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Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making

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ABSTRACT

Heronian mean (HM) operator has the advantages of considering the interrelationships between parameters, and linguistic intuitionistic fuzzy number (LIFN), in which the membership and non-membership are expressed by linguistic terms, can more easily describe the uncertain and the vague information existing in the real world. In this paper, we propose the partitioned Heronian mean (PHM) operator which assumes that all attributes are partitioned into several parts and members in the same part are interrelated while in different parts there are no interrelationships among members, and develop some new operational rules of LIFNs to consider the interactions between membership function and non-membership function, especially when the degree of non-membership is zero. Then we extend PHM operator to LIFNs based on new operational rules, and propose the linguistic intuitionistic fuzzy partitioned Heronian mean (LIFPHM) operator, the linguistic intuitionistic fuzzy weighted partitioned Heronian mean (LIFWPHM) operator, the linguistic intuitionistic fuzzy partitioned geometric Heronian mean (LIFPGHM) operator and linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator. Further, we develop two methods to solve multi-attribute group decision making (MAGDM) problems with the linguistic intuitionistic fuzzy information. Finally, we give some examples to verify the effectiveness of two proposed methods by comparing with the existing

1. Introduction

Multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM) problems widely exist in real life. In the process of decision making, we are facing many difficulties such as how to describe the fuzzy and uncertain attributes [14,18,20]. In order to solve these problems, Zadeh [27] proposed the fuzzy sets (FSs), which are very convenient to express fuzzy information. However, because FS only has the membership function, it may be difficult to describe some complex situation. For example, for a problem of judgment, if the result received that five people agreed, three people disagreed and two people were absent. It is obvious that FS cannot express the results of this example. Then Atanassov [1,2] presented the theory of intuitionistic fuzzy sets (IFSs) to express some complex situations. The IFS is composed of a membership function and a non-membership function. Obviously, IFS is better than FS to describe the fuzzy information. However, in real decision environment, there are many situations in which the attributes of decision making problems cannot be presented by the exact quantitative form, but they may be described by qualitative form. In this case, decision makers usually describe their preferences by linguistic terms. For instance, when the speed of cars is evaluated by observation, terms like "very slow", "slow", "fast", "very fast" are usually used to express the experts' preference. In order to deal with the linguistic information better in decision making process, many linguistic computational models have been developed [6–11]. For example, Bordogna et al. [4] and Pei et al. [16] proposed some the linguistic weighted aggregation operators [21,23,24].

Although IFS can easily describe the fuzzy information [15,17], however, because its membership function and non-membership function are expressed by crisp numbers, under some conditions, such as some qualitative indexes, it may not be convenient for decision makers to

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express their preference. On the basis of the IFS, Chen [5] defined linguistic intuitionistic fuzzy numbers (LIFNs) in which the membership degree and the non-membership degree are expressed by linguistic variables. Obviously, the LIFNs can take the advantages of linguistic term sets and IFNs, and can deal with vague or imprecise information more convenient.

For aggregation operators, in general, we need to consider two aspects. One is for functions and the other is for operations. In the functions, the traditional aggregation operators [25,26] can only aggregate a set of real values into a real value. In order to solve the complex real decision problems by aggregation operators, there are many complex aggregation operators which are developed for the different functions. For example, in order to consider the interrelationships between attitudes, the Bonferroni mean (BM) operator [3] was proposed by Bonferroni and the Heronian mean [19] was proposed by Sykora. About the difference of them, the BM operator can consider the relationship between any two criteria, but it generally ignores the interrelationship between the criteria and itself, and the aggregation result is with redundancy. Although the HM operator has the similar expression with BM operator, it can solve the weaknesses of BM operator. Now, the HM operator has extensively been applied to the decision making process, however, HM operator supposed that each attribute has relationship with the other attributes. But in real decision process, the situation may not always be existed. For example, when selecting a production, we may consider some attributes as follows: basic function (C_1), ease of use (C_2), production quality (C_3) and production price (C_4). The attributes can be partitioned into two parts: $P_1 = \{C_1, C_2\}$ and $P_2 = \{C_3, C_4\}$. It is obvious thatC_1 is related with C_2, so they are in the same partP_1. However, there is no relationship between the parts P_1 and P_2. We can classify the attributes into the different groups according to the different aspects of the impact of different attributes. For example, when we go to rent a house, we may consider many factors like price, transportation, comfort level, infrastructure and so on. As we all know, the higher the price is, the better the comfort level is. Similarly, the convenience of transport will indirectly determine whether the infrastructure is complete. So we can divide the above four factors into two groups i.e., the first group are price and comfort level, while the second group are transportation and infrastructure. Obviously, HM operator cannot deal with this type of decision making problem.

In the operations, the traditional operations of LIFSs in [5] do not consider the interactions between membership function and non-membership function, so in some special cases, they don't get reasonable aggregating results, especially when exist zero in the subscripts of the membership or non-membership of LIFSs. For example, let$\gamma_1 = \left(s_{a_1}, s_{b_1}\right)$,$\gamma_2 = \left(s_{a_2}, s_{b_2}\right)$be two LIFNs, and $b_1 = 0$, $b_2 \neq 0$, then by the addition operation in Chen [5], then we can get $\gamma = (\gamma_1 \oplus \gamma_2) = (s_c, s_d)$ and $d = 0$. Obviously, this aggregation result is unreasonable.

From the above discussion, we can know the existing HM operator for LIFNs has two main drawbacks. (1) The HM operator can consider the interrelationships between the aggregating parameters, but it assumed that each argument is related to the rest arguments, and it cannot deal with the decision making problem with interrelationships for the attributes only in same a partition and with no interrelationships in different partitions. (2) Most of the existing HM operators adopted the traditional operational rules proposed by Chen [5], so they didn't consider the interactions between membership function and non-membership function, and could get the unreasonable results sometimes, especially when the subscript of the non-membership degree is zero. So the research goal and contributions of this paper are (1) to propose some new HM operators, which are called partitioned Heronian mean (PHM) operators, to overcome the first drawback; (2) to define new operational rules of LIFNs to overcome the second drawback; (3) to develop new decision making methods to solve the decision making problems with interrelationships for the attributes only in same a partition and with no interrelationships in different partitions.

The rest of this paper is organized as follows. In Section 2, we briefly review some basic concepts of IFSs, linguistic term sets, and LIFNs. In Section 3, we propose the new partitioned Heronian mean operators for the LIFNs, including the linguistic intuitionistic fuzzy partitioned Heronian mean (LIFPHM) operator, the linguistic intuitionistic fuzzy weighted partitioned Heronian mean (LIFWPHM) operator, the linguistic intuitionistic fuzzy partitioned geometric Heronian mean (LIFPGHM) operator and the linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator. In Section 4, we propose two MAGDM methods based on the proposed operators with the linguistic intuitionistic fuzzy information. In Section 5, we give some examples to illustrate the effectiveness of the proposed MAGDM methods and compare them with the existing methods, and to show the advantages of our methods. The conclusions are given inSection 6.

2. Preliminaries

2.1. The intuitionistic fuzzy numbers

Definition 1. [1]

Let Y be a universe of discourse with a generic element in$Y$denoted by$y$. An intuitionistic fuzzy set (IFS) B in Y is given by

$$B = \{\ll y, u_B(y), \nu_B(y) \gg > y \in Y\} \tag{1}$$

where $u_B(y)$ is the membership function, and$\nu_B(y)$is the non-membership function. For each point y in Y, we have $u_B(y)$ $\nu_B(y) \in [0, 1]$ and $0 \le u_B$ &tpcheck; $y + \nu_B(y) \le 1$.

For each IFS B in Y, let $\pi(y) = 1 - u_B(y) - \nu_B(y)$, $\forall y \in Y$, and we call $\pi(y)$ the indeterminacy degree of the element$y$to the set B [1,2]. It can be easily proved that $0 \le \pi(y) \le 1$, $\forall y \in Y$.

To the given element y, the pair $(u_B(y), \nu_B(y))$is called an intuitionistic fuzzy number (IFN). For convenience, we use$\tilde{\alpha} = (u_\alpha, \nu_\alpha)$to represent an IFN, which meets $u_\alpha \in [0, 1]$, $\nu_\alpha \in [0, 1]$ and $0 \le u_\alpha + \nu_\alpha \le 1$.

2.2. The linguistic term set

Let us consider a complete and finite ordered discrete linguistic term set $S = \{s_0, s_1, \cdots, s_t\}$, where,$t$is the even value. For example, when $t = 4$, the linguistic term set$S$and their semantics can be defined as$S = \left\{s_0\,(\text{very low}); s_1\,(\text{low}); s_2\,(\text{medium}); s_3\,(\text{high}); s_4\,(\text{very high})\right\}$.

The characteristics and the operational laws of linguistic variables can obtain from the references [2,16–19]. For convenience, we use$S_{[0,t]}$to express the set of continuous linguistic sets in the intervals [0,t] which was extended from the discrete linguistic terms$S = \{s_0, s_1, \cdots, s_t\}$in order to relieve the information loss in the calculating process.

2.3. Linguistic intuitionistic fuzzy numbers

Definition 2. [5]
Suppose $s_a, s_b \in S_{[0,t]}$ and $\gamma = (s_a, s_b)$, if $a + b \le t$, then γ can be called the linguistic intuitionistic fuzzy numbers (LIFNs).

Remark 1. If $s_a \in S_{[0,t]}$, then $(s_a, neg(s_a))$ is also a LIFN, where $neg(s_\alpha) = s_{t-\alpha}$.

Remark 2. The uncertain linguistic variables (ULV) [16] can be converted to the LIFNs. Suppose $\tilde{S} = [s_a, s_b]$ is an ULV, where $a, b \in [0, t]$, s_a and s_b are the lower and the upper limits of the ULV, then we can get that (s_a, s_{t-b}) is a LIFN which has the same information as the ULV $[s_a, s_b]$.

For convenience, we use $\Gamma_{[0,t]}$ to express the set of all LIFNs.

