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Wealth transfers and the role of collateral when lifetimes are uncertain

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Abstract We develop a general equilibrium model of wealth transfers in the presence of uncertain lifetimes and default. Without introducing exogenous debt constraints, agents are allowed to make collateral-backed promises at any state of their life span.

Keywords: Uncertain lifetimes · Collateralized assets · Donations · Wills

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1. Introduction

From a general equilibrium point of view, the analysis of finite–horizon economies with heterogeneous agents has been extended to an infinite horizon introducing either households with an infinite lifetime (see, for instance, Magill and Quinzii (1994) and Hernandez and Santos (1996)) or overlapping generations of finitely–lived agents (as in Schmachtenberg (1988) and Geanakoplos and Polemarchakis (1991)).

In these models, altruistic motives had appeared as polar cases of our real world behavior. While finitely-lived agents are interpreted as totally selfish individuals, infinitely-lived households are considered as dynasties of finitely-lived generations, that care about their descendants as much as they care about themselves.

In the other side, all the models cited above assume that agents' lifetimes are deterministic, assets are free of default and commodities are perishable. In addition, when individuals are infinitely–lived, the successive postponements of their commitments, through the appeal to new credits – Ponzi schemes – were ruled out using exogenous debt constraints or transversality conditions. Models with finitely–lived agents assume that individuals do not have access to credit markets at their terminal dates, as there are no mechanisms to assure that debts, made at the end of life, will be repaid.

These assumptions are strong simplifications of the financial markets practices, restrict agents' behavior and do not follow from individual rationality. Thus, we want to address, in a general equilibrium framework, the actual market practices that allow agents to acquire loans, even when lifetimes are uncertain and assets are subject to default. Moreover, we aim to consider agents' altruistic behavior in a more realistic manner, allowing for wealth transfers, such as *donations* and *bequests*.[1]

To achieve these goals, we develop a model in which lifetimes are uncertain and physical bundles of durable commodities can be used to collateralize assets. We essentially show that collateral plays a crucial role when agents' lifetimes are uncertain. Indeed, the existence of physical guarantees creates a natural form to allow agents to make promises at *all* nodes of their life span, without introducing any exogenous credit constraint.[2]

[1]The importance of donations and bequests was extensively highlighted by economic research on capital accumulation, social security systems or public deficits (see, for instance, Kotlikoff and Summers (1981), Fuster (2000) and Cardia and Michel (2004)). Moreover, was proving that decisions about allocation of consumption or determination of amounts for savings and bequests are strongly affected by the expected life duration (see, for instance, Leung (1994), Fuster (1999), Dynan, Skinner and Zeldes (2002) and d'Albis (2006)).

[2]Other important roles of collateral were previously addressed in the literature. When agents are infinitely–lived, commodities are durable and assets are collateralized, Araujo, Páscoa and Torres-Martínez (2002) show equilibrium existence, without imposing debt constraints or transversality conditions to avoid Ponzi schemes. In a similar

Since individuals may have physical and financial wealths left over when they pass away, we introduce some mechanisms to regulate wealth reallocations. First, agents can write wills in order to determine their bequests. Second, each individual can make nominal donations during his lifetime. In particular, an agent can make donations to disinherit some agents who will have legal rights over his estate when he passes away.

When an agent passes away, the market seizes his estate and honors his commitments before beneficiaries of his testament receive their bequests. Moreover, when an agent's estate is not totally distributed after both the payments of his debts and the delivery of his bequests, his *intestate estate*[3] will be divided among his heirs according to rules determined by exogenous *inheritance laws*.

In addition to utility benefits from consumption, agents may receive, according to their degree of altruism, utility gains from both the amount of their donations and the structure of their testamentary rights. Consequently, unlike the classical Overlapping Generations model in which individuals are totally selfish, agents may be interested in purchasing financial assets even at their terminal nodes, in order to increase the value of their estate and, therefore, assure higher transfers for those agents who will hold testamentary rights. On the other hand, selfish agents may also leave accidental bequests to their heirs as lifetimes are uncertain.

The paper is organized as follows. In the second section, we present the model. The assumptions and our equilibrium existence results are presented in Section 3. Section 4 is devoted to some examples of bequest functions illustrating how optimal testamentary transfers may vary as functions of agents' wealth. Finally, we make some comments on the optimal level of donations and we discuss some possible extensions of our analysis. Proofs are given in the Appendices.

2. The model

Stochastic structure. The stochastic structure is described by an infinite event-tree with a unique root. There is a countable set of time periods, $\{0, 1, \ldots\}$, and there is no uncertainty at $t = 0$. Thus, denoting by s_0 the unique state of nature at the first period, we suppose that given a history of realization of uncertainty $\overline{s}_t = (s_0, \ldots, s_{t-1})$, there exists a finite set $S(\overline{s}_t)$ of states of nature at period t. An information set $\xi = (t, \overline{s}_t, s)$, where $t \geqslant 1$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. Let ξ_0 be the initial node, at $t = 0$. The set of nodes in the economy is called the *event-tree* and is denoted by D.

We refer to the nodes $\xi = (t, \overline{s}_t, s)$, with $t \geqslant 1$, as successors of ξ_0. Moreover, given $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of ξ, and we write $\mu \geqslant \xi$, if both $t' \geqslant t \geqslant 1$ and

framework, Kubler and Schmedders (2003) prove that collateral rationalizes tight borrowing limits in computational stationary equilibria.

[3] When a valid will (or testament) has been made, but only applies to part of the estate, the remaining wealth forms the *intestate estate*.

$(\bar{s}_{t'}, s') = (\bar{s}_t, s, \dots)$. Let $t(\xi) = t$ be the period associated to $\xi = (t, \bar{s}_t, s)$ and let ξ^- be its (unique) predecessor, that is, $\xi^- \leqslant \xi$ and $t(\xi^-) = t(\xi) - 1$. Now, we denote the set of *immediate successors* of ξ by $\xi^+ := \{\mu \in D : \mu \geqslant \xi, t(\mu) = t(\xi) + 1\}$. Finally, let $D(\xi) = \{\mu \in D : \mu \geqslant \xi\}$ be the set of successors of ξ, and $D_T(\xi) := \{\mu \in D(\xi) : t(\mu) \leqslant t(\xi) + T\}$.

Demographic structure and physical markets. Letting I be the set of agents in the economy, the set of nodes at which an agent $i \in I$ can trade is denoted by $D^i \subset D$. Thus, we allow lifetime durations to be affected by uncertainty. Note that, the traditional overlapping generations model and the infinitely-lived households model can be obtained as particular cases of our demographic structure.

Let $I(\xi) := \{i \in I : \xi \in D^i\}$ be the non-empty set of agents who are alive at node $\xi \in D$. We suppose that the number of agents who are alive at ξ, $n(\xi) := \#I(\xi)$, is finite. When the set D^i is finite, agent i is said to be finitely–lived. Otherwise, agent i will have at least one infinite–life path through the event-tree.

Without loss of generality, we assume that agents do not exit the economy at a given node and reappear afterward on the markets. That is, for each $i \in I$, $D(\mu) \cap D^i = \emptyset, \ \forall \mu \in D \setminus D^i$.

At each node, there is a finite ordered set, G, of physical goods that are traded on spot markets by the alive consumers. Let $p = (p(\xi); \xi \in D)$ be the commodity price process, where $p(\xi) = (p(\xi, g); g \in G) \in \mathbb{R}^G_+ \setminus \{0\}$ denotes the spot price of commodities at ξ.

Each agent i has an endowment process $w^i := (w^i(\xi, g), \ (g, \xi) \in G \times D^i) \in \mathbb{R}^{G \times D^i}_+$ and chooses a consumption plan $x^i := \big(x^i(\xi); \xi \in D^i\big) \in \mathbb{R}^{G \times D^i}_+$, where $x^i(\xi) \in \mathbb{R}^G_+$ denotes the consumption bundle at node ξ. A plan x^i gives a utility level $U^i(x^i)$, where $U^i : \mathbb{R}^{G \times D^i}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ represents agent i's preferences over physical consumptions.

Commodities may be durable and suffer depreciation. The depreciation structure is given by a collection of non-zero $G \times G$ matrices, $(Y_\xi; \xi \in D)$, with non-negative entries. So, when agent i uses the services of a bundle $x \in \mathbb{R}^G_+$ at $\xi \in D^i$, he receives, at each immediate successor $\mu \in \xi^+ \cap D^i$, a bundle $Y_\mu x$. To simplify notations, agent i's accumulated endowment up to node $\xi \in D^i$ will be denoted by $W^i(\xi) := w^i(\xi) + Y_\xi W^i(\xi^-)$, with $W^i(\xi^-) = 0$ for $\xi^- \notin D^i$.

Financial markets. At each $\xi \in D$, there is a finite ordered set, $J(\xi)$, of one–period real assets, available for inter-temporal transaction and insurance. As in Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (2002), assets are subject to default and backed by physical collateral requirements.[4]

[4]We consider one–period assets for ease of notations. In the presence of collateral requirements, equilibrium existence can be proved even with long–lived assets. Such a financial structure, in a model with infinitely–lived agents, was studied by Araujo, Páscoa and Torres-Martínez (2005).

More precisely, an asset $j \in J(\xi)$ is characterized by a vector of real promises $A(\mu, j) \in \mathbb{R}^G_+$, at each $\mu \in \xi^+$, and by a vector of unitary collateral requirements $C(\xi, j) \in \mathbb{R}^G_+$, which is held and consumed by the borrowers, for each unit of asset j that they sold at ξ.

Let $q = (q(\xi);\, \xi \in D) \in \prod_{\xi \in D} \mathbb{R}^{J(\xi)}_+$ be the financial price process, where $q(\xi) = (q(\xi, j);\, j \in J(\xi))$ denotes the asset price vector at ξ. The set of state-contingent assets in the economy is denoted by $D(J) = \{(\xi, j) :\ \xi \in D,\ j \in J(\xi)\}$.

As there are no extra economic default penalties and the unique enforcement in case of default is the seizure of the constituted collateral, a seller i of one unit of $j \in J(\xi)$ at $\xi \in D^i$, pays at each immediate successor $\mu \in \xi^+ \cap D^i$, the minimum between the depreciated value of the collateral and the original promises; $R_{\mu,j}(p(\mu)) :=\ \min \{p(\mu)A(\mu, j), p(\mu)Y_\mu\, C(\xi, j)\}$.[5]

Since promises are backed by physical collateral, agent i is allowed to sell assets at any node of his life span. As will be explained more precisely hereafter, given $\xi \in D^i$, we suppose that, at each $\mu \in \xi^+ \setminus D^i$, the market seizes agent i's estate, pays the commitments made at ξ and delivers the remained value of wealth to individuals who have testamentary or inheritance rights over agent i's estate.

On the other hand, although financial transactions are anonymous, each lender knows that the market will enforce contracts even when borrowers pass away. In addition, we assume that each lender believes that (i) borrowers are rational and are aware of the market rules in case of default, (ii) all agents in the economy have monotonic preferences. Thus, each buyer i of one unit of asset $j \in J(\xi)$, expects to receive, at each $\mu \in \xi^+$, the amount $R_{\mu,j}(p(\mu))$.

Let us denote by $\theta^i(\xi) = (\theta^i(\xi, j);\, j \in J(\xi))$ and by $\varphi^i(\xi) = (\varphi^i(\xi, j);\, j \in J(\xi))$, respectively, the long and short positions of agent i at $\xi \in D^i$. When agent i chooses a financial process $(\theta^i, \varphi^i) := \big((\theta^i(\xi), \varphi^i(\xi));\, \xi \in D^i\big)$, he pays (or receives), at each $\xi \in D^i$, an amount $q(\xi)\left(\theta^i(\xi) - \varphi^i(\xi)\right)$; and expects to receive (or delivers), at any $\mu \in \xi^+ \cap D^i$, the effective payment $R_\mu(p(\mu))\left(\theta^i(\xi) - \varphi^i(\xi)\right)$, where $R_\mu(p(\mu)) := (R_{\mu,j}(p(\mu)); j \in J(\xi))$. Moreover, the consumption allocation chosen by agent i satisfies the collateral constraint: $x^i(\xi) \geqslant \sum_{j \in J(\xi)} C(\xi, j)\varphi^i(\xi, j)$.

Bequests. In our model, agents can prevent the disappearance of their terminal physical and financial allocations from the economy through intergenerational transfers. Thus, individuals can devolve their properties and assets upon other agents through a *will*. In such a testament, an agent chooses bequests that other agents will receive when he passes away.

[5]Note that, for simplicity, collateral repossession is the only enforcement mechanism in case of default. However, without technical problems, linear utility penalties for default, as in Dubey, Geanakoplos and Shubik (2005), can be added to our model when agents are finitely–lived. When some agent has at least one infinite–life path, the presence of utility penalties may lead to Ponzi schemes. However, the introduction of upper bounds on the penalty coefficient would assure the existence of equilibrium (see Pascoa and Seghir (2006)).

Formally, for each $i \in I(\xi)$, let $I_{-i}(\xi) := I(\xi) \setminus \{i\}$. Define by $D^i_{-i} = \{\xi \in D^i : I_{-i}(\xi) \neq \emptyset\}$ the subset of D^i in which there is at least one alive agent to be beneficiary of agent i's bequests. Also, let $\overline{D}^i := \{\xi \in D^i : \xi^+ \setminus D^i \neq \emptyset\}$ be the set of nodes in which agent i has a positive probability to pass away in the next period.

We suppose that there is a set $\overline{D}^i_\star \subset \overline{D}^i \cap D^i_{-i}$ in which agent i has a *bequest motive* that incites him to write a will in order to predetermine the distribution of his estate in case of death. Furthermore, at any node in $\overline{D}^i \setminus \overline{D}^i_\star$, the inheritance laws, defined below, will be applied in order to distribute agent i's estate, in case of death in the next period. Thus, due to lifetime uncertainty, an agent i can leave *accidental bequests* as he may accumulate savings up to a node $\xi \in \overline{D}^i \setminus \overline{D}^i_\star$ in order to improve his future consumption.

The amount and the distribution of bequests among the beneficiaries may positively affect agents' preferences, as will be detailed at the end of this section. Thus, unlike classical overlapping generations models, agents may be interested in buying assets at their terminal nodes.

We suppose that each agent i chooses, at each $\xi \in \overline{D}^i_\star$, the rights over his future estate writing a will $b^i_\xi := (b^i(\mu); \mu \in \xi^+ \setminus D^i)$, where $b^i(\mu) \in \mathbb{R}^{I_{-i}(\xi)}_+$ represents the nominal bequests that individuals in $I_{-i}(\xi)$ will receive from agent i at $\mu \in \xi^+ \setminus D^i$.

