Received: 18 May 2016 | Accepted: 24 February 2017

DOI: 10.1002/mana.201600214

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Periodic solutions of neutral fractional differential equations

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Funding Information
FONDECYT, Grant Number: 1130144; PAIFAC; ENL016/15.

Abstract

We characterize the existence of periodic solutions for some abstract neutral functional fractional differential equations with finite delay when the underlying space is a UMD space.

KEYWORDS

Operator-valued Fourier multipliers, R-boundedness, periodic vector-valued Lebesgue spaces, neutral functional differential equations

MSC (2010)

34G10, 34K13, 35A02, 47D06

1 | INTRODUCTION

It is well known that neutral functional differential equations are used to represent important physical systems. We refer to [20,24,28] for a discussion about this aspect of the theory. Similarly, motivated by the fact that abstract neutral functional differential equations (abbreviated, ANFDE) arise in many areas of applied mathematics, this type of equations has received much attention in recent years ([11,21,38]). In particular, the problem of existence of solutions having a periodicity property has been considered by several authors. On the other hand, because several important physical phenomena are modeled by abstract fractional differential equations this type of equations have been studied extensively last time for many authors. We refer the reader to the works [5,29,33–35] and the references listed therein for recent information on this subject. For some applications of fractional differential equations, see [4,10,17,27].

Let X be a Banach space. Throughout this paper $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ are closed linear operators with $D(A) \subseteq D(B)$.

The aim of this paper is to characterize existence of periodic solutions for the classes of linear abstract neutral functional fractional differential equations described in the form

$$D^\alpha(x(t) - Bx(t-r)) = Ax(t) + L(x_t) + f(t), \quad t \in \mathbb{R},\ \alpha > 0, \tag{1.1}$$

and

$$D^\alpha(x(t) - Bx(t-r)) = A(x(t) - Bx(t-r)) + L(x_t) + f(t), \quad t \in \mathbb{R},\ \alpha > 0. \tag{1.2}$$

In these equations, $r > 0$ is a fixed number, the function x_t given by $x_t(\theta) = x(t+\theta)$ for θ in an appropriate domain, denotes the *history* of the function $x(\cdot)$ at t. The delay operator L is a bounded linear map defined on an appropriate space. Without loss of generality, we will concentrate on the existence of 2π-periodic solutions. For this reason, throughout this work we assume that $0 < r < 2\pi$ and that $f : \mathbb{R} \to X$ is a locally p-integrable and 2π-periodic function for $1 \leq p < \infty$.

Next we denote by $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, and we shall identify the spaces of vector or operator-valued functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, throughout this paper $L^p(\mathbb{T}; X)$, $1 \leq p < \infty$, denotes the space consisting of all 2π-periodic Bochner measurable X-valued functions f defined on $\mathbb{R}$ such that the restriction of f to $[0, 2\pi]$ is p-integrable. We consider $L^p(\mathbb{T}; X)$ endowed with the norm $\|f\|_p = \left(\int_0^{2\pi} \|f(t)\|^p dt\right)^{1/p}$.

We study Equations (1.1) and (1.2) in the framework of the so-called Liouville–Grünwald–Letnikov fractional derivative. This concept was introduced in [19,30] and has subsequently been studied by several authors. In [19,30] the fractional derivative is defined directly as a limit of a fractional difference quotient (see also [36]). In [9], the authors apply this approach based on fractional differences to study fractional differentiation of periodic scalar functions. This idea has been used in [26] to extend the definition of fractional differentiation to vector-valued functions. In the case of periodic functions this concept enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as provides a connection with the classical Weyl fractional derivative (see [36]).

In the next section we recall the basic concepts and properties to establish our results, in the Section 3 we establish our results about the existence of periodic solutions, and finally, in the Section 4 we present an application of our results.

2 | PRELIMINARIES

In this paper we use the following notations. Let Y, Z be Banach spaces. In what follows, we denote by $\mathcal{L}(Y, Z)$ the Banach space of bounded linear operators from Y into Z endowed with the norm of operators. We abbreviate this notation to $\mathcal{L}(Y)$ in the case $Y = Z$. The space $D(A)$ endowed with the graph norm becomes a Banach space denoted by $[D(A)]$. It is follows from the closed graph theorem that the inclusion $[D(A)] \hookrightarrow [D(B)]$ is continuous.

Let $\alpha > 0$. Given $f \in L^p(\mathbb{T}; X)$ for $1 \le p < \infty$ the Riemann difference

$$\Delta_t^\alpha f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj) \tag{2.1}$$

(where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ is the binomial coefficient) exists almost everywhere, and

$$\|\Delta_t^\alpha f\|_{L^p(\mathbb{T};X)} \le \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \|f\|_{L^p(\mathbb{T};X)} \tag{2.2}$$

since $\sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| < \infty$ (see [9]).

The following definition is the direct extension of [9, Definition 2.1] to the vector-valued case. See also [26] for their connection with fractional differential equations.

Definition 2.1. *Let X be a complex Banach space, $\alpha > 0$ and $1 \le p < \infty$. If for $f \in L^p(\mathbb{T}; X)$ there exists $g \in L^p(\mathbb{T}; X)$ such that $\lim_{t\to 0^+} t^{-\alpha}\Delta_t^\alpha f = g$ in the $L^p(\mathbb{T}; X)$ norm, then g is called the α^{th}*–Liouville–Grünwald–Letnikov derivative *of f in the mean of order p.*

In this text we abbreviate this terminology by α^{th}-derivative of f and we use the notation $g = D^\alpha f$. Note that D^α is lineal.

Example 2.2. *The α^{th}-derivative of e^{iax} for any real a is given by $(ia)^\alpha e^{iax}$. In particular, $D^\alpha \sin x = \sin\left(x + \frac{\pi}{2}\alpha\right)$ and $D^\alpha \cos x = \cos\left(x + \frac{\pi}{2}\alpha\right)$.*

We also mention here a few properties, the proof of which follows the same routine as in the scalar case given in [9, Proposition 4.1]. By this reason we will omit them. We denote by Ψ_α to the function given by

$$\Psi_\alpha(\xi) = 2\sum_{k=1}^{\infty} \frac{\cos(k\xi - \alpha\pi/2)}{k^\alpha}.$$

Moreover, for $g \in L^p(\mathbb{T}; X)$,

$$I^\alpha g(t) = \frac{1}{2\pi}\int_0^{2\pi} g(t-\xi)\Psi_\alpha(\xi)\, d\xi.$$

Proposition 2.3. *Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}; X)$. For $\alpha, \beta > 0$ the following properties hold:*

(i) *If $D^\alpha f \in L^p(\mathbb{T}; X)$, then $D^\beta f \in L^p(\mathbb{T}; X)$ for all $0 < \beta < \alpha$.*

(ii) *$D^\alpha D^\beta f = D^{\alpha+\beta} f$ whenever one of the two sides is well defined.*

(iii) *If $g \in L^p(\mathbb{T}; X)$ and $\alpha > 1$, then $I^\alpha g$ is a continuous function.*

Next we recall some basic concepts of harmonic analysis necessary to obtain our results. We begin with the concept of R-boundedness.

Definition 2.4. *A family of operators $\mathcal{T} = \{T_i : i \in I\} \subseteq \mathcal{L}(Y, Z)$ is said to be R-*bounded *if there exist a constant $C > 0$ and $p \in [1, \infty)$ such that for each finite set $J \subseteq I$, $T_i \in \mathcal{T}$, $y_i \in Y$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality*

$$\left\| \sum_{i\in J} \varepsilon_i T_i y_i \right\|_{L^p(\Omega;Z)} \leq C \left\| \sum_{i\in J} \varepsilon_i y_i \right\|_{L^p(\Omega;Y)}$$

*is verified. The smallest of the constant C is called R-*bound *of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.*

Several properties of R-bounded families can be founded in the monograph of Denk–Hieber–Prüss [12, Section 3]. We remark here that large classes of operators are R-bounded (cf. [18] and references therein).

For $f \in L^1(\mathbb{T}; Y)$ we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$, its k-th Fourier coefficient, which is given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)\, dt.$$

In what follows we denote by $H^{\alpha,p}(\mathbb{T}; X)$ the space consisting of vector-valued functions $u \in L^p(\mathbb{T}; X)$ for which there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = (ik)^\alpha \hat{u}(k)$ for all $k \in \mathbb{Z}$, where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2}\operatorname{sgn} k}$.