Definition 3. [5]
Let $\gamma = (s_a, s_b)$, $\gamma_1 = (s_{a_1}, s_{b_1})$, $\gamma_2 = (s_{a_2}, s_{b_2}) \in \Gamma_{[0,t]}$, $\lambda \gg> 0$, then the operational rules of the LIFNs can be defined as follows:

(1) $\gamma_1 \oplus \gamma_2 = \left(s_{a_1}, s_{b_1}\right) \oplus \left(s_{a_2}, s_{b_2}\right) = \left(s_{a_1+a_2-\frac{a_1a_2}{t}}, s_{\frac{b_1b_2}{t}}\right)$ (2)

(2) $\gamma_1 \otimes \gamma_2 = \left(s_{a_1}, s_{b_1}\right) \otimes \left(s_{a_2}, s_{b_2}\right) = \left(s_{\frac{a_1a_2}{t}}, s_{b_1+b_2-\frac{b_1b_2}{t}}\right)$ (3)

(3) $\lambda\gamma = \lambda(s_a, s_b) = \left(s_{t-t\left(1-\frac{a}{t}\right)^\lambda}, s_{t\left(\frac{b}{t}\right)^\lambda}\right)$ (4)

(4) $\gamma^\lambda = (s_a, s_b)^\lambda = \left(s_{t\left(\frac{a}{t}\right)^\lambda}, s_{t-t\left(1-\frac{b}{t}\right)^\lambda}\right)$ (5)

Example 1. Three teachers $T_i\,(i = 1, 2, 3)$ want to evaluate a student γ, the support and opposition ratings from three teachers are given by using linguistic intuitionistic fuzzy information $\gamma_i = (s_{a_i}, s_{b_i})(i = 1, 2, 3) \in \Gamma_{[0,8]}$. Suppose $\gamma_1 = (s_5, s_2)$, $\gamma_2 = (s_3, s_4)$, $\gamma_3 = (s_6, s_0)$, and $w = (0.25, 0.35, 0.4)^T$ is the weight vector of teachers. According to Definition 3, we can get the aggregation result of these intuitionistic fuzzy information is $\gamma = (s_c, s_d) = \overset{3}{\underset{i=1}{\oplus}} w_i\gamma_i = (s_{5.1013}, s_0)$. As we can see $s_d = s_{b_3} = 0$, in other words, $s_{b_j}\,(j = 1, 2)$ have no influences on the aggregation result. Obviously, this is unreasonable.

Definition 4. [5]
Let $\gamma = (s_a, s_b) \in \Gamma_{[0,t]}$, then

the score function Ls of the LIFN γ is

$$Ls(\gamma) = a - b \tag{6}$$

and its accuracy function is

$$Lh(\gamma) = a + b \tag{7}$$

Definition 5. [5]
Let $\gamma_1 = (s_{a_1}, s_{b_1})$, $\gamma_2 = (s_{a_2}, s_{b_2}) \in \Gamma_{[0,t]}$, then we have

(1) If $Ls(\gamma_1) \ll Ls(\gamma_2)$, then $\gamma_1 \ll \gamma_2$;
(2) If $Ls(\gamma_1) = Ls(\gamma_2)$, then
If $Lh(\gamma_1) \ll Lh(\gamma_2)$, then $\gamma_1 \ll \gamma_2$;
If $Lh(\gamma_1) = Lh(\gamma_2)$, then $\gamma_1 = \gamma_2$.

Obviously, for any $(s_a, s_b) \in \Gamma_{[0,t]}$, we have $(s_0, s_t) \le (s_a, s_b) \le (s_t, s_0)$. Suppose $\gamma_1 = \left(s_{a_1}, s_{b_1}\right)$, $\gamma_2 = \left(s_{a_2}, s_{b_2}\right) \in \Gamma_{[0,t]}$, if $a_1 \ge a_2$ and $b_1 \le b_2$, then $\gamma_1 \ge \gamma_2$.

Example 2. If $\gamma_1 = (s_4, s_3) \in \Gamma_{[0,8]}$ and $\gamma_2 = (s_4, s_2) \in \Gamma_{[0,8]}$, then based on the Definitions 4 and 5, we can get the following results.

$$\text{Ls}(\gamma_1) = 4 - 3 = 1,\ Ls(\gamma_2) = 4 - 2 = 2.$$

Since $Ls(\gamma_1) \ll Ls(\gamma_2)$, then we can get $\gamma_1 \ll \gamma_2$.

Example 3. If $\gamma_1 = (s_5, s_3) \in \Gamma_{[0,8]}$ and $\gamma_2 = (s_4, s_2) \in \Gamma_{[0,8]}$, then based on the Definitions 4 and 5, we can get the following results.

$$\text{Ls}(\gamma_1) = 5 - 3 = 2,\ Ls(\gamma_2) = 4 - 2 = 2.$$

$$\text{Lh}(\gamma_1) = 5 + 3 = 8,\ Lh(\gamma_2) = 4 + 2 = 6.$$

Since $Ls(\gamma_1) = Ls(\gamma_2)$ and $Lh(\gamma_1) \gg> Lh(\gamma_2)$, so we can get $\gamma_1 \gg> \gamma_2$.

3. The linguistic intuitionistic fuzzy aggregation operators

3.1. Partitioned heronian mean

Heronian mean (HM) can capture the interrelationships of the input data [9] [14], which is an important tool to solve the MADM problems. It can be defined as follows:

Definition 6. [9]

Let $I=[0, 1]$, $p, q \geq 0$, $H^{p,q}: I^m \to I$, then the Heronian mean (HM) operator $H^{p,q}$ can be defined as:

$$HM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \left(\frac{2}{m(m+1)} \sum_{i=1}^{m} \sum_{j=i}^{m} \gamma_i^p \gamma_j^q \right)^{\frac{1}{p+q}} \tag{8}$$

Obviously, the HM operator has some properties of idempotency, monotonicity and boundedness.

Definition 7. [12]

Let $I=[0, 1]$, $p, q \geq 0$, $H^{p,q}: I^m \to I$, then the geometric Heronian mean (GHM) operator$H^{p,q}$can be defined as:

$$GHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \frac{1}{p+q} \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(p\gamma_i + p\gamma_j \right) \right)^{\frac{2}{m(m+1)}} \tag{9}$$

We know that HM and GHM have the advantage of considering the interrelationships between any two input arguments. But in many situations, input arguments can be partitioned into many distinct sorts and the members in same sort are interrelated, and the members in the different sorts are independent. In this background, we propose the partitioned Heronian mean operator and the partitioned geometric Heronian mean operator.

Definition 8. Let$\gamma_i \, (i = 1, 2, \ldots, m)$be a collection of inputs, which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where

$P_h = \left\{ \gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|} \right\} (\mathrm{h} = 1, 2, \cdots, \mathrm{d})$, $\sum_{h=1}^{d} |P_h| = m$and$|P_h|$denotes the cardinality of P_h. For any $p, q \geq 0$, the aggregation functions:

$$PHM\left(\gamma_1, \gamma_2, \cdots, \gamma_m\right) = \frac{1}{d} \left(\sum_{h=1}^{d} \left(\frac{2}{|P_h|\left(|P_h|+1\right)} \sum_{i=1}^{|P_h|} \sum_{j=i}^{|P_h|} \gamma_{hi}^{P} \gamma_{hj}^{q} \right)^{\frac{1}{p+q}} \right) \tag{10}$$

is called partitioned Heronian mean(PHM) operator.

Definition 9. Let$\gamma_i \, (i = 1, 2, \ldots, m)$be a collection of inputs, which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where$P_h = \left\{ \gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|} \right\} (\mathrm{h} = 1, 2, \cdots, \mathrm{d})$, $\sum_{h=1}^{d} |P_h| = m$ and$|P_h|$denotes the cardinality of P_h. For any $p, q \geq 0$, the aggregation functions:

$$PGHM\left(\gamma_1, \gamma_2, \cdots, \gamma_m\right) = \left(\prod_{h=1}^{d} \left(\frac{1}{p+q} \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(p\gamma_{hi} + p\gamma_{hj} \right) \right)^{\frac{2}{|P_h|\left(|P_h|+1\right)}} \right) \right)^{1/d} \tag{11}$$

is called partitioned geometric Heronian mean(PGHM) operator.

3.2. New operational rules for LIFNs

In order to solve the problem in Example 1, we considered the interactions between the membership function and the non-membership function of LIFNs and proposed the new operational rules as follows.

Definition 10. Let$\gamma = \left(s_a, s_b\right)$, $\gamma_1 = \left(s_{a_1}, s_{b_1}\right)$, $\gamma_2 = \left(s_{a_2}, s_{b_2}\right) \in \Gamma_{[0,t]}$, $\lambda \gg> 0$, then the new operational rules of the LIFNs are defined as follows:

$$(1)\ \gamma_1 \oplus \gamma_2 = \left(s_{a_1}, s_{b_1}\right) \oplus \left(s_{a_2}, s_{b_2}\right) = \left(s_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{a_i}{t}\right)\right)}, s_{t\left(\prod_{i=1}^{2}\left(1-\frac{a_i}{t}\right)-\prod_{i=1}^{2}\left(1-\frac{a_i}{t}-\frac{b_i}{t}\right)\right)} \right) \tag{12}$$

$$(2)\ \gamma_1 \otimes \gamma_2 = \left(s_{a_1}, s_{b_1}\right) \otimes \left(s_{a_2}, s_{b_2}\right) = \left(s_{t\left(\prod_{i=1}^{2}\left(1-\frac{b_i}{t}\right)-\prod_{i=1}^{2}\left(1-\frac{a_i}{t}-\frac{b_i}{t}\right)\right)}, s_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{b_i}{t}\right)\right)} \right) \tag{13}$$

$$(3)\ \lambda\gamma = \lambda\left(s_a, s_b\right) = \left(s_{t\left(1-\left(1-\frac{a}{t}\right)^{\lambda}\right)}, s_{t\left(\left(1-\frac{a}{t}\right)^{\lambda}-\left(1-\frac{a}{t}-\frac{b}{t}\right)^{\lambda}\right)} \right) \tag{14}$$

(4) $\gamma^{\lambda} = (s_a, s_b)^{\lambda} = \left(s_{t\left(\left(1-\frac{b}{t}\right)^{\lambda} - \left(1-\frac{a}{t}-\frac{b}{t}\right)^{\lambda} \right)}, s_{t\left(1-\left(1-\frac{b}{t}\right)^{\lambda}\right)} \right)$ (15)

Example 4. Let us re-consider the problem in Example 1 with the new interaction operational laws in Definition 10, we can get the aggregation result of the linguistic intuitionistic fuzzy information which is$\gamma = (s_c, s_d) = \overset{3}{\underset{i=1}{\oplus}} w_i \gamma_i = (s_{4.9498}, s_{1.7307})$, and meetsmin $\left\{s_{b_1}, s_{b_2}, s_{b_3}\right\} \le s_d \le \max\left\{s_{b_1}, s_{b_2}, s_{b_3}\right\}$. Obviously, this result is reasonable.