Note that wills take into account future contingencies. Moreover, at each $\xi \in \overline{D}^i_\star$, agent i determines testamentary rights only among agents who are alive at this node, since he does not know the demographic structure at the successors $\mu \in \xi^+ \setminus D^i$.

On the other hand, as pointed out earlier, we assume that (i) markets can enforce contracts even when borrowers pass away, and (ii) markets secure that lenders are paid back before the distribution of the testamentary rights among the beneficiaries. Moreover, the market will try to honor the original contracts that lenders has subscribed, maximizing simultaneously the value of the wealth delivered to the beneficiaries of testamentary rights. Therefore, as the physical estate includes the depreciated value of collateral requirements, lenders will receive their entire expected returns.

Therefore, given a price process p and an allocation $(x^i, \theta^i, \varphi^i)$, the value of agent i's estate at $\mu \in \xi^+ \setminus D^i$, after the payments of the debts induced by his sales at $\xi \in \overline{D}^i$, is given by:

$$e_\mu\left(p, \left(x^i, \theta^i, \varphi^i\right)\right) \;\; := \;\; p(\mu) Y_\mu x^i(\xi) + R_\mu(p(\mu))\left(\theta^i(\xi) - \varphi^i(\xi)\right).$$

The first term of the right-hand side of the previous equality represents the depreciated value of agent i's consumption that served as collateral or not. The second term represents the net returns of his portfolios.

Now, to make wealth transfers consistent with the amount of estate, we suppose that when agent i writes a will at $\xi \in \overline{D}^i_\star$, the bequests $b^i(\mu) = (b^i_k(\mu); k \in I_{-i}(\xi))$, with $\mu \in \xi^+ \setminus D^i$, satisfy the following conditions:

$$(1) \qquad \sum_{k\in I_{-i}(\xi)} b_k^i(\mu) \quad \leqslant \quad e_\mu\left(p,\left(x^i,\theta^i,\varphi^i\right)\right),$$

$$(2) \qquad \alpha_k^i(\xi)\, e_\mu\left(p,\left(x^i,\theta^i,\varphi^i\right)\right) \quad \leqslant \quad b_k^i(\mu), \quad \forall k\in I_{-i}(\xi),$$

where $b_k^i(\mu)$ denotes the amount of wealth that agent k will receive, if he is alive, at node μ and $\alpha_k^i(\xi)\in[0,1]$ represents the forced shares or legitime, that is, the portion of his estate from which agent i cannot disinherit agent k.[6] Inequality (1) states that the total bequest made by an agent cannot exceed his estate. Inequality (2) conveys that the bequest that an agent k receives from agent i is greater than or equal to the minimal amount guaranteed by the forced shares.

As mentioned above, agents' preferences may be positively affected through their *bequest motives*, which reflect their altruism toward their descendants. More precisely, the objective function of agent i includes a function $G^i : \prod_{\xi\in\overline{D}^i_\star} \mathbb{R}_+^{(\xi^+\setminus D^i)\times I_{-i}(\xi)} \to \mathbb{R}_+\cup\{+\infty\}$, such that, given a commodity price process p, if he chooses a plan (x^i,θ^i,φ^i) and writes wills $b^i := (b^i_\xi;\xi\in\overline{D}^i_\star)$, he receives utility gains given by:

$$G^i\left(\frac{b^i(\mu)}{p(\mu)v(\mu)};\mu\in\xi^+\setminus D^i,\xi\in\overline{D}^i_\star\right),$$

where $v(\mu) := \big(v(\mu,g),\ g\in G\big)\in\mathbb{R}^G_{++}$, which is exogenously given, allows us to transform nominal bequests into real terms.

Inheritance laws. Given $\xi\in\overline{D}^i$, agent i's *intestate estate* at a node $\mu\in\xi^+\setminus D^i$ is defined as the amount of his estate that was not distributed after the payments of his debts and the delivery of his testamentary rights, i.e.

$$\tau^i_\mu\left(p,\left(x^i,\theta^i,\varphi^i,b^i\right)\right) = \begin{cases} e_\mu\left(p,\left(x^i,\theta^i,\varphi^i\right)\right), & \text{if } \xi\in\overline{D}^i\setminus\overline{D}^i_\star; \\ e_\mu\left(p,\left(x^i,\theta^i,\varphi^i\right)\right) - \sum\limits_{k\in I(\mu)\cap I_{-i}(\xi)} b_k^i(\mu), & \text{if } \xi\in\overline{D}^i_\star. \end{cases}$$

More precisely, if agent i does not make a will at the predecessor node ξ, his intestate estate at $\mu\in\xi^+\setminus D^i$ is equal to the depreciated value of his wealth. Nevertheless, when agent i writes a will at ξ, his intestate estate is equal to his depreciated wealth net of the bequests that alive beneficiaries receive.

In order to avoid the disappearance of these resources from the economy and to protect agents from their (selfish) parents, we introduce a structure of *inheritance laws*. Formally, for each $i\in I$,

[6] In civil and Roman law, the legitime, or forced share, of a decedent's estate is that portion of the estate from which he cannot disinherit his children or his wife, for instance, without sufficient legal cause. The word comes from French *héritier légitime*, meaning rightful heir. Some countries adopt this system to protect the inherence of the legitime wife.

the civil law jurisdictions on inheritance determine the rights that agents $k \in I(\mu)$ have over agent i's intestate estate at each $\mu \in \xi^+ \setminus D^i$, where $\xi \in \overline{D}^i$. These rights are given by a vector of shares $\left(\beta^i_k(\mu);\, k \in I(\mu)\right) \in \Delta^{n(\mu)}$.[7] Thus, at each $\mu \in \xi^+ \setminus D^i$, the set, $\{k \in I(\mu) :\, \beta^i_k(\mu) > 0\}$, of agent i's heirs is nonempty.

Donations. We allow agents to make intra-generational transfers through donations. An individual can make gifts either for altruistic motives or to disinherit agents who will have, by law, rights over his estate when he dies.

Each agent i could be interested in making donations at nodes $\xi \in D^i_{-i}$, and has utility gains *only* when these transfers are received by agents in a set $I^\star_{-i}(\xi) \subset I_{-i}(\xi)$. In this way, we do not exclude agents who are uninterested in making donations, as some of the sets $(I^\star_{-i}(\xi);\, \xi \in D^i_{-i})$ may be empty. In order to simplify notations we define $I^\star_{-i}(\xi)$ as the empty set for nodes in $D^i \setminus D^i_{-i}$.

In order to avoid that an individual receives back his donations through a chain of wealth transfers; we assume that, at each $\xi \in D$ and for each $n \in \mathbb{N}$, given a chain $(i_1, \ldots, i_n) \in I(\xi)^n$:

$$\text{if } i_j \in I^\star_{-i_{j+1}}(\xi),\ \forall j \in \{1, \ldots, n-1\},\ \text{then } i_1 \neq i_n.$$

Note that, the condition above is introduced to bound the amount of donations in equilibrium.

Now, when $I^\star_{-i}(\xi) \neq \emptyset$, agent i can transfer his wealth to other individuals choosing a vector of nominal donations $d^i(\xi) := \left(d^i_k(\xi); k \in I^\star_{-i}(\xi)\right) \in \mathbb{R}_+^{I^\star_{-i}(\xi)}$, where $d^i_k(\xi)$ denotes the wealth that agent k receives from agent i at ξ. Let $d^i := \left(d^i(\xi);\, \xi \in D^i\,,\ I^\star_{-i}(\xi) \neq \emptyset\right)$ be the agent i's donation plan.

Agent i's *gains* from donations are measured by a function

$$F^i : \prod_{\{\xi \in D^i\,,\ I^\star_{-i}(\xi) \neq \emptyset\}} \mathbb{R}_+^{I^\star_{-i}(\xi)} \to \mathbb{R}_+ \cup \{+\infty\},$$

in such a form that, given a commodity price process p, his objective function, which depends on his consumption plan and bequest motive, also includes a term $F^i\left(\frac{d^i(\xi)}{p(\xi)v(\xi)}; \xi \in D^i,\, I^\star_{-i}(\xi) \neq \emptyset\right)$, that depends on the *real* amount of transfers.

Although each alive agent may know the identity of the other agents in the markets, individual allocations are anonymous. Therefore, agents are unaware of the donations they may receive, as well as their rights over the estate of deceased agents and the value of the associated intestate estate. For this reason, we need to introduce variables representing the expected nominal transfers that agents anticipate to receive.

[7] The simplex $\Delta^n := \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+ : \sum\limits_{i=1}^{n} z_i = 1\}$.

More precisely, we suppose that each individual $i \in I$ takes as given, at each $\xi \in D^i$, an *anonymous nominal transfer* $s^i(\xi) \in \mathbb{R}_+$, representing the amount of wealth that he expects to receive as donations or as inheritances through wills or via civil law jurisdictions. This variable will be determined endogenously in equilibrium. Agent i's vector of nominal transfers is denoted by $s^i := (s^i(\xi); \xi \in D^i)$.

Before defining the budget sets and the equilibrium of our model, let us denote agent i's set of admissible plans by Γ^i, and the space of prices by $\mathbb{P}$.[8]

Definition 1. Given prices and anonymous nominal transfers $\left(p, q, s^i\right) \in \mathbb{P} \times \mathbb{R}_+^{D^i}$, the *budget set*, $B^i(p, q, s^i)$, of an agent $i \in I$, is the set of plans $(x^i, \theta^i, \varphi^i, d^i, b^i) \in \Gamma^i$ that satisfy the following constraints:

• At each initial node $\xi \in \underline{D}^i := \left\{\eta \in D^i \setminus \{\xi_0\} : \eta^- \notin D^i\right\} \cup \left(\{\xi_0\} \cap D^i\right)$,

$$p(\xi)x^i(\xi) + q(\xi)\left(\theta^i(\xi) - \varphi^i(\xi)\right) + \sum_{k \in I^\star_{-i}(\xi)} d^i_k(\xi) \leqslant p(\xi)w^i(\xi) + s^i(\xi)\,;$$

• At each $\xi \in D^i \setminus \underline{D}^i$,

$$\begin{aligned} p(\xi)x^i(\xi) + q(\xi)\left(\theta^i(\xi) - \varphi^i(\xi)\right) &+ \sum_{k \in I^\star_{-i}(\xi)} d^i_k(\xi) \\ &\leqslant p(\xi)\left(w^i(\xi) + Y_\xi x^i(\xi^-)\right) + s^i(\xi) + R_\xi(p(\xi))\left(\theta^i(\xi^-) - \varphi^i(\xi^-)\right); \end{aligned}$$

• At each $\xi \in D^i$,

$$x^i(\xi) \geqslant \sum_{j \in J(\xi)} C(\xi, j)\varphi^i(\xi, j).$$

• Given $\xi \in \overline{D}^i_\star$, for each $\mu \in \xi^+ \setminus D^i$,

$$\begin{aligned} \sum_{k \in I_{-i}(\xi)} b^i_k(\mu) &\leqslant e_\mu\left(p, \left(x^i, \theta^i, \varphi^i\right)\right), \\ \alpha^i_k(\xi)\, e_\mu\left(p, \left(x^i, \theta^i, \varphi^i\right)\right) &\leqslant b^i_k(\mu), \quad \forall k \in I_{-i}(\xi). \end{aligned}$$

Definition 2. An *equilibrium* of our economy is given by a plan of prices and anonymous transfers $\left[(\overline{p}, \overline{q}); (\overline{s}^i)_{i \in I}\right]$, jointly with allocations $\left(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{d}^i, \overline{b}^i\right)_{i \in I}$ in $\Gamma := \prod_{i \in I} \Gamma^i$ such that:

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[8] It follows from the previous definitions that,

$$\Gamma^i := \mathbb{R}_+^{G \times D^i} \times \prod_{\xi \in D^i} \mathbb{R}_+^{J(\xi)} \times \prod_{\xi \in D^i} \mathbb{R}_+^{J(\xi)} \times \prod_{\{\xi \in D^i :\, I^\star_{-i}(\xi) \neq \emptyset\}} \mathbb{R}_+^{I^\star_{-i}(\xi)} \times \prod_{\xi \in \overline{D}^i_\star} \mathbb{R}_+^{(\xi^+ \setminus D^i) \times I_{-i}(\xi)}.$$

In addition, $\mathbb{P} := \prod_{\xi \in D} \left(\mathbb{R}^G_+ \setminus \{0\}\right) \times \mathbb{R}_+^{D(J)}$.

(i) For each $i \in I$, the allocation $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{d}^i, \overline{b}^i)$ maximizes the objective function,

$$V^i\Big(\overline{p}, (x^i, d^i, b^i)\Big) := U^i(x^i) + F^i\left(\frac{d^i(\xi)}{\overline{p}(\xi)v(\xi)}; \xi \in D^i,\ I^\star_{-i}(\xi) \neq \emptyset\right) + G^i\left(\frac{b^i(\mu)}{\overline{p}(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i, \xi \in \overline{D}^i_\star\right),$$

over the plans $(x^i, \theta^i, \varphi^i, d^i, b^i) \in B^i(\overline{p}, \overline{q}, \overline{s}^i)$.

(ii) In financial and physical markets, the aggregate demand must be equal to the aggregate supply, node by node. That is, for each $\xi \in D$,

$$\sum_{i \in I(\xi)} \overline{\theta}^i(\xi) = \sum_{i \in I(\xi)} \overline{\varphi}^i(\xi)\,, \qquad \sum_{i \in I(\xi)} \overline{x}^i(\xi) = \sum_{i \in I(\xi)} w^i(\xi) + \sum_{i \in I(\xi^-)} Y_\xi \overline{x}^i(\xi^-),$$

where $I(\xi_0^-) = \emptyset$.