Let $f \in L^p(\mathbb{T}; X)$. It has been shown by Butzer and Westphal [9, Theorem 4.1] that $D^\alpha f \in L^p(\mathbb{T}; X)$ if and only if there exists $g \in L^p(\mathbb{T}; X)$ such that $(ik)^\alpha \hat{f}(k) = \hat{g}(k)$, and in this case we have that $D^\alpha f = g$. As a consequence, the characterization

$$H^{\alpha,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}; X) : D^\alpha u \in L^p(\mathbb{T}; X)\}$$

is fulfilled. Moreover, if $g = D^\alpha f \in L^p(\mathbb{T}; X)$, then

$$f(t) - \hat{f}(0) = I^\alpha g(t), \quad \text{a.e.}$$

Definition 2.5. *For $1 \leq p < \infty$, a sequence $(M_k)_{k\in\mathbb{Z}}$ in $\mathcal{L}(Y, Z)$ is said to be an L^p-*multiplier *if for each $f \in L^p(\mathbb{T}; Y)$ there exists $u \in L^p(\mathbb{T}; Z)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.*

To complete these concepts we define the UMD spaces. But, since we just will use some results from the literature, it is enough for us to present a simple definition of UMD space. A Banach space Z is said to be *UMD* if the Hilbert transform is bounded on $L^p(\mathbb{R}; Z)$ for some (and then for all) p, with $1 < p < \infty$.

The following theorem, due to Arendt and Bu [7, Theorem 1.3], is the discrete analogue of the operator-valued version of Mikhlin's theorem due to Weis [37].

Theorem 2.6. *Let $1 < p < \infty$. Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X, Y)$. If the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

3 | RESULTS

In this section, we assume that both, Equation (1.1) and Equation (1.2), have finite delay $h > 0$ and that $0 < r \leq h \leq 2\pi$. Consequently, we will assume that the delay operator $L : L^p([-h, 0]; X) \to X$ is a bounded linear map, and the function x_t is defined by $x_t(\theta) = x(t + \theta)$ for $-h \leq \theta \leq 0$.

Example 3.1. *Let $\eta : [-h, 0] \to \mathcal{L}(X)$ be a strongly continuous function. Let $L : L^p([-h, 0]; X) \to X$ be the bounded linear operator given by the Riemann–Stieltjes integral*

$$L(\phi) = \int_{-h}^{0} \eta(\theta)\phi(\theta)\, d\theta. \tag{3.1}$$

For $k \in \mathbb{Z}$, we define $e_k(\theta) = e^{ik\theta}$ for $-h \leq \theta \leq 0$. Moreover, we write $L_k x = L(e_k x)$ for $x \in X$ and $B_k = e^{-ikr} B$. It is clear that $L_k \in \mathcal{L}(X)$.

We study initially existence of periodic solutions of Equation (1.1). This equation is an extension of the equation

$$\frac{d}{dt}(x(t) - Bx(t-r)) = Ax(t) + L(x_t) + f(t), \quad t \in \mathbb{R},$$

which has been widely studied. To complete the remarks we made in the Introduction, we only mention here to [13] and the references listed in this paper for information on recent results on the existence of solutions of the above equation having a periodicity property.

For $k \in \mathbb{Z}$, we consider the operator

$$\Delta_k = (ik)^\alpha I - (ik)^\alpha B_k - L_k - A$$

defined on $[D(A)]$, with $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i\alpha}{2}\mathrm{sgn}(k)}$.

We call spectrum of Equation (1.1) to the set

$$\sigma_{\mathbb{Z}}(\Delta) = \left\{k \in \mathbb{Z} : \Delta_k \text{ has no continuous inverse}\right\}.$$

Next we define the notion of strong solution of the fractional neutral differential Equation (1.1) with delay.

Definition 3.2. *Let $1 \le p < \infty$ and let $f \in L^p(\mathbb{T}; X)$. A function $x(\cdot)$ is called a* strong L^p-solution *of Equation* (1.1) *if $x \in L^p(\mathbb{T}; [D(A)]) \cap H^{\alpha,p}(\mathbb{T}; [D(B)])$ and Equation* (1.1) *holds for almost all $t \in \mathbb{R}$.*

The following property is a consequence of [7, Proposition 1.11].

Proposition 3.3. *Let $1 \le p < \infty$ and suppose that $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$. If the sequence*

$$\left((ik)^\alpha((ik)^\alpha I - (ik)^\alpha B_k - L_k - A)^{-1}\right)_{k\in\mathbb{Z}}$$

is an L^p-multiplier, then the family $\left\{(ik)^\alpha((ik)^\alpha I - (ik)^\alpha B_k - L_k - A)^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded.

Lemma 3.4. *Let $1 \le p < \infty$. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$ and that for every $f \in L^p(\mathbb{T}; X)$ there exists a strong L^p-solution u of Equation* (1.1)*. Then u is the unique strong L^p-solution of Equation* (1.1)*.*

Proof. *Assume that $u(\cdot)$ is a strong L^p-solution of Equation* (1.1) *corresponding to $f = 0$. Since Au and Bu are locally integrable functions, we get that $\hat{u}(k) \in D(A)$, and taking Fourier transform in Equation* (1.1), *we obtain*

$$(ik)^\alpha(I - B_k)\hat{u}(k) = (L_k + A)\hat{u}(k), \quad k \in \mathbb{Z}.$$

It follows that $\hat{u}(k) = 0$ for every $k \in \mathbb{Z}$ and, therefore, $u = 0$. □

Lemma 3.5. *Let $1 \le p < \infty$. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists an unique strong L^p-solution of Equation* (1.1), *and that one of the following conditions is satisfied:*

(a) *The linear operator B is bounded with domain X.*

(b) *For each $k \in \mathbb{Z}$, the operator $(ik)^\alpha I - L_k - A$ has bounded inverse.*

Then

(i) *The spectrum $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$.*

(ii) *The sequence $\left((ik)^\alpha\left((ik)^\alpha I - (ik)^\alpha B_k - L_k - A\right)^{-1}\right)_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. *Let $k \in \mathbb{Z}$ and $x \in X$. For $t \in \mathbb{R}$, we define $f(t) = e^{ikt}x$. By our hypotheses there exists an unique strong L^p-solution $u(\cdot)$ of Equation* (1.1)*. Taking Fourier transform on both sides of Equation* (1.1) *we get*

$$(ik)^\alpha(I - B_k)\hat{u}(k) = (L_k + A)\hat{u}(k) + x,$$

which implies that the operator Δ_k is surjective. We assume that $x \in ker(\Delta_k)$. Substituting $u(t) = e^{ikt}x$ in Equation (1.1), *we obtain that u is a strong L^p-solution of Equation* (1.1) *corresponding to the function $f(t) = 0$. Consequently, $u(t) = 0$ and $x = 0$. Combining these assertions, we conclude that Δ_k has inverse.*

Assume first that condition (a) is fulfilled. In this case, the operator Δ_k is closed which implies that Δ_k^{-1} is a bounded linear operator. If we assume now that condition (b) holds, we write

$$N_k = \left((ik)^\alpha I - L_k - A\right)^{-1}.$$

Then

$$\Delta_k = \left(I - (ik)^\alpha B_k N_k\right) \left((ik)^\alpha I - L_k - A\right).$$

Since Δ_k and $((ik)^\alpha I - L_k - A)$ are invertible operators, we have that $\left(I - (ik)^\alpha B_k N_k\right)$ has inverse, and

$$\Delta_k^{-1} = N_k \left(I - (ik)^\alpha B_k N_k\right)^{-1}. \tag{3.2}$$

Moreover, $B_k N_k : X \to X$ is a closed linear operator and therefore is bounded. Consequently, we obtain that Δ_k^{-1} is a bounded linear operator. Hence $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$.