In the following, on the basis of the new operational laws of LIFNs, we extend the partitioned Heronian mean (PHM) to the LIFNs and propose the linguistic intuitionistic fuzzy PHM (LIFPHM) operator, the linguistic intuitionistic fuzzy weighted PHM (LIFWPHM) operator, the linguistic intuitionistic fuzzy partitioned geometric Heronian mean (LIFPGHM) operator and the linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator. Then we will research some properties and some special cases of these new operators.

3.3. The LIFPHM and LIFWPHM operators

Definition 11. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)(i = 1, 2, \cdots, m)$is a collection of the LIFNs which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$,where $P_h = \left\{\gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|}\right\}$ (h = 1, 2, ···, d), $\sum_{h=1}^{d} |P_h| = m$and$|P_h|$denotes the cardinality of P_h. For any $p, q \ge 0$,$LIFPHM : \Gamma_{[0,t]}^{m} \to \Gamma_{[0,t]}$, if

$$LIFPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \frac{1}{d}\left(\sum_{h=1}^{d} \left(\frac{2}{|P_h|(|P_h|+1)} \sum_{i=1}^{|P_h|} \sum_{j=i}^{|P_h|} \gamma_{hi}^{p} \otimes \gamma_{hj}^{q} \right)^{\frac{1}{p+q}} \right) \tag{16}$$

where$\Gamma_{[0,t]}$is the set of all LIFNs, then *LIFPHM* is called the linguistic intuitionistic fuzzy partitioned Heronian mean (LIFPHM) aggregation operator.

Based on the operational laws of the LIFNs proposed in formulas (12–15), from (16), we can get the aggregation result shown as theorem 1.

Theorem 1. Suppose$\gamma_i = \left(s_{a_i}, s_{b_i}\right)(i = 1, 2, \cdots, m)$is a collection of the LIFNs, and for any$p, q \ge 0$, then, the result aggregated from (16) is still a LIFN, and even

$$LIFPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)}, \right.$$

$$\left. s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\prod_{h=1}^{d}\left(\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)} \right) \tag{17}$$

where, $\alpha_{hi} = 1 - v_{hi}$, $\alpha_{hj} = 1 - v_{hj}$, $\beta_{hi} = 1 - u_{hi} - v_{hi}$, $\beta_{hj} = 1 - u_{hj} - v_{hj}$ and $u_{hi} = \frac{a_{hi}}{t}$, $u_{hj} = \frac{a_{hj}}{t}$, $v_{hi} = \frac{b_{hi}}{t}$, $v_{hj} = \frac{b_{hj}}{t}$.

Proof. Since$\gamma_{hi}^{p} = \left(s_{t\left(\left(1-\frac{b_{hi}}{t}\right)^{p}-\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)}, s_{t\left(1-\left(1-\frac{b_{hi}}{t}\right)^{p}\right)} \right)and\gamma_{hj}^{q} = \left(s_{t\left(\left(1-\frac{b_{hj}}{t}\right)^{q}-\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)}, s_{t\left(1-\left(1-\frac{b_{hj}}{t}\right)^{q}\right)} \right)$,

let $\frac{a_{hi}}{t} = u_{hi}\,\frac{a_{hj}}{t} = u_{hj}$, $\frac{b_{hi}}{t} = v_{hi}$, $\frac{b_{hj}}{t} = v_{hj}$,
then

$$\gamma_{hi}^{p} = \left(s_{t\left((1-v_{hi})^{p}-(1-u_{hi}-v_{hi})^{p}\right)}, s_{t\left(1-(1-v_{hi})^{p}\right)} \right),\ \gamma_{hj}^{q} = \left(s_{t\left((1-v_{hj})^{q}-(1-u_{hj}-v_{hj})^{q}\right)}, s_{t\left(1-(1-v_{hj})^{q}\right)} \right),$$

and

$$\gamma_{hi}^{p} \otimes \gamma_{hj}^{q} = \left(s_{t\left((1-v_{hi})^{p}(1-v_{hj})^{q}-(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)}, s_{t\left(1-(1-v_{hi})^{p}(1-v_{hj})^{q}\right)} \right)$$

then

$$\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}=\left(s_{t\left(1-\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-(1-v_{hi})^{p}(1-v_{hj})^{q}+(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)\right)},s_{t\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-(1-v_{hi})^{p}(1-v_{hj})^{q}+(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)-\prod_{\substack{i=1\\j=i}}^{|P_h|}(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)}\right)$$

Further,

$$\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}=\left(s_{t\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-(1-v_{hi})^{p}(1-v_{hj})^{q}+(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-(1-v_{hi})^{p}(1-v_{hj})^{q}+(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}(1-u_{hi}-v_{hi})^{p}(1-u_{hj}-v_{hj})^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)}\right),$$

and let $1-v_{hi}=\alpha_{hi}$, $1-v_{hj}=\alpha_{hj}$, $1-u_{hi}-v_{hi}=\beta_{hi}$, $1-u_{hj}-v_{hj}=\beta_{hj}$,
we can get

we can get $$\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}=\left(s_{t\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)},s_{t\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)}\right)$$

$$\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}\right)^{\frac{1}{p+q}}=\left(s_{t\left(\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}-\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)},\right.$$

$$\left.s_{t\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)}\right),$$

then

$$\sum_{h=1}^{d}\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}\right)^{\frac{1}{p+q}}=\left(s_{t\left(1-\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)},\right.$$

$$\left.s_{t\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)-\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)}\right),$$

So, $$\frac{1}{d}\left(\sum_{h=1}^{d}\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}\gamma_{hi}^{P}\otimes\gamma_{hj}^{q}\right)^{\frac{1}{p+q}}\right)=$$

then

$$\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)}, \right.$$

$$\left. s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)} \right) \tag{18}$$

Example 5. Let $\gamma_1 = (s_6, s_1)$, $\gamma_2 = (s_5, s_2)$, $\gamma_3 = (s_3, s_4)$, $\gamma_4 = (s_7, s_0)$ and $\gamma_5 = (s_5, s_1) \in \Gamma_{[0,8]}$ be five LIFNs, and suppose that we partition the five LIFNs into two sorts, i.e., $P_1 = \{\gamma_{11}, \gamma_{12}, \gamma_{13}\} = \{\gamma_1, \gamma_2, \gamma_3\}$ and $P_2 = \{\gamma_{24}, \gamma_{25}\} = \{\gamma_4, \gamma_5\}$, then we can use the *LIFPHM* operator to aggregate them. Suppose the aggregated result is $\gamma = (s_a, s_b)$, without loss of generality, let $p = q = 1$, then we get

(1) When $h = 1$, we get

$$\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\left(1-\frac{b_{1i}}{t}\right)^p\left(1-\frac{b_{1j}}{t}\right)^q+\left(1-\frac{a_{1i}}{t}-\frac{b_{1i}}{t}\right)^p\left(1-\frac{a_{1j}}{t}-\frac{b_{1j}}{t}\right)^q\right)\right)^{\frac{2}{|P_1|(|P_1|+1)}}$$

$$=\left(\left(1-\left(1-\frac{1}{8}\right)^1\left(1-\frac{1}{8}\right)^1+\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\right)\times\left(1-\left(1-\frac{1}{8}\right)^1\left(1-\frac{3}{8}\right)^1+\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\right)\times\right.$$

$$\left(1-\left(1-\frac{1}{8}\right)^1\left(1-\frac{4}{8}\right)^1+\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\right)\times\left(1-\left(1-\frac{2}{8}\right)^1\left(1-\frac{2}{8}\right)^1+\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\right)\times$$

$$\left.\left(1-\left(1-\frac{2}{8}\right)^1\left(1-\frac{4}{8}\right)^1+\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\right)\times\left(1-\left(1-\frac{4}{8}\right)^1\left(1-\frac{4}{8}\right)^1+\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\right)\right)^{\frac{2}{3\times 4}}$$

$$=0.8282,$$

and $\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\frac{a_{1i}}{t}-\frac{b_{1i}}{t}\right)^p\left(1-\frac{a_{1j}}{t}-\frac{b_{1j}}{t}\right)^q\right)^{\frac{2}{|P_1|(|P_1|+1)}}=\left(\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\times\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\times\right.$

$$\left(1-\frac{6}{8}-\frac{1}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\times\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\times\left(1-\frac{5}{8}-\frac{2}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1$$

$$\left.\times\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\left(1-\frac{3}{8}-\frac{4}{8}\right)^1\right)^{\frac{2}{3\times 4}}=0.01563,$$

then

$$\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\left(1-\frac{b_{1i}}{t}\right)^p\left(1-\frac{b_{1j}}{t}\right)^q+\left(1-\frac{a_{1i}}{t}-\frac{b_{1i}}{t}\right)^p\left(1-\frac{a_{1j}}{t}-\frac{b_{1j}}{t}\right)^q\right)\right)^{\frac{2}{|P_1|(|P_1|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\frac{a_{1i}}{t}-\frac{b_{1i}}{t}\right)^p\left(1-\frac{a_{1j}}{t}-\frac{b_{1j}}{t}\right)^q\right)^{\frac{2}{|P_1|(|P_1|+1)}}\right)^{1/(p+q)}$$

$$=(1-0.8282+0.01563)^{\frac{1}{1+1}}=0.4330,$$

and $\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\frac{a_{1i}}{t}-\frac{b_{1i}}{t}\right)^p\left(1-\frac{a_{1j}}{t}-\frac{b_{1j}}{t}\right)^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}=0.01563^{\frac{1}{1+1}}=0.125$