(iii) For each agent $i \in I$, and at each $\xi \in D^i$, expected anonymous transfers must match the effective transfers that he receives. That is,

$$\overline{s}^i(\xi) = \sum_{\{k \in I(\xi):\ i \in I^\star_{-k}(\xi)\}} \overline{d}^k_i(\xi) + \sum_{\{k \in I(\xi^-)\setminus I(\xi):\ \xi^- \in \overline{D}^k_\star\}} \overline{b}^k_i(\xi) + \sum_{k \in I(\xi^-)\setminus I(\xi)} \beta^k_i(\xi)\, \tau^k_\xi\left(\overline{p}, (\overline{x}^k, \overline{\theta}^k, \overline{\varphi}^k, \overline{b}^k)\right).$$

Note that we focus on default and bequest to stress an important role that collateralized assets play when lifetimes are uncertain in a *rational expectation setting*, where agents perfectly foresee prices and monetary transfers. On one hand, when economies are stationary, this hypothesis is not too demanding, as each agent can, through past experience, develop a reliable intuition and form correct expectations about these endogenous variables. On the other hand, in the non-stationary case, which is allowed in our framework, the perfect foresight assumption appears as restrictive, at least from an *intuitive* point of view. In a critical approach to rational expectations, Daher, Martins da Rocha, Páscoa and Vailakis (2006) prove that, when default is penalized through collateral repossession and utility penalties, solvency problems associated to temporary equilibrium models disappear, without restricting agents' believes about future variables. It follows from this result that, in two–period models, when assets are collateralized, it is not necessary to assume that individuals perfectly foresee future variables. Future research in the direction of Daher, Martins da Rocha, Páscoa and Vailakis (2006) might lead to a refinement of the perfect foresight assumption in OLG models, particulary, when assets are collateralized and lifetimes are uncertain.

3. Equilibrium existence

Our first result assures the existence of an equilibrium when all agents are *finitely–lived*. It extends the classical equilibrium existence results in Overlapping Generations models to allow for stochastic lifetimes, wealth transfers, default and, fundamentally, the access to credit markets at all states of the life span.[9]

Theorem 1. *Suppose that all agents are finitely–lived and that:*

[A1] *For each agent i and for each $\xi \in D^i$, the accumulated endowments $W^i(\xi) \in \mathbb{R}^G_{++}$. The forced-shares satisfy* $\sum_{k \in I_{-i}(\mu)} \alpha^i_k(\mu) < 1,\ \forall \mu \in \overline{D}^i_\star,\ \forall i \in I.$

[A2] *For each $(\xi, n) \in D \times \mathbb{N}$, if $(i_1, \ldots, i_n) \in I(\xi)^n$ satisfies $i_j \in I^\star_{-i_{j+1}}(\xi)$, for each $j \in \{1, \ldots, n-1\}$, then $i_1 \neq i_n$;*

[A3] *For each $\xi \in D$, unitary collateral requirements $C(\xi, j) \neq 0$, for all $j \in J(\xi)$;*

[A4] *For each agent $i \in I$, functions U^i, F^i and G^i have finite values, are concave and continuous in all variables. Moreover, the function U^i is strictly increasing in all variables and the function G^i is nondecreasing.*

Then, there is an equilibrium.

Proof. See Appendix A.

The first assumption is required to guarantee that budget-set correspondences are lower hemi-continuous.[10] Assumption [A2] rules out donation cycles and hypothesis [A3] assures that short

[9]The existing literature of OLG models in General Equilibrium Theory assumes that: (i) default is not allowed, (ii) lifetimes do not depend on the states of the nature and (iii) agents cannot have access to the financial markets at the last period of their lifetime. Under these conditions, Florenzano, Gourdel and Pascoa (2001) show the equilibrium existence with perishable goods and real assets. Therefore, they had to impose, a priori, bounds on the short–sales. Recently, Seghir (2006) proves the existence of equilibrium for an OLG model with numeraire assets and perishable goods.

[10]The requirement on forced shares made in Assumption [A1] can be relaxed to: $\sum_{k \in I_{-i}(\mu)} \alpha^i_k(\mu) \leqslant 1$, for each $i \in I$ and for any $\mu \in \overline{D}^i_\star$. Note that, in this case, agent i cannot make any choice on his bequests at the nodes μ in which $\sum_{k \in I_{-i}(\mu)} \alpha^i_k(\mu) = 1$, because he is constrained to choose bequests $b^i_k(\eta) = \alpha^i_k(\mu) e^i_\eta(p, (x^i, \theta^i, \varphi^i))$, where $\eta \in \mu^+ \setminus D^i$ and $k \in I_{-i}(\mu)$. However, we can (i) redefine agent' i budget set, avoiding bequests at node μ as variables that can be chosen by agent i, and (ii) redefine agent i's objective function taking into account this change, i.e. replacing $b^i_k(\eta)$ by $\alpha^i_k(\mu) e^i_\eta(p, (x^i, \theta^i, \varphi^i))$. With these changes we can apply the same technique that we will develop to prove equilibrium when forced shares do not exhaust the entire agents estate.

Now, legal systems usually allow agents to have a free percentage of their estate, that can be allocated to any individual that is alive, at the moment in which bequests are determined. Thus, to shorten notations, avoiding sales will be bounded in equilibrium, as physical resources are scarce. Finally, Assumption [A4] is required to assure that consumers' maximization problems have a solution.

Given a set A, let $\ell^{\infty}_{+}(\mathbb{R}^A) = \{x = (x(a); a \in A) \in \mathbb{R}^A_+ : \max_{a\in A} x(a) < +\infty\}$. We refer to an element x in $\ell^{\infty}_{+}(\mathbb{R}^A)$ as a *bounded plan.*

Our next result shows that, when agents have at least one infinite–life path through the event–tree, Ponzi schemes can be ruled out in our economy, without need to impose any exogenous debt constraint. In particular, we extend the equilibrium existence result of Araujo, Páscoa and Torres-Martínez (2002) to a model with uncertain lifetimes and wealth transfers.

Theorem 2. *Suppose that Assumptions [A1]-[A3] hold and that,*

[A5] *The sequence* $\left(n(\xi), \sum_{i\in I(\xi)} W^i(\xi), \sum_{j\in J(\xi)} C(\xi,j)\right)_{\xi\in D}$ *belongs to* $\ell^{\infty}_{+}(\mathbb{R}^D \times \mathbb{R}^{D\times G} \times \mathbb{R}^{D\times G})$. *There is* $\underline{v} > 0$ *such that, for each* $(\xi,g) \in D \times G$, $v(\xi,g) \geqslant \underline{v}$.

[A6] *For each* $i \in I$, *the utility function* $U^i : \mathbb{R}^{G\times D^i}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ *is separable in time and states of nature, in the sense that,*

$$U^i(x^i) = \sum_{\xi\in D^i} u^i(\xi, x^i(\xi)),$$

where $u^i(\xi,\cdot) : \mathbb{R}^G_+ \to \mathbb{R}_+$ *is continuous, strictly increasing and concave. Moreover, for each plan* $x^i \in \ell^{\infty}_{+}(\mathbb{R}^{G\times D^i})$, *the associated utility,* $U^i(x^i)$, *is finite.*

[A7] *For each* $i \in I$,

$$F^i\left(\frac{d^i(\xi)}{p(\xi)v(\xi)}; \xi \in D^i,\, I^{\star}_{-i}(\xi) \neq \emptyset\right) = \sum_{\{\xi\in D^i:\, I^{\star}_{-i}(\xi)\neq\emptyset\}} f^i\left(\xi, \frac{d^i(\xi)}{p(\xi)v(\xi)}\right),$$

where $f^i(\xi,\cdot) : \mathbb{R}^{I^{\star}_{-i}(\xi)}_+ \to \mathbb{R}_+$ *is continuous, non-decreasing and concave. Moreover,* F^i *has a finite value at any bounded plan.*

[A8] *For each agent* i,

$$G^i\left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i, \xi \in \overline{D}^i_{\star}\right) = \sum_{\xi\in\overline{D}^i_{\star}} g^i\left(\xi, \left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i\right)\right),$$

redefinitions of budget sets and objective functions, we will maintain along the paper the restriction on forced shares given by Assumption [A1].

where $g^i(\xi,\cdot):\mathbb{R}_+^{(\xi^+\setminus D^i)\times I_{-i}(\xi)}\to\mathbb{R}_+$ *is continuous, non-decreasing and concave. In addition, gains from bequest are finite at any bounded plan.*

Then, there is an equilibrium.

Proof. See Appendix B.

Remark 1. *Non-arbitrage conditions and Ponzi schemes.*

When promises are backed by physical collateral, short–sales are endogenously bounded, node by node, and this is sufficient to assure equilibrium existence when agents are finitely–lived.

Nevertheless, when agents are infinitely-lived, they could enter into Ponzi schemes by increasing sequentially their loans and postponing *ad-eternum* the payment of their debts. However, as in Araujo, Páscoa and Torres-Martínez (2002), the "haircut", $p(\xi)C(\xi,j)-q(\xi,j)$, will be strictly positive for each (ξ,j).[11] Thus, given prices (p,q) and nominal transfers $(s^i)_{i\in I}$, each market feasible allocation $(x^i,\theta^i,\varphi^i,d^i,b^i)_{i\in I}$ satisfies:

$$q(\xi)\varphi^i(\xi)\leqslant p(\xi)\sum_{j\in J(\xi)}C(\xi,j)\,\varphi^i(\xi,j)\leqslant p(\xi)\sum_{i\in I(\xi)}x^i(\xi).$$

Therefore, under Assumption [A5] there is $\overline{W}=(\overline{W}(g);g\in G)\in\mathbb{R}^G_{++}$ such that, any market feasible allocation satisfies the following *endogenous* debt constraint:

$$\frac{q(\xi)\varphi^i(\xi)}{\sum_{g\in G}p(\xi,g)}\leqslant\max_{g\in G}\overline{W}(g),$$

which rules out schemes consisting of a sequential increase, *ad infinitum*, of the debt without repayment.

4. On bequest and wills

In this section, we give some simple examples of bequest functions that allow us to find optimal testamentary transfers as a function of the amount of agents' estate.

We will use the following property to find optimal bequests: Given equilibrium prices and nominal transfers $(\overline{p},\overline{q},\overline{s}^i)$, an allocation $\left(\overline{x}^i,\overline{\theta}^i,\overline{\varphi}^i,\overline{d}^i,\overline{b}^i\right)\in\Gamma^i$ is optimal for agent i only if:

$$G^i\left(\frac{\overline{b}^i(\mu)}{\overline{p}(\mu)v(\mu)};\mu\in\xi^+\setminus D^i,\xi\in\overline{D}^i_\star\right)\geqslant G^i\left(\frac{b^i(\mu)}{\overline{p}(\mu)v(\mu)};\mu\in\xi^+\setminus D^i,\xi\in\overline{D}^i_\star\right),$$

[11] In fact, the returns of the joint financial operation of short-selling an asset and constituting the required collateral are non-negative, since borrowers will pay (or the market will deliver to lenders) only the minimum between the value of the depreciated collateral and the value of the debt. Therefore, as borrowers hold and consume the collateral bundles, individual optimality assures that $p(\xi)C(\xi,j)-q(\xi,j)>0$, for each $(\xi,j)\in D(J)$ (see Lemma 2 in Appendix A).

for all vectors $b^i \in \prod_{\xi \in \overline{D}^i_\star} \mathbb{R}_+^{(\xi^+ \setminus D^i) \times I_{-i}(\xi)}$ such that, for each $\xi \in \overline{D}^i_\star$ and $\mu \in \xi^+ \setminus D^i$:

$$\begin{aligned} \sum_{k \in I_{-i}(\xi)} b^i_k(\mu) \quad &\leqslant \quad e_\mu\left(\overline{p}, \left(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i\right)\right), \\ \alpha^i_k(\xi)\, e_\mu\left(\overline{p}, \left(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i\right)\right) \quad &\leqslant \quad b^i_k(\mu), \quad \forall k \in I_{-i}(\xi). \end{aligned}$$

In order to simplify the examples below, we suppose that agent i is not forced by law to deliver a minimum percentage of his wealth to another agent (i.e.: $\alpha^i_k(\xi) = 0$ for all $k \in I_{-i}(\xi)$ and $\xi \in \overline{D}^i$). Moreover, agent i's objective function is given by:

$$(3) \quad V^i(p, (x^i, d^i, b^i)) = \sum_{\xi \in D^i} \beta^{t(\xi)} \rho^i(\xi) u^i(x^i(\xi)) + \sum_{\{\xi \in D^i :\, I^\star_{-i}(\xi) \neq \emptyset\}} \beta^{t(\xi)} \rho^i(\xi) f^i_\xi\left(\frac{d^i(\xi)}{p(\xi)v(\xi)}\right)$$

$$+ \sum_{\xi \in \overline{D}^i_\star} \beta^{t(\xi)} \eta^i(\xi) g^i_\xi\left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i\right),$$

where $\beta \in (0,1)$ is a discount factor, $\rho^i(\xi) \in (0,1)$ and $\eta^i(\xi) = \rho^i(\xi) - \sum_{\mu \in \xi^+ \cap D^i} \rho^i(\mu)$. Also, agent i expects to be alive at ξ with probability $\rho^i(\xi)$. Thus, the parameter $\eta^i(\xi) \in (0,1)$ represents the probability of reaching node $\xi \in \overline{D}^i$ and passing away in the next period. We suppose that $\rho^i(\xi) = 1$, for each $\xi \in \underline{D}^i$,

$$\rho^i(\xi) = \sum_{\mu \in \xi^+} \rho^i(\mu), \quad \forall \xi \in D^i \setminus \overline{D}^i, \qquad \text{and} \qquad \rho^i(\xi) > \sum_{\mu \in \xi^+} \rho^i(\mu), \quad \forall \xi \in \overline{D}^i.$$

Example 1. *For each $\xi \in \overline{D}^i_\star$, fix an agent $k(\xi) \in I_{-i}(\xi)$ and a scalar $A(\xi) > 0$. If bequest functions are given by:*

$$g^i_\xi\left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i\right) := \sum_{\mu \in \xi^+ \setminus D^i} \left[2 \min\left\{A(\xi), \frac{b^i_{k(\xi)}(\mu)}{p(\mu)v(\mu)}\right\} + \min_{\{k \in I_{-i}(\xi) :\, k \neq k(\xi)\}} \frac{b^i_k(\mu)}{p(\mu)v(\mu)}\right],$$

then, equilibrium bequests depend on the value of the future estate.

In fact, given $\xi \in \overline{D}^i_\star$, at the nodes $\mu \in \xi^+ \setminus D^i$ in which the real value of his estate is less than or equal to $A(\xi)$, agent i bequeaths all of his estate to agent $k(\xi)$. On the other hand, when the real value of agent i's estate is greater than $A(\xi)$ at μ, agent $k(\xi)$ receives, if alive, a real bequest equal to $A(\xi)$, while the other individuals are entitled to receive the same real transfer, $\frac{1}{n(\xi)-1}\left(\frac{e_\mu(p,(x^i,\theta^i,\varphi^i))}{p(\mu)v(\mu)} - A(\xi)\right)$.

