

The weak equilibrium (x^*,p,q) cannot be supported by a rationing equilibrium. Indeed, defining

$$K_i(x_i^*)=\{\xi-x_i^*:\xi\in X_i,\ u_i(\xi)>u_i(x_i^*)\},\ i\in I, \tag{5}$$

we would need the existence of a salient cone K such that $K_i(x_i^*)\subseteq K$, for all $i\in I$. However, as $0_3\neq x_3^*-x_2^*\in K_2(x_2^*)$ and $-(x_3^*-x_2^*)=(x_2^*-x_3^*)\in K_3(x_3^*)$, it follows that such a cone K cannot be salient.

Using the allocation in (4), the definition in (5) and assuming that $m_3=m_4$, by setting

$$K(x)=\operatorname{pos}\left(\cup_{i\in I}K_i(x_i)\right),$$

it follows that $(x,p,q,K(x))$ is a rationing equilibrium.[8] Note that for all $i\in I\setminus\{4\}$, x_i is a maximal element in the budget set, and thus for those consumers the cone $K(x)$ — which was in fact introduced in order to define a concept of demand less restrictive than the Walras demand, but more restrictive than the weak demand-is somehow irrelevant. Consumer $i=4$ could just about afford $x_4'=(0,0,1)$ by selling all his initial endowment in goods and fiat money. The vector $z=x_4'-x_4=(-1,0,1)$ is in $K(x)$, and we could think of z, or $(z,-m_4)$ if we include the amount of money of consumer $i=4$, as a net trade direction (in terms of goods and paper money) for which consumers might be rationed. In fact, in the present example it is the only affordable net trade direction in $K(x)$ which matters here.

Remark 3.3. The salient cone K in the rationing equilibrium definition will be determined endogenously as part of the equilibrium, and summarizes the information that each consumer needs to have in addition to market prices (and their own characteristics) in order to formulate a demand, leading to a stable economic situation, in the sense that no further trading can take place making all participants in a second round of trading strictly better off (see Florig and Rivera (2010)).[9] It seems natural to us to impose that K is a salient cone, since if there is some rationing for a net trade direction $z\in\mathbb{R}^L$, say due to some sort of excessive-demand, then it should be easy to find counterpart for the opposite net trade direction, $-z$.

Florig and Rivera (2010) show that the allocation of a rationing equilibrium (x,p,q,K) of an exchange economy $\mathcal{E}=(\{X_i,P_i,e_i\}_{i\in I},m,\{T_i\}_{i\in I})$ can be supported as a Walras equilibrium with an appropriate distribution of fiat money. Building on this, below we show that x can also be supported by a Walras equilibrium with a price system close to that of the rationing equilibrium, provided the initial distribution in fiat money is perturbed in an appropriate way.

Proposition 3.2. *Let (x,p,q,K), with $q>0$, be a rationing equilibrium of an exchange economy $\mathcal{E}=(\{X_i,P_i,e_i\}_{i\in I},m,\{T_i\}_{i\in I})$. Then, for all $\epsilon>0$ there exists $(p_\epsilon,m_\epsilon)\in\mathbb{R}^L\setminus\{0\}\times L^1(\mathcal{I},\mathbb{R}_{++})$ such that (x,p_ϵ,q) is a Walras equilibrium of economy $\mathcal{E}^\epsilon$ when m is replaced by m_ϵ, with $\|p-p_\epsilon\|\leq\epsilon$ and*

$$\int_{\mathcal{I}}|m(t)-m_\epsilon(t)|dt\ \leq\ \epsilon.$$

5 Furthermore, under general assumptions, Florig and Rivera (2015) show that weak (and rationing) equilibria converge to standard competitive equilibria, provided the consumption and production sets converge to convex polyhedral sets.

6 At equilibrium, allocations must be individually rational. At an equilibrium allocation candidate, each consumer consumes at least one unit of one good, and thus, by feasibility, one consumer consumes two units. If (x,p,q) is a Walras equilibrium, then consumers 2, 3, 4 have all the same budget set. Hence, consumers 3 and 4 must obtain the same level of utility. First, assume that consumer 1 consumes only one unit. Then, individual rationality imposes $x_1=(1,0,0)$. In this case, individual 3 (or 4) must be consuming $(0,0,1)$, and then 4 (or 3) would need to consume $(1,1,0)$ to attain the same level of utility. Then, feasibility would imply that consumer 2 obtains $(1,0,0)$. However, $(1,1,0)$ would also need to be within consumer 2's budget set - a contradiction. Now, we will differentiate two cases where consumer 1 consumes at equilibrium two units of goods. Case (a): if at equilibrium $(0,0,1)$ were in the budget set of consumer 3 or 4, then the other agent would need to consume at least $(1,1,0)$ so that both obtain the same level of utility at equilibrium. Then nothing would be left to consume for consumer 2 - a contradiction. Case (b): at equilibrium, $(0,0,1)$ is not in the budget set of consumer 3 or 4. Hence $p_1+qm<p_3$ and, since at the equilibrium we have $qm\geq 0$, it follows that $p_1<p_3$. In consequence, only consumer 1 can afford good three. However, given the price system, it would contradict utility maximization if he consumed good three. Thus Case (b) cannot lead to a Walras equilibrium.

7 In order to verify that (x^*,p,q) is a weak equilibrium, note first that $x_1^*\in d_1(p,q)$ and therefore $x_1^*\in D_1(p,q)$. For all $\varepsilon>0$, set $p(\varepsilon,2)=p+(0,\varepsilon,0)$ and $p(\varepsilon,3)=p+(0,0,\varepsilon)$. Hence, for all $\varepsilon>0$, $x_2^*\in d_2(p(\varepsilon,2),q)$ and $x_i^*\in d_i(p(\varepsilon,3),q)$ for $i=3,4$. Therefore, for $i=2,3,4$ we also have $x_i^*\in D_i(p,q)$.

8 The cone $K(x)$ is salient, since it is the positive hull of a finite number of points $z\in\mathbb{R}^3$ satisfying $p\cdot z>0$.

9 When studying the existence of a Walras equilibrium without transfers, Hammond (2003) uses an exogenous closed convex set containing zero to restrict the possible trades among consumers. We do not follow this approach because we do not have ex ante information regarding the directions of trade that we would like to favor in order to exclude unstable allocations that a weak equilibrium allocation could produce.

Proof. By Proposition 1 in Florig and Rivera (2010), x is in the rejective core (as defined therein) of the exchange economy $\mathcal{E}$. Then, by proposition 2 in Florig and Rivera (2010), there exists $(p', m') \in \mathbb{R}^L \setminus \{0\} \times L^1(\mathcal{I}, \mathbb{R}_{++})$ such that $(x, p', q' = 1)$ is a Walras equilibrium with money of the exchange economy $\mathcal{E}' = \left(\{X_i, P_i, e_i\}_{i\in I}, m', \{T_i\}_{i\in I}\right)$. For $\mu \in]0, 1[$, let us define $p_\mu = \mu p' + (1-\mu)p$, $q_\mu = q$ and

$$m_\mu = \frac{\mu}{q} m' + (1-\mu)m \in L^1(\mathcal{I}, \mathbb{R}_{++}).$$

For a.e. $t \in \mathcal{I}$ and all $\xi \in P_{i(t)}(x(t))$, it is clear that $p \cdot \xi \geq p \cdot e_{i(t)} + qm(t) \geq p \cdot x(t)$ and $p' \cdot \xi > p' \cdot e_{i(t)} + q\frac{m'(t)}{q} \geq p' \cdot x(t)$, implying that for all $\mu \in]0, 1[$,

$$p_\mu \cdot \xi > p_\mu \cdot e_{i(t)} + q\left(\mu\frac{m'(t)}{q} + (1-\mu)m(t)\right) \geq p_\mu \cdot x(t).$$

Hence, (x, p_μ, q) is a Walras equilibrium of the exchange economy $\mathcal{E}^\mu = \left(\{X_i, P_i, e_i\}_{i\in I}, m_\mu, \{T_i\}_{i\in I}\right)$. Finally, as $\lambda(\mathcal{I})$ is finite, it follows that for all $\epsilon > 0$ there exists $\bar{\mu} \in]0, 1[$ such for all $\mu \in]0, \bar{\mu}]$, $\int_{\mathcal{I}} |m(t) - m_\mu(t)|dt \leq \epsilon$. Moreover, if $\bar{\mu} > 0$ is chosen small enough, we also have $\|p - p_\mu\| \leq \epsilon$ for all $\mu \in]0, \bar{\mu}]$. □

4. Existence of equilibrium

The existence of a weak equilibrium will be an important building block to establish our main result, Theorem 4.1 below.