(2) When h=2, we have

$$\left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\left(1-\frac{b_{2i}}{t}\right)^p\left(1-\frac{b_{2j}}{t}\right)^q+\left(1-\frac{a_{2i}}{t}-\frac{b_{2i}}{t}\right)^p\left(1-\frac{a_{2j}}{t}-\frac{b_{2j}}{t}\right)^q\right)\right)^{\frac{2}{|P_2|(|P_2|+1)}}$$

$$=\left(\left(1-\left(1-\frac{0}{8}\right)^1\left(1-\frac{0}{8}\right)^1+\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\right)\times\left(1-\left(1-\frac{0}{8}\right)^1\left(1-\frac{1}{8}\right)^1+\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\right)\times\right.$$

(2) When $h = 2$, we have

$$\left(1-\left(1-\frac{1}{8}\right)^1\left(1-\frac{1}{8}\right)^1+\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\right)\Bigg)^{\frac{2}{2\times3}}=0.9682,$$

and $\left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\frac{a_{2i}}{t}-\frac{b_{2i}}{t}\right)^p\left(1-\frac{a_{2j}}{t}-\frac{b_{2j}}{t}\right)^q\right)^{\frac{2}{|P_2|(|P_2|+1)}}=\left(\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\times\left(1-\frac{7}{8}-\frac{0}{8}\right)^1\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\times\right.$

$$\left.\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\left(1-\frac{5}{8}-\frac{1}{8}\right)^1\right)^{\frac{2}{2\times3}}=0.0313,$$

then

$$\left(\begin{array}{l}1-\left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\left(1-\frac{b_{2i}}{t}\right)^p\left(1-\frac{b_{2j}}{t}\right)^q+\left(1-\frac{a_{2i}}{t}-\frac{b_{2i}}{t}\right)^p\left(1-\frac{a_{2j}}{t}-\frac{b_{2j}}{t}\right)^q\right)\right)^{\frac{2}{|P_2|(|P_2|+1)}}+\\ \left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\frac{a_{2i}}{t}-\frac{b_{2i}}{t}\right)^p\left(1-\frac{a_{2j}}{t}-\frac{b_{2j}}{t}\right)^q\right)^{\frac{2}{|P_2|(|P_2|+1)}}\end{array}\right)^{1/(p+q)}$$

$$=(1-0.9682+0.0313)^{\frac{1}{1+1}}=0.2512,$$

and $\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\frac{a_{2i}}{t}-\frac{b_{2i}}{t}\right)^p\left(1-\frac{a_{2j}}{t}-\frac{b_{2j}}{t}\right)^q\right)^{\frac{2}{|P_2|(|P_2|+1)}}\right)^{1/(p+q)}=0.0313^{\frac{1}{1+1}}=0.1768.$

So, $a=8\times\left(1-((1-0.4330+0.125)\times(1-0.2512+0.1768))^{\frac{1}{2}}\right)=1.5973$ (3).

Similar to the process of calculating a, we can get b = 5.2134.

Thus$LIFPHM^{1,1}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_4)=\gamma=(s_a,s_b)=(s_{1.5973},s_{5.2134})$.

In the following, we will explore some characteristics of theLIFPHMoperator.

Theorem 2 ((Idempotency).). Let$(\gamma_1, \gamma_2, \ldots, \gamma_m)$be a collection of LIFNs, if all γ_k (k = 1, 2, ..., m) are equal, i.e., $\gamma_k=\gamma=(s_a, s_b)$, $k=1,2,\cdots,m$, then

$$LIFPHM^{p,q}(\gamma_1,\gamma_2,\cdots,\gamma_m)=\gamma \quad (19)$$

Proof. Let $LIFPHM^{p,q}(\gamma_1,\gamma_2,\cdots,\gamma_m)=(s_\vartheta,s_\rho)$, since the formula (17) is equal to the formula (18), we can only prove the formula (18).

(a) We firstly prove that $s_\vartheta=s_a$, i.e.,$\vartheta=a$.

As mentioned above, we can knows$s_{a_{hi}}=s_{a_{hj}}=s_aands_{b_{hi}}=s_{b_{hj}}=s_b$, let$\frac{a_{hi}}{t}=u_{hi}$,$\frac{a_{hj}}{t}=u_{hj}$,$\frac{b_{hi}}{t}=v_{hi}$,$\frac{b_{hj}}{t}=v_{hj}$, and $1-v_{hi}=\alpha_{hi}$, $1-v_{hj}=\alpha_{hj}$, $1-u_{hi}-v_{hi}=\beta_{hi}$, $1-u_{hj}-v_{hj}=\beta_{hj}$, then $u=u_{hi}=u_{hj}$, $v=v_{hi}=v_{hj}$, $\alpha=\alpha_{hi}=\alpha_{hj}$ and $\beta=\beta_{hi}=\beta_{hj}$, we can get

$$\vartheta=t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)$$

$$=t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha^{p+q}+\beta^{p+q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta^{p+q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta^{p+q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)$$

$$=t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(1-\alpha^{p+q}+\beta^{p+q}\right)+\beta^{p+q}\right)^{1/(p+q)}+\left(\beta^{p+q}\right)^{1/(p+q)}\right)\right)^{1/d}\right)$$

$$=t\left(1-\left(\prod_{h=1}^{d}(1-\alpha+\beta)\right)^{1/d}\right)=t\left(1-\left(\prod_{h=1}^{d}(1-(1-v)+(1-u-v))\right)^{1/d}\right)=t(1-(1-u))=t\frac{a}{t}=a.$$

i.e., $s_\vartheta=s_a$.

(b) We also prove that $s_\rho = s_b$.
Similar to (a), we can also prove that $s_\rho = s_b$.
According to (a) and (b), we have $(s_\vartheta, s_\rho) = (s_a, s_b)$, i.e., $LIFPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \gamma$. **Q. E. D.**

Theorem 3 ((Commutativity)). Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)$ and $\gamma'_i = \left(s'_{a_i}, s'_{b_i}\right)$ $(i = 1, 2, \cdots m)$ are two collections of IFNs, if $(\gamma'_1, \gamma'_2, \cdots, \gamma'_m)$ is any permutation of $(\gamma_1, \gamma_2, \cdots, \gamma_m)$, then

$$LIFPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = LIFPHM^{p,q}(\gamma'_1, \gamma'_2, \cdots, \gamma'_m) \tag{20}$$

Proof. Based on Eq. (18), we have

$$LIFPHM^{p,q}(\gamma_1,\gamma_2,\cdots,\gamma_m)=\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)}\right),$$

$$LIFPHM^{p,q}(\gamma'_1,\gamma'_2,\cdots,\gamma'_m)=$$

$$\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{\prime p}\alpha_{hj}^{\prime q}+\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{\prime p}\alpha_{hj}^{\prime q}+\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{\prime p}\beta_{hj}^{\prime q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)}\right),$$

Since $(\gamma'_1, \gamma'_2, \cdots, \gamma'_m)$ is any permutation of $(\gamma_1, \gamma_2, \cdots, \gamma_m)$, then we can get $LIFPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = LIFPHM^{p,q}(\gamma'_1, \gamma'_2, \cdots, \gamma'_m)$.
In the following, we will study some special cases of the $LIFPHM^{p,q}$ operator in regard to the parameters p and q.
(i) When $q \to 0$, then we can get

$$LIFPHM^{p,0}(\gamma_1,\gamma_2,\cdots,\gamma_m)=\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1}^{|P_h|}\left(1-\left(1-\frac{b_{hi}}{t}\right)^{p}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}+\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1}^{|P_h|}\left(1-\left(1-\frac{b_{hi}}{t}\right)^{p}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}+\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)^{1/d}\right)}\right) \tag{21}$$

(ii) When $p = 1$, $q \to 0$, then we can get

$$LIFPHM^{1,0}(\gamma_1,\gamma_2,\cdots,\gamma_m)=\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}\right)},\; s_{t\left(\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}\right)}\right) \tag{22}$$

(iii) When $p \to 0$, then we can get

$$LIFPHM^{0,q}(\gamma_1,\gamma_2,\cdots,\gamma_m)=$$

$$\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{b_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{b_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)^{1/d}\right)}\right) \tag{23}$$

(iv) When $p = q = 1$, then we can get

$$LIFPBM^{1,1}(\gamma_1,\gamma_2,\cdots,\gamma_m)=$$

$$\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{b_{hi}}{t}\right)\left(1-\frac{b_{hj}}{t}\right)+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{b_{hi}}{t}\right)\left(1-\frac{b_{hj}}{t}\right)+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)^{1/d}\right)}\right) \tag{24}$$

As we all know, the $LIFPHM^{p,q}$ operator only considers the interrelationships between input parameters; however, it doesn't take into account the self-importance of parameters. In order to overcome this drawback, we will further develop a linguistic intuitionistic fuzzy weighted PHM (LIFWPHM) operator.

Definition 12. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)(i = 1, 2, \cdots, m)$ is a collection of the LIFNs partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where $P_h = \left\{\gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|}\right\}(h = 1, 2, \cdots, d)$, $\sum_{h=1}^{d}|P_h| = m$ and $|P_h|$ denotes the cardinality of P_h. For any $p, q \geq 0$, $LIFWPBM : \Gamma_{[0,t]}^{m} \to \Gamma_{[0,t]}$, if

$$LIFWPHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \frac{1}{d}\left(\sum_{h=1}^{d}\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p \otimes \left(mw_{hj}\gamma_{hj}\right)^q\right)^{\frac{1}{p+q}}\right) \tag{25}$$

where $\Gamma_{[0,t]}$ is the set of all LIFNs, and $w = (w_1, w_2, \cdots, w_m)^T$ is the weight vector of $(\gamma_1, \gamma_2, \cdots, \gamma_m)$, $w_j \in [0, 1]$, $\sum_{j=1}^{m} w_j = 1$, then *LIFPBM* is called the linguistic intuitionistic fuzzy partitioned Heronian mean (LIFPHM) aggregation operator.