The following examples show that, when bequest functions take into account only the distribution of wealth, optimal amounts of bequest can be found as *fixed shares* of agents' estate.

Example 2. *For each $\xi \in \overline{D}^i_\star$, let us fix a vector $\pi(\xi) = (\pi(\xi,k); k \in I_{-i}(\xi)) \in \mathbb{R}^{I_{-i}(\xi))}_+ \setminus \{0\}$. Let $\Lambda(\xi) = \{k \in I_{-i}(\xi) : \pi(\xi,k) > 0\}$ and consider the following bequest function:*

$$g^i_\xi\left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i\right) := \sum_{\mu\in\xi^+\setminus D^i} \min_{k\in\Lambda(\xi)} \pi(\xi,k)\,\frac{b^i_k(\mu)}{p(\mu)v(\mu)}.$$

Then, in equilibrium, agent i writes a will that gives to agent $k \in \Lambda(\xi)$, at any node $\mu \in \xi^+ \setminus D^i$, the following share of his estate:

$$a^i_k(\xi) = \frac{1}{\pi(\xi,k) \sum\limits_{k'\in\Lambda(\xi)} \pi(\xi,k')^{-1}}.$$

In this case, the total intestate estate, at $\mu \in \xi^+ \setminus D^i$, is equal to a $\sum\limits_{k\in\Lambda(\xi)\setminus I(\mu)} a^i_k(\xi)$ percent of the estate value, $e_\mu(p,(x^i,\theta^i,\varphi^i))$. In the particular case in which, for all pairs $(k,k') \in \Lambda(\xi)\times\Lambda(\xi)$, $\pi(\xi,k) = \pi(\xi,k')$, all agents in $I(\mu)\cap\Lambda(\xi)$ receive, as testamentary rights at node μ, the same percentage, $\frac{1}{\#\Lambda(\xi)}$, of agent i's estate. In this case, the intestate estate at μ is equal to $\left(1 - \frac{\#(\Lambda(\xi)\cap I(\mu))}{\#\Lambda(\xi)}\right)$ percent of agent i's wealth at μ.

Example 3. *With the notations of the previous example, consider the following bequest function:*

$$g^i_\xi\left(\frac{b^i(\mu)}{p(\mu)v(\mu)}; \mu \in \xi^+ \setminus D^i\right) := \sum_{\mu\in\xi^+\setminus D^i} \sum_{k\in\Lambda(\xi)} \pi(\xi,k)\,\frac{b^i_k(\mu)}{p(\mu)v(\mu)}.$$

Then, an agent $k \in \Lambda(\xi)$ will receive a bequest from agent i, at $\mu \in \xi^+ \setminus D^i$, only if $\pi(\xi,k) = \overline{\pi}(\xi)$, where $\overline{\pi}(\xi) = \max\limits_{k\in\Lambda(\xi)} \pi(\xi,k)$. In addition, when agent i write a will, he is indifferent between all distributions of his estate among the agents $k \in \Lambda(\xi)$ for which $\pi(\xi,k) = \overline{\pi}(\xi)$. Thus, if there is a unique agent $k \in \Lambda(\xi)$ such that $\pi(\xi,k) = \overline{\pi}(\xi)$, the whole estate of i is received by k, at the nodes $\mu \in \xi^+ \setminus D^i$ in which k is alive.

5. About the equilibrium level of donations

In this section, we briefly comment on the optimality of the equilibrium level of donations. To simplify our analysis, we assume that the objective functions have the functional form given by equation (3) and that the functions f^i_ξ are concave and continuous. In addition, as in Theorem 2, the functions $(u^i; i \in I)$ are supposed to be continuous, concave and strictly increasing in all variables.

In equilibrium, the optimality of agent i's allocation assures that, for each node $\xi \in D^i$, with $I^{\star}_{-i}(\xi) \neq \emptyset$, there is a *strictly positive* Kuhn-Tucker multiplier γ^i_ξ and super-gradients[12]

$$(u'(i,\xi), f'(i,\xi)) \in \partial u^i(\overline{x}^i(\xi)) \times \partial f^i_\xi \left(\frac{\overline{d}^i(\xi)}{\overline{p}(\xi)v(\xi)} \right),$$

such that:

$$\gamma^i_\xi\, \overline{p}(\xi) \geqslant \beta^{t(\xi)}\rho(\xi) u'(i,\xi) \qquad \text{and} \qquad \gamma^i_\xi\, \overline{d}^i_k(\xi) = \frac{\beta^{t(\xi)}\rho(\xi)}{\overline{p}(\xi)v(\xi)} f'_k(i,\xi)\, \overline{d}^i_k(\xi), \quad \forall k \in I^{\star}_{-i}(\xi),$$

where $f'(i,\xi) = (f'_k(i,\xi); k \in I^{\star}_{-i}(\xi))$.

Thus, when agent i makes donations to k (i.e. $\overline{d}^i_k(\xi) > 0$), we have that $f'_k(i,\xi) \neq 0$, as the functions u^i are strictly monotonic. Furthermore, if we suppose that $\overline{d}^i(\xi) \gg 0$ and that f^i_ξ is differentiable on the interior of its domain, then there is always a vector $\tilde{d}^i(\xi) \gg 0$ such that $f^i_\xi \left(\frac{\overline{d}^i(\xi)}{\overline{p}(\xi)v(\xi)} \right) < f^i_\xi \left(\frac{\tilde{d}^i(\xi)}{\overline{p}(\xi)v(\xi)} \right)$. In fact, differentiability assures that $\nabla f^i_\xi \left(\frac{\overline{d}^i(\xi)}{\overline{p}(\xi)v(\xi)} \right) = f'(i,\xi) \neq 0$.

Therefore, as a consequence of the tradeoff between consumption and altruism, although individual plans, $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{d}^i)$, are (globally) optimal, the equilibrium level of donations can be, in many cases, sub-optimal.

6. A final remark on altruistic behavior

In our model, agents may care about their descendants and, therefore, they may be interested in accumulating wealth in order to leave bequests to their offsprings. On the other hand, an individual who cares about his parents could make donations to them during his lifetime. Of course, when receiving bequests, descendants do not have any incentive to pay ancestors' debts. In fact, it is not realistic, from a pure economic point of view, to assume that descendants are urged to pay the debts of their antecedents.

However, non-economic motives may lead to altruism toward ancestors. In this case, when receiving bequests, agents may be interested in paying more than the minimum between the value of the depreciated collateral and their antecedents' debt.

If ancestors do not perfectly foresee the attitude of their descendants, the collateral cost will still be greater than the asset price. Thus, short sales will be bounded and, even when agents have at least one infinite–life path through the event–tree, Ponzi schemes are ruled out.

[12]Given a concave function $u : \mathbb{R}^n_+ \to \mathbb{R}_+$, the *super-differential* of u at $x \in \mathbb{R}^n_+$ is defined by:

$$\partial u(x) = \{p \in \mathbb{R}^n : u(y) - u(x) \leqslant p(y-x), \ \forall y \in \mathbb{R}^n_+\}.$$

Any element of the super-differential set is called a *super-gradient* of u at x. When the function u is differentiable at x, we have that $\partial u(x) = \{\nabla u(x)\}$.

Nevertheless, if agents perfectly foresee that their descendants have incentives to pay more than the minimum between the value of the original promise and the value of the depreciated collateral, then, unlike our model, loans may be greater than collateral costs (as the joint operation of selling an asset and constituting the required collateral will no longer have nonnegative returns). In such a case, individuals' degree of altruism toward their ancestors may act as utility penalties for default, and infinitely–lived agents may end up doing Ponzi schemes. In fact, in a recent work, Pascoa and Seghir (2006) show that Ponzi schemes become possible in the presence of collateral and harsh utility penalties, as borrowers may pay more than the value of the depreciated collateral and the value of their debt.

However, in a model with collateralized assets, in which each agent perfectly foresees the altruistic behavior that his descendants have toward him when he passes away, equilibrium may still exist. Indeed, by analogy with the results of Páscoa and Seghir (2006), we presume that, for an equilibrium to exist, it is sufficient that some (infinitely–lived) agents are not too altruistic toward their antecedents, but the existence argument will have to be carefully redone.

APPENDIX A. PROOF OF THEOREM 1.

Theorem 1 will be proved using a generalized game approach.

Without loss of generality, we assume that, at each $\xi \in D$, if $J(\xi) \neq \emptyset$, then for each $j \in J(\xi)$, there is at least one node $\mu \in \xi^+$ such that $\min\{\|A(\mu, j)\|_\Sigma \,; \|Y_\mu C(\xi, j)\|_\Sigma\} > 0$.[13]

Now, given a vector $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, with $m \geqslant 1$, we will denote by $\|z\|_{\max} := \max_{r\in\{1,\ldots,m\}} |z_r|$, the max-norm of z. The norm of the sum will be denoted by $\|z\|_\Sigma := \sum_{r=1}^{m} |z_r|$.

The following lemma provides a characterization of agents in terms of their donation motives and will be used to prove that individual allocations and prices are bounded in equilibrium.

Lemma 1. *Under Assumption [A2], for each node $\xi \in D$,*

$$I(\xi) = \bigcup_{r=1}^{n(\xi)} I^r(\xi),$$

where the collection of disjoint sets $\{I^r(\xi) : 1 \leqslant r \leqslant n(\xi)\}$ is defined, recursively, via,

$$\begin{aligned} I^1(\xi) &= \{i \in I(\xi) : i \notin I^\star_{-k}(\xi),\ \forall k \in I_{-i}(\xi)\}; \\ I^r(\xi) &= \left\{ i \in I(\xi) : i \in I^\star_{-k}(\xi) \ \Rightarrow k \in \bigcup_{r'<r} I^{r'}(\xi) \right\} \setminus \bigcup_{r'<r} I^{r'}(\xi), \quad \forall r > 1. \end{aligned}$$

[13]In fact, otherwise, independently of the value of commodity prices, asset j delivers no payments at equilibrium, and therefore, either $\overline{q}(\xi, j) = 0$ or $\overline{\theta}^i(\xi, j) = 0$, for each $i \in I(\xi)$. Thus, such an asset can be eliminated from the economy, without changing the space of financial transfers.

Moreover, the set, $I^1(\xi)$, of agents who do not receive donations at ξ (independently of the price level), is non-empty.

Proof. By definition $\bigcup_{r=1}^{n(\xi)} I^r(\xi) \subset I(\xi)$. Thus, let us suppose that there is $i_1 \in I(\xi)$ such that $i_1 \notin \bigcup_{r=1}^{n(\xi)} I^r(\xi)$. Since $i_1 \notin I^1(\xi)$, the set of agents $i \in I_{-i_1}(\xi)$, with $i_1 \in I^\star_{-i}(\xi)$, is non-empty. Moreover, there is an agent $i_2 \in I_{-i_1}(\xi)$ who satisfies both $i_1 \in I^\star_{-i_2}(\xi)$ and $i_2 \notin I^1(\xi)$, since otherwise, $i_1 \in I^2(\xi)$, which leads to a contradiction.

It follows that the set of agents $i \in I_{-i_2}(\xi)$ for which $i_2 \in I^\star_{-i}(\xi)$ is also non-empty. Therefore, by analogous arguments, there is $i_3 \in I_{-i_2}(\xi)$ such that both $i_2 \in I^\star_{-i_3}(\xi)$ and $i_3 \notin I^1(\xi)$. Moreover, by Assumption [A2], $i_3 \notin \{i_1, i_2\}$.

With this process, we can construct a family $\{i_1, \ldots, i_{n(\xi)+1}\}$ with $n(\xi)+1$ *different* agents that satisfies $i_j \in I^\star_{-i_{j+1}}(\xi)$, for each j, but this contradicts Assumption [A2], since $\#I(\xi) = n(\xi)$.

Finally, by construction, if $I^r(\xi) \neq \emptyset$, then $I^{r'}(\xi) \neq \emptyset$, for each $r' < r$. Therefore, as $\bigcup_{r=1}^{n(\xi)} I^r(\xi) = I(\xi) \neq \emptyset$, we conclude that $I^1(\xi)$ is a non-empty set. $\square$

The following lemma assures that individual allocations and prices are bounded in equilibrium.

Lemma 2. *Under assumptions [A3] and [A4], for each $\xi \in D$, there is a vector $(m(\xi), \Omega(\xi), M(\xi)) \gg 0$ such that given an equilibrium $\left[(\overline{p}, \overline{q}); (\overline{s}^i)_{i\in I}; \left(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{d}^i, \overline{b}^i\right)_{i\in I}\right]$ in which prices $\|(\overline{p}(\xi), \overline{q}(\xi))\|_\Sigma = 1$, at each $\xi \in D$, we have:*

(a) $\|\overline{p}(\xi)\|_\Sigma > m(\xi)$.
(b) *For each agent $i \in I(\xi)$, $\left\|\left(\overline{x}^i(\xi), \overline{\theta}^i(\xi), \overline{\varphi}^i(\xi), \overline{d}^i(\xi), \overline{b}^i_\xi\right)\right\|_{\max} < \Omega(\xi)$.*
(c) *For each $i \in I(\xi)$, $\overline{s}^i(\xi) < M(\xi)$.*

Proof. The arguments are similar to those made in Araujo, Páscoa and Torres-Martínez (2002) (Lemma 1, pp. 1621). Indeed, the joint operation of short selling an asset and purchasing the associated collateral yields to nonnegative returns. So, it follows from Assumption [A4] that the *financial haircut*, $\overline{p}(\xi)C(\xi, j) - \overline{q}(\xi, j)$, is strictly positive, for each $j \in J(\xi)$ (see Proposition 1, pp. 1624, in Araujo, Páscoa and Torres-Martínez (2002)).

On the other hand, as prices $(\overline{p}(\xi), \overline{q}(\xi))$ are in the simplex, we have that:

$$\sum_{j\in J(\xi)} \overline{p}(\xi)C(\xi, j) > \sum_{j\in J(\xi)} \overline{q}(\xi, j) = 1 - \|\overline{p}(\xi)\|_\Sigma.$$

For each node $\xi \in D$, let us define $\overline{C}(\xi) = \max_{g\in G} \sum_{j\in J(\xi)} C(\xi, j, g) > 0$. Then, it follows from the previous arguments that $\|\overline{p}(\xi)\|_\Sigma > m(\xi) := \frac{1}{1+\overline{C}(\xi)}$, $\forall \xi \in D$.