Proposition 4.1. *If Assumptions **F**, **C**, **S** and **M** hold, then there exists a weak equilibrium for economy $\mathcal{E}$, with a strictly positive price for fiat money.*

The weak equilibrium also leads to the existence of a Walras equilibrium provided the initial allocation of fiat money is dispersed.

Corollary 4.1. *If Assumptions **F**, **C**, **S**, **M** and **D** hold, then there exists a Walras equilibrium for economy $\mathcal{E}$, with a strictly positive price for fiat money.*

Proof. Let (x, y, p, q), with $q > 0$, be a weak equilibrium of $\mathcal{E}$, which exists by Proposition 4.1. By Proposition 3.1 for a.e. $t \in \mathcal{I}$, $\inf\{p \cdot P_{i(t)}(x(t))\} \geq w_t(p, q)$. If (x, y, p, q) is not a Walras equilibrium of this economy, there would exist $i \in I$ and $T_i' \subseteq T_i$, with $\lambda(T_i') > 0$, such that for all $t \in T_i'$, $\inf\{p \cdot P_{i(t)}(x(t))\} = w_t(p, q)$. By the finiteness of X_i, we can choose T_i' such that $x(t)$ is constant on T_i'. Therefore there exists $\xi \in X_i$, such that for all $t \in T_i'$, $\xi \in P_i(x(t))$ and $p \cdot \xi = w_t(p, q)$. It follows that

$$m(t) = \frac{1}{q}\left(p \cdot \xi - p \cdot e_{i(t)} - \sum_{j\in J} \theta_{i(t)j}\pi_j(p)\right)$$

is constant on T_i', contradicting Assumption **D**. □

Remark 4.1. A dividend equilibrium, or in the terms of Mas-Colell (1992) a Walras equilibrium with slack, is a Walras equilibrium with money where the initial distribution of fiat money is not given a priori as part of the fundamentals of the economy, but rather as part of the equilibrium specification (Kajii, 1996). Indeed, at a dividend equilibrium, for each agent a parameter of additional income is determined endogenously. This is equivalent to an endogenous determination of $q \geq 0$ and $m \in L^1(\mathcal{I}, \mathbb{R}_{++})$. Since it is always possible to construct an initial endowment of fiat money which satisfies assumption **D**, it follows that Corollary 4.1 implies the existence of a dividend equilibrium.

The next Theorem uses the two results just established, starting with a weak equilibrium in order to construct the existence of a rationing equilibrium through an iterative argument.

Theorem 4.1. *If Assumptions **F**, **C**, **S** and **M** hold, then there exists a rationing equilibrium for economy $\mathcal{E}$, with a strictly positive price for fiat money.*

Appendix A

A.1. Proof of Proposition 3.1

Proof. For part (i), let $t \in \mathcal{I}$, $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$ and

$$a_t(p, q) = \left\{\xi \in B_t(p, q) : \inf\left\{p \cdot P_{i(t)}(\xi)\right\} \geq w_t(p, q),\ \xi \notin \operatorname{conv} P_{i(t)}(\xi)\right\}.$$

By definition (cf. Section 2)

$$D_t(p, q) = \left\{\xi \in \mathbb{R}^L : \exists \mathrm{N} \in \mathbb{N}_\infty^*, \exists (p_n, q_n) \to_{\mathrm{N}} (p, q), \exists \xi_n \to_{\mathrm{N}} \xi \text{ with } \xi_n \in d_t(p_n, q_n)\right\}. \quad (6)$$

Let $\xi \in D_t(p, q)$, and let ξ_n, p_n, q_n and N from the identity in (6). For all $n \in \mathrm{N}$, and all $x' \in P_{i(t)}(\xi_n)$, one has $p_n \cdot x' > w_t(p_n, q_n) \geq p_n \cdot \xi_n$, and therefore $\xi_n \notin \operatorname{conv} P_{i(t)}(\xi_n)$. Assumption **F** implies trivially that $P_{i(t)}$ has an open graph in $X_{i(t)}$ and therefore $\xi \notin \operatorname{conv} P_{i(t)}(\xi)$. Moreover, as w_t is well defined and continuous in (p, q), $\inf\left\{p \cdot P_{i(t)}(\xi)\right\} \geq w_t(p, q)$, and therefore $D_t(p, q) \subseteq a_t(p, q)$. For the opposite inclusion, let $\xi \in a_t(p, q)$. If $p \cdot \xi < w_t(p, q)$, then for $\varepsilon > 0$ small enough, $\xi \in d_t(p, q - \varepsilon)$, and therefore $\xi \in D_t(p, q)$. Thus, assume $p \cdot \xi = w_t(p, q)$. Since $\xi \notin \operatorname{conv} P_{i(t)}(\xi)$, from the Hahn–Banach separation theorem (see, for example, Rockafellar and Wets (1998)), and the finiteness of $P_{i(t)}(\xi)$, there exists $\bar{p} \in \mathbb{R}^L$ such that $\inf\left\{\bar{p} \cdot P_{i(t)}(\xi)\right\} > \bar{p} \cdot \xi$. For $\varepsilon > 0$ small enough, define $p_\varepsilon = p + \varepsilon\,\bar{p}$ and

$$q_\varepsilon = \frac{p_\varepsilon \cdot (\xi - e_{i(t)}) - \sum_{j\in J} \theta_{i(t)j}\pi_j(p_\varepsilon)}{m(t)}.$$

Note that $\lim_{\varepsilon\to 0}(p_\varepsilon, q_\varepsilon) = (p, q)$. Moreover, for small $\varepsilon > 0$, $\inf\left\{p_\varepsilon \cdot P_{i(t)}(\xi)\right\} > p_\varepsilon \cdot \xi = w_t(p_\varepsilon, q_\varepsilon)$ and therefore $\xi \in d_t(p_\varepsilon, q_\varepsilon)$, this leading to $\xi \in D_t(p, q)$.