Based on the operational laws of the LIFNs described in formulas (12–15), from Eq. (25), we can get the aggregation result shown as theorem 4.

Theorem 4. Suppose $\gamma_k = \left(s_{a_k}, s_{b_k}\right)$ $(k = 1, 2, \cdots, m)$ is a collection of the LIFNs, and for any $p, q \geq 0$, then, the result aggregated from (25) is still a LIFN, and even

$$LIFWPHM^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_m) = \left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1,j=i}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{i=1,j=i}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)}, \right.$$
$$\left. s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1,j=i}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{i=1,j=i}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{i=1,j=i}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)} \right) \tag{26}$$

where, $\alpha_{hi} = 1 - u_{hi}^{mw_{hi}} + v_{hi}^{mw_{hi}}$, $\alpha_{hj} = 1 - u_{hj}^{mw_{hj}} + v_{hj}^{mw_{hj}}$, $\beta_{hi} = v_{hi}^{mw_{hi}}$, $\beta_{hj} = v_{hj}^{mw_{hj}}$, and $u_{hi} = 1 - \frac{a_{hi}}{t}$, $u_{hj} = 1 - \frac{a_{hj}}{t}$, $v_{hi} = 1 - \frac{a_{hi}}{t} - \frac{b_{hi}}{t}$, $v_{hj} = 1 - \frac{a_{hj}}{t} - \frac{b_{hj}}{t}$.

Proof.

Since $mw_{hi}\gamma_{hi} = \left(s_{t-t\left(1-\frac{a_{hi}}{t}\right)^{mw_{hi}}}, s_{t\left(1-\frac{a_{hi}}{t}\right)^{mw_{hi}}-t\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{mw_{hi}}}\right)$, $mw_{hj}\gamma_{hj} = \left(s_{t-t\left(1-\frac{a_{hj}}{t}\right)^{mw_{hj}}}, s_{t\left(1-\frac{a_{hj}}{t}\right)^{mw_{hj}}-t\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{mw_{hj}}}\right)$, and

$$\left(mw_{hi}\gamma_{hi}\right)^p = \left(s_{t\left(1-\left(1-\frac{a_{hi}}{t}\right)^{mw_{hi}}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{mw_{hi}}\right)^p-t\left(\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{mw_{hi}}\right)^p}, s_{t-t\left(1-\left(1-\frac{a_{hi}}{t}\right)^{mw_{hi}}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{mw_{hi}}\right)^p}\right),$$

$$\left(mw_{hj}\gamma_{hj}\right)^q = \left(s_{t\left(1-\left(1-\frac{a_{hj}}{t}\right)^{mw_{hj}}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{mw_{hj}}\right)^q-t\left(\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{mw_{hj}}\right)^q}, s_{t-t\left(1-\left(1-\frac{a_{hj}}{t}\right)^{mw_{hj}}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{mw_{hj}}\right)^q}\right),$$

let $1-\frac{a_{hi}}{t}=u_{hi}$, $1-\frac{a_{hj}}{t}=u_{hj}$, $1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}=v_{hi}$, $1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}=v_{hj}$,

then $(mw_{hi}\gamma_{hi})^p=\left(s_{t\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p-t\left(v_{hi}^{mw_{hi}}\right)^p},s_{t-t\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p}\right)$, $(mw_{hj}\gamma_{hj})^q=\left(s_{t\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q-t\left(v_{hj}^{mw_{hj}}\right)^q},s_{t-t\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q}\right)$,

$$(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q=\left(s_{t\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q-t\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q},s_{t-t\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q}\right)$$

and

$$\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q=\left(s_{t\left(1-\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q+\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)\right)},\right.$$
$$\left.s_{t\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q+\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)-\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)}\right),$$

Further,

$$\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q=\left(s_{t\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q+\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)},\right.$$
$$\left.s_{t\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}\right)^p\left(1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}\right)^q+\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(v_{hi}^{mw_{hi}}\right)^p\left(v_{hj}^{mw_{hj}}\right)^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)}\right),$$

and let $1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}=\alpha_{hi}$, $1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}=\alpha_{hj}$, $v_{hi}^{mw_{hi}}=\beta_{hi}$, $v_{hj}^{mw_{hj}}=\beta_{hj}$,

we can get

$$\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q=$$

$$\left(s_{t\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)},s_{t\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)}\right)$$

$$\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q\right)^{\frac{1}{p+q}}=$$

$$\left(s_{t\left(\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}-\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)},\ s_{t\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)}\right),$$

then

$$\sum_{h=1}^{d}\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q\right)^{\frac{1}{p+q}}=$$

$$\left(s_{t\left(1-\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)},\right.$$

$$\left.s_{t\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)-\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)}\right),$$

So, $\frac{1}{d}\left(\sum_{h=1}^{d}\left(\frac{2}{|P_h|(|P_h|+1)}\sum_{i=1}^{|P_h|}\sum_{j=i}^{|P_h|}(mw_{hi}\gamma_{hi})^p\otimes(mw_{hj}\gamma_{hj})^q\right)^{\frac{1}{p+q}}\right)=$

$$\left(s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^p\alpha_{hj}^q+\beta_{hi}^p\beta_{hj}^q\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^p\beta_{hj}^q\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)}\right)$$

So we get that (26) is kept.

Example 6. Let $\gamma_1=(s_6,s_1)$, $\gamma_2=(s_5,s_2)$, $\gamma_3=(s_3,s_4)$, $\gamma_4=(s_7,s_0)$ and $\gamma_5=(s_5,s_1)\in\Gamma_{[0,8]}$ be five LIFNs, and $w=(0.25,0.2,0.22,0.15,0.18)^T$ be the weight vector of $\gamma_i\,(i=1,2,3,4,5)$. Suppose that we partition the five numbers into two sorts, i.e., $P_1=\{\gamma_{11},\gamma_{12},\gamma_{13}\}=\{\gamma_1,\gamma_2,\gamma_3\}$ and $P_2=\{\gamma_{24},\gamma_{25}\}=\{\gamma_4,\gamma_5\}$, then we can use the *LIFPBM* operator to aggregate the five LIFNs. Suppose the comprehensive value is $\gamma=(s_a,s_b)$, without loss of generality, let $p=q=1$, then we get

$$u_{11}=1-\frac{a_{11}}{t}=0.25,\ u_{12}=1-\frac{a_{12}}{t}=0.375,\ u_{13}=1-\frac{a_{13}}{t}=0.625,\ u_{24}=1-\frac{a_{24}}{t}=0.125,\ u_{25}=1-\frac{a_{25}}{t}=0.375$$

$$v_{11}=1-\frac{a_{11}}{t}-\frac{b_{11}}{t}=0.125, v_{12}=1-\frac{a_{12}}{t}-\frac{b_{12}}{t}=0.125, v_{13}=1-\frac{a_{13}}{t}-\frac{b_{13}}{t}=0.125, v_{24}=1-\frac{a_{24}}{t}-\frac{b_{24}}{t}=0.125$$

$$v_{25}=1-\frac{a_{25}}{t}-\frac{b_{25}}{t}=0.25 \text{and} \alpha_{11}=1-u_{11}^{mw_{11}}+v_{11}^{mw_{11}}=0.8975, \alpha_{12}=1-u_{12}^{mw_{12}}+v_{12}^{mw_{12}}=0.75,$$

$$\alpha_{13}=1-u_{13}^{mw_{13}}+v_{13}^{mw_{13}}=0.5052, \alpha_{24}=1-u_{24}^{mw_{24}}+v_{24}^{mw_{24}}=1, \alpha_{25}=1-u_{25}^{mw_{25}}+v_{25}^{mw_{25}}=0.8735, \beta_{11}=v_{11}^{mw_{11}} =0.0743, \beta_{12}=v_{12}^{mw_{12}}=0.125, \beta_{13}=v_{13}^{mw_{13}}=0.1015, \beta_{24}=v_{24}^{mw_{24}}=0.2102, \beta_{25}=v_{25}^{mw_{25}}=0.2872$$

then

$$\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\alpha_{1i}^{p}\alpha_{1j}^{q}+\beta_{1i}^{p}\beta_{1j}^{q}\right)\right)^{\frac{2}{|P_1|(|P_1|+1)}}=\left(\left(1-0.8975^1 0.8975^1+0.0743^1 0.0743^1\right)\times\left(1-0.8975^1 0.75^1+0.0743^1 0.125^1\right)\right.$$

$$\times\left(1-0.8975^1 0.5052^1+0.0743^1 0.1015^1\right)\times\left(1-0.75^1 0.75^1+0.125^1 0.125^1\right)$$

$$\times\left(1-0.75^1 0.5052^1+0.125^1 0.1015^1\right)\times\left(1-0.5052^1 0.5052^1+0.1015^1 0.1015^1\right)^{\frac{2}{3\times 4}}=0.4479,$$

and $\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\beta_{1i}^{p}\beta_{1j}^{q}\right)^{\frac{2}{|P_1|(|P_1|+1)}}$

$$=\left(0.0743^1 0.0743^1\times 0.0743^1 0.125^1\times 0.0743^1 0.1015^1\times 0.125^1 0.125^1\times 0.125^1 0.1015^1\times 0.1015^1 0.1015^1\right)^{\frac{2}{3\times 4}}=0$$

further

$$\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\left(1-\alpha_{1i}^{p}\alpha_{1j}^{q}+\beta_{1i}^{p}\beta_{1j}^{q}\right)\right)^{\frac{2}{|P_1|(|P_1|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\beta_{1i}^{p}\beta_{1j}^{q}\right)^{\frac{2}{|P_1|(|P_1|+1)}}\right)^{1/(p+q)}=(1-0.4479+0.0096)^{\frac{1}{1+1}}=0.7495,$$

and $\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_1|}\beta_{1i}^{p}\beta_{1j}^{q}\right)^{\frac{2}{|P_1|(|P_1|+1)}}\right)^{1/(p+q)}=0.0096^{\frac{1}{1+1}}=0.0981$