Moreover, as the feasibility conditions in item (ii) of Definition 2 hold, individual consumption bundles are bounded, node by node, by the aggregate resources. Thus, Assumption [A3] and collateral constraints guarantee that agents' short-sales are bounded, node by node. Long positions are bounded too, due to financial market feasibility in equilibrium. So, budgetary constraints and physical-financial feasibility conditions assure that bequests are bounded, node by node, as prices $(\overline{p}(\xi), \overline{q}(\xi)) \in \Delta^{\#G+\#J(\xi)}$.

It follows that, at each $\xi \in D$, nominal transfers received by agents in $I^1(\xi)$ are bounded. In fact, these agents do not receive donations from other individuals, bequests are bounded and feasibility condition (iii) of Definition 2 holds. Thus, nominal donations made by agents in $I^1(\xi)$ are also bounded, as $\overline{p}(\xi)C(\xi,j) - \overline{q}(\xi,j) > 0$.

Using recursive arguments, one can easily show that: (i) the nominal transfers received by agents in $I^r(\xi)$, $r > 1$, are bounded, because donations made by the agents in $\bigcup_{r'<r} I^{r'}(\xi)$ have an upper bound; (ii) donations made by agents in $I^r(\xi)$, $r > 1$ are bounded, as prices are in the simplex (their nominal transfers were previously bounded and $\overline{p}(\xi)C(\xi,j) - \overline{q}(\xi,j) > 0$).

Therefore, for each $\xi \in D$, there exist $\Omega(\xi) > 0$, such that, for any $i \in I(\xi)$,

$$\left\| \left(\overline{x}^i(\xi), \overline{\theta}^i(\xi), \overline{\varphi}^i(\xi), \overline{d}^i(\xi), \overline{b}^i_\xi \right) \right\|_{\max} < \Omega(\xi).$$

Now, for each $\xi\eta \in D$, there exists $M(\eta) > 0$ such that, for each $\left[(p,q); (s^i)_{i\in I}; \left(x^i, \theta^i, \varphi^i, d^i, b^i\right)_{i \in I}\right] \in \mathbb{P} \times \Gamma$, satisfying

$$\begin{aligned} \|(p(\xi), q(\xi))\|_\Sigma &= 1, \quad \forall \xi \in D; \\ \| \left(x^i(\xi), \theta^i(\xi), \varphi^i(\xi), d^i(\xi), b^i_\xi\right) \|_{\max} &< \Omega(\xi), \quad \forall \xi \in D^i, \forall i \in I; \end{aligned}$$

we have:

$$\sum_{\{k \in I(\eta):\ i \in I^\star_{-k}(\eta)\}} d^k_i(\eta) + \sum_{\{k \in I(\eta^-)\setminus I(\eta):\, \eta^- \in \overline{D}^k_\star\}} b^k_i(\eta) + \sum_{k \in I(\eta^-)\setminus I(\eta)} \beta^k_i(\eta)\, \tau^k_\eta \left(p, (x^k, \theta^k, \varphi^k, b^k)\right) < M(\eta).$$

Therefore, feasibility condition (iii) of Definition 2 implies that $\overline{s}^i(\eta) < M(\eta)$, for each $i \in I(\eta)$. $\square$

The game $\mathcal{G}$. In order to prove the equilibrium existence, we introduce a game and we show that (i) this game always has a (pure strategy) Cournot-Nash equilibrium and (ii) each Cournot-Nash equilibrium is an equilibrium for our economy.

The generalized game $\mathcal{G}$ that we consider is characterized by:

- <u>*A set of players.*</u> There is a countable set of players constituted by:

(i) The set of agents, $i \in I$, of the original economy,

(ii) A player $h(\xi)$ for each node $\xi \in D$,

(iii) A player $h(i,\xi)$ for each pair $(i,\xi) \in I(D) := \{(k,\eta) \in I \times D : \eta \in D^k\}$.

To shorten notations below, we denote the set of players by $H = I \cup H(D) \cup H(I(D))$, where the set $H(D) := \{h(\xi) : \xi \in D\}$ and $H(I(D)) := \{h(i,\xi) : (i,\xi) \in I(D)\}$.

• *Strategies.*

(i) For each player $h \in I$, the set of strategies, $\overline{\Gamma}^h$, is given by the collection of plans

$$(x^h(\xi), \theta^h(\xi), \varphi^h(\xi), d^h(\xi), b^h_\xi)_{\xi \in D^i} \in \Gamma^h$$

such that $\|(x^h(\xi), \theta^h(\xi), \varphi^h(\xi), d^h(\xi), b^h_\xi)\|_{\max} \leqslant \Omega(\xi)$, for each $\xi \in D^i$.

(ii) For $h = h(\xi)$, $\overline{\Gamma}^h = \{(p(\xi), q(\xi)) \in \Delta^{\#G + \#J(\xi)} : \|p(\xi)\|_\Sigma \geqslant m(\xi)\}$.

(iii) If $h = h(i,\xi)$, then $\overline{\Gamma}^h := \{s^i(\xi) \in \mathbb{R}_+ : s^i(\xi) \leqslant M(\xi)\}$.

For simplicity, let $\eta^h = (x^h, \theta^h, \varphi^h, d^h, b^h) \in \overline{\Gamma}^h$ be a generic vector of strategies for a player $h \in I$ and $\eta := (\eta^h; h \in I)$ a plan of strategies for the agents in I. Moreover, $(p,q) := ((p(\xi), q(\xi)); \xi \in D)$ will denote a generic plan of strategies for the players $h \in H(D)$ and $s := (s^i(\xi); (\xi, i) \in I(D))$ a plan of strategies for the players $h \in H(I(D))$.

Finally, let $\overline{\Gamma} = \prod_{h \in H} \overline{\Gamma}^h$ be the space of strategies of the game $\mathcal{G}$, in which a generic element is denoted by (p,q,s,η).

• *Admissible strategies.* The strategies that can be effectively chosen for a player $h \in H$ may depend on the actions taken by the other agents, through a correspondence of admissible strategies $\Phi^h : \overline{\Gamma}_{-h} \twoheadrightarrow \overline{\Gamma}^h$, where $\overline{\Gamma}_{-h} = \prod_{h' \neq h} \overline{\Gamma}^{h'}$. Thus, denoting by $(p,q,s,\eta)_{-h}$ a generic element of $\overline{\Gamma}_{-h}$, we suppose that:

(i) If $h = i \in I$, $\Phi^h[(p,q,s,\eta)_{-h}] = C^h(p,q,s^h)$, where $C^h(p,q,s^h)$ denotes the set of strategies $\eta^h \in \overline{\Gamma}^h$ that satisfy the budget set restrictions at nodes $\xi \in D^h$, at prices (p,q), given the nominal transfers $s^i := (s^i(\xi); \xi \in D^i)$ chosen by the players $h(i,\xi)$, with $\xi \in D^i$.

(ii) If $h \in H(D) \cup H(I(D))$, $\Phi^h[(p,q,s,\eta)_{-h}] = \overline{\Gamma}^h$.

• *Objective functions*. Each player $h \in H$ is also characterized by his objective function, denoted by $K^h : \overline{\Gamma}^h \times \overline{\Gamma}_{-h} \to \mathbb{R}_+$. We assume that:

(i) If $h = h(\xi) \in H(D)$ and $(\tilde{p}(\xi), \tilde{q}(\xi)) \in \overline{\Gamma}^h$, then

$$K^h((\tilde{p}(\xi), \tilde{q}(\xi)); (p,q,s,\eta)_{-h}) := \tilde{p}(\xi)\left(\sum_{i\in I(\xi)} (x^i(\xi) - w^i(\xi)) - Y_\xi \sum_{i \in I(\xi^-)} W^i(\xi^-)\right) + \tilde{q}(\xi) \sum_{i\in I(\xi)} \left(\theta^i(\xi) - \varphi^i(\xi)\right).$$

(ii) If $h = h(k,\xi) \in H(I(D))$ and $\tilde{s}^k(\xi) \in \overline{\Gamma}^h$, then

$$K^h(\tilde{s}^k(\xi); (p,q,s,\eta)_{-h}) := -\left[\tilde{s}^k(\xi) - \left(\sum_{\{i\in I(\xi):\, k \in I^\star_{-i}(\xi)\}} d^i_k(\xi) + \sum_{\{i \in I(\xi^-)\setminus I(\xi): \xi^- \in \overline{D}^i_\star\}} b^i_k(\xi)\right) - \sum_{i\in I(\xi^-)\setminus I(\xi)} \beta^i_k(\xi)\tau^i_\xi(p,(x^i,\theta^i,\varphi^i,b^i))\right]^2.$$

(iii) If $h = i \in I$ and $\tilde{\eta}^i = (\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{d}^i, \tilde{b}^i) \in \overline{\Gamma}^h$, then

$$K^h(\tilde{\eta}^i; (p,q,s,\eta)_{-h}) := V^i(p, (\tilde{x}^i, \tilde{d}^i, \tilde{b}^i)).$$

For each $h \in H$, we define the correspondence of *optimal strategies* as follows:

$$\Psi^h((p,q,s,\eta)_{-h}) := \text{Argmax}\left\{K^h(y;(p,q,s,\eta)_{-h}) : y \in \Phi^h((p,q,s,\eta)_{-h})\right\}.$$

Let $\Psi : \overline{\Gamma} \twoheadrightarrow \overline{\Gamma}$, be the correspondence defined by $\Psi(p,q,s,\eta) = \prod_{h\in H} \Psi^h((p,q,s,\eta)_{-h})$.

Definition 3. A Cournot-Nash equilibrium for the generalized game $\mathcal{G}$ is a plan of strategies $(\overline{p},\overline{q},\overline{s},\overline{\eta}) \in \overline{\Gamma}$ such that $(\overline{p},\overline{q},\overline{s},\overline{\eta}) \in \Psi(\overline{p},\overline{q},\overline{s},\overline{\eta})$.

Lemma 3. *Under assumptions [A1]-[A2] admissible correspondences, $(\Phi^h; h \in H)$, are continuous and compact-valued in the product topology. Moreover, these correspondences have non-empty and convex values.*

Proof. When $h \in H(D) \cup H(I(D))$, $\Phi^h((p,q,s,\eta)_{-h}) = \overline{\Gamma}^h$, for all $(p,q,s,\eta)_{-h} \in \overline{\Gamma}_{-h}$. So, it follows that the four properties stated in the lemma hold.

For each $h \in I$, Assumptions [A1] and the definition of the budget constraints assure that Φ^h has non-empty, compact and convex values. In addition, the upper-hemicontinuity of Φ^h follows from the fact that Φ^h has a closed graph and compact values.

Now, we define the *interior correspondence* $int(\Phi^h) : \overline{\Gamma}_{-h} \twoheadrightarrow \overline{\Gamma}^h$ as follows:

$$int(\Phi^h)((p,q,s,\eta)_{-h}) = int(C^h(p,q,s^h)),$$

where $int(C^h(p,q,s^h))$ is the set of allocations $\eta^h = (x^h, \theta^h, \varphi^h, d^h, b^h) \in \overline{\Gamma}^h$ that satisfy all the budget restrictions of agent h *as strict inequalities*.

The definition of our price space and Assumption [A1] imply that $int(\Phi^h)$ has non-empty values. Moreover, $int(\Phi^h)$ has open graph and, therefore, is lower-hemicontinuous. As Φ^h has convex values, one gets that Φ^h is lower hemicontinuous too (see Lemma 16.22 in Aliprantis and Border (1999)). $\square$

Lemma 4. *Under assumptions [A1], [A2] and [A4], there is a Cournot-Nash equilibrium for $\mathcal{G}$.*

Proof. It follows from Assumptions [A4] that each objective function in the game is continuous in all variables and quasi-concave with respect to its own strategy.

Furthermore, since the sets of strategies are compact and admissible correspondences, $(\Phi^h; h \in H)$, are continuous with non-empty, convex and compact-values, it follows from Berge's Maximum Theorem (see Theorem 16.31 in Aliprantis and Border (1999)) that, for each $h \in H$, the correspondence of optimal strategies, Ψ^h, is upper-hemicontinuous with non-empty, convex and compact values. Thus, the correspondence Ψ has non-empty and convex values.

Since $\forall h \in H$, Ψ^h has a closed graph, then Ψ is also closed in the product topology. Moreover, it follows from Tychonoff's Theorem (see Theorem 2.57 in Aliprantis and Border (1999)) that Ψ has compact values. Applying Kakutani's Fixed Point Theorem to Ψ, we conclude the proof. $\square$

Finally, Theorem 1 is a direct consequence of the following result:

Lemma 5. *Under assumptions [A1]-[A4], a Cournot-Nash equilibrium for $\mathcal{G}$ is an equilibrium of our original economy.*

Proof. Let us fix a Cournot-Nash equilibrium $(\overline{p}, \overline{q}, \overline{s}, \overline{\eta})$. It follows from the definition of $M(\xi)$ (see proof of Lemma 2) that, for each pair $(\xi, k) \in I(D)$, one has:

$$(4)\quad \overline{s}^k(\xi) = \sum_{\{i \in I(\xi):\, k \in I^\star_{-i}(\xi)\}} \overline{d}^i_k(\xi) + \sum_{\{i \in I(\xi^-) \setminus I(\xi): \xi^- \in \overline{D}^i_\star\}} \overline{b}^i_k(\xi) + \sum_{i \in I(\xi^-) \setminus I(\xi)} \beta^i_k(\xi) \tau^i_\xi(\overline{p}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{b}^i)).$$

Since for each $h = i \in I$, the collection $\overline{\eta}^h = (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{d}^h, \overline{b}^h)$ satisfies the budget constraints, it follows from the inequality above and the physical-financial budget constraints that:

$$\overline{p}(\xi_0) \sum_{i \in I(\xi_0)} \left[\overline{x}^i(\xi_0) - w^i(\xi_0)\right] + \overline{q}(\xi_0) \sum_{i \in I(\xi_0)} (\overline{\theta}^i(\xi_0) - \overline{\varphi}^i(\xi_0)) \quad \leqslant \quad 0; \tag{5}$$

$$\overline{p}(\xi) \left[\sum_{i \in I(\xi)} (\overline{x}^i(\xi) - w^i(\xi)) - Y_\xi \sum_{i \in I(\xi^-)} \overline{x}^i(\xi^-) \right] + \overline{q}(\xi) \sum_{i \in I(\xi)} (\overline{\theta}^i(\xi) - \overline{\varphi}^i(\xi)) \tag{6}$$
$$\leqslant \sum_{i \in I(\xi^-)} R_\xi(\overline{p}(\xi))(\overline{\theta}(\xi^-) - \overline{\varphi}(\xi^-));$$

As the left-hand side of equation (5) represents the objective function of player $h = h(\xi_0)$, its optimal value is less than or equal to zero. Thus,

$$\sum_{i \in I(\xi_0)} \left[\overline{x}^i(\xi_0) - w^i(\xi_0)\right] \leqslant 0. \tag{7}$$

Then, it follows from the proof of Lemma 2 that $\overline{x}^i(\xi_0, g) < \Omega(\xi_0)$, for each $g \in G$. Moreover, collateral constraints assure that the short sales satisfy $\overline{\varphi}^i(\xi_0, j) < \Omega(\xi_0)$, for all $j \in J(\xi_0)$. Therefore, strict monotonicity of u^i implies that, for each asset $j \in J(\xi_0)$, $\overline{p}(\xi_0)C(\xi, j) - \overline{q}(\xi_0, j) > 0$. So, we guarantee that $||\overline{p}(\xi_0)||_\Sigma > m(\xi_0)$.