For part (ii), let $K \in \mathcal{C}_L$ and define

$$b_t(p, q, K) = \left\{\xi \in B_t(p, q) : \inf\{p \cdot P_{i(t)}(\xi)\} \geq w_t(p, q),\ P_{i(t)}(\xi) - \{\xi\} \subseteq K\right\}.$$

By the definition of weak demand and part (i) of the proposition, it is clear that $\delta_t(p, q, K) \subseteq b_t(p, q, K)$. For the converse inclusion, let $\xi \in b_t(p, q, K)$ and suppose $P_{i(t)}(\xi) = \{z_1, \ldots, z_m\}$. By the part (i) above it is enough to show that $\xi \notin \operatorname{conv} P_i(\xi)$. We proceed by contraposition to show that result. Let $\lambda_j \geq 0$, $j = 1, \ldots, m$, $\lambda_1 + \cdots + \lambda_m = 1$ such that $\sum_{i=1}^m \lambda_i z_i = \xi$. By Assumption **C**, $m > 1$. Assuming $\lambda_1 > 0$, it follows that $-\lambda_1(z_1 - \xi) = \sum_{i=2}^m \lambda_i(z_i - \xi) \in K$, and then $-(z_1 - \xi) \in K$. Since $z_1 - \xi \in K$, by the properties of K we obtain $z_1 - \xi = 0$, a contradiction with Assumption **C**. Hence, $\xi \notin \operatorname{conv} P_i(\xi)$. □

A.2. Existence of rationing equilibria

The proof of Proposition 4.1, the existence of a weak equilibrium, is given in Appendix A.2.2. For this result we need some technical lemmata provided in Appendix A.2.1. Once these results are established, the proof of Theorem 4.1, the existence of rationing equilibrium, is given in Appendix A.2.3.

A.2.1. Technical results

The next lemma is a straightforward extension of the well-known Debreu–Gale–Nikaido lemma.

Lemma A.1. *Let $\varepsilon \in]0, 1]$ and $\varphi : \mathbb{B}(0_L, \varepsilon) \rightrightarrows \mathbb{R}^L$ be an upper hemi-continuous correspondence, with nonempty, convex, compact values. If there exists $b > 0$ such that for every $p \in \mathbb{B}(0_L, \varepsilon)$,*

$$\sup_{z\in\varphi(p)} p \cdot z \le b(1 - \|p\|),$$

then there exists $\bar{p} \in \mathbb{B}(0_L, \varepsilon)$ such that, either (i) $0_L \in \varphi(\bar{p})$ or (ii) $\|\bar{p}\| = \varepsilon$ and $\exists\, \xi \in \varphi(\bar{p})$ such that ξ and $\bar{p}$ are linearly dependent, with $\|\xi\| \le b\,\frac{1-\varepsilon}{\varepsilon}$.

Proof. From the properties of φ, there is a convex compact subset $K \subset \mathbb{R}^L$ such that $\varphi(p) \subset K,\ p \in \mathbb{B}(0_L, \varepsilon)$. Let us now consider the correspondence $F : \mathbb{B}(0_L, \varepsilon) \times K \rightrightarrows \mathbb{B}(0_L, \varepsilon) \times K$ such that

$$F(p, z) = \left\{q \in \mathbb{B}(0_L, \varepsilon) : \forall q' \in \mathbb{B}(0_L, \varepsilon),\ q \cdot z \ge q' \cdot z\right\} \times \varphi(p).$$

From Kakutani's Fixed Point Theorem, F has a fixed point, $(\bar{p}, \xi)$. If $\|\bar{p}\| < \varepsilon$, then we clearly have that $\xi = 0_L$. If $\|\bar{p}\| = \varepsilon$, then from the definition of F, we have that $\bar{p}$ and ξ are linearly dependent (in case $\xi \neq 0_L$, they are collinear). Finally, by the condition over φ and standard inequalities, it follows that $\|\xi\| \le b\,\frac{1-\varepsilon}{\varepsilon}$. □

We will use hierarchic prices and lexicographic ordering for limit arguments similar to Florig (2001). Hereinafter, by convenience regarding the notation, vectors of $\mathbb{R}^m$ are supposed to be columns, and for $r \in \mathbb{N}$, $[\psi_1, \ldots, \psi_r] \in \mathbb{R}^{m\times r}$ is the matrix with columns $\psi_1, \ldots, \psi_r \in \mathbb{R}^m$. For $x \in \mathbb{R}^m$, $[\psi_1, \ldots, \psi_r]^{\mathrm{t}}x = (\psi_1 \cdot x, \ldots, \psi_r \cdot x)^{\mathrm{t}} \in \mathbb{R}^r$, and for $K \subset \mathbb{R}^m$, we set

$$[\psi_1, \ldots, \psi_r]^{\mathrm{t}}K = \left\{[\psi_1, \ldots, \psi_r]^{\mathrm{t}}\, x : x \in K\right\}.$$

Furthermore, for $m \in \mathbb{N}$, the lexicographic order on $\mathbb{R}^m$ is denoted by $\le_{lex}$.[10] The maximum and the argmax with respect to this order are denoted by $\max_{lex}$ and $\operatorname{argmax}_{lex}$, respectively.

Definition A.1. For a positive integer $k \le m$, a set of orthonormal vectors $\{\psi_1, \ldots, \psi_k\} \subset \mathbb{R}^m$ coupled with sequences $\varepsilon_r : \mathbb{N} \to \mathbb{R}_{++}$, $r \in \{1, \ldots, k\}$, is called a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$, if there exists $\mathrm{N} \in \mathbb{N}^*_\infty$ such that following hold:

(a) for all $r \in \{1, \ldots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$,
(b) for all $n \in \mathrm{N}$, $\psi(n) = \sum_{r=1}^{k} \varepsilon_r(n)\psi_r$.

In the following, a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$ as before is denoted by $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathrm{N}\}$.

Lemma A.2. *Every sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ admits a lexicographic decomposition.*

Proof. By setting $\widehat{\psi}_1(n) = \psi(n)$, $n \in \mathbb{N}$, there are $\psi_1 \in \mathbb{R}^m$, with $\|\psi_1\| = 1$, and $\mathrm{N}_1 \in \mathbb{N}^*_\infty$, such that

$$\frac{\widehat{\psi}_1(n)}{\|\widehat{\psi}_1(n)\|} \to_{\mathrm{N}_1} \psi_1 \quad \text{and} \quad \widehat{\psi}_1(n) \cdot \psi_1 > 0,\ \forall n \in \mathrm{N}_1.$$

Recursively, for $r \in \{1, \ldots, m-1\}$, given $\psi_r \in \mathbb{R}^m$, $\|\psi_r\| = 1$, and $\mathrm{N}_r \in \mathbb{N}^*_\infty$, we define $\mathcal{H}^r = \psi_r^\perp$ and we set

$$\widehat{\psi}_{r+1}(n) = \operatorname{proj}_{\mathcal{H}^r}\left(\widehat{\psi}_r(n)\right),\ n \in \mathrm{N}_r.$$

If there exists $\mathrm{N}' \subseteq \mathrm{N}_r$, $\mathrm{N}' \in \mathbb{N}^*_\infty$, such that $\widehat{\psi}_{r+1}(n) = 0_m$ for all $n \in \mathrm{N}'$, then we set $\mathrm{N} = \mathrm{N}'$, otherwise choose $\mathrm{N}_{r+1} \subseteq \mathrm{N}_r$, $\mathrm{N}_{r+1} \in \mathbb{N}^*_\infty$, such that

$$\frac{\widehat{\psi}_{r+1}(n)}{\|\widehat{\psi}_{r+1}(n)\|} \to_{\mathrm{N}_{r+1}} \psi_{r+1} \in \mathbb{R}^m \quad \text{and}$$
$$\widehat{\psi}_{r+1}(n) \cdot \psi_{r+1} > 0,\ \forall n \in \mathrm{N}_{r+1},$$

and define $\mathcal{H}^{r+1} = \psi_{r+1}^\perp$. Therefore, by construction, for each $r \in \{1, \ldots, m-1\}$, the subset $\left\{\widehat{\psi}_{r+1}(n) : n \in \mathrm{N}_{r+1}\right\}$ is contained in a subspace of dimension $m-r$. Thus, there are $k \in \{1, \ldots, m\}$ and $\mathrm{N}'' \in \mathbb{N}^*_\infty$, $\mathrm{N}'' \subseteq \mathrm{N}_k$, such that $\widehat{\psi}_{k+1}(n) = 0_m$ and $\widehat{\psi}_k(n) \neq 0_m$ on N''. We set $\mathrm{N} = \mathrm{N}''$, and then, by construction, we have that the set of vectors $\{\psi_1, \ldots, \psi_k\}$ is orthonormal. Hence, for all $n \in \mathrm{N}$ and all $r \in \{1, \ldots, k\}$, ψ_r is orthogonal to $\widehat{\psi}_{r+1}(n)$, and the following holds for each $r \in \{1, \ldots, k-1\}$:

$$\widehat{\psi}_r(n) = \widehat{\psi}_{r+1}(n) + (\widehat{\psi}_r(n) \cdot \psi_r)\psi_r \quad \text{and} \quad \frac{\|\widehat{\psi}_{r+1}(n)\|}{\|\widehat{\psi}_r(n)\|} \to_{\mathrm{N}} 0.$$