On the other hand, let $f \in L^p(\mathbb{T}; X)$. Turning to use our hypotheses we can assert that there exists an unique function $u \in L^p(\mathbb{T}; [D(A)]) \cap H^{\alpha,p}(\mathbb{T}; [D(B)])$ for which Equation (1.1) is verified a.e. Applying Fourier transform on both sides of this equation, we obtain

$$(ik)^\alpha (I - B_k)\hat{u}(k) = (L_k + A)\hat{u}(k) + \hat{f}(k).$$

Hence,

$$(ik)^\alpha \hat{u}(k) = (ik)^\alpha \left[(ik)^\alpha I - (ik)^\alpha B_k - L_k - A\right]^{-1} \hat{f}(k)$$

for all $k \in \mathbb{Z}$. Since $u \in H^{\alpha,p}(\mathbb{T}; X)$, there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = (ik)^\alpha \hat{u}(k)$, which shows the assertion (ii). □

Next, we consider the reverse assertion. We establish that assuming appropriate conditions a type of converse of Lemma 3.5 is fulfilled. In what follows, we assume that X is an UMD space, $1 < p < \infty$, and the family $\left\{(ik)^\alpha N_k : k \in \mathbb{Z}\right\}$ is R-bounded.

Under these assumptions, according to [32, Theorem 3.5], we have that for every $g \in L^p(\mathbb{T}; X)$ there exists an unique strong L^p-solution v of the equation

$$D^\alpha v(t) = Av(t) + L(v_t) + g(t). \tag{3.3}$$

Applying the Fourier transform to above equality, we obtain

$$\hat{v}(k) = N_k \hat{g}(k), \quad k \in \mathbb{Z}. \tag{3.4}$$

Moreover, the map $\Gamma : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ given by

$$\Gamma(g) = D^\alpha v \tag{3.5}$$

is a bounded linear operator (see [32, Corollary 3.6]).

In the next lemma, we use the notations introduced in (3.3), (3.4) and (3.5).

Lemma 3.6. *Let X be an UMD space and $1 < p < \infty$. Assume that the family $\left\{(ik)^\alpha N_k : k \in \mathbb{Z}\right\}$ is R-bounded and $B \in \mathcal{L}(X)$. If $|k|^\alpha \|B\| \|N_k\| < 1$ for all $k \in \mathbb{Z}$ and $\|B\| \|\Gamma\| < 1$, then the sequence $\left(\left(I - (ik)^\alpha e^{-ikr} B N_k\right)^{-1}\right)_{k \in \mathbb{Z}}$ is an L^p-multiplier.*

Proof. *It follows from our hypotheses that $S_k = \left(I - (ik)^\alpha e^{-ikr} B N_k\right)^{-1} \in \mathcal{L}(X)$.*

Let $f \in L^p(\mathbb{T}; X)$. We define the map $\mathcal{A} : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ by

$$\mathcal{A}(\varphi)(t) = B\Gamma(\varphi)(t - r) + f(t).$$

Clearly $\mathcal{A}$ is a contraction. Therefore, there exists $g \in L^p(\mathbb{T}; X)$ such that

$$g(t) = B\Gamma(g)(t - r) + f(t) = BD^\alpha v(t - r) + f(t). \tag{3.6}$$

Using (3.4), we get

$$\hat{g}(k) = e^{-ikr}(ik)^\alpha B\hat{v}(k) + \hat{f}(k) = e^{-ikr}(ik)^\alpha B N_k \hat{g}(k) + \hat{f}(k),$$

which implies that $\hat{g}(k) = S_k \hat{f}(k)$. □

The following result is a consequence of Lemma 3.6 and [32, Theorem 3.5].

Theorem 3.7. *Let X be a UMD space and $1 < p < \infty$. Assume that the family $\{(ik)^\alpha N_k : k \in \mathbb{Z}\}$ is R-bounded and $B \in \mathcal{L}(X)$. If $|k|^\alpha \|B\| \|N_k\| < 1$ for all $k \in \mathbb{Z}$ and $\|B\|\|\Gamma\| < 1$, then for every $f \in L^p(\mathbb{T}; X)$ there exists an unique strong L^p-solution of Equation* (1.1).

Proof. *Let $f \in L^p(\mathbb{T}; X)$. According to Lemma 3.6 we have that the sequence $(S_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. Therefore, there exists $g \in L^p(\mathbb{T}; X)$ such that $\hat{g}(k) = S_k \hat{f}(k)$. It follows from* (3.6) *that*

$$g(t) = BD^\alpha v(t-r) + f(t),$$

where the function v satisfies Equation (3.3). *On the other hand, from Theorem 3.5 in [32], we assert that there exists a unique $u \in H^{\alpha,p}(\mathbb{T}; X) \cap L^p(\mathbb{T}; [D(A)])$ such that $\hat{u}(k) = N_k \hat{g}(k)$ and*

$$D^\alpha u(t) = Au(t) + L(u_t) + g(t), \ a.e.$$

Hence $u(t) = v(t)$, and

$$D^\alpha u(t) = Au(t) + L(u_t) + BD^\alpha u(t-r) + f(t),$$

which implies that

$$D^\alpha u(t) - BD^\alpha u(t-r) = Au(t) + L(u_t) + f(t).$$

Consequently, u is a strong L^p-solution of Equation (1.1). *Since* (3.2) *is also valid in this case, we obtain that $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$ and the uniqueness of u follows from the Lemma 3.4.* □

In the case of a Hilbert space, Theorem 3.7 takes a particularly simple form.

Corollary 3.8. *Let H be Hilbert space and $1 < p < \infty$. Assume that $\sup_{k\in\mathbb{Z}} \|(ik)^\alpha N_k\| < \infty$ and $B \in \mathcal{L}(X)$. If $|k|^\alpha \|B\| \|N_k\| < 1$ for all $k \in \mathbb{Z}$ and $\|B\|\|\Gamma\| < 1$, then for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong L^p-solution of Equation* (1.1).

Proof. *This is a consequence of Plancherel's Theorem.* □

In our next statements, we define $\Gamma_0 : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ by $\Gamma_0(g) = \Gamma(g) - g$. It is clear that Γ_0 is a bounded linear operator (see [32, Corollary 3.6]). We will establish a similar result to the previous theorem, considering that the operator B is not continuous.

Lemma 3.9. *Let X be a UMD space and $1 < p < \infty$. Assume that the family $\{(ik)^\alpha N_k : k \in \mathbb{Z}\}$ is R-bounded and $B^{-1} \in \mathcal{L}(X)$. Assume further that $\left\|B^{-1}\right\| + \|\Gamma_0\| < 1$ and $|k|^\alpha \|BN_k\| < 1$ for all $k \in \mathbb{Z}$, then the sequence $\left(\left(I - (ik)^\alpha e^{-ikr} BN_k\right)^{-1}\right)_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. *Since $BN_k \in \mathcal{L}(X)$ we obtain that $S_k = \left(I - ike^{-ikr} BN_k\right)^{-1} \in \mathcal{L}(X)$. Using the notations introduced in* (3.3) *and* (3.5)*, we now argue as in the proof of Lemma 3.6. Let $f \in L^p(\mathbb{T}; X)$. If g satisfies* (3.6)*, then*

$$g(t) - B(Av(t-r) + L(v_{t-r}) + g(t-r)) = f(t)$$

which implies that

$$B^{-1} g(t) - (Av(t-r) + L(v_{t-r})) - g(t-r) = B^{-1} f(t),$$

for $t \in \mathbb{R}$. Substituting t by $t + r$,

$$\begin{aligned} g(t) &= B^{-1} g(t+r) - (Av(t) + L(v_t)) - B^{-1} f(t+r) \\ &= B^{-1} g(t+r) - \Gamma_0(g)(t) - B^{-1} f(t+r). \end{aligned} \tag{3.7}$$

Since the map $g \mapsto B^{-1} g(t+r) - \Gamma_0(g)(t)$ is a contraction from $L^p(\mathbb{T}; X)$ into $L^p(\mathbb{T}; X)$, there exists $g \in L^p(\mathbb{T}; X)$ that satisfies (3.7). *We complete the proof arguing as in the proof of Lemma 3.6.* □

As a consequence of the Lemma 3.9 we have the following result.