Now, it follows from the monotonicity of agents' objective functions that inequality (5) holds as an equality. Thus, as a consequence of the best response of player $h(\xi_0)$, we have that

$$\sum_{i \in I(\xi_0)} \left(\overline{\theta}^i(\xi_0) - \overline{\varphi}^i(\xi_0)\right) \leqslant 0,$$

which guarantees that $\overline{\theta}^i(\xi_0, j) < \Omega(\xi_0)$, for each asset $j \in J(\xi_0)$.

Furthermore, Assumption [A4] assures that $\overline{p}(\xi_0) \gg 0$. Thus, physical market feasibility holds at ξ_0. In addition, the monotonicity of the preferences guarantees that financial markets clear, at the initial node.

The same arguments can be applied to prove that financial markets clear at each $\xi > \xi_0$ (using equation (6) and the market feasibility at ξ^-). Thus, a Cournot-Nash equilibrium of the game $\mathcal{G}$ satisfies feasibility conditions of items (ii) and (iii) of Definition 2.

On the other side, the definition of Cournot-Nash equilibrium guarantees that, for each agent $i \in I$, the plan $\overline{\eta}^i$ belongs to $B^i(\overline{p}, \overline{q}, \overline{s}^i)$ and

$$V^i(\overline{p}, (\overline{x}^i, \overline{d}^i, \overline{b}^i)) \;\geqslant\; V^i(\overline{p}, (x^i, d^i, b^i)), \qquad \forall\, \eta^i = (x^i, \theta^i, \varphi^i, d^i, b^i) \in C^i(\overline{p}, \overline{q}, \overline{s}^i) \subset B^i(\overline{p}, \overline{q}, \overline{s}^i).$$

Finally, as D^i is a finite set and $\left\| \left(\overline{x}^i(\xi), \overline{\theta}^i(\xi), \overline{\varphi}^i(\xi), \overline{d}^i(\xi), \overline{b}^i_\xi\right) \right\|_{\max} < \Omega(\xi)$, for each $\xi \in D^i$, the quasi-concavity of V^i implies that $V^i(\overline{p}, (\overline{x}^i, \overline{d}^i, \overline{b}^i)) \geqslant V^i(\overline{p}, (x^i, d^i, b^i))$, for each $(x^i, \theta^i, \varphi^i, d^i, b^i) \in B^i(\overline{p}, \overline{q}, \overline{s}^i)$, which assures the optimality of individual allocations on the budget set. □

Appendix B. Proof of Theorem 2.

Let $\mathcal{I} := \{i \in I : \#D^i = +\infty\}$ be the set of infinitely–lived agents in the economy.

For each $T \in \mathbb{N}$, $T \geqslant 3$, let us consider an abstract economy, $\mathcal{E}^T$, populated only by *finitely–lived* agents, where each $i \in \mathcal{I}$ is *replaced* by an agent who has a maximal lifetime of T. Thus, we suppose that, for each $i \in \mathcal{I}$, the associated agent, also denoted by i, is alive only at nodes in $D^{i,T} := \left(\bigcup_{\mu \in \underline{D}^i} D_T(\mu)\right) \cap D^i$, where $\underline{D}^i = \left\{\eta \in D^i \setminus \{\xi_0\} : \eta^- \notin D^i\right\} \cup \left(\{\xi_0\} \cap D^i\right)$. For each $i \in I \setminus \mathcal{I}$, we set $D^{i,T} = D^i$.

When we make this truncation, it is possible that, depending on the original demographic structure, some nodes of D disappear from the abstract economy $\mathcal{E}^T$. In fact, the set of nodes in which agents trade commodities and assets will be given by $D^T = \bigcup_{i \in I} D^{i,T} \subset D$.[14]

However, for our purposes, we only need the set D^T to be, asymptotically, equal to the original event-tree D. Note that, this condition is satisfied, as for each $T \geqslant 3$, $D_T(\xi_0) \subset D^T$ and $D^{i,T} \subset D^{i,T+1}$, which implies that $\bigcup_{T \geqslant 3} D^T = D$.

In this context, the set of agents who are alive at node $\xi \in D^T$ is given by $I(\xi, T) := \{i \in I(\xi) : \xi \in D^{i,T}\}$. Analogously, given $i \in I$, we define the following sets:

$$
\begin{aligned}
I_{-i}(\xi, T) &= I_{-i}(\xi) \cap I(\xi, T), \quad \forall i \in I(\xi, T),\\
I^{\star}_{-i}(\xi, T) &= I^{\star}_{-i}(\xi) \cap I(\xi, T), \quad \forall i \in I(\xi, T),\\
D^{i,T}_{-i} &= \{\xi \in D^{i,T} : I_{-i}(\xi, T) \neq \emptyset\}\\
\overline{D}^{i,T} &= \{\xi \in D^{i,T-1} : (\xi^+ \setminus D^{i,T}) \cap D^T \neq \emptyset\}\\
\overline{D}^{i,T}_{\star} &= \overline{D}^{i}_{\star} \cap \overline{D}^{i,T} \cap D^{i,T}_{-i}.
\end{aligned}
$$

Thus, in $\mathcal{E}^T$, agent $i \in I$ is restricted to make bequests only at the first $T-1$ periods of his life span. Given $\xi \in D$, if $T \geqslant t(\xi)$ then $I(\xi, T) = I(\xi)$ and, therefore, $I_{-i}(\xi, T) = I_{-i}(\xi)$ and $I^{\star}_{-i}(\xi, T) = I^{\star}_{-i}(\xi)$.

In the truncated economy $\mathcal{E}^T$, agent i receives nominal transfers $s^{i,T} := (s^{i,T}(\xi); \xi \in D^{i,T})$ and, given prices (p, q), he can choose any plan in the truncated budget set, $B^{i,T}(p^T, q^T, s^{i,T})$, which is defined as the collection of vectors $(x^{i,T}, \theta^{i,T}, \varphi^{i,T}, d^{i,T}, b^{i,T})$ in

$$
\Gamma^{i,T} := \mathbb{R}_+^{G \times D^{i,T}} \times \prod_{\xi \in D^{i,T}} \mathbb{R}_+^{J(\xi)} \times \prod_{\xi \in D^{i,T}} \mathbb{R}_+^{J(\xi)} \times \prod_{\{\xi \in D^{i,T} : I^{\star}_{-i}(\xi,T) \neq \emptyset\}} \mathbb{R}_+^{I^{\star}_{-i}(\xi,T)} \times \prod_{\xi \in \overline{D}^{i,T}_{\star}} \mathbb{R}_+^{\left(\xi^+ \setminus D^i\right) \times I_{-i}(\xi,T)},
$$

that satisfy the budgetary restrictions at nodes $\xi \in D^{i,T}$.

[14] For instance, if the economy is populated only by infinitely–lived households, who are born at ξ_0, the set $D^T = D_T(\xi_0)$.

Furthermore, agent i's objective function is given by,

$$V^{i,T}(p,(x^{i,T},d^{i,T},b^{i,T})) := \sum_{\xi\in D^{i,T}} u^i(\xi,x^{i,T}(\xi)) + \sum_{\{\xi\in D^{i,T}: I^\star_{-i}(\xi,T)\neq\emptyset\}} f^{i,T}\left(\xi,\frac{d^{i,T}(\xi)}{p(\xi)v(\xi)}\right)$$

$$+\sum_{\xi\in\overline{D}^{i,T}_\star} g^{i,T}\left(\xi,\left(\frac{b^{i,T}(\mu)}{p(\mu)v(\mu)};\mu\in\xi^+\setminus D^{i,T}\right)\right),$$

where $f^{i,T}(\xi,\cdot):\mathbb{R}_+^{I^\star_{-i}(\xi,T)}\to\mathbb{R}_+$ is defined as,

$$f^{i,T}\left(\xi,\frac{d^{i,T}(\xi)}{p(\xi)v(\xi)}\right) = f^i\left(\xi,\left(\frac{d^{i,T}_k(\xi)}{p(\xi)v(\xi)};k\in I^\star_{-i}(\xi,T)\right),0\right),$$

and the bequest function $g^{i,T}(\xi,\cdot):\mathbb{R}^{(\xi^+\setminus D^{i,T})\times I_{-i}(\xi,T)}\to\mathbb{R}_+$ satisfies,

$$g^{i,T}\left(\xi,\left(\frac{b^{i,T}(\mu)}{p(\mu)v(\mu)};\mu\in\xi^+\setminus D^{i,T}\right)\right) = g^i\left(\xi,\left(\frac{b^{i,T}_k(\mu)}{p(\mu)v(\mu)};\mu\in\xi^+\setminus D^{i,T},\, k\in I_{-i}(\xi,T)\right),0\right).$$

Under the assumptions of Theorem 2, we can construct, as in the proof of Theorem 1, a generalized game in which the set of Cournot-Nash equilibria coincides with the set of equilibria for $\mathcal{E}^T$ (in the sense of Definition 2, restricting feasibility conditions to nodes in D^T). To this end, it is sufficient to redefine the generalized game $\mathcal{G}$, of Appendix A, taking into account the new definition of event-tree D^T, the new demographic structure and agents' characteristics defined above.

Therefore, our truncated economy $\mathcal{E}^T$ will always have an equilibrium

$$\left(\overline{p}^T,\overline{q}^T,\overline{s}^T;(\overline{x}^{i,T},\overline{\theta}^{i,T},\overline{\varphi}^{i,T},\overline{d}^{i,T},\overline{b}^{i,T})_{i\in I}\right),$$

with $\|(\overline{p}^T(\xi),\overline{q}^T(\xi))\|_\Sigma=1$, for each $\xi\in D^{i,T}$.

Moreover, for each $T\in\mathbb{N}$, the optimality of $\overline{\eta}^{i,T}:=(\overline{x}^{i,T},\overline{\theta}^{i,T},\overline{\varphi}^{i,T},\overline{d}^{i,T},\overline{b}^{i,T})$ assures that there exist non-negative multipliers:

$$\begin{aligned}
\overline{\lambda}^{i,T} &= \left(\overline{\lambda}^{i,T}_\xi;\xi\in D^{i,T}\right)\in\mathbb{R}_+^{D^{i,T}}\\
\overline{\phi}^{i,T} &= \left(\overline{\phi}^{i,T}_\mu;\mu\in\xi^+\setminus D^{i,T},\,\xi\in\overline{D}^{i,T}_\star\right)\in\prod_{\xi\in\overline{D}^{i,T}_\star}\mathbb{R}_+^{\xi^+\setminus D^{i,T}}\\
\overline{\psi}^{i,T} &= \left(\overline{\psi}^{i,T}_{\mu,k};\mu\in\xi^+\setminus D^{i,T},\xi\in\overline{D}^{i,T}_\star,\,k\in I_{-i}(\xi,T)\right)\in\prod_{\xi\in\overline{D}^{i,T}_\star}\mathbb{R}_+^{(\xi^+\setminus D^{i,T})\times I_{-i}(\xi,T)}
\end{aligned}$$

such that, for each plan $\eta^{i,T}=(x^{i,T},\theta^{i,T},\varphi^{i,T},d^{i,T},b^{i,T})\in\Gamma^{i,T}$ that satisfies the collateral constraints,

$$x^{i,T}(\xi)\geqslant\sum_{j\in J(\xi)}C(\xi,j)\varphi^{i,T}(\xi,j),\quad\forall i\in I,\,\forall\xi\in D^{i,T},$$

we have,

$$\mathcal{L}^{i,T}\left(\left(\overline{p}^T,\overline{q}^T,\overline{s}^{i,T},\overline{\lambda}^{i,T},\overline{\phi}^{i,T},\overline{\psi}^{i,T}\right);\eta^{i,T}\right) \leqslant V^{i,T}(\overline{p}^T,(\overline{x}^{i,T},\overline{d}^{i,T},\overline{b}^{i,T}));\tag{8}$$

$$\overline{\lambda}_\xi^{i,T} L_\xi^i\left(\left(\overline{p}^T,\overline{q}^T,\overline{s}^{i,T}\right);\overline{\eta}^{i,T}\right) = 0, \qquad \forall \xi\in D^{i,T};\tag{9}$$

$$\overline{\psi}_{\mu,k}^{i,T} L_{1,k}\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\overline{\eta}^{i,T}\right) = 0, \qquad \forall \mu\in\xi^+\setminus D^{i,T},\ \forall\xi\in\overline{D}_\star^{i,T},\ \forall k\in I_{-i}(\xi,T);\tag{10}$$

$$\overline{\phi}_\mu^{i,T} L_2\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\overline{\eta}^{i,T}\right) = 0, \qquad \forall \mu\in\xi^+\setminus D^{i,T},\ \forall\xi\in\overline{D}_\star^{i,T};\tag{11}$$

where the lagrangian function, $\mathcal{L}^{i,T}$, is given by:

$$\begin{aligned}\mathcal{L}^{i,T}\left(\left(\overline{p}^T,\overline{q}^T,\overline{s}^{i,T},\overline{\lambda}^{i,T},\overline{\phi}^{i,T},\overline{\psi}^{i,T}\right);\eta^{i,T}\right) &:= V^{i,T}(\overline{p}^T,(x^{i,T},d^{i,T},b^{i,T}))\\ &+\sum_{\xi\in D^{i,T}}\overline{\lambda}_\xi^{i,T} L_\xi^i\left(\left(\overline{p}^T,\overline{q}^T,\overline{s}^{i,T}\right);\eta^{i,T}\right)\\ &+\sum_{\xi\in\overline{D}_\star^{i,T}}\ \sum_{\mu\in\xi^+\setminus D^{i,T}}\ \sum_{k\in I_{-i}(\xi,T)}\overline{\psi}_{\mu,k}^{i,T}L_{1,k}\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\eta^{i,T}\right)\\ &+\sum_{\xi\in\overline{D}_\star^{i,T}}\ \sum_{\mu\in\xi^+\setminus D^{i,T}}\overline{\phi}_\mu^{i,T}L_2\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\eta^{i,T}\right);\end{aligned}$$

and

$$\begin{aligned}L_\xi^i\left(\left(\overline{p}^T,\overline{q}^T,\overline{s}^{i,T}\right);\eta^{i,T}\right) &\geqslant 0, \quad \text{is agent } i\text{'s physical-financial budget constraint at node } \xi;\\ L_{1,k}\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\eta^{i,T}\right) &= b_k^{i,T}(\mu)-\alpha_k^i(\xi)\,e_\mu(\overline{p}^T,(x^{i,T},\theta^{i,T},\varphi^{i,T})),\\ L_2\left(\mu,\left(\overline{p}^T,\overline{q}^T\right);\eta^{i,T}\right) &= e_\mu(\overline{p}^T,(x^{i,T},\theta^{i,T},\varphi^{i,T}))-\sum_{k\in I_{-i}(\xi,T)} b_k^{i,T}(\mu).\end{aligned}$$