Therefore

$$\widehat{\psi}_r(n) \cdot \psi_r = \left(\|\widehat{\psi}_r(n)\|^2 - \|\widehat{\psi}_{r+1}(n)\|^2\right)^{\frac{1}{2}},$$

from which we have $\psi(n) = \sum_{r=1}^{k} \varepsilon_r(n)\,\psi_r$, with $\varepsilon_r(n) = \widehat{\psi}_r(n) \cdot \psi_r > 0$. Developing $\varepsilon_r(n)$, we have

$$\frac{\varepsilon_{r+1}(n)}{\varepsilon_r(n)} = \left(\frac{\left\|\widehat{\psi}_{r+1}(n)\right\|^2 - \left\|\widehat{\psi}_{r+2}(n)\right\|^2}{\left\|\widehat{\psi}_r(n)\right\|^2 - \left\|\widehat{\psi}_{r+1}(n)\right\|^2}\right)^{1/2} = \frac{\left\|\widehat{\psi}_{r+1}(n)\right\|}{\left\|\widehat{\psi}_r(n)\right\|}\left(\frac{1 - \frac{\|\widehat{\psi}_{r+2}(n)\|^2}{\|\widehat{\psi}_{r+1}(n)\|^2}}{1 - \frac{\|\widehat{\psi}_{r+1}(n)\|^2}{\|\widehat{\psi}_r(n)\|^2}}\right)^{1/2},$$

which implies that $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$. □

Lemma A.3. *Let $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathrm{N}\}$ be a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ and let $z \in \mathbb{R}^m$. There exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:*

$$[\psi_1, \ldots, \psi_k]^{\mathrm{t}} z \le_{lex} 0_k \iff \psi(n) \cdot z \le 0.$$

Proof. For $r \in \{0, 1, \ldots, k\}$ we set $\Psi(r) = [\psi_1, \ldots, \psi_r]^{\mathrm{t}}$ if $r > 0$, and $\Psi(r) = 0_m^{\mathrm{t}}$ when $r = 0$. Let $z \in \mathbb{R}^m$. If $\Psi(k)z = 0_k$, then $\psi(n) \cdot z = 0$, $n \in \mathrm{N}$. Hence, for the sequel assume $\Psi(k)z \neq 0_k$. Then, there exists $s \in \{1, \ldots, k\}$ such that

$$\Psi(s-1)z = 0_{\max\{1,s-1\}} \quad \text{and} \quad \psi_s \cdot z = \delta \neq 0.$$

Since

$$\psi(n) \cdot z = \sum_{r=1}^{k} \varepsilon_r(n)\,\psi_r \cdot z,$$

we have that

$$\frac{1}{\varepsilon_s(n)}\psi(n) \cdot z = a_n + b_n,$$

with

$$a_n = \frac{1}{\varepsilon_s(n)} \sum_{r=1}^{s} \varepsilon_r(n)\,\psi_r \cdot z \quad \text{and}$$

$$b_n = \frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \sum_{r=s+1}^{k} \frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)}\,\psi_r \cdot z,$$

[10] We recall, for $(s, t) \in \mathbb{R}^m \times \mathbb{R}^m$, $s \le_{lex} t$, if $s_r > t_r$ for some $r \in \{1, \ldots, m\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho < t_\rho$. We write $s <_{lex} t$ if $s \le_{lex} t$, but not $t \le_{lex} s$. Of course, $t \ge_{lex} s$ means that $s \le_{lex} t$ (similarly for $>_{lex}$).

and $b_n = 0$ if $s = k$. By the properties above, for all $n \in \mathrm{N}$, $a_n = \delta$ and b_n converges to 0, which imply that for all large $n \in \mathrm{N}$,

$$\frac{1}{\varepsilon_s(n)} \psi(n) \cdot z \in \left[\delta - \frac{|\delta|}{2}, \delta + \frac{|\delta|}{2}\right].$$

Therefore, on the one hand, if $\Psi(k)z <_{lex} 0_k$, then $\delta < 0$ implying that for all large $n \in \mathrm{N}$, $\psi(n) \cdot z < 0$, and, on the other hand, if $\Psi(k)z >_{lex} 0_k$, then $\delta > 0$ and for all large $n \in \mathrm{N}$, $\psi(n) \cdot z > 0$, establishing the converse statement of the lemma. □

Lemma A.4. *Let Z be a finite subset of $\mathbb{R}^m$, and let $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathrm{N}\}$ be a lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$. Then, there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathrm{N}$:*

$$\mathrm{argmax}_{lex} [\psi_1, \ldots, \psi_k]^{\mathrm{t}} Z = \mathrm{argmax}\ \psi(n) \cdot Z.$$

Proof. For $n \in \mathrm{N}$, denote $F(n) = \mathrm{argmax}\ \psi(n) \cdot Z$, and $F = \mathrm{argmax}_{lex} [\psi_1, \ldots, \psi_k]^{\mathrm{t}} Z$. Assume now that there exists $\mathrm{N}_0 \in \mathbb{N}_\infty^*$, $\mathrm{N}_0 \subseteq \mathrm{N}$, such that for all $n \in \mathrm{N}_0$, $F(n) \neq F$. Since Z is a finite set, we can choose $\overline{\mathrm{N}} \in \mathbb{N}_\infty^*$, $\overline{\mathrm{N}} \subseteq \mathrm{N}_0$, such that $F(n)$ is constant on $\overline{\mathrm{N}}$.[11] Given that, let $\bar{\xi} \in F$ and $\xi' \in F(n)$ for $n \in \overline{\mathrm{N}}$. If $\xi' \notin F$ then there exists $\rho \in \{1, \ldots, k\}$, such that $\psi_\rho \cdot \xi' < \psi_\rho \cdot \bar{\xi}$, and for $r \in \{1, \ldots, \rho - 1\}$, $\psi_r \cdot \xi' = \psi_r \cdot \bar{\xi}$. As for all $r \in \{1, \ldots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathrm{N}} 0$, this contradicts the fact that $\psi(n) \cdot \xi' \geq \psi(n) \cdot \bar{\xi}$, $n \in \overline{\mathrm{N}}$. Therefore $\xi' \in F$, and then

$$[\psi_1, \ldots, \psi_k]^{\mathrm{t}} \xi' = [\psi_1, \ldots, \psi_k]^{\mathrm{t}} \bar{\xi}.$$

Furthermore, this last identity implies that for all $n \in \overline{\mathrm{N}}$, $\psi(n) \cdot \bar{\xi} = \psi(n) \cdot \xi'$, and then $\bar{\xi} \in F(n)$ for all $n \in \overline{\mathrm{N}}$, a contradiction. Therefore, there exists $\bar{n}$ such that for all $n \in \mathrm{N}$, $n > \bar{n}$, $F(n) = F$. □