Theorem 3.10. *Let X be a UMD space and $1 < p < \infty$. Assume that the family $\{(ik)^\alpha N_k : k \in \mathbb{Z}\}$ is R-bounded and $B^{-1} \in \mathcal{L}(X)$. Assume further that $\|B^{-1}\| + \|\Gamma_0\| < 1$ and $|k|^\alpha \|BN_k\| < 1$ for all $k \in \mathbb{Z}$, then for every $f \in L^p(\mathbb{T}; X)$ there exists an unique strong L^p-solution of Equation* (1.1).

Now, we consider a special case. Specifically, we will take $r = h = 2\pi$. As we are looking for 2π-periodic solutions of Equation (1.1), the problem is reduced to finding 2π-periodic solutions of the equation

$$D^\alpha(I - B)x(t) = Ax(t) + L(x_t) + f(t), \quad t \in \mathbb{R}, \ 1 < \alpha \leq 2. \tag{3.8}$$

The following result characterizes the existence and uniqueness of solutions of the abstract neutral fractional differential equation (3.8).

Theorem 3.11. *Let X be a UMD space, $1 < p < \infty$ and $1 < \alpha \leq 2$. Assume further that $I - B$ has bounded inverse. The following assertions are equivalent:*

- **(i)** *For every $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of Equation* (1.1).
- **(ii)** *For every $k \in \mathbb{Z}$ the operator $(ik)^\alpha(I - B) - L_k - A$ has a bounded inverse and the families*

$$\left\{(ik)^\alpha((ik)^\alpha(I - B) - L_k - A)^{-1} : k \in \mathbb{Z}\right\} \quad \text{and} \quad \left\{(ik)^\alpha B((ik)^\alpha(I - B) - L_k - A)^{-1} : k \in \mathbb{Z}\right\}$$

are R-bounded.

Proof. *Assume that (i) holds and let $g \in L^p(\mathbb{T}; [D(B)])$. The function $f = (I - B)g \in L^p(\mathbb{T}; X)$. The solution $x(\cdot)$ of Equation* (1.1) *also solves Equation* (3.8) *and, therefore is a strong L^p-solution of the equation*

$$D^\alpha x(t) = (I - B)^{-1}Ax(t) + (I - B)^{-1}L(x_t) + g(t), \tag{3.9}$$

for $t \in \mathbb{R}$, $1 < \alpha \leq 2$. We consider Equation (3.9) *with values in the space $[D(B)]$. Let $E_k = (ik)^\alpha I - (I - B)^{-1}A - (I - B)^{-1}L_k$. Note that $(I - B)^{-1}A : D(A) \to [D(B)]$ is a closed operator and $(I - B)^{-1}L : L^p([-2\pi, 0], [D(B)]) \to [D(B)]$ is continuous. According to [32, Therem 3.5] we obtain that E_k has a bounded inverse, and the family $\left\{(ik)^\alpha E_k^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded with the norm in $[D(B)]$. Since $\Delta_k^{-1} = E_k^{-1}(I - B)^{-1}$, the family $\left\{(ik)^\alpha \Delta_k^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded.*

Assume now that (ii) holds. Let $f \in L^p(\mathbb{T}; X)$. It follows from (ii) that the family $\left\{(ik)^\alpha E_k^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded for the norm in $[D(B)]$. Using again [32, Theorem 3.5], with $g(t) = (I - B)^{-1}f(t)$, we obtain that the equation

$$D^\alpha x(t) = (I - B)^{-1}Ax(t) + (I - B)^{-1}L(x_t) + g(t)$$

has a unique strong L^p-solution in the space $[D(B)]$. It is clear that $D^\alpha x(t) \in D(B)$ and

$$BD^\alpha x(t) = D^\alpha x(t) - Ax(t) - L(x_t) - f(t).$$

Since $[D(A)] \hookrightarrow [D(B)]$ is continuous, it follow that $BD^\alpha x(t) = D^\alpha Bx(t)$. Then $x(\cdot)$ is a strong L^p-solution 2π periodic of Equation (3.8)*, which in turn implies that $x(\cdot)$ is a strong L^p-solution of Equation* (1.1). □

As we hoped, the case $B = 0$ reproduces [32, Theorem 3.5]. Moreover, the following consequences are interesting.

Corollary 3.12. *Let X be a UMD space, $1 < p < \infty$ and $1 < \alpha \leq 2$. Assume further that $I - B$ has bounded inverse. The following assertions are equivalent:*

- **(i)** *For every $f \in L^p(\mathbb{T}; X)$, $t \in \mathbb{R}$, there exists a unique strong L^p-solution of the equation*

$$D^\alpha(x(t) - Bx(t - 2\pi)) = Ax(t) + f(t).$$

- **(ii)** *For every $k \in \mathbb{Z}$ the operator $(ik)^\alpha(I - B) - A$ has a bounded inverse and the families*

$$\left\{(ik)^\alpha((ik)^\alpha(I - B) - A)^{-1} : k \in \mathbb{Z}\right\} \quad \text{and} \quad \left\{(ik)^\alpha B((ik)^\alpha(I - B) - A)^{-1} : k \in \mathbb{Z}\right\}$$

are R-bounded.

Corollary 3.13. *Let $1 < p < \infty$, $1 < \alpha \leq 2$ and $|b| < 1$. For every $f \in L^p(\mathbb{T})$ there exists a unique strong L^p-solution of the scalar equation*

$$D^\alpha(x(t) - bx(t - 2\pi)) = ax(t) + f(t).$$

Now, we consider Equation (1.2). This equation is a generalization of the equation

$$\frac{d}{dt}(x(t) - Bx(t - r)) = A(x(t) - Bx(t - r)) + L(x_t) + f(t), \quad t \in \mathbb{R}.$$

This special type of equations have also been used to model interesting physical systems. In particular Wu and Xia [39,40] have shown that a ring array of resistively coupled transmission lines can be modeled by a neutral functional differential equation. The limit case of a continuous array of lossless transmission lines leads to Hale [23,25] (see also [38]) to study a partial neutral functional differential equation of the special form

$$\frac{d}{dt}Du_t = K\frac{\partial^2}{\partial\xi^2}Du_t + f(u_t),$$

where $K > 0$ is a constant, and

$$Du_t u(\cdot, t) - qu(\cdot, t - r)$$

where $r > 0$, q is a constant satisfying $0 < q < 1$ and $f(u_t) = g(u(\cdot, t), u(\cdot, t - r))$ for some function g. Subsequently, the abstract version

$$\frac{d}{dt}Dx_t = ADx_t + f(t, x_t), \tag{3.10}$$

has been studied by several authors. We mention to [1–3], [21] for well-posedness, existence of solutions, stability, and existence of periodic and almost periodic solutions, [14] for the existence of asymptotically almost automorphic solutions and [6,16] for the existence of pseudo almost automorphic solutions. In addition, control systems modeled by these equations have been studied recently in [8,15,22].

To establish our results, we adapt the concept of solution given in the Definition 3.2.

Definition 3.14. *Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{T}; X)$. A function x is called a* strong L^p-solution *of Equation* (1.1) *if $x \in L^p(\mathbb{T}; [D(B)])$ and $y \in L^p(\mathbb{T}; [D(A)]) \cap H^{\alpha,p}(\mathbb{T}; X)$, where $y(t) = x(t) - Bx(t - r)$, and Equation* (1.2) *holds for almost all $t \in \mathbb{R}$.*

Initially we establish a result of existence of periodic solutions of perturbation type. Next we keep the notations introduced in (3.3) and (3.4). We define the application $\Upsilon : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ by

$$\Upsilon(g) = v, \tag{3.11}$$

where v is a strong L^p-solution of equation $D^\alpha v(t) = Av(t) + L(v_t) + g(t)$.

We have that Υ is a bounded linear operator ([32, Corollary 3.6]). We consider the following condition for the operator B.

(H1) Let $v \in L^p(\mathbb{T}; X)$. The equation

$$u(t) - Bu(t - r) = v(t), \ t \in \mathbb{R},$$

admits a unique solution $u \in L^p(\mathbb{T}; X)$.

If the assumption (H1) holds, we denote by $P : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ the map defined by $u = Pv$. It follows from the closed graph theorem that P is a bounded linear map.

Next we exhibit some cases where the assumption (H1) is fulfilled.