In addition, we have

$$\left(\prod_{\substack{i=1\\j=i}}^{|P_2|}\left(1-\alpha_{2i}^{p}\alpha_{2j}^{q}+\beta_{2i}^{p}\beta_{2j}^{q}\right)\right)^{\frac{2}{|P_2|(|P_2|+1)}}=$$

$$=\left(\left(1-1^1 1^1+0.2102^1 0.2102^1\right)\times\left(1-1^1 0.8735^1+0.2102^1 0.2872^1\right)\times\left(1-0.8735^1 0.8735^1+0.2872^1 0.2872^1\right)\right)^{\frac{2}{2\times 3}}=0.1382,$$

and $\left(\prod_{\substack{i=1\\j=i}}^{|P_2|} \beta_{2i}^{p}\beta_{2j}^{q}\right)^{\frac{2}{|P_2|(|P_2|+1)}} = \left(0.2102^1 0.2102^1 \times 0.2102^1 0.2872^1 \times 0.2872^1 0.2872^1\right)^{\frac{2}{2\times 3}} = 0.0604$

then $\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_2|} \left(1-\alpha_{2i}^{p}\alpha_{2j}^{q}+\beta_{2i}^{p}\beta_{2j}^{q}\right)\right)^{\frac{2}{|P_2|(|P_2|+1)}} + \left(\prod_{\substack{i=1\\j=i}}^{|P_2|} \beta_{2i}^{p}\beta_{2j}^{q}\right)^{\frac{2}{|P_2|(|P_2|+1)}}\right)^{1/(p+q)} = (1-0.1382+0.0604)^{\frac{1}{1+1}} =$ 0.9603,

and $\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_2|} \beta_{2i}^{p}\beta_{2j}^{q}\right)^{\frac{2}{|P_2|(|P_2|+1)}}\right)^{1/(p+q)} = 0.0604^{\frac{1}{1+1}} = 0.2457$

So, $a = 8 \times \left(1-((1-0.7495+0.0981)\times(1-0.9603+0.2457))^{\frac{1}{2}}\right) = 5.4767.$

Similar to the process of calculating a, we can get $b = 1.2814$.

Thus $LIFPHM^{1,1}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_4) = \gamma = (s_a, s_b) = (s_{5.4767}, s_{1.2814})$.

3.4. LIFPGHM and LIFWPGHM operators

The geometric aggregation operators are the dual operators of arithmetic aggregation operators, and they can stress the role of the individual data and do not allow a "short board" while arithmetic aggregation operators emphasize the impact of the overall data, and allow the strong complementarity among the various data [12]. So it is necessary to propose LIFPGHM and LIFWPGHM operators shown as follows.

Definition 13. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)$ $(i = 1, 2, \cdots, m)$ is a collection of the LIFNs partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where $P_h = \left\{\gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|}\right\}$ (h = 1, 2, ⋯, d), $\sum_{h=1}^{d}|P_h| = m$ and $|P_h|$ denotes the cardinality of P_h. For any $p, q \geq 0$, $LIFPGHM : \Gamma_{[0,t]}^{m} \to \Gamma_{[0,t]}$, if

$$LIFPGHM^{p,q}(\gamma_1,\gamma_2,\cdots,\gamma_m) = \left(\prod_{h=1}^{d}\left(\frac{1}{p+q}\left(\prod_{\substack{i,j\,\in\, P_h\\ i\neq j}}\left(p\gamma_{hi}\oplus q\gamma_{hj}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)\right)^{1/d} \tag{27}$$

where $\Gamma_{[0,t]}$ is the set of all LIFNs, then *LIFPGHM* is called the linguistic intuitionistic fuzzy partitioned geometric Heronian mean (LIFPGHM) operator.

Based on the operational laws of the LIFNs described in formulas (12–15), from (27), we can get the aggregation result shown as theorem 5.

Theorem 5. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)$ $(i = 1, 2, \cdots, m)$ is a collection of the LIFNs, and for any $p, q \geq 0$, then, the result aggregated from (27) is still a LIFN, and even

$$LIFPGHM^{p,q}(\gamma_1,\gamma_2,\cdots,\gamma_m) = \left(s_{t(x-y)}, s_{t(1-x_v)}\right) \tag{28}$$

where,

$$x_u = \left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d},$$

$$y_u = \left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d},$$

$$x_v = \left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}.$$

The process to prove Theorem 5 is equal to Theorem 1, it is omitted here.

In the following, we will explore some characteristics of the LIFPGHM operator.

Theorem 6 ((Idempotency)). Let$(\gamma_1, \gamma_2, ..., \gamma_m)$be a collection of LIFNs, if all γ_k $(k = 1, 2, ..., m)$are equal, i.e.$\gamma_k = \gamma = (s_a, s_b)$, $k = 1, 2, \cdots, m$, then

$$LIFPGHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = \gamma \tag{29}$$

The proof of this theorem is similar to that of theorem 2, it is omitted here.

Theorem 7 ((Commutativity)). Suppose$\gamma_i = \left(s_{a_i}, s_{b_i}\right)and\gamma'_i = \left(s'_{a_i}, s'_{b_i}\right)$ $(i = 1, 2, \cdots m)$are two collections of IFNs, if$(\gamma'_1, \gamma'_2, \cdots, \gamma'_m)$is any permutation of $(\gamma_1, \gamma_2, \cdots, \gamma_m)$, then

$$LIFPGHM^{p,q}(\gamma_1, \gamma_2, \cdots, \gamma_m) = LIFPGHM^{p,q}(\gamma'_1, \gamma'_2, \cdots, \gamma'_m) \tag{30}$$

The proof of this theorem is similar to that of theorem 3, it is omitted here.

In the following, we will study some special cases of the $LIFPGBM^{p,q}$operator in regard to the parameters p and q.

(i) When $q \to 0$, then we can get

$$LIFPHM^{p,0}(\gamma_1, \gamma_2, \quad, \gamma_m) = \left(s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)^{p}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}+\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)^{1/d}\right)}, s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{i=1}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)^{p}+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}+\left(\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)^{p}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/p}\right)\right)^{1/d}\right)} \right) \tag{31}$$

(ii) When $p = 1, q \to 0$, then we can get

$$LIFPHM^{1,0}(\gamma_1, \gamma_2, \quad, \gamma_m) = \left(s_{t\left(\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{b_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}\right)}, s_{t\left(1-\left(\prod_{h=1}^{d}\left(\prod_{i=1}^{|P_h|}\left(1-\frac{b_{hi}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/d}\right)} \right) \tag{32}$$

(iii) When$p \to 0$, then we can get

$$LIFPHM^{0,q}(\gamma_1, \gamma_2, \quad, \gamma_m) = \left(s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}+\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)^{1/d}\right)}, s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hj}}{t}\right)^{q}+\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}+\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/q}\right)\right)^{1/d}\right)} \right) \tag{33}$$

(iv) When $p = q = 1$, then we can get

$$LIFPHM^{1,1}(\gamma_1,\gamma_2,\ \ ,\gamma_m)=$$

$$\left(s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}\right)+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}+\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)^{1/d}\right)},\right.$$

$$\left.s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\left(1-\frac{a_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}\right)+\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}+\left(\left(\prod_{\substack{i=1\\ j=i}}^{|P_h|}\left(1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}\right)\left(1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/2}\right)\right)^{1/d}\right)}\right) \tag{34}$$

As we all know, the $LIFPGHM^{p,q}$ operator only considers the interrelationships between input parameters, and does not take the weight of each input parameter into account. In order to overall consider these two aspects, we will further propose a linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator.

Definition 14. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)$ $(i = 1, 2, \cdots, m)$ is a collection of the LIFNs partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where $P_h = \left\{\gamma_{h1}, \gamma_{h2}, \cdots, \gamma_{h|P_h|}\right\}$ $(h = 1, 2, \cdots, d)$, $\sum_{h=1}^{d}|P_h| = m$ and $|P_h|$ denotes the cardinality of P_h. For any $p, q \geq 0$, $LIFWPGHM : \Gamma_{[0,t]}^{m} \to \Gamma_{[0,t]}$, if

$$LIFWPGHM^{p,q}\left(\gamma_1,\gamma_2,\cdots,\gamma_m\right)=\left(\prod_{h=1}^{d}\left(\frac{1}{p+q}\left(\prod_{\substack{i,j\in P_h\\ i\neq j}}\left(p\left(\gamma_{hi}\right)^{mw_{hi}}{}_{i}\oplus q\left(\gamma_{hj}\right)^{mw_{hj}}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)\right)^{1/d} \tag{35}$$

where $\Gamma_{[0,t]}$ is the set of all LIFNs, and $w = (w_1, w_2, \cdots, w_m)^T$ is the weight vector of $(\gamma_1, \gamma_2, \cdots, \gamma_m)$, $w_j \in [0, 1]$, $\sum_{j=1}^{m} w_j = 1$, then $LIFWPGHM$ is called the linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator.

Based on the operational laws of the LIFNs described in formulas (12–15) and the formula (35), the aggregation result is shown as theorem 8.