Lemma A.5. *Let $\{(p_n, q_n) \in \mathbb{R}^m \times \mathbb{R}\}_{n \in \mathbb{N}}$ be a sequence for which $\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1}^k, \mathrm{N}\}$ is a lexicographic decomposition. Assume that $(\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}} \neq 0_k$, and let ρ the smallest $r \in \{1, \ldots, k\}$ such that $\mathrm{q}_r \neq 0$. For $n \in \mathrm{N}$, define*

$$(\bar{p}_n, \bar{q}_n) = \sum_{r=1}^{\rho} \varepsilon_r(n)(\mathrm{p}_r, \mathrm{q}_r) \in \mathbb{R}^m \times \mathbb{R}.$$

Let $z \in \mathbb{R}^m$ and for $T \subseteq \mathbb{R}$, let $\mu : T \to \mathbb{R}$, such that for all $t \in T$,

$$[\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}} z \leq_{lex} \mu(t)(\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}}$$

and $\mathrm{q}_\rho\, \mu(t)$ is bounded from below on T. Then there exists $\bar{n} \in \mathbb{N}$, such that for all $t \in T$ and all $n \in \mathrm{N}$, $n > \bar{n}$,

$$\bar{p}_n \cdot z \leq \bar{q}_n\, \mu(t).$$

Proof. For $r \in \{1, \ldots, k\}$, we set $\mathcal{P}(r) = [\mathrm{p}_1, \ldots, \mathrm{p}_r]^{\mathrm{t}}$ and $\mathcal{Q}(r) = (\mathrm{q}_1, \ldots, \mathrm{q}_r)^{\mathrm{t}}$. Using this, we notice $\mathcal{P}(k)z \leq_{lex} \mu(t)\, \mathcal{Q}(k)$ implies that $\mathcal{P}(\rho)z \leq_{lex} \mu(t)\, \mathcal{Q}(\rho)$. We will split the remainder into two cases.

Case A. $\mathcal{P}(\rho - 1)z = \mu(t)\, \mathcal{Q}(\rho - 1)$.

In this case $\mathcal{P}(\rho)z \leq_{lex} \mu(t)\, \mathcal{Q}(\rho)$ implies that for all $r \in \{1, \ldots, \rho\}$, $\mathrm{p}_r \cdot z \leq \mathrm{q}_r\, \mu(t)$, from which we have that for all $n \in \mathrm{N}$ and all $t \in T$, $\bar{p}_n \cdot z \leq \bar{q}_n\, \mu(t)$.

Case B. There exists $s \in \{1, \ldots, \rho - 1\}$ such that $\mathrm{p}_s \cdot z < \mathrm{q}_s\, \mu(t) = 0$ and $\mathcal{P}(s-1)z = \mu(t)\, \mathcal{Q}(s-1)$.

Let $\delta = -\mathrm{p}_s \cdot z > 0$ and let

$$\underline{\mu} = -\inf_{t \in T} \mathrm{q}_\rho\, \mu(t).$$

For $n \in \mathrm{N}$ we set

$$a_n = \frac{1}{\varepsilon_s(n)} \sum_{r=1}^{s} \varepsilon_r(n) \left(\mathrm{p}_r \cdot z - \mathrm{q}_r\, \mu(t)\right) = -\delta,$$

and

$$b_n = \frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \sum_{r=s+1}^{\rho-1} \frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)} \left(\mathrm{p}_r \cdot z - \mathrm{q}_r \mu(t)\right) + \frac{\varepsilon_\rho(n)}{\varepsilon_s(n)} \left(\mathrm{p}_\rho \cdot z + \underline{\mu}\right).$$

By the fact that $\frac{\varepsilon_{s+1}(n)}{\varepsilon_s(n)} \to_{\mathrm{N}} 0$, and for $r = s+1, \ldots, \rho$, $\frac{\varepsilon_r(n)}{\varepsilon_{s+1}(n)}$ converges to zero, or is identically equal to 1, then we have $b_n \to_{\mathrm{N}} 0$. Hence, there exists $\bar{n}$ such that for all $n \in \mathrm{N}$, $n > \bar{n}$, $a_n + b_n < -\delta/2 < 0$, and then, for all $t \in T$, $\bar{p}_n \cdot z - \bar{q}_n\, \mu(t) \leq \varepsilon_s(n)(a_n + b_n) < 0$ for n as stated. □

A.2.2. *Proof of Proposition 4.1: existence of weak equilibrium*

Proof. The proof will be split into different steps.

Step 1. *Perturbed equilibrium.*

For $n \in \mathbb{N}$, $n > 1$, we set $\varepsilon_n = 1 - 1/n$ and we define the excess of demand correspondence

$$\varphi : \mathbb{B}(0_L, \varepsilon_n) \rightrightarrows \sum_{i \in I} \mathrm{conv}\left(\lambda(T_i) X_i\right) - \sum_{j \in J} \mathrm{conv}\left(\lambda(T_j) Y_j\right) - \{e\}$$

such that

$$\varphi(p) = \int_{\mathcal{I}} D_t(p, 1 - \|p\|) dt - \sum_{j \in J} \mathrm{conv}\left(\lambda(T_j) S_j(p)\right) - \{e\},$$

which is nonempty, convex and compact valued, and upper hemi-continuous (the convexity of the integral of the weak demand is due to Theorem 8.6.3 in Aubin and Frankowska (1990)). Furthermore, for each $n \in \mathbb{N}$, $n > 1$, and $p \in \mathbb{B}(0_L, \varepsilon_n)$, we have

$$p \cdot \varphi(p) \leq (1 - \|p\|) \int_{\mathcal{I}} m(t) dt,$$

and therefore we can use Lemma A.1 to conclude that for each $n \in \mathbb{N}$, $n > 1$, there exists

$$(x_n, y_n, p_n, q_n) \in \prod_{i \in I} L^1(T_i, X_i) \times \prod_{j \in J} L^1(T_j, Y_j) \times \mathbb{B}(0, \varepsilon_n) \times \mathbb{R}_{++} \quad (7)$$

such that for a.e. $t \in \mathcal{I}$, $x_n(t) \in D_t(p_n, q_n)$, and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}(p_n)$, with $q_n = 1 - \|p_n\|$,

$$\int_{\mathcal{I}} x_n(t) dt - \int_{\mathcal{J}} y_n(t) dt - e \in \varphi(p_n)$$

and (see (ii) in Lemma A.1)

$$\left\| \int_{\mathcal{I}} x_n(t) dt - \int_{\mathcal{J}} y_n(t) dt - e \right\| \leq \frac{1}{n-1} \int_{\mathcal{I}} m(t) dt.$$

By Lemma A.1, if for some n, $\|p_n\| < 1 - 1/n$, then $\int_{\mathcal{I}} x_n(t) dt - \int_{\mathcal{J}} y_n(t) dt - e = 0_L$, and (x_n, y_n, p_n, q_n) would therefore be a weak equilibrium with $q_n > 0$. For the sequel we will assume $q_n = 1/n$.

Step 2 . *Lexicographic price decomposition.*

Since $q_n > 0$, by Lemma A.2 there exists $\{\{(\mathrm{p}_r, \mathrm{q}_r), \varepsilon_r\}_{r=1}^k, \mathrm{N}\}$, a lexicographic decomposition of $\{(p_n, q_n) \in \mathbb{R}^L \times \mathbb{R}\}_{n \in \mathbb{N}, n > 1}$. Given that, we set

$$(\bar{p}_n, \bar{q}_n) = \sum_{r=1}^{\rho} \varepsilon_r(n)(\mathrm{p}_r, \mathrm{q}_r),$$

with ρ being the smallest $r \in \{1, \ldots, k\}$ such that $\mathrm{q}_r \neq 0$. As $q_n > 0$, such $\rho \leq k$ exists and $\mathrm{q}_\rho > 0$.