- **(i)** Let $B \in \mathcal{L}(X)$ with $\|B\| < 1$. In this case, $\|I - P\| \leq \dfrac{\|B\|}{1 - \|B\|}$.
- **(ii)** The operator $B^{-1} \in \mathcal{L}(X)$ and $\|B^{-1}\| < 1$. In this case, $\|I - P\| \leq \dfrac{1}{1 - \|B^{-1}\|}$.
- **(iii)** The sequence $\left((I - B_k)^{-1}\right)_{k\in\mathbb{Z}}$ is an L^p-multiplier.

The argument used in the proof of the Lemma 3.5 shows that if the condition (H1) holds, then $I - B_k$ has a bounded inverse for each $k \in \mathbb{Z}$.

Lemma 3.15. *Let X be a UMD space and $1 < p < \infty$. Assume that the assumption (H1) holds, the family $\{(ik)^{\alpha} N_k : k \in \mathbb{Z}\}$ is R-bounded and $\|L\|\|I - P\|\|\Upsilon\| < 1$, then the sequence*

$$\left(\left(I - L_k B_k (I - B_k)^{-1} N_k\right)^{-1}\right)_{k\in\mathbb{Z}}$$

is an L^p-multiplier.

Proof. *Let $f \in L^p(\mathbb{T}; X)$. In view of that*

$$\|L(x_t)\| \le \|L\| \, \|x\|_{L^p(\mathbb{T};X)}$$

for all $x \in L^p(\mathbb{T}; X)$, the equation

$$g(t) = L([(P - I)\Upsilon g]_t) + f(t) \tag{3.12}$$

has a unique solution $g \in L^p(\mathbb{T}; X)$. Defining $v = \Upsilon g$ and $u = Pv$, we can write

$$g(t) = L(u_t) - L(v_t) + f(t),$$

and taking Fourier transform in this equality, we get

$$\hat{g}(k) = L_k \hat{u}(k) - L_k \hat{v}(k) + \hat{f}(k) = L_k\left((I - B_k)^{-1} - I\right)\hat{v}(k) + \hat{f}(k) = L_k B_k (I - B_k)^{-1} N_k \hat{g}(k) + \hat{f}(k). \tag{3.13}$$

Hence

$$\left(I - L_k B_k (I - B_k)^{-1} N_k\right) \hat{g}(k) = \hat{f}(k). \tag{3.14}$$

Let $m \in \mathbb{Z}$ be fixed. For $y \in X$, choosing $f(t) = e^{imt} y$ we obtain that $\left(I - L_m B_m (I - B_m)^{-1} N_m\right)\hat{g}(m) = y$, which shows that $\left(I - L_m B_m (I - B_m)^{-1} N_m\right)$ is a surjective operator. Similarly, if for some $x \in X$, $\left(I - L_m B_m (I - B_m)^{-1} N_m\right)x = 0$ and we define $g(t) = e^{imt} x$, then $\left(I - L_k B_k (I - B_k)^{-1} N_k\right)\hat{g}(k) = 0$ for all $k \in \mathbb{Z}$. From Equation (3.12) *follows that $g = 0$, which implies that* $\ker\left(I - L_m B_m (I - B_m)^{-1} N_m\right) = \{0\}$. *We complete the proof by combining these remarks with* (3.14). ☐

The following result is a consequence of the Lemma 3.15 and [32, Theorem 3.5].

Theorem 3.16. *Let X be a UMD space and $1 < p < \infty$. Assume that the assumption (H1) holds, the set $\{(ik)^{\alpha} N_k : k \in \mathbb{Z}\}$ is R-bounded and $\|L\|\|I - P\|\|\Upsilon\| < 1$, then for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong L^p-solution of Equation* (1.2).

Proof. *Let $f \in L^p(\mathbb{T}; X)$. From Lemma 3.15 we have that the sequence $\left((I - L_k B_k (I - B_k)^{-1} N_k)^{-1}\right)_{k\in\mathbb{Z}}$ is an L^p-multiplier. Therefore, there exists $g \in L^p(\mathbb{T}; X)$ such that $\hat{g}(k) = \left(I - L_k B_k (I - B_k)^{-1} N_k\right)^{-1} \hat{f}(k)$.*

Using again [32, Theorem 3.5] we can affirm that there exists a unique strong L^p-solution v of Equation (3.3). *Hence $\hat{v}(k) = N_k \hat{g}(k)$. We set $u = Pv$. It follows from* (3.13) *that*

$$\hat{g}(k) + L_k \hat{v}(k) = L_k \hat{u}(k) + \hat{f}(k), \;\; k \in \mathbb{Z},$$

which implies that

$$g(t) + L(v_t) = f(t) + L(u_t), \;\; a.e.\, t \in \mathbb{R}.$$

Substituting g and v in (3.3), *we get that u is a strong L^p-solution of Equation* (1.2). ☐

Theorem 3.16 can be interpreted as a perturbation result for Equation (1.2). Before formally establish this property, we will study some properties of R-bounded families, which are immediate consequences of the definition.

Lemma 3.17. *Let $C = \{C_k : k \in \mathbb{Z}\} \subseteq \mathcal{L}(X)$ be a R-bounded family. Let $\varepsilon > 0$ such that $\varepsilon \mathcal{R}(C) < 1$. Then the family $\left\{(I - \varepsilon C_k)^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded.*

Proof. *Let $x_k \in X$ and r_k, $k \in \mathbb{Z}$, be independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$. For $J \subset \mathbb{Z}$ finite,*

$$\left\|\sum_{k\in J} r_k (I-\varepsilon C_k)^{-1} x_k\right\|_{L^p(\Omega;X)} = \left\|\sum_{k\in J} r_k \sum_{j=0}^{\infty} \varepsilon^j C_k^j x_k\right\|_{L^p(\Omega;X)} = \left\|\sum_{j=0}^{\infty} \varepsilon^j \sum_{k\in J} r_k C_k^j x_k\right\|_{L^p(\Omega;X)}$$
$$\leq \sum_{j=0}^{\infty} \varepsilon^j \left\|\sum_{k\in J} r_k C_k^j x_k\right\|_{L^p(\Omega;X)} \leq \sum_{j=0}^{\infty} \varepsilon^j \mathcal{R}(C)^j \left\|\sum_{k\in J} r_k x_k\right\|_{L^p(\Omega;X)}$$

which establishes the assertion. □

Corollary 3.18. *Let $C \in \mathcal{L}(X)$ with $\|C\| < 1$, and let $a_k \in \mathbb{C}$ with $|a_k| \leq 1$ for all $k \in \mathbb{Z}$. Then the family $\left\{(I - a_k C)^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded.*

Proof. *Taking $C_k = a_k C$ and $\varepsilon = 1/2$, the assertion is an immediate consequence of Lemma 3.17 and the contraction principle of Kahane [12].* □

Lemma 3.19. *Let C be a closed linear operator such that $1 \in \rho(C)$ and $\|C(I-C)^{-1}\| < 1/2$. Then the family $\left\{(I - e^{-ikr} C)^{-1} : k \in \mathbb{Z}\right\}$ is R-bounded.*

Proof. *From the identity*

$$I - e^{-ikr} C = (I - C) + \left(1 - e^{-ikr}\right) C = \left[I + \frac{1-e^{-ikr}}{2} 2C(I-C)^{-1}\right](I-C)$$

follows

$$\left(I - e^{-ikr} C\right)^{-1} = (I-C)^{-1}\left[I + \frac{1-e^{-ikr}}{2} 2C(I-C)^{-1}\right]^{-1},$$

and the assertion is a direct consequence of Corollary 3.18. □

Lemma 3.20. *If the family $\{C_k : k \in \mathbb{Z}\}$ is R-bounded, then the families $\{L_k C_k : k \in \mathbb{Z}\}$ and $\{C_k L_k : k \in \mathbb{Z}\}$ are R-bounded.*