Theorem 8. Suppose $\gamma_i = \left(s_{a_i}, s_{b_i}\right)$ $(i = 1, 2, \cdots, m)$ is a collection of the LIFNs, and for any $p, q \geq 0$, then, the result aggregated from (35) is still a LIFN, and even

$$LIFWPGHM^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_m)=$$

$$\left(s_{t\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}-\left(\prod_{h=1}^{d}\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)^{1/d}\right)}, s_{t\left(1-\left(\prod_{h=1}^{d}\left(1-\left(1-\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\left(1-\alpha_{hi}^{p}\alpha_{hj}^{q}+\beta_{hi}^{p}\beta_{hj}^{q}\right)\right)^{\frac{2}{|P_h|(|P_h|+1)}}+\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}+\left(\left(\prod_{\substack{i=1\\j=i}}^{|P_h|}\beta_{hi}^{p}\beta_{hj}^{q}\right)^{\frac{2}{|P_h|(|P_h|+1)}}\right)^{1/(p+q)}\right)\right)^{1/d}\right)} \right) \tag{36}$$

where $\alpha_{hi}=1-u_{hi}^{mw_{hi}}+v_{hi}^{mw_{hi}}$, $\alpha_{hj}=1-u_{hj}^{mw_{hj}}+v_{hj}^{mw_{hj}}$, $\beta_{hi}=v_{hi}^{mw_{hi}}\beta_{hj}=v_{hj}^{mw_{hj}}$ and $u_{hi}=1-\frac{b_{hi}}{t}$, $u_{hj}=1-\frac{b_{hj}}{t}$, $v_{hi}=1-\frac{a_{hi}}{t}-\frac{b_{hi}}{t}$, $v_{hj}=1-\frac{a_{hj}}{t}-\frac{b_{hj}}{t}$.

The process to prove Theorem 8 is equal to Theorem 4, it is omitted here.

4. The MAGDM method based on the proposed operators

In the section, the decision approaches based on LIFWPHM operator or LIFWPGHM operator will be proposed to deal with the MAGDM problems with linguistic intuitionistic fuzzy information.

4.1. Description of the MAGDM problems

Suppose that $X=\{X_1,X_2,\cdots,X_m\}$ denotes the collection of all alternatives, $C=\{C_1,C_2,\cdots,C_n\}$ denotes the collection of all attributes, and $D=\{D_1,D_2,\cdots,D_p\}$ denotes the collection of all decision makers. We use the LIFN $\gamma_{ij}^{k}=\left(s_{a_{ij}}^{k},s_{b_{ij}}^{k}\right)$ to represent the evaluation value of the alternative X_i with respect to the attribute C_j given by the decision maker D_k. The decision matrix is expressed by $\Upsilon^{k}=\left(\gamma_{ij}^{k}\right)_{m\times n}$. $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_p)$ is the weight vector of decision makers $D_k(k=1,2,\ldots p)$ and $\lambda_k\in[0,1]$, $\sum_{k=1}^{p}\lambda_k=1$, $w=(w_1,w_2,\cdots,w_n)$ is the weight vector of attributes $\{C_1,C_2,\cdots,C_n\}$ and $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$. By above information, we should rank the alternatives.

4.2. The decision making methods based on LIFWPHM and LIFWPGBM operators

Step 1: Standardize the decision-making information

From common senses, it is necessary to convert the different types of the attributes to same type. So, we can convert the attribute values of cost type into benefit type, and the transformed decision matrices are expressed by $R^{k}=\left(r_{ij}^{k}\right)_{m\times n}$ (k = 1, 2, ..., p), $(i=1,2,\cdots,m, j=1,2,\cdots,n)$, where

$$r_{ij}^{k}=\left(s_{\alpha_{ij}}^{k},s_{\beta_{ij}}^{k}\right)=\begin{cases}\left(s_{a_{ij}}^{k},s_{b_{ij}}^{k}\right) & \text{for benefit attribute} C_j\\ \left(s_{b_{ij}}^{k},s_{a_{ij}}^{k}\right) & \text{for cost attribute} C_j\end{cases} \tag{37}$$

Step 2: Utilize the *LIFWPHM* operator

$$r_{i}^{k}=\left(s_{\alpha_i}^{k},s_{\beta_i}^{k}\right)=LIFWPHM(r_{i1}^{k},r_{i2}^{k},\cdots,r_{in}^{k}) \tag{38}$$

or the *LIFWPGHM* operator

$$r_{i}^{k}=\left(s_{\alpha_i}^{k},s_{\beta_i}^{k}\right)=LIFWPGHM(r_{i1}^{k},r_{i2}^{k},\cdots,r_{in}^{k}) \tag{39}$$

to derive all evaluation values of each alternative to the overall preference value $r_i^k(i=1,2,\cdots,m;k=1,2,\cdots,p)$.

Table 1
Linguistic intuitionistic fuzzy decision matrixR^1given byD_1.

	C_1	C_2	C_3	C_4	C_5
X_1	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)	(s_7, s_1)	(s_5, s_2)
X_2	(s_6, s_2)	(s_5, s_2)	(s_6, s_1)	(s_6, s_2)	(s_7, s_1)
X_3	(s_6, s_1)	(s_5, s_3)	(s_7, s_1)	(s_5, s_1)	(s_3, s_4)
X_4	(s_5, s_2)	(s_7, s_1)	(s_4, s_3)	(s_6, s_1)	(s_4, s_4)

Table 2
Linguistic intuitionistic fuzzy decision matrix R^2 given byD_2.

	C_1	C_2	C_1	C_2	C_1
X_1	(s_7, s_1)	(s_4, s_4)	(s_6, s_2)	(s_5, s_2)	(s_3, s_5)
X_2	(s_7, s_1)	(s_5, s_1)	(s_6, s_1)	(s_5, s_2)	(s_4, s_3)
X_3	(s_5, s_2)	(s_6, s_1)	(s_7, s_1)	(s_5, s_3)	(s_4, s_4)
X_4	(s_6, s_2)	(s_4, s_3)	(s_5, s_2)	(s_7, s_1)	(s_5, s_3)

Table 3
Linguistic intuitionistic fuzzy decision matrixR^3given byD_3.

	C_1	C_2	C_1	C_2	C_1
X_1	(s_6, s_1)	(s_5, s_2)	(s_3, s_4)	(s_7, s_1)	(s_5, s_2)
X_2	(s_7, s_1)	(s_6, s_2)	(s_7, s_1)	(s_6, s_2)	(s_5, s_1)
X_3	(s_5, s_3)	(s_5, s_2)	(s_6, s_1)	(s_4, s_3)	(s_3, s_1)
X_4	(s_6, s_2)	(s_7, s_1)	(s_5, s_1)	(s_5, s_2)	(s_5, s_3)

Table 4
Linguistic intuitionistic fuzzy decision matrix R^4 given by D_4.

	C_1	C_2	C_1	C_2	C_1
X_1	(s_5, s_3)	(s_4, s_4)	(s_7, s_1)	(s_5, s_1)	(s_4, s_2)
X_2	(s_6, s_1)	(s_7, s_1)	(s_6, s_1)	(s_5, s_2)	(s_6, s_1)
X_3	(s_5, s_2)	(s_3, s_4)	(s_6, s_2)	(s_3, s_3)	(s_5, s_2)
X_4	(s_4, s_3)	(s_5, s_1)	(s_4, s_2)	(s_6, s_2)	(s_5, s_2)

Step 3: Utilize the *LIFWPHM* operator

$$r_i = \left(s_{\alpha_i}, s_{\beta_i}\right) = LIFWPHM(r_i^1, r_i^2, \cdots, r_i^t) \tag{40}$$

or the *LIFWPGHM* operator

$$r_i = \left(s_{\alpha_i}, s_{\beta_i}\right) = LIFWPGHM(r_i^1, r_i^2, \cdots, r_i^t) \tag{41}$$

to get the collective overall preference value $r_i (i = 1, 2, \cdots, m)$.

Step 4: Calculate the score function $Ls(r_i)$ and accuracy function $Lh(r_i)$ of the all values $r_i (i = 1, 2, \cdots, m)$ cccording to the formulas (6) and (7).

Step 5: Rank the alternatives

Based on the score function and the accuracy function, we can rank all the alternatives$\left\{X_1, X_2, \cdots, X_m\right\}$and choose the best one(s).

Step 6: End.

5. An application example

In this section, we will give an example [5] about searching the best global supplier with linguistic intuitionistic fuzzy information to verify the effectiveness of the proposed methods.

Example 7. [5]

There are four potential global suppliers $X_i (i = 1, 2, 3, 4)$ which are evaluated by four decision makers $D_k (k = 1, 2, 3, 4)$ with respect to five attributes $C_j (j = 1, 2, 3, 4, 5)$. These attributes include: overall cost of the product C_1, quality of the product C_2, service performance of supplier C_3, supplier's profile C_4 and risk factor C_5, where the weight vector of the attributes is $w = (0.25, 0.20, 0.15, 0.18, 0.22)^T$and the weight vector of decision makers is$\lambda = (0.25, 0.30, 0.20, 0.25)^T$. Based on the interrelationship pattern, we can assume two partition structures for the attribute sets:$P_1 = \left\{C_1, C_2, C_5\right\}andP_2 = \left\{C_3, C_4\right\}$. The decision makers $D_k (k = 1, 2, 3, 4)$ evaluate the suppliers $X_i (i = 1, 2, 3, 4)$ with respect to the attributes $C_j (j = 1, 2, 3, 4, 5)$ by LIFNs based on the linguistic term set:$S = \left\{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, \ s_3 = \text{slightly poor}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\right\}$ and obtain following four decision matrices$R^k = \left[\gamma_{ij}^k\right]_{4\times 5} (k = 1, 2, 3, 4)$which are listed in Tables 1–4 , whereγ_{ij}^kcan be expressed as$\left(s_{a_{ij}}^k, s_{b_{ij}}^k\right)$.

5.1. Rank the alternatives by the proposed methods

(1) Rank the alternatives by the *LIFWPHM* operator

Step 1: Normalize the decision-making matrices

All the measured values are benefit type, so they do not need to do the standardization.