In the sequel, without loss of generality we assume $\mathrm{N} = \mathbb{N}$, and for $r \in \{1, \ldots, k\}$ we denote

$$\mathcal{P}(r) = [\mathrm{p}_1, \ldots, \mathrm{p}_r]^{\mathrm{t}} \text{ and } \mathcal{Q}(r) = (\mathrm{q}_1, \ldots, \mathrm{q}_r)^{\mathrm{t}}.$$

[11] If $Z = \{\zeta_1, \ldots, \zeta_f\}$, then for every $i \in \{1, \ldots, f\}$, either there exists $n_i \in \mathrm{N}_0$ such that for all $n \in \mathrm{N}_0$, $n > n_i$, $\zeta_i \notin F(n)$, in which case we define $\mathrm{N}_i = \mathrm{N}_{i-1} \cap \{n \in \mathbb{N} : n > n_i\}$, or for some $\mathrm{N}_i \in \mathbb{N}_\infty^*$, $\mathrm{N}_i \subseteq \mathrm{N}_{i-1}$, $\zeta_i \in F(n)$ for each $n \in \mathrm{N}_i$. Then, $F(n)$ is constant for $n \in \overline{\mathrm{N}} = \mathrm{N}_f$.

Step 3. *Supply: There exists $n_J \in \mathbb{N}$ such that for all $j \in J$ and $n \in \mathbb{N}$, $n > n_J$,*

$$S_j(p_n) \subseteq S_j(\bar{p}_n) \subseteq Y_j.$$

Applying Lemma A.4 twice for each $j \in J$, we have $n_j \in \mathbb{N}$ such that for all $n > n_j$,

$$S_j(p_n) = \text{argmax}_{lex}\, \mathcal{P}(k)\, Y_j \quad \text{and}$$
$$S_j(\bar{p}_n) = \text{argmax}_{lex}\, \mathcal{P}(\rho)\, Y_j.$$

Note that $\text{argmax}_{lex}\mathcal{P}(k)\,Y_j \subseteq \text{argmax}_{lex}\mathcal{P}(\rho)\,Y_j$, and that for n large enough, $S_j(\bar{p}_n)$ does not depend on n. Set $n_J = \max_{j\in J} n_j$.

Step 4. *Income: For all $i \in I$, there exists $z_i \in \mathbb{R}^L$ such that for all $t \in T_i$ and $n > n_J$ with $n \in \mathbb{N}$,*

$$w_t(p_n, q_n) = p_n \cdot z_i + q_n m(t) \quad \text{and}$$
$$w_t(\bar{p}_n, \bar{q}_n) = \bar{p}_n \cdot z_i + \bar{q}_n\, m(t).$$

For all $j \in J$, let $\zeta_j \in \text{argmax}_{lex}\mathcal{P}(k)Y_j$. Step 3 establishes the result by setting for all $i \in I$

$$z_i = e_i + \sum_{j\in J} \lambda(T_j)\theta_{ij}\, \zeta_j.$$

Step 5. *There exists $\bar{n} > n_J$ such that for all $t \in \mathcal{I}$, $\xi \in X_{i(t)}$ and $n > \bar{n}$ with $n \in \mathbb{N}$,*

$$\mathcal{P}(k)(\xi - z_{i(t)}) \leq_{lex} m(t)\, \mathcal{Q}(k) \Rightarrow \bar{p}_n \cdot (\xi - z_{i(t)}) \leq \bar{q}_n m(t),$$
$$\mathcal{P}(k)(\xi - z_{i(t)}) \geq_{lex} m(t)\, \mathcal{Q}(k) \Rightarrow \bar{p}_n \cdot (\xi - z_{i(t)}) \geq \bar{q}_n m(t).$$

For $i \in I$ and $\xi \in X_i$, consider the subsets:

$$T_i^1(\xi) = \left\{t \in T_i : \mathcal{P}(k)(\xi - z_{i(t)}) \leq_{lex} m(t)\, \mathcal{Q}(k)\right\},$$
$$T_i^2(\xi) = \left\{t \in T_i : \mathcal{P}(k)(\xi - z_{i(t)}) \geq_{lex} m(t)\, \mathcal{Q}(k)\right\}.$$

Since $m(\cdot)$ is bounded on every subset of $\mathcal{I}$, we can apply Lemma A.5 to the vector $(\xi - z_{i(t)})$ for the set $T_i^1(\xi)$ leading to the existence of $n_i^1(\xi)$ such that for all $n > n_i^1(\xi)$,

$$\bar{p}_n \cdot (\xi - z_{i(t)}) \leq \bar{q}_n\, m(t)$$

on $T_i^1(\xi)$. Moreover, we can apply as well the same lemma to the vector $-(\xi - z_{i(t)})$ coupled with $\widetilde{m} : T_i^2(\xi) \to \mathbb{R}$ with $\widetilde{m}(t) = -m(t)$, leading to the existence of $n_i^2(\xi)$ such that for all $n > n_i^2(\xi)$,

$$\bar{p}_n \cdot (\xi - z_{i(t)}) \geq \bar{q}_n m(t)$$

on $T_i^2(\xi)$. As I is finite and the consumption sets are finite, by choosing

$$\bar{n} = \max_{\xi\in X_i,\, i\in I} \left\{n_i^1(\xi), n_i^2(\xi), n_J\right\} \in \mathbb{N}$$

we can establish the desired result.

Step 6. *Budget: For all $n > \bar{n}$, for all $t \in \mathcal{I}$, $\limsup_{\nu\to\infty} B_t(p_\nu, q_\nu) \subseteq B_t(\bar{p}_n, \bar{q}_n)$.*

Let $t \in \mathcal{I}$ and $\xi \in \limsup_{\nu\to\infty} B_t(p_\nu, q_\nu)$. Then there are $\{\xi_\nu\}_{\nu\in\mathbb{N}}$ with $\xi_\nu \in B_t(p_\nu, q_\nu) \subseteq X_{i(t)}$ for all $\nu \in \mathbb{N}$, and $\mathrm{N}_\xi(t) \in \mathbb{N}_\infty^*$, such that $\xi_\nu \to_{\mathrm{N}_\xi(t)} \xi$. As $X_{i(t)}$ is finite, we can choose $\mathrm{N}_\xi(t)$ such that for all $\nu \in \mathrm{N}_\xi(t)$, $\xi_\nu = \xi$ and $\nu > \bar{n}$. Hence, for all $\nu \in \mathrm{N}_\xi(t)$, $p_\nu \cdot (\xi - z_{i(t)}) \leq q_\nu\, m(t)$, and then, by Lemma A.3, we have that $\mathcal{P}(k)(\xi - z_{i(t)}) \leq_{lex} m(t)\, \mathcal{Q}(k)$. This implies by Step 5 that for all $n \in \mathbb{N}$, $n > \bar{n}$, $\xi \in B_t(\bar{p}_n, \bar{q}_n)$.