Proof. *Let $x_k \in X$ and r_k, $k \in \mathbb{Z}$, be independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$. For $J \subset \mathbb{Z}$ finite, we have*

$$\left\|\sum_{j=1}^{N} r_j L_j C_j x_j\right\|^p_{L^p(\Omega;X)} = \int_\Omega \left\|\sum_{j=1}^{N} r_j(t) L_j(C_j x_j)\right\|^p_X dt = \int_\Omega \left\|\sum_{j=1}^{N} r_j(t) L(e_j C_j x_j)\right\|^p_X dt$$
$$\leq \|L\|^p \int_\Omega \left\|\sum_{j=1}^{N} r_j(t) e_j C_j x_j)\right\|^p_{L^p([-h,0];X)} dt \leq \|L\|^p \int_\Omega \int_{-h}^{0} \left\|\sum_{j=1}^{N} r_j(t) e_j(s) C_j x_j)\right\|^p_X ds\, dt$$
$$= \|L\|^p \int_{-h}^{0} \int_\Omega \left\|\sum_{j=1}^{N} r_j(t) e_j(s) C_j x_j)\right\|^p_X dt\, ds = \|L\|^p \int_{-h}^{0} \left\|\sum_{j=1}^{N} r_j e_j(s) C_j x_j)\right\|^p_{L^p(\Omega;X)} ds.$$

Applying again the contraction principle of Kahane [12], and the fact that $\{C_k : k \in \mathbb{Z}\}$ is an R-bounded family, we obtain

$$\left\|\sum_{j=1}^{N} r_j L_j C_j x_j\right\|^p_{L^p(\Omega;X)} \leq 2\|L\|^p \int_{-h}^{0} \left\|\sum_{j=1}^{N} r_j C_j x_j)\right\|^p_{L^p(\Omega;X)} ds \leq 2\|L\|^p h C \left\|\sum_{j=1}^{N} r_j x_j)\right\|^p_{L^p(\Omega;X)},$$

for some constant $C \geq 0$, hence the family $\{L_k C_k : k \in \mathbb{Z}\}$ is R-bounded. In similar form, we show that the family $\{C_k L_k : k \in \mathbb{Z}\}$ is R-bounded. □

Remark 3.21. *Suppose that X is a UMD space, $1 < p < \infty$ and the condition (H1) holds. Assume in further that the family $\{(ik)^\alpha((ik)^\alpha I - A)^{-1} : k \in \mathbb{Z}\}$ is R-bounded. It follows from [32] that for every $g \in L^p(\mathbb{T}; X)$ the equation*

$$D^\alpha v(t) = Av(t) + g(t)$$

has a unique strong L^p-solution. Moreover, the map $\mathcal{A} : L^p(\mathbb{T}; X) \to L^p(\mathbb{T}; X)$ given by

$$\mathcal{A}(g) = v$$

is a bounded linear operator. Let $\varepsilon > 0$. If the family $\{(ik)^\alpha N_k : k \in \mathbb{Z}\}$ is R-bounded, then

$$(ik)^\alpha I - \varepsilon L_k - A = [(ik)^\alpha I - A]\left[I - \varepsilon((ik)^\alpha I - A)^{-1} L_k\right].$$

Hence, for ε small enough,

$$N_k^\varepsilon := \left[(ik)^\alpha I - \varepsilon L_k - A\right]^{-1} = \left[I - \varepsilon((ik)^\alpha I - A)^{-1} L_k\right]^{-1}\left[(ik)^\alpha I - A\right]^{-1},$$

and combining Lemma 3.17 with Lemma 3.20, we get that the family $\{(ik)^\alpha N_k^\varepsilon : k \in \mathbb{Z}\}$ is R-bounded. Consequently, it follows from [32] that for each $g \in L^p(\mathbb{T}; X)$ there exists a unique strong L^p-solution $v^\varepsilon(\cdot)$ to equation

$$D^\alpha v(t) = Av(t) + \varepsilon L(v_t) + g(t). \tag{3.15}$$

Moreover, it is clear that $v^\varepsilon = \mathcal{A}(\varepsilon L(v^\varepsilon_\cdot) + g(\cdot))$, which implies that

$$\|v^\varepsilon\|_{L^p(\mathbb{T};X)} \le \varepsilon\, h^{1/p}\,\|\mathcal{A}\|\,\|L\|\|v^\varepsilon\|_{L^p(\mathbb{T};X)} + \|\mathcal{A}\|\|g\|_{L^p(\mathbb{T};X)}.$$

It follows that $\|v^\varepsilon\|_{L^p(\mathbb{T};X)} \le C\|g\|_{L^p(\mathbb{T};X)}$ for some constant $C > 0$. Hence, if Υ_ε denotes the map Υ corresponding Equation (3.15)*, we have that $\|\Upsilon_\varepsilon\| \le C$ and the condition $\varepsilon\|L\|\|I - P\|\|\Upsilon_\varepsilon\| < 1$ is fulfilled for $\varepsilon > 0$ enough small. This shows that Theorem 3.16 can be applied to establish the existence of periodic solutions to the equation*

$$D^\alpha(x(t) - Bx(t - r)) = A(x(t) - Bx(t - r)) + \varepsilon L(x_t) + f(t)$$

for $\varepsilon > 0$ sufficiently small.

We are in a position to establish the following result.

Theorem 3.22. *Let X be a UMD space and $1 < p < \infty$. Assume that the families $\{(I - B_k)^{-1} : k \in \mathbb{Z}\}$ and $\left\{(ik)^\alpha\left((ik)^\alpha I - L_k(I - B_k - A)^{-1}\right)^{-1} : k \in \mathbb{Z}\right\}$ are R-bounded. Then for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong L^p-solution of Equation* (1.2).

Proof. *Let $f \in L^p(\mathbb{T}; X)$. To abbreviate the text, for $k \in \mathbb{Z}$, we denote*

$$a_k = (ik)^\alpha \ \text{ and } \ Q_k = \left(a_k I - A - L_k(I - B_k)^{-1}\right)^{-1}.$$

Therefore, the family $\{a_k Q_k : k \in \mathbb{Z}\}$ is R-bounded. Moreover,

$$\begin{aligned} k(a_{k+1}Q_{k+1} - a_k Q_k) &= kQ_{k+1}\left[a_k L_{k+1}(I - B_{k+1})^{-1} - a_{k+1} L_k (I - B_k)^{-1}\right] Q_k \\ &= kQ_{k+1}(L_{k+1} - L_k)(I - B_{k+1})^{-1} a_k Q_k \\ &\quad + kQ_{k+1}\, L_k(I - B_{k+1})^{-1} a_k Q_k - a_{k+1} Q_{k+1} L_k (I - B_k)^{-1} k Q_k. \end{aligned} \tag{3.16}$$

The three terms on the right hand side of this decomposition define R-bounded families by our hypothesis and Lemma 3.20. Hence, from the properties of R-bounded families (see [12]) we conclude that the family $\{k(a_{k+1}Q_{k+1} - a_k Q_k) : k \in \mathbb{Z}\}$ is R-bounded. It follows from Theorem 2.6 that $\{a_k Q_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. The argument used in [32, Lemma 2.6] shows *that there exists $v \in H^{\alpha,p}(\mathbb{T};[D(A)])$ such that $\hat{v}(k) = Q_k \hat{f}(k)$. Since $\{a_k Q_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, there exists $u \in L^p(\mathbb{T};X)$ such that*

$$\hat{u}(k) = \frac{1}{(ik)^\alpha}(I - B_k)^{-1}(ik)^\alpha Q_k \hat{f}(k), \quad k \neq 0.$$

Therefore, $u \in L^p(\mathbb{T};[D(B)])$ and

$$(I - B_k)\hat{u}(k) = Q_k \hat{f}(k) = \hat{v}(k), \quad k \in \mathbb{Z}.$$

Hence we obtain that $v(t) = u(t) - Bu(t-r)$. Combining these assertions, we get

$$(ik)^\alpha \hat{v}(k) = A\hat{v}(k) + L_k \hat{u}(k) + \hat{f}(k), \quad k \in \mathbb{Z},$$

which in turn implies that

$$D^\alpha v(t) = Av(t) + L(u_t) + f(t), \quad t \in \mathbb{R},$$

and completes the proof. □

The decomposition (3.16) allows us to establish another result of existence of strong L^p-solution Equation (1.2). We will use the following property.