Lemma 6. *Under the assumptions on Theorem 2, for each $\xi\in D$, the sequence*

$$\left(\left(\overline{p}^T(\xi),\overline{q}^T(\xi),\overline{s}^{i,T}(\xi);\overline{\eta}^{i,T}(\xi),\overline{\lambda}_\xi^{i,T}\right);i\in I(\xi)\right)_{T\geqslant t(\xi)},$$

where $\overline{\eta}^{i,T}(\xi):=(\overline{x}^{i,T}(\xi),\overline{\theta}^{i,T}(\xi),\overline{\varphi}^{i,T}(\xi),\overline{d}^{i,T}(\xi),\overline{b}_\xi^{i,T})$, is bounded.

Proof. Under Assumption [A5], and using the same arguments as in Lemma 2, market feasibility assures that consumption allocations are uniformly bounded along the event-tree. Thus, Assumption [A3] guarantees that short sales are bounded, node by node. Moreover, financial feasibility of equilibrium implies that long positions are bounded from above, node by node. Note that, by construction, all of these bounds are independent of the value of T.

In addition, Assumption [A5] implies that there exists $\overline{W}\in\mathbb{R}_{++}^G$ such that $\sum_{i\in I(\xi)} W^i(\xi)\leqslant\overline{W}$, for each $\xi\in D$. Thus, physical-financial feasibility conditions and the definition of assets effective payments guarantee that:

$$\begin{aligned}\sum_{k\in I_{-i}(\xi,T)} \overline{b}_k^{i,T}(\mu) &\leqslant e_\mu(\overline{p}^T,(\overline{x}^{i,T},\overline{\theta}^{i,T},\overline{\varphi}^{i,T}))\\ &\leqslant 2\overline{p}^T(\mu)\overline{W} \leqslant 2\|\overline{W}\|_\Sigma,\end{aligned}$$

where the last inequality follows from the fact that $(\overline{p}^T(\xi),\overline{q}^T(\xi)) \in \Delta^{\#G+\#J(\xi)}$. Thus, bequests are uniformly bounded, along the event-tree and independent of the value of T.

Using the same recursive argument of Lemma 2, it follows from the inequalities above and the feasibility conditions that nominal transfers, $(\overline{s}^{i,T})_{i\in I}$, and donations, $(\overline{d}^{i,T})_{i\in I}$, are *uniformly bounded* along the event-tree, as $(n(\xi);\xi\in D)$ is uniformly bounded and Assumption [A2] holds.

Therefore, the sequence $\left(\left(\overline{p}^T(\xi),\overline{q}^T(\xi),\overline{s}^{i,T}(\xi);\overline{\eta}^{i,T}(\xi)\right);i\in I(\xi)\right)_{T\geqslant t(\xi)}$ is bounded.

On the other hand, by Assumption [A5], the sum of commodity prices is uniformly bounded away from zero along the event-tree. In fact, with the notation of Lemma 2, $m(\xi)\geqslant \frac{1}{1+\overline{C}}$, where the upper bound $\overline{C} := \sup_{\xi\in D}\max_{g\in G}\sum_{j\in J(\xi)} C(\xi,j,g) < +\infty$. As $v(\xi,g)\geqslant \underline{v}>0$, for each $g\in G$, it follows that, $\overline{p}^T(\xi)v(\xi)\geqslant \frac{\underline{v}}{1+\overline{C}}$.

Now, as consumption allocations, bequests and donations are uniformly bounded, it follows from Assumptions [A5]-[A8] that there is $\overline{V}>0$ such that,

$$V^{i,T}(\overline{p}^T,(\overline{x}^{i,T},\overline{d}^{i,T},\overline{b}^{i,T}))\leqslant \overline{V}, \quad \forall T>t(\xi).$$

Thus, applying equation (8) to the plan $\eta^{i,T}=(x^{i,T},0,0,0,b^{i,T})$ where,

$$x^{i,T}(\mu)=\begin{cases} W^i(\mu), & \text{if } \mu<\xi\\ 0, & \text{otherwise};\end{cases}$$

$$b_k^{i,T}(\mu)=\alpha_k^i(\varrho)\,e_\mu(\overline{p}^T,(x^{i,T},0,0)),\ \ \forall\mu\in\varrho^+\setminus D^{i,T},\ \forall\varrho\in\overline{D}_\star^{i,T},\ \forall k\in I_{-i}(\varrho,T);$$

one gets $\overline{\lambda}_\xi^{i,T}\overline{p}^T(\xi)W^i(\xi)\leqslant\overline{V}$. Therefore, as $\|\overline{p}^T(\xi)\|_\Sigma > m(\xi)$, we conclude, using Assumption [A1], that,

$$\overline{\lambda}_\xi^{i,T}\leqslant\frac{\overline{V}}{m(\xi)\min\limits_{g\in G}W^i(\xi,g)}<+\infty,$$

which implies that the sequence of multipliers $\left(\overline{\lambda}_\xi^{i,T};i\in I(\xi)\right)_{T\geqslant t(\xi)}$ is also bounded. □

Lemma 7. *Under the assumptions on Theorem 2, given $\xi\in\overline{D}_\star^i$, the sequences*

$$(\overline{\zeta}_\xi^{i,T})_{T>t(\xi)} := \left(\overline{\psi}_{\mu,k}^{i,T}\overline{p}^T(\mu)Y_\mu W^i(\xi);\ \ \mu\in\xi^+\setminus D^i,\ \ k\in I_{-i}(\xi)\right)_{T>t(\xi)},$$

$$(\overline{\varsigma}_\xi^{i,T})_{T>t(\xi)} := \left(\overline{\phi}_\mu^{i,T}\,\overline{p}^T(\mu)Y_\mu W^i(\xi);\ \ \mu\in\xi^+\setminus D^i\right)_{T>t(\xi)},$$

are bounded.

Proof. Given $T > t(\xi)$, consider the plan $\eta^{i,T} = (x^{i,T}, 0, 0, 0, b^{i,T})$, where

$$\begin{aligned} x^{i,T}(\mu) &= W^i(\mu), \ \ \forall \mu \in D^{i,T}\,; \\ b_k^{i,T}(\mu) &= \tilde{\alpha}_k^i(\varrho)\, e_\mu(\overline{p}^T, (x^{i,T}, \theta^{i,T}, \varphi^{i,T})), \quad \forall \mu \in \varrho^+ \setminus D^{i,T}, \ \ \forall \varrho \in \overline{D}_\star^{i,T}, \ \ \forall k \in I_{-i}(\varrho, T), \end{aligned}$$

and $\alpha_k^i(\varrho) < \tilde{\alpha}_k^i(\varrho)$, $\sum_{k \in I_{-i}(\varrho)} \tilde{\alpha}_k^i(\varrho) < 1$, $\forall i \in I(\varrho)$. Note that, by Assumption [A1], it is always possible to find constants $\tilde{\alpha}_k^i(\varrho)$ satisfying the condition above.

Now, evaluating equation (8) at $\eta^{i,T}$, we obtain that,

$$\sum_{k \in I_{-i}(\xi)} \overline{\psi}_{\mu,k}^{i,T} \overline{p}^T(\mu) Y_\mu W^i(\xi) \left(\tilde{\alpha}_k^i(\xi) - \alpha_k^i(\xi)\right) + \overline{\phi}_\mu^{i,T} \overline{p}^T(\mu) Y_\mu W^i(\xi) \left(1 - \sum_{k \in I_{-i}(\xi)} \tilde{\alpha}_k^i(\xi)\right) \leqslant \overline{V}.$$

□

Lemma 8. *Under the assumptions on Theorem 2, given $\xi \in D$,*

$$\overline{\lambda}_\xi^{i,T} \overline{p}^T(\xi, g) \geqslant u^i(\xi, \overline{x}^{i,T}(\xi) + 1_g) - u^i(\xi, \overline{x}^{i,T}(\xi)) > 0, \ \ \forall T > t(\xi),$$

where $1_g = (\chi_g(g'); g' \in G) \in \mathbb{R}_+^G$ and

$$\chi_g(g') := \begin{cases} 1, & \textit{if } g' = g; \\ 0, & \textit{if } g' \neq g. \end{cases}$$

Proof. Given $T > t(\xi)$, evaluating equation (8) at $\eta^{i,T} = (x^{i,T}, \overline{\theta}^{i,T}, \overline{\varphi}^{i,T}, \overline{d}^{i,T}, b^{i,T})$ where,

$$x^{i,T}(\mu, g') = \begin{cases} \overline{x}^{i,T}(\mu, g'), & \text{if } (\mu, g') \neq (\xi, g), \\ \overline{x}^{i,T}(\mu, g') + 1, & \text{if } (\mu, g') = (\xi, g), \end{cases}$$

and

$$b_k^{i,T}(\mu) = \alpha_k^i(\varrho)\, e_\mu(\overline{p}^T, (x^{i,T}, \overline{\theta}^{i,T}, \overline{\varphi}^{i,T})), \ \ \forall \mu \in \varrho^+ \setminus D^{i,T}, \forall \varrho \in \overline{D}_\star^{i,T}, \ \ \forall k \in I_{-i}(\varrho, T);$$

we obtain that,

$$-\overline{\lambda}_\xi^{i,T} \overline{p}^T(\xi, g) + \sum_{\mu \in \xi^+ \cap D^i} \overline{\lambda}_\mu^{i,T} \sum_{g' \in G} \overline{p}^T(\mu, g')\, Y_\mu(g', g) \leqslant u^i(\xi, \overline{x}^{i,T}(\xi)) - u^i(\xi, \overline{x}^{i,T}(\xi) + 1_g).$$

Finally, as matrices Y_μ have non-negative entries, we conclude the proof. □

Lemma 9. *Under assumptions [A1]-[A3] and [A5]-[A8], there exists a subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that:*

$$\begin{aligned} \left(\overline{p}^{T_k}, \overline{q}^{T_k}, \overline{s}^{T_k}; (\overline{x}^{i,T_k}, \overline{\theta}^{i,T_k}, \overline{\varphi}^{i,T_k}, \overline{d}^{i,T_k}, \overline{b}^{i,T})_{i \in I}\right) &\longrightarrow_{k \to +\infty} \left(\overline{p}, \overline{q}, \overline{s}; (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{d}^i, \overline{b}^i)_{i \in I}\right) \\ (\overline{\lambda}^{i,T}, \overline{\psi}^{i,T_k}, \overline{\phi}^{i,T_k})_{i \in I} &\longrightarrow_{k \to +\infty} (\overline{\lambda}^i, \overline{\psi}^i, \overline{\phi}^i)_{i \in I}. \end{aligned}$$

Proof. It follows from lemmas 6 and 7 that, for each $\xi \in D$, the sequence

$$Z^T(\xi) := \begin{cases} \left(\left(\overline{p}^T(\xi), \overline{q}^T(\xi), \overline{s}^{i,T}(\xi); \overline{\eta}^{i,T}(\xi), \overline{\lambda}_\xi^{i,T}, \overline{\zeta}_\xi^{i,T}, \overline{\varsigma}_\xi^{i,T}\right); i \in I(\xi)\right), & \text{if } T > t(\xi), \\ 0 & \text{otherwise,} \end{cases}$$

is bounded. Since the event-tree D is countable, Tychonoff's Theorem guarantees the existence of a common subsequence $(T_k)_{k\geqslant 1} \subset \mathbb{N}$ such that, for each ξ,

$$\lim_{k\to+\infty} Z^{T_k}(\xi) = \left(\left(\overline{p}(\xi), \overline{q}(\xi), \overline{s}^i(\xi); \overline{\eta}^i(\xi), \overline{\lambda}_\xi^i, \overline{\zeta}_\xi^i, \overline{\varsigma}_\xi^i\right); i \in I(\xi)\right).$$

Moreover, Lemma 8 assures that, for each $\xi \in D$, $\overline{\lambda}_\xi^i \overline{p}(\xi, g) \geqslant u^i(\xi, \overline{x}^i(\xi) + 1_g) - u^i(\xi, \overline{x}^i(\xi)) > 0$. So, given $i \in I$, for each $\mu \in \xi^+ \setminus D^i$, $\overline{p}^T(\mu) Y_\mu W^i(\xi)$ converges to $\overline{p}(\mu) Y_\mu W^i(\xi) > 0$. Finally, Lemma 7 implies that, for each $i \in I$ and $\xi \in \overline{D}_\star^i$, we have:

$$\left(\overline{\psi}_{\mu,i'}^{i,T_k};\quad \mu \in \xi^+ \setminus D^i,\quad i' \in I_{-i}(\xi)\right) \to_{k\to+\infty} \frac{\overline{\zeta}_\xi^i}{\overline{p}(\mu) Y_\mu W^i(\xi)} < +\infty\,,$$

$$\left(\overline{\phi}_\mu^{i,T_k};\quad \mu \in \xi^+ \setminus D^i\right) \to_{k\to+\infty} \frac{\overline{\varsigma}_\xi^i}{\overline{p}(\mu) Y_\mu W^i(\xi)} < +\infty.$$