Step 7. *Demand: For all $n > \bar{n}$, for all $t \in \mathcal{I}$ with $m(t) > 0$, $\limsup_{\nu\to\infty} D_t(p_\nu, q_\nu) \subseteq D_t(\bar{p}_n, \bar{q}_n)$.*

Let $t \in \mathcal{I}$ with $m(t) > 0$ and $\xi \in \limsup_{\nu\to\infty} D_t(p_\nu, q_\nu)$. Then, similar to previous step, there are $\{\xi_\nu\}_{n\in\mathbb{N}}$ with $\xi_\nu \in D_t(p_\nu, q_\nu) \subseteq B_t(p_\nu, q_\nu)$ for all $\nu \in \mathbb{N}$, and $\mathrm{N}_\xi(t) \in \mathbb{N}_\infty^*$, such that $\xi_\nu \to_{\mathrm{N}_\xi(t)} \xi$. Since $X_{i(t)}$ is finite, we can choose $\mathrm{N}_\xi(t)$ such that for all $\nu \in \mathrm{N}_\xi(t)$, $\xi_\nu = \xi$ and $\nu > \bar{n}$.

For all $\nu \in \mathrm{N}_\xi(t)$, $q_\nu m(t) > 0$ and $\xi \in D_t(p_\nu, q_\nu)$. Then, Proposition 3.1 implies, on the one hand that $\xi \notin \text{conv}P_{i(t)}(\xi)$ and, on the other hand, that for all $\nu \in \mathrm{N}_\xi(t)$ and all $\bar{\xi} \in P_{i(t)}(\xi)$, $p_\nu \cdot (\bar{\xi} - z_{i(t)}) \geq q_\nu\, m(t)$. Thus, by Lemma A.3, we have that $\mathcal{P}(k)(\bar{\xi} - z_{i(t)}) \geq_{lex} m(t)\, \mathcal{Q}(k)$. This implies by Step 5 that for all $n \in \mathbb{N}$, $n > \bar{n}$, $w_t(\bar{p}_n, \bar{q}_n) \leq \bar{p}_n \cdot \bar{\xi}$. Since $\xi \in \limsup_{\nu\to\infty} B_t(p_\nu, q_\nu)$ we have also that for all $n \in \mathbb{N}$, $n > \bar{n}$, $\bar{p}_n \cdot \xi \leq w_t(\bar{p}_n, \bar{q}_n)$. Therefore, for all $t \in \mathcal{I}$ with $m(t) > 0$, all $n \in \mathbb{N}$, $n > \bar{n}$, and all $\xi \in \limsup_{\nu\to\infty} D_t(p_\nu, q_\nu)$, we have $\bar{p}_n \cdot \xi \leq w_t(\bar{p}_n, \bar{q}_n) \leq \inf\{\bar{p}_n \cdot P_{i(t)}(\xi)\}$ and $\xi \notin \text{conv}P_{i(t)}(\xi)$. Hence, by Proposition 3.1 it follows that $\xi \in D_t(\bar{p}_n, \bar{q}_n)$, for $n > \bar{n}$.

Step 8. *Equilibrium allocation.*

From (7), using Artstein's (1979) version of Fatou's lemma, there exists $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$, $x^*(t) \in \text{acc}\{x_n(t)\}_{n\in\mathbb{N}}$ and $y^*(t') \in \text{acc}\{y_n(t')\}_{n\in\mathbb{N}}$.

Step 9. *Conclusion: For all $n \in \mathbb{N}$ with $n > \bar{n}$, $(x^*, y^*, \bar{p}_n, \bar{q}_n)$ is a weak equilibrium with $\bar{q}_n > 0$.*

Indeed, let $n \in \mathbb{N}$ with $n > \bar{n}$. By Step 3, we have for all $t' \in \mathcal{J}$, $\text{acc}\{y_\nu(t')\}_{\nu\in\mathbb{N}} \subseteq S_{j(t')}(\bar{p}_n)$.

By Step 7, we have for all $t \in \mathcal{I}$ with $m(t) > 0$, $\text{acc}\{x_\nu(t)\}_{\nu\in\mathbb{N}} \subseteq D_t(\bar{p}_n, \bar{q}_n)$. Of course $\bar{q}_n > 0$ and the last property is valid for a.e. $t \in \mathcal{I}$ since, by Assumption **M**, $\lambda(\{t \in \mathcal{I} : m(t) \leq 0\}) = 0$. □

A.2.3. *Proof of Theorem 4.1: existence of a rationing equilibrium*

Proof. Let (x_0, y_0, p_0, q_0) be a weak equilibrium of $\mathcal{E}$ with $q_0 > 0$. If $p_0 = 0_L$, then for a.e. $t \in \mathcal{I}$, $P_{i(t)}(x_0(t)) = \emptyset$ and then, for $K = \{0_L\}$, (x_0, y_0, p_0, q_0, K) is a rationing equilibrium of $\mathcal{E}$. Hereinafter we suppose that $p_0 \neq 0_L$, and let $m^1 : \mathcal{I} \to \mathbb{R}_{++}$ be a mapping strictly increasing and bounded. As the consumption sets and the number of types of consumers are finite, we can define a finite set of types of consumers $A = \{1, \ldots, A\}$ satisfying the following:

(i) $\{T_a\}_{a\in A}$ is a finer partition of $\mathcal{I}$ than $\{T_i\}_{i\in I}$,
(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_0(t) = x_a$.

For $a \in A$ we set

$$X_a^1 = (P_a(x_a) \cup \{x_a\}) \cap (\{x_a\} + p_0^{\perp}).$$

In a similar manner as done for consumers, we can define a finite set of types of producers $B = \{1, \ldots, B\}$ satisfying the following:

(i) $\{T_b\}_{b\in B}$ is a finer partition of $\mathcal{J}$ than $\{T_j\}_{j\in J}$,
(ii) for every $b \in B$, there exists y_b such that for every $t \in T_b$, $y_0(t) = y_b$.

For $b \in B$, let

$$Y_b^1 = (Y_b - \{y_b\}) \cap p_0^{\perp},$$

and then, denoting $e_a^1 = x_a$, we define the following auxiliary economy

$$\mathcal{E}^1 = \Big(\{X_a^1, P_a^1, e_a^1\}_{a\in A}, \{Y_b^1\}_{b\in B},$$
$$\{\theta_{ab}\}_{(a,b)\in A\times B}, m^1, \{T_a\}_{a\in A}, \{T_b\}_{b\in B}\Big),$$

where m^1 defines the initial endowments of fiat money for each consumer, and θ_{ab} satisfying the conditions for the privately ownership of the firms and P_a^1 is the restriction of P_a to X_a^1.

Clearly the economy $\mathcal{E}^1$ satisfies the assumptions of Proposition 4.1, hence there exists a weak equilibrium for this economy, with the price of fiat money strictly positive. Moreover, since m^1 satisfies Assumption **D**, by Corollary 4.1 there exists a Walras equilibrium (with fiat money) for the economy $\mathcal{E}^1$, which is denoted by (x_1, y_1, p_1, q_1), with $q_1 > 0$.

In the following, we denote

$$\mathcal{P} = [p_0, p_1]^t \in \mathbb{R}^{2\times L},$$

and for $t \in \mathcal{I}$ we set $w_t = (w_t^0, w_t^1)^t$, with

$$w_t^0 = p_0 \cdot e_{i(t)} + q_0 m(t) + \sum_{j\in J} \theta_{i(t)j}\, \pi_j(p_0)$$

and

$$w_t^1 = p_1 \cdot e_{a(t)}^1 + q_1 m^1(t) + \sum_{b\in B} \theta_{a(t)b}\, \lambda(T_b)\, \max p_1 \cdot Y_b^1,$$

where $a(t) \in A$ such that $t \in T_{a(t)}$.

Claim A.1. *For a.e. $t \in \mathcal{I}$, $\mathcal{P}x_1(t) \leq_{lex} w_t$.*

By definition of X_a^1, $a \in A$, $p_0 \cdot x_0(t) = p_0 \cdot x_1(t)$, a.e. $t \in \mathcal{I}$. Since for every $r \in \{0, 1\}$, $p_r \cdot x_r(t) \leq w_t^r$, we conclude $\mathcal{P}x_1(t) \leq_{lex} w_t$, for a.e. $t \in \mathcal{I}$.