Lemma 3.23. *Let L be the map defined by* (3.1)*, where $\eta : [-h,0] \to \mathcal{L}(X)$ is a strongly differentiable operator-valued map. Then the family $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded.*

Proof. *For $x \in X$ and $k \in \mathbb{Z}$, $k \neq 0$, integrating by parts, we obtain*

$$kL_k x = k\int_{-h}^{0} e^{ik\theta}\eta(\theta)x\,d\theta = i\left(e^{-ikh}\eta(-h) - \eta(0)\right) + i\int_{-h}^{0} e^{ik\theta}\eta(\theta)'x\,d\theta.$$

Define $V : L^p([-h,0];X) \to X$ by $V(\varphi) = \int_{-h}^{0}\eta'(\theta)\varphi(\theta)\,d\theta$.

Hence V_k is the operator given by $V_k x = \int_{-h}^{0} e^{ik\theta}\eta(\theta)'x\,d\theta$. Since $\{V_k : k \in \mathbb{Z}\}$ is a R-bounded family ([31, Proposition 3.2]), our assertion is a consequence of the previous representation for kL_k. □

In the next statement we continue using the notations introduced in Theorem 3.22 and Lemma 3.23.

Proposition 3.24. *Let X be a UMD space and $1 < p < \infty$. Let L be the map defined by* (3.1) *with $\eta : [-h,0] \to \mathcal{L}(X)$ a strongly differentiable operator-valued map. Assume that the families $\left\{(I - B_k)^{-1} : k \in \mathbb{Z}\right\}$ and $\left\{|k|^{\alpha/2} Q_k : k \in \mathbb{Z}\right\}$ are R-bounded. Then for every $f \in L^p(\mathbb{T};X)$ there exists a unique strong L^p-solution of Equation* (1.2).

Proof. *It follows from Lemma 3.23 that the family $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded. Using* (3.16) *we can write*

$$\begin{aligned} k(a_{k+1}Q_{k+1} - a_k Q_k) &= s_k |k|^{\alpha/2} Q_{k+1} k(L_{k+1} - L_k)(I - B_{k+1})^{-1}|k|^{\alpha/2} Q_k \\ &\quad + s_k |k|^{\alpha/2} Q_{k+1}\, kL_k (I - B_{k+1})^{-1}|k|^{\alpha/2} Q_k \\ &\quad - s_{k+1}|k+1|^{\alpha/2} Q_{k+1} kL_k (I - B_k)^{-1}|k+1|^{\alpha/2} Q_k, \end{aligned}$$

where $s_k = e^{i\frac{\pi}{2}\alpha sgn(k)}$. This implies that the family $\{k(a_{k+1}Q_{k+1} - a_k Q_k) : k \in \mathbb{Z}\}$ is R-bounded. We complete the demonstration arguing as in the proof of Theorem 3.22. □

Example 3.25. Set $X = \mathbb{C}$ and $1 < \alpha < 2$. For $\rho \in \mathbb{R} \setminus \{-1\}$, consider the neutral fractional differential equation with finite delay

$$D^\alpha(x(t) - Bx(t-r)) = \rho x(t) + x(t - 2\pi) + f(t), \quad t \in [0, 2\pi]. \tag{3.17}$$

Writing $A = \rho I$ and $Lx_{\cdot} = x(\cdot - 2\pi)$, we obtain the abstract form of Equation (1.1). Note that $L_k = e^{2\pi ik} = 1$ for all$k \in \mathbb{Z}$. Since $\rho \in \mathbb{R} \setminus \{-1\}$, we observe that$(ik)^\alpha - 1 - \rho \neq 0$ for all $k \in \mathbb{Z}$. Hence the family$\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}} = \left\{(ik)^\alpha((ik)^\alpha - 1 - \rho)^{-1} : k \in \mathbb{Z}\right\}$ is bounded. Let $f \in L^p(\mathbb{T})$. Let Γ be the map defined in (3.5). From *[32, Corollary 3.6]* there exists a constant$C > 0$

independent of f such that $\|\Gamma\| < C$. Suppose that $B \in \mathcal{L}(X)$, and $\|B\| < \min\left\{1, \frac{1}{C}\right\}$. According to Corollary 3.8 there exists a unique solution $x \in L^p(\mathbb{T})$ of Equation (3.17).

Remark 3.26. *The condition that $L : L^p([-h,0]; X) \to X$ is a bounded linear map used in our development is demanding. However, our results can be extended to include equations of type*

$$D^{\alpha}(x(t) - Bx(t-r)) = Ax(t) + L(x_t) + \sum_{i=1}^{m} C_i x(t - r_i) + f(t), \;\; t \in \mathbb{R}, \;\; \alpha > 0,$$

and

$$D^{\alpha}(x(t) - Bx(t-r)) = A(x(t) - Bx(t-r)) + L(x_t) + \sum_{i=1}^{m} C_i x(t - r_i) + f(t), \;\; t \in \mathbb{R}, \;\; \alpha > 0,$$

where $0 < r_i \le r$. We have avoided this approach to simplify the reading of the text.

4 | APPLICATIONS

In this last section we present an application of our results to partial neutral functional differential equations. As we have already mentioned, equations of type (3.10) have been studied by several authors to model important physical systems. Next we consider a fractionary version of the equation studied in [3] (see also [6,16]). Specifically, we consider the equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}[u(t,\xi) - qu(t-r,\xi)] = \frac{\partial^2}{\partial \xi^2}[u(t,\xi) - qu(t-r,\xi)] + \int_{-r}^{0} \gamma(\theta) u(t+\theta,\xi)\, d\theta + \tilde{f}(t,\xi), \quad t \in \mathbb{R},\ \xi \in [0,\pi], \tag{4.1}$$

$$u(t,\xi) - qu(t-r,\xi) = 0, \quad \xi = 0, \pi, t \in \mathbb{R}. \tag{4.2}$$

We assume that $\gamma : [-r, 0] \to \mathbb{R}$ is a continuous function, $\tilde{f} : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ satisfies appropriated conditions and $q \in (0,1)$. We model this equation in the space $X = L^2([0,\pi])$, and the operators A, B and L are given by

$$Az = \frac{d^2}{d\xi^2} z(\xi) \quad \text{with domain} \quad D(A) = \left\{ z \in X : z \in H^2([0,\pi]), \;\; z(0) = z(\pi) = 0 \right\},$$

$$Bz = qz, \quad z \in X,$$

and

$$L(\varphi) = \int_{-r}^{0} \gamma(\theta)\varphi(\theta)\, d\theta, \quad \varphi \in L^2([-r,0]; X).$$

We assume that $\tilde{f}$ satisfies the Carathéodory conditions of order 2 and that $\tilde{f}(t,\xi)$ is 2π-periodic at the variable t. Let $f(t) = \tilde{f}(t,\cdot)$. It follows that $f \in L^2(\mathbb{T}; X)$. Let $x(t) = u(t,\cdot)$. With these notations, the problem (4.1)-(4.2) adopt the abstract form of Equation (1.2). Since $0 < q < 1$, then the condition (H1) is fulfilled. It is also clear that $\|(I - B_k)^{-1}\| \le (1-q)^{-1}$ and $\|L\| \le C$, where $C = \left(\int_{-r}^{0} \gamma(\theta)^2 d\theta\right)^{1/2}$.

On the other hand, the spectrum of A consists of eigenvalues $-n^2$ for $n \in \mathbb{N}$. Their associated eigenvectors are

$$z_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi).$$

Furthermore, the set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X. In particular,

$$Ax = \sum_{n=1}^{\infty} -n^2 \langle x, z_n \rangle z_n$$

for $x \in D(A)$. Consequently, $(ik)^\alpha$ belongs to the resolvent set of A and

$$((ik)^\alpha I - A)^{-1}x = \sum_{n=1}^{\infty} \frac{1}{(ik)^\alpha + n^2} \langle x, z_n \rangle z_n \tag{4.3}$$

for all $x \in X$. Using (4.3) we can estimate $\|((ik)^\alpha I - A)^{-1}\|$. We distinguish two cases. First, if $0 < \alpha \leq 1$ then $Re((ik)^\alpha) \geq 0$, which implies that

$$\|((ik)^\alpha I - A)^{-1}\| \leq \frac{1}{\sqrt{1 + |k|^{2\alpha}}}.$$

Second, if $1 < \alpha < 2$ then $Re((ik)^\alpha) < 0$ for $k \neq 0$. In this case, for $k \neq 0$ and $n \in \mathbb{N}$, we have

$$|(ik)^\alpha + n^2| \geq |Im((ik)^\alpha)| = |k|^\alpha \sin\left(\frac{\pi}{2}\alpha\right).$$

It follows that

$$\|((ik)^\alpha I - A)^{-1}\| \leq \frac{1}{|k|^\alpha \sin\left(\frac{\pi}{2}\alpha\right)}$$

for $k \neq 0$. Since X is a Hilbert space, the following result is an immediate consequence of Theorem 3.22.