□

Lemma 10. *Under assumptions [A1]-[A3] and [A5]-[A8], the following transversality condition holds for each $i \in \mathcal{I}$:*

$$\lim_{N\to+\infty} \sum_{\{\xi\in D^i:\, t(\xi)=N\}} \overline{\lambda}_\xi^i \left(\overline{p}(\xi)\overline{x}^i(\xi) + \sum_{r\in I_{-i}^\star(\xi)} \overline{d}_r^i(\xi) + \overline{q}(\xi)\left(\overline{\theta}^i(\xi) - \overline{\varphi}^i(\xi)\right)\right) = 0. \tag{12}$$

Proof. Given $\xi \in D^{i,T_k}$, consider the following allocation:

$$\eta^{i,T_k}(\mu) := (x^{i,T_k}(\mu), \theta^{i,T_k}(\mu), \varphi^{i,T_k}(\mu), d^{i,T_k}(\mu), b_\mu^{i,T_k}) = \begin{cases} 0\,, & \text{if } \mu = \xi\,; \\ \overline{\eta}^{i,T_k}(\mu)\,, & \text{otherwise,} \end{cases}$$

where $b_\mu^{i,T_k} := (b^{i,T_k}(\zeta), \zeta \in \mu^+ \setminus D^{i,T_k})$. It follows from equations (8)-(11) that,

$$u^i(\xi, \overline{x}^{i,T_k}(\xi)) + f^{i,T_k}\left(\xi, \frac{\overline{d}^{i,T_k}(\xi)}{\overline{p}^{T_k}(\xi) v(\xi)}\right) \chi^{i,T_k}(\xi) + g^{i,T_k}\left(\xi, \overline{b}_\xi^{i,T_k}\right) \chi_{\overline{D}_\star^{i,T_k}}(\xi)$$

$$\geqslant -\sum_{\mu\in\xi^+\setminus D^{i,T_k}} \overline{\lambda}_\mu^{i,T_k} \left(\overline{p}^{T_k}(\mu)\overline{x}^{i,T_k}(\mu) + \sum_{r\in I_{-i}^\star(\mu)} \overline{d}_r^{i,T_k}(\mu) + \overline{q}^{T_k}(\mu)\left(\overline{\theta}^{i,T_k}(\mu) - \overline{\varphi}^{i,T_k}(\mu)\right)\right)$$

$$+\overline{\lambda}_\xi^{i,T_k} \left(\overline{p}^{T_k}(\xi)\overline{x}^{i,T_k}(\xi) + \sum_{r\in I_{-i}^\star(\xi)} \overline{d}_r^{i,T_k}(\xi) + \overline{q}^{T_k}(\xi)\left(\overline{\theta}^{i,T_k}(\xi) - \overline{\varphi}^{i,T_k}(\xi)\right)\right),$$

where $\chi^{i,T_k}(\xi)=1$, if $I^{\star}_{-i}(\xi,T)\neq\emptyset$, and equal to zero otherwise. Also, $\chi_{\overline{D}^{i,T_k}_{\star}}(\xi)=1$, if $\xi\in\overline{D}^{i,T_k}_{\star}$, and equal to zero otherwise. Therefore, it follows that,

$$\overline{\lambda}^{i,T_k}_{\xi}\left(\overline{p}^{T_k}(\xi)\overline{x}^{i,T_k}(\xi)+\sum_{r\in I^{\star}_{-i}(\xi)}\overline{d}^{i,T_k}_{r}(\xi)+\overline{q}^{T_k}(\xi)\,(\overline{\theta}^{i,T_k}(\xi)-\overline{\varphi}^{i,T_k}(\xi))\right)$$

$$\leqslant\sum_{\{\mu\in D^{i,T_k};\ \mu\geqslant\xi\}}\left(u^i(\mu,\overline{x}^{i,T_k}(\mu))+f^{i,T_k}\left(\mu,\frac{\overline{d}^{i,T_k}(\mu)}{\overline{p}^{T_k}(\mu)v(\mu)}\right)\chi^{i,T_k}(\xi)+g^{i,T_k}\left(\mu,\overline{b}^{i,T_k}_{\mu}\right)\chi_{\overline{D}^{i,T_k}_{\star}}(\xi)\right).$$

Now, Lemma 6 and assumptions [A6]-[A8] assure the existence of a summable plan $(a^i_{\xi};\xi\in D^i)\subset\mathbb{R}^{D^i}_{+}$ such that the right hand side of equation above is, for each $k\geqslant 1$, less than or equal to $\sum_{\{\mu\in D^i:\mu\geqslant\xi\}}a^i_{\mu}$.

Thus, taking the limit as k goes to infinity, Lemma 9 implies that:

$$\overline{\lambda}^{i}_{\xi}\left(\overline{p}(\xi)\overline{x}^{i}(\xi)+\sum_{r\in I^{\star}_{-i}(\xi)}\overline{d}^{i}_{r}(\xi)+\overline{q}(\xi)\,(\overline{\theta}^{i}(\xi)-\overline{\varphi}^{i}(\xi))\right)\leqslant\sum_{\{\mu\in D^i:\mu\geqslant\xi\}}a^i_{\mu}.$$

Summing on $\xi\in D^i$, with $t(\xi)=N$, and taking the limit as N goes to infinity, we conclude that:

$$\lim_{N\to+\infty}\sum_{\{\xi\in D^i:\,t(\xi)=N\}}\overline{\lambda}^{i}_{\xi}\left(\overline{p}(\xi)\overline{x}^{i}(\xi)+\sum_{r\in I^{\star}_{-i}(\xi)}\overline{d}^{i}_{r}(\xi)+\overline{q}(\xi)\,(\overline{\theta}^{i}(\xi)-\overline{\varphi}^{i}(\xi))\right)\leqslant 0.$$

□

The existence of an equilibrium is a direct consequence of the following result.

Lemma 11. *Under assumptions [A1]-[A3] and [A5]-[A8], the allocation* $\left(\overline{p},\overline{q},\overline{s};(\overline{x}^i,\overline{\theta}^i,\overline{\varphi}^i,\overline{d}^i,\overline{b}^i)_{i\in I}\right)$ *is an equilibrium for our economy.*

Proof. Conditions (ii) and (iii) in Definition 2 are satisfied, as $\left(\overline{p},\overline{q},\overline{s};(\overline{x}^i,\overline{\theta}^i,\overline{\varphi}^i,\overline{d}^i,\overline{b}^i)_{i\in I}\right)$ is a limit of equilibria of truncated economies. Thus, it is sufficient to assure the individual optimality of plans $(\overline{\eta}^i)_{i\in I}:=(\overline{\eta}^i(\xi);\,\xi\in D^i\,)_{i\in I}\in\prod_{i\in I}\Gamma^i$.

Now, using the notations of Appendix A, it follows from Lemma 2 that, for each finitely–lived agent $i\in I$, one has $\overline{\eta}^{i,T_k}\in\Psi^i((\overline{p}^{T_k},\overline{q}^{T_k},\overline{s}^{T_k},\overline{\eta}^{T_k})_{-i})$, for each k high enough. Since the optimal strategies correspondence, Ψ^i, is closed, taking the limit as k goes to infinity, we obtain that $\overline{\eta}^i\in\Psi^i((\overline{p},\overline{q},\overline{s},\overline{\eta})_{-i})$. Thus, using the same arguments made in Lemma 5, assumptions [A6]-[A8] assure that $\overline{\eta}^i$ is an optimal plan for agent i.

In order to finish our proof, we have to assure that plans $\overline{\eta}^i$ are also optimal for infinitely–lived agents.

Fix $i \in \mathcal{I}$. Suppose, by contradiction, that there exists $\delta > 0$ and $\eta^i = (x^i, \theta^i, \varphi^i, d^i, b^i) \in B^i(\overline{p}, \overline{q}, \overline{s}^i)$, such that, $V^i(\overline{p}, (x^i, d^i, b^i)) - V^i(\overline{p}, (\overline{x}^i, \overline{d}^i, \overline{b}^i)) > \delta$.

Fix $N \in \mathbb{N}$ such that $N > t(\xi)$, for some $\xi \in D^i$. Consider, for each $T_k > N$ the allocation,

$$\eta^{i,T_k}(\mu) := (x^{i,T_k}(\mu), \theta^{i,T_k}(\mu), \varphi^{i,T_k}(\mu), d^{i,T_k}(\mu), b_\mu^{i,T_k}) = \begin{cases} \eta^i(\mu)\,, & \text{if } t(\mu) < N\,; \\ \overline{\eta}^{i,T_k}(\mu)\,, & \text{in other case.} \end{cases}$$

It follows from equations (8)-(11) that,

$$\begin{aligned} V^{i,N-1}(\overline{p}^{T_k}, (\overline{x}^{i,T_k}, \overline{d}^{i,T_k}, \overline{b}^{i,T_k})) &- V^{i,N-1}(\overline{p}^{T_k}, (x^i, d^i, b^i)) \\ &\geqslant \sum_{\{\mu \in D^{i,T_k}:\, t(\mu) \leqslant N\}} \overline{\lambda}_\mu^{i,T_k} L_\mu^i((\overline{p}^{T_k}, \overline{q}^{T_k}, \overline{s}^{i,T_k}),\, \eta^{i,T_k}). \end{aligned}$$

Since $\eta^i \in B^i(\overline{p}, \overline{q}, \overline{s}^i)$, taking the limit, as k goes to infinity, we obtain that:

$$\begin{aligned} V^{i,N-1}(\overline{p}, (x^i, d^i, b^i)) - V^{i,N-1}(\overline{p}, (\overline{x}^i, \overline{d}^i, \overline{b}^i)) &\leqslant \sum_{\{\mu \in D^i:\, t(\mu) = N\}} \overline{\lambda}_\mu^i \left(\overline{p}(\mu)\overline{x}^i(\mu) + \sum_{r \in I_{-i}^\star(\mu)} \overline{d}_r^i(\mu) \right) \\ &+ \sum_{\{\mu \in D^i:\, t(\mu) = N\}} \overline{\lambda}_\mu^i\, \overline{q}(\mu)\, (\overline{\theta}^i(\mu) - \overline{\varphi}^i(\mu)). \end{aligned}$$

As we know that there exists $N^\star$ such that:

$$V^{i,N-1}(\overline{p}, (x^i, d^i, b^i)) - V^{i,N-1}(\overline{p}, (\overline{x}^i, \overline{d}^i, \overline{b}^i)) \geqslant \frac{\delta}{2}, \quad \forall N > N^\star,$$

we have,

$$\frac{\delta}{2} \leqslant \lim_{N \to +\infty} \sum_{\{\mu \in D^i:\, t(\mu) = N\}} \overline{\lambda}_\mu^i \left(\overline{p}(\mu)\overline{x}^i(\mu) + \sum_{r \in I_{-i}^\star(\mu)} \overline{d}_r^i(\mu) + \overline{q}(\mu)\, (\overline{\theta}^i(\mu) - \overline{\varphi}^i(\mu)) \right) = 0,$$

a contradiction. □

References

Aliprantis, C.D., Border, K.C.: Infinite dimensional analysis: a hitchhiker's guide, 2nd edn.. Berlin Heidelberg New York: Springer 1999.

Araujo. A.P., Páscoa, M.R., Torres-Martínez, J.P.: Collateral avoids Ponzi schemes in incomplete markets. Econometrica **70**, 1613-1638 (2002).

Araujo. A.P., Páscoa, M.R., Torres-Martínez, J.P.: Bubbles, Collateral and Monetary Equilibrium. PUC-Rio working papers (2005).

Cardia, E., Michel P.: Altruism, intergenerational transfers of time and bequests. J. Econ. Dynam. Control **28**, 1681-1701 (2004).

Daher, W., Martins da Rocha V.F., Páscoa M., Vailakis, Y.: Sequential trading without perfect foresight: the role of default and collateral. Working paper FEUNL, Portugal (2006).

d'Albis, H.: Demographic structure and capital accumulation. J. Econ. Theory **132**, 411-434 (2007).

Dubey, P., Geanakoplos J., Shubik, M.: Default and punishment in general equilibrium. Econometrica, **73**, 1-37 (2005).

Dubey, P., Geanakoplos J., Zame, W.R.: Default, Collateral and Derivatives. Yale University working paper (1995).

Dynan, K.E, Skinner J., Zeldes,S.P.: The Importance of Bequest and Life–Cycle Saving in Capital Accumulation: A New Answer. Amer. Econ. Rev. **92**, 274-278 (2002).

Florenzano, M., Gourdel, P., Pascoa, M.R.: Overlapping Generations Model with Incomplete Markets. J. Math. Econ. **36,** 201-217 (2001).

Fuster, L.: Effects of uncertain lifetime and annuity insurance on capital accumulation and growth. Econ. Theory **13**, 429-445 (1999).

Fuster, L.: Capital accumulation in an Economy with dynasties and uncertain Lifetimes. Rev. Econ. Dynam. **4**, 650-674 (2000).

Geanakoplos, J., Zame, W.R.: Collateral and the enforcement of intertemporal contracts. Yale University working paper (2002).

Geanakoplos, J., Polemarchakis, H.M.: Overlapping Generations. Handbook of Mathematical Economics, vol. IV, 1899-1960. Amsterdam, Elsevier 1991.

Hernandez, A., Santos, M.: Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets. J. Econ. Theory **71**, 102-130 (1996).

Kotlikoff, L.J., Summers, L.H.: The Role of Intergenerational transfers in Aggregate Capital Accumulation. J. Polit. Economy **89,** 706-732 (1981).

Kubler, F., Schmedders, K.: Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral. Econometrica **71,** 1767-1793 (2003).

Leung, S.F: Uncertain lifetime, the theory of the consumer, and the life cycle hypothesis. Econometrica **62**, 1233-1239 (1994).

Magill, M., Quinzii, M. : Infinite Horizon Incomplete Markets. Econometrica **62**, 853-880 (1994).

Páscoa, M.R, Seghir, A.: Harsh default penalties lead to Ponzi schemes. Working paper No 1, IFE, American University of Beirut (2006). Forthcoming in Games and Economic Behavior.

Schmachtenberg, R.: Stochastic Overlapping Generations Model with Incomplete Markets 1: Existence of Equilibria. Discussion Paper No. 363-88, Department of Economics, University of Mannheim (1988).

Seghir, A.: An Overlapping Generations Model with Non–ordered Preferences and Numeraire Incomplete Markets, Decis. Econ. Finance, **28**, 95-112 (2006).