Claim A.2. *For a.e. $t \in \mathcal{I}$, $\xi(t) \in P_{i(t)}(x_1(t))$ implies $\mathcal{P}x_1(t) <_{lex} \mathcal{P}\xi(t)$.*

Let $\xi(t) \in P_{i(t)}(x_1(t))$. By construction, for a.e. $t \in \mathcal{I}$, $x_1(t) \neq x_0(t)$ implies $x_1(t) \in P_{i(t)}(x_0(t))$ and then, by transitivity of the preferences, $\xi(t) \in P_{i(t)}(x_0(t))$. Therefore, for a.e. $t \in \mathcal{I}$, we have $\xi(t) \in P_{i(t)}(x_0(t))$, which implies $p_0 \cdot x_1(t) = p_0 \cdot x_0(t) \leq p_0 \cdot \xi(t)$. For $t \in \mathcal{I}$, the claim is thus satisfied for $\xi(t) \in P_{i(t)}(x_1(t))$ such that $p_0 \cdot x_1(t) < p_0 \cdot \xi(t)$. For $t \in \mathcal{I}$ and $\xi(t) \in P_{i(t)}(x_1(t))$ such that $p_0 \cdot x_1(t) = p_0 \cdot \xi(t)$ we have $\xi(t) \in X_{a(t)}^1$ and as (x_1, y_1, p_1, q_1) is a Walras equilibrium of $\mathcal{E}^1$, it follows that $p_1 \cdot x_1(t) < p_1 \cdot \xi(t)$ for a.e. $t \in \mathcal{I}$ satisfying $p_0 \cdot x_1(t) = p_0 \cdot \xi(t)$.

To conclude the proof of Theorem 4.1 we will show that $(\bar{x}, \bar{y}, \bar{p}, \bar{q}) = (x_1, y_0 + y_1, p_0, q_0)$, coupled with the salient cone

$$K = \{0_L\} \cup \{\xi \in \mathbb{R}^L \mid 0_2 <_{lex} \mathcal{P}\xi\}$$

is a rationing equilibrium of $\mathcal{E}$. Let

$$\mathcal{I}' = \left\{ t \in \mathcal{I} \mid \mathcal{P}x_1(t) <_{lex} \min_{lex} \mathcal{P}P_{i(t)}(x_1(t)) \right\},$$

and note that by Claim A.2 $\lambda(\mathcal{I} \setminus \mathcal{I}') = 0$. Again by Claim A.2, K contains[12]

$$K' = \{0_L\} \cup \left\{ \bigcup_{t\in\mathcal{I}'} P_{i(t)}(\bar{x}(t)) - \{\bar{x}(t)\} \right\}.$$

Hence, by the construction of K and the characterization of the rationing demand in Proposition 3.1, it follows that for a.e. $t \in \mathcal{I}$, $\bar{x}(t) \in \delta_t(\bar{p}, \bar{q}, K)$. Finally, by the construction of the iteration of weak equilibria, for a.e. $t \in \mathcal{J}$ and all $z \in Y_j$, $\mathcal{P}(z - \bar{y}(t)) \leq_{lex} 0_2$ and therefore

$$\{0_L\} = \left(Y_{j(t)} - \{\bar{y}(t)\}\right) \cap K.$$

From this we can conclude that for a.e. $t \in \mathcal{J}$, $\bar{y}(t) \in \sigma_t(\bar{p}, \bar{q}, K)$. □

[12] As the consumption sets are finite, the set conv K' generates a closed salient cone contained in K. Thus we could impose the cone in the rationing equilibrium to be closed.

References

Artstein, Z., 1979. A note on Fatou's lemma in several dimensions. J. Math. Econom. 6, 277–282.
Aubin, J.P., Frankowska, H., 1990. Set-Valued Analysis. Birkhäuser, Basel.
Aumann, R., Drèze, J., 1986. Values of markets with satiation or fixed prices. Econometrica 54, 1271–1318.
Balasko, Y., 1982. Equilibria and efficiency in the fixprice setting. J. Econom. Theory 28, 113–127.
Bikhchandani, S., Mamer, J.W., 1997. Competitive equilibrium in an exchange economy with indivisibilities. J. Econom. Theory 74, 385–413.
Bobzin, H., 1998. Indivisibilities: Microeconomic Theory with Respect To Indivisible Goods and Factors. Physica Verlag, Heidelberg.
Broome, J., 1972. Approximate equilibrium in economies with indivisible commodities. J. Econom. Theory 5, 224–249.
Dierker, E., 1971. Analysis of exchange economies with indivisible commodities. Econometrica 39, 997–1008.
Drèze, J., Müller, H., 1980. Optimality properties of rationing schemes. J. Econom. Theory 23, 150–159.
Florig, M., 2001. Hierarchic competitive equilibria. J. Math. Econom. 35, 515–546.
Florig, M., Rivera, J., 2010. Core equivalence and welfare properties without divisible goods. J. Math. Econom. 46, 467–474.
Florig, M., Rivera, J., 2015. Walrasian equilibrium as limit of a competitive equilibrium without divisible goods. In: Serie De Documentos De Trabajo, SDT404. Departamento de Economía, Universidad de Chile.
Hammond, P., 2003. Equal rights to trade and mediate. Soc. Choice Welf. 21, 181–193.
Henry, C., 1970. Indivisibilités dans une économie d'échange. Econometrica 38, 542–558.
Hildenbrand, W., 1974. Core and Equilibria of a Large Economy, Princeton University Press. Princeton, New Jersey.
Inoue, T., 2008. Indivisible commodities and the nonemptiness of the weak core. J. Math. Econom. 44, 96–111.
Inoue, T., 2014. Indivisible commodities and an equivalence theorem on the strong core. J. Math. Econom. 54, 22–35.
Kajii, A., 1996. How to discard non-satiation and free-disposal with paper money. J. Math. Econom. 25, 75–84.
Khan, A., Yamazaki, A., 1981. On the cores of economies with indivisible commodities and a continuum of traders. J. Econom. Theory 24, 218–225.
Konishi, H., Quint, T., Wako, J., 2001. On the Shapley-Scarf economy: the case of multiple types of indivisible goods. J. Math. Econom. 35, 1–15.
Konovalov, A., 2005. The core of an economy with satiation. Econom. Theory 25, 711–719.
Makarov, V., 1981. Some results on general assumptions about the existence of economic equilibrium. J. Math. Econom. 8, 87–100.
Mas-Colell, A., 1977. Indivisible commodities and general equilibrium theory. J. Econom. Theory 16, 443–456.
Mas-Colell, A., 1992. Equilibrium theory with possibly satiated preferences. In: Majumdar, M. (Ed.), Equilibrium and Dynamics: Essays in Honour of David Gale. MacMillan, London, pp. 201–213.
Quinzii, M., 1984. Core and competitive equilibria with indivisibilities. Internat. J. Game Theory 13, 41–60.
Rockafellar, R., Wets, R., 1998. Variational Analysis. Springer Verlag, Berlin.
Shapley, L., Scarf, H., 1974. On core and indivisibility. J. Math. Econom. 1, 23–28.
Sönmez, T., 1996. Implementation in generalized matching problems. J. Math. Econom. 26, 429–439.
van der Laan, G., Talman, D., Yang, Z., 2002. Existence and welfare properties of equilibrium in an exchange economy with multiple divisible and indivisible commodities and linear production technologies. J. Econom. Theory 103, 411–428.
Wako, J., 1984. A note on the strong core of a market with indivisible goods. J. Math. Econom. 13, 189–194.