Corollary 4.1. *Under the above conditions, if one of the following situations is verified:*

(a) $0 < \alpha \leq 1$ *and* $C + q < 1$, *or*

(b) $1 < \alpha < 2$ *and* $\frac{C}{\sin\left(\frac{\pi}{2}\alpha\right)} + q < 1$,

then there exists a unique strong L^2*-solution of the problem* (4.1)–(4.2).

Proof. *We set*

$$d(k) = \begin{cases} \dfrac{1}{\sqrt{1 + |k|^{2\alpha}}}, & 0 < \alpha \leq 1, \\ \dfrac{1}{|k|^\alpha \sin\left(\frac{\pi}{2}\alpha\right)}, & 1 < \alpha < 2. \end{cases}$$

In view of that, in either case (a) or (b),

$$\|((ik)^\alpha I - A)^{-1} L_k (I - B_k)^{-1}\| \leq \frac{C d(k)}{1 - q} < 1,$$

and writing

$$\left((ik)^\alpha I - L_k(I - B_k - A)^{-1}\right)^{-1} = \left(I - ((ik)^\alpha I - A)^{-1} L_k (I - B_k)^{-1}\right)^{-1} ((ik)^\alpha I - A)^{-1}$$

we infer that the family

$$\left\{(ik)^\alpha \left((ik)^\alpha I - L_k (I - B_k)^{-1} - A\right)^{-1} : k \in \mathbb{Z}\right\}$$

is bounded in $\mathcal{L}(X)$. *This completes the proof.* □

ACKNOWLEDGEMENTS

The first author is partially supported by FONDECYT 1130144. The second author is partially supported by PAIFAC and ENL016/15.

REFERENCES

[1] M. Adimy and K. Ezzinbi, *Existence and linearized stability for partial neutral functional differential equations with nondense domains*, Differential Equations Dynam. Systems **7**(4) (1999), 371–417.

[2] M. Adimy, K. Ezzinbi, and M. Laklach, *Spectral decomposition for partial neutral functional differential equations*, Can. Appl. Math. Q. **9**(1) (2001), 1–34.

[3] M. Adimy, A. Elazzouzi, and K. Ezzinbi, *Bohr–Neugebauer type theorem for some partial neutral functional differential equations*, Nonlinear Anal. **66**(5) (2007), 1145–1160.

[4] O. P. Agrawal, *Application of fractional derivatives in thermal analysis of disk brakes*, Nonlinear Dynam. **38** (2004), 191–206.

[5] R. P. Agarwal, M. Benchohra, and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973–1033.

[6] M. Alia, K. Ezzinbi, and S. Fatajou, *Exponential dichotomy and pseudo almost automorphy for partial neutral functional differential equations*, Nonlinear Anal. **71** (2009), 2210–2226.

[7] W. Arendt and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), 311–343.

[8] H. Bounit and S. Hadd, *Regular linear systems governed by neutral FDEs*, J. Math. Anal. Appl. **320** (2006), 836–858.

[9] P. Butzer and U. Westphal, *An access to fractional differentiation via fractional difference quotients*, Lecture Notes in Mathematics **457**, pp. 116–145, Springer, Berlin, 1975.

[10] M. Concezzi, R. Garra, and R. Spigler, *Fractional relaxation and fractional oscillation models involving Erdélyi–Kober integrals*, Fract. Calc. Appl. Anal. **18**(5) (2015), 1212–1231.

[11] R. Datko, *Linear autonomous neutral differential equations in a Banach space*, J. Differential Equations **25** (1977), 258–274.

[12] R. Denk, M. Hieber, and J. Prüss, *R-boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Memoirs of the American Mathematical Society **166** (788) 2003.

[13] T. Diagana, *Existence of almost automorphic solutions to some neutral functional differential equations with infinite delay*, Electron. J. Differential Equations **2008**(129) (2008), 1–14.

[14] T. Diagana, E. Hernández, and M. J. P. dos Santos, *Existence of asymptotically almost automorphic solutions to some abstract partial neutral integro-differential equations*, Nonlinear Anal. **71** (2009), 248–257.

[15] A. Elharfi, H. Bounit, and S. Hadd, *Representation of infinite-dimensional neutral non-autonomous control systems*, J. Math. Anal. Appl. **323** (2006), 497–512.

[16] K. Ezzinbi, S. Fatajou, and G. N'Guérékata, *Pseudo-almost-automorphic solutions to some neutral partial functional differential equations in Banach spaces*, Nonlinear Anal. **70** (2009), 1641–1647.

[17] M. Fabrizio, *Some remarks on the fractional Cattaneo–Maxwell equation for the heat propagation*, Fract. Calc. Appl. Anal. **18**(4) (2015), 1074–1079.

[18] M. Girardi, and L. Weis, *Criteria for R-boundedness of operator families*, Lecture Notes in Pure and Applied Mathematics **234**, M. Dekker, New York, 2003, pp. 203–221.

[19] A. Grünwald, *Über begrenzte Derivationen und deren Anwendung*, Z. Angew. Math. Phys. **12** (1867), 441–480.

[20] S. Guo, *Equivariant normal forms for neutral functional differential equations*, Nonlinear Dynam. **61** (2010), 311–329.

[21] S. Hadd, *Singular functional differential equations of neutral type in Banach spaces*, J. Funct. Anal. **254** (2008), 2069–2091.

[22] S. Hadd and A. Rhandi, *Feedback theory for neutral equations in infinite dimensional state spaces*, Note Mat. **28**(1) (2008), 43–68.

[23] J. Hale, *Partial neutral functional differential equations*, Rev. Roumaine Math. Pures Appl. **39**(4) (1994), 339–344.

[24] J. Hale and S. Verduyn, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[25] J. Hale, *Coupled oscillators on a circle*, Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo **1**(4) (1994), 441–457.

[26] V. Keyantuo and C. Lizama, *A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications*, Math. Nach. **284**(4) (2011), 494–506.

[27] M. Klimek and O. P. Agrawal, *Fractional Sturm–Liouville problem*, Comput. Math. Appl. **66**(5) (2013), 795–812.

[28] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Acad. Publ., Dordrecht, 1999.

[29] V. Lakshmikantham and J. Vasundhara, *Theory of fractional differential equations in a Banach space*, Eur. J. Pure Appl. Math. 1, **1** (2008), 38–45.

[30] A. Letnikov, *Theory and differentiation of fractional order*, Mat. Sb. **3** (1868), 1–66.

[31] C. Lizama, *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl. **324** (2006), 921–933.

[32] C. Lizama and V. Poblete, *Periodic solutions of fractional differential equations with delay*, J. Evol. Equ. **11** (2011), 57–70.

[33] G. Mophou and G. N'Guérékata, *On integral solutions of some nonlocal fractional differential equations with nondense domain*, Nonlinear Anal. **71** (2009), 4668–4675.

[34] D. Pandey, A. Ujlayan, and D. Bahuguna, *On a solution to fractional order integrodifferential equations with analytic semigroups*, Nonlinear Anal. **71** (2009), 3690–3698.

[35] Y. Ren, Y. Qin, R. Sakthivel, *Existence results for fractional order semilinear integro-differential evolution equations with infinite delay*, Integral Equuations Operator Theory **67** (2010), 33–49.

[36] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[37] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), 735–758.

[38] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

[39] J. Wu and H. Xia, *Self-sustained oscillations in a ring array of coupled lossless transmission lines*, J. Differential Equations **124** (1996), 247–278.

[40] J. Wu and H. Xia, *Rotating waves in neutral partial functional differential equations*, J. Dynam. Differential Equations **11**(2) (1999), 209–238.

How to cite this article: Henríquez H, Poblete V. Periodic solutions of neutral fractional differential equations. *Mathematische Nachrichten*. 2017;290:2858–2873. https://doi.org/10.1002/mana.201600214