Step2: Calculate the comprehensive attribute value $\tilde{r}_i^k$ of each alternative determined by the k-th decision maker by the *LIFWPHM* operator in (38) (suppose $p = q = 1$), we can get

$r_1^1 = (s_{5.8202}, s_{2.1798})$, $r_2^1 = (s_{5.9454}, s_{2.0546})$, $r_3^1 = (s_{5.4425}, s_{2.5575})$, $r_4^1 = (s_{4.6523}, s_{3.3477})$,
$r_1^2 = (s_{4.8814}, s_{3.1186})$, $r_2^2 = (s_{5.2720}, s_{2.2728})$, $r_3^2 = (s_{5.2654}, s_{2.7346})$, $r_4^2 = (s_{5.3411}, s_{2.6589})$,
$r_1^3 = (s_{5.0919}, s_{2.9081})$, $r_2^3 = (s_{6.0569}, s_{1.9431})$, $r_3^3 = (s_{4.5805}, s_{3.4195})$, $r_4^3 = (s_{4.9615}, s_{3.0385})$,
$r_1^4 = (s_{5.2209}, s_{2.7791})$, $r_2^4 = (s_{5.7509}, s_{2.2491})$, $r_3^4 = (s_{4.2819}, s_{3.7181})$, $r_4^4 = (s_{4.7685}, s_{3.2315})$,

Step 3: Obtain the collective preference values by the *LIFWPHM* operator in (40) (suppose $p = q = 1$), and we get

$$r_1 = (s_{6.2428}, s_{1.7572}),\ r_2 = (s_{6.6900}, s_{1.3100}),\ r_3 = (s_{5.9206}, s_{2.0794}),\ r_4 = (s_{5.9325}, s_{2.0675}).$$

Step 4: Calculate the score function $Ls(r_i)(i = 1, 2, 3, 4, 5)$ of $r_i(i = 1, 2, 3, 4)$, we can get

$$\mathrm{Ls}(r_1) = 4.4857, \mathrm{Ls}(r_2) = 5.3799, \mathrm{Ls}(r_3) = 3.8413, Ls(r_4) = 3.8650.$$

Step 5: Rank the alternatives

Based on the score functions $Ls(r_i)$ ($i = 1, 2, 3, 4$), we can get the sorting of the alternatives $\{X_1, X_2, X_3, X_4\}$ shown as follows

$X_2 \succ X_1 \succ X_4 \succ X_3$.

So, the best alternative is X_2

Step 6: End.

(2) Rank the alternatives by the *LIFWPGHM* operator

Step 1: Normalize the decision-making matrices

All the measured values are benefit type, so they do not need to do the standardization.

Step2: Calculate the comprehensive attribute value $\tilde{r}_i^k$ of each alternative determined by the k-th decision maker by the *LIFWPGHM* operator in (39) (suppose $p = q = 1$), we can get

$r_1^1 = (s_{6.0555}, s_{1.9445})$, $r_2^1 = (s_{6.2163}, s_{1.7837})$, $r_3^1 = (s_{5.5970}, s_{2.4030})$, $r_4^1 = (s_{6.1672}, s_{1.8328})$,
$r_1^2 = (s_{5.2262}, s_{2.7738})$, $r_2^2 = (s_{6.3715}, s_{1.6285})$, $r_3^2 = (s_{5.7076}, s_{2.2924})$, $r_4^2 = (s_{5.6171}, s_{2.3829})$,
$r_1^3 = (s_{5.9570}, s_{2.0430})$, $r_2^3 = (s_{6.5463}, s_{1.4537})$, $r_3^3 = (s_{5.8379}, s_{2.1621})$, $r_4^3 = (s_{6.5806}, s_{1.4194})$,
$r_1^4 = (s_{5.3831}, s_{2.6169})$, $r_2^4 = (s_{7.0269}, s_{0.9731})$, $r_3^4 = (s_{5.3326}, s_{2.6674})$, $r_4^4 = (s_{5.8273}, s_{2.1727})$.

Step 3: Obtain the collective preference values by the *LIFWPGHM* operator in (41) (suppose $p = q = 1$), and we get

$$r_1 = (s_{4.9679}, s_{3.0321}),\ r_2 = (s_{6.1047}, s_{1.8953}),\ r_3 = (s_{4.9637}, s_{3.0363}),\ r_4 = (s_{5.4374}, s_{2.5626}).$$

Step 4: Calculate the score function $Ls(r_i)(i = 1, 2, 3, 4)$ of $r_i(i = 1, 2, 3, 4)$, we can get

$$\mathrm{Ls}(r_1) = 1.9357,\ Ls(r_2) = 4.2095,\ \mathrm{Ls}(r_3) = 1.9273,\ \mathrm{Ls}(r_4) = 2.8748.$$

Step 5: Rank the alternatives

Based on the score function $Ls(r_i)(i = 1, 2, 3, 4)$, we can get the sorting of the alternatives $\{X_1, X_2, X_3, X_4\}$ shown as follows

$$X_2 \succ X_4 \succ X_1 \succ X_3.$$

So, the best alternative is X_2

Step 6: End.

Obviously, these two methods produced the same ranking results.

5.2. The influence of the parameters p and q on decision making results

Further, we discuss the influence of the parameters p and q on decision making results based on the *LIFWPHM* operator and LIFWPGHM operator, the ranking results with the different values p and q are shown in Figs. 1–8.

From Figs. 1–8, we can see that the ranking results may be different for the different parameters p, q by *LIFWPHM* and *LIFWPGHM* operators. However, the best alternative is X_2. For *LIFWPHM* operator, there is a difference in ranking position of X_3 and X_4, when $p \le 1$ or $q \le 1$, we get $X_4 \succ X_3$, and when $p \gg> 1$ or $q \gg> 1$, we have $X_3 \succ X_4$. While for *LIFWPGHM* operator, there is a difference in ranking position of X_1 and X_3. When $p \le 1$ or $q \le 1$, we have $X_1 \succ X_3$, and when $p \gg> 1$ or $q \gg> 1$, then $X_3 \succ X_1$. In addition, the expected values of the aggregated results based on the *LIFWPHM* operator are becoming bigger and bigger when the values of parameters p, q is increasing, while for the *LIFWPGHM* operator, they becomes smaller and smaller.

5.3. The verification of the effectiveness

To further prove the effectiveness of the developed methods in this paper, we solve the same example 7 by comparing with the two existing MAGDM methods based on different aggregation operators under linguistic intuitionistic fuzzy environment: the uncertain linguistic ordered weighted averaging (ULOWA) and uncertain linguistic hybrid aggregation (ULHA) operators proposed by Xu [22] and the generalized uncertain linguistic weighted Heronian mean (GULWHM) operator proposed by Liu et al. [13]. To begin, as mentioned in remark 2 of Section 2.3, we can transform the uncertain linguistic information into the LIFN form (see Remark 2). Therefore, we can compare our methods with Xu [22] and Liu et al. [13] by transforming the LIFNs into ULNs. The results of transformation are listed in Tables 5–8 .

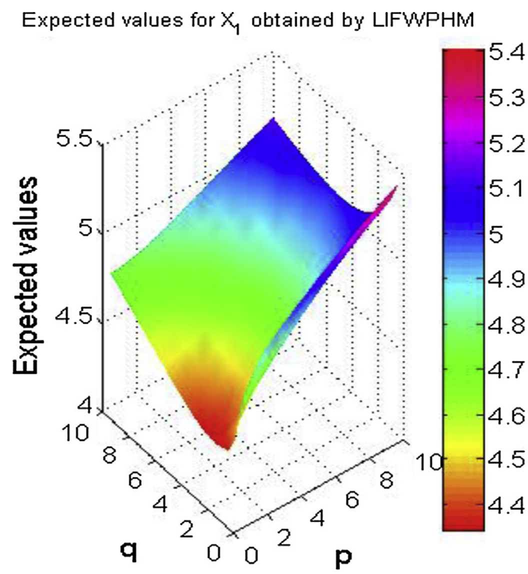

Fig. 1. Expected values of alternative$X_1$when $p, q \in (0, 10)$based on *LIFWPHM* operator.

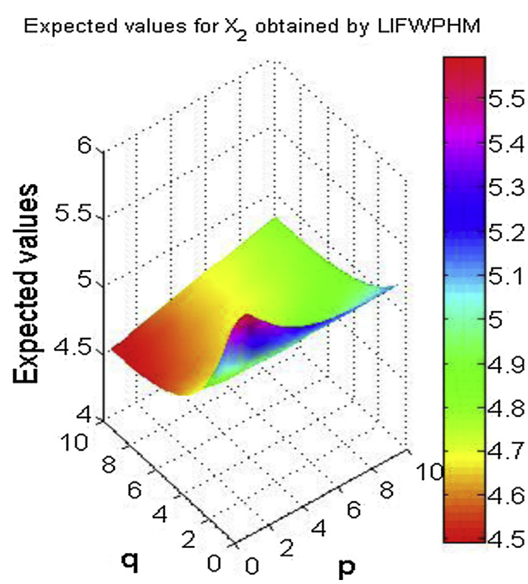

Fig. 2. Expected values of alternative X_2 when $p, q \in (0, 10)$based on *LIFWPHM* operator.

Table 5
Uncertain linguistic decision matrixR^1 given by D_1.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_7, s_7]$	$[s_6, s_6]$	$[s_4, s_5]$	$[s_7, s_7]$	$[s_5, s_6]$
X_2	$[s_6, s_6]$	$[s_5, s_6]$	$[s_6, s_7]$	$[s_6, s_6]$	$[s_7, s_7]$
X_3	$[s_6, s_7]$	$[s_5, s_5]$	$[s_7, s_7]$	$[s_5, s_7]$	$[s_3, s_4]$
X_4	$[s_5, s_6]$	$[s_7, s_7]$	$[s_4, s_5]$	$[s_6, s_7]$	$[s_4, s_4]$

Table 6
Uncertain linguistic decision matrix R^2 given byD_2.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_7, s_7]$	$[s_4, s_4]$	$[s_6, s_6]$	$[s_5, s_6]$	$[s_3, s_3]$
X_2	$[s_7, s_7]$	$[s_5, s_7]$	$[s_6, s_7]$	$[s_5, s_6]$	$[s_4, s_5]$
X_3	$[s_5, s_6]$	$[s_6, s_7]$	$[s_7, s_7]$	$[s_5, s_5]$	$[s_4, s_4]$
X_4	$[s_6, s_6]$	$[s_4, s_5]$	$[s_5, s_6]$	$[s_7, s_7]$	$[s_5, s_5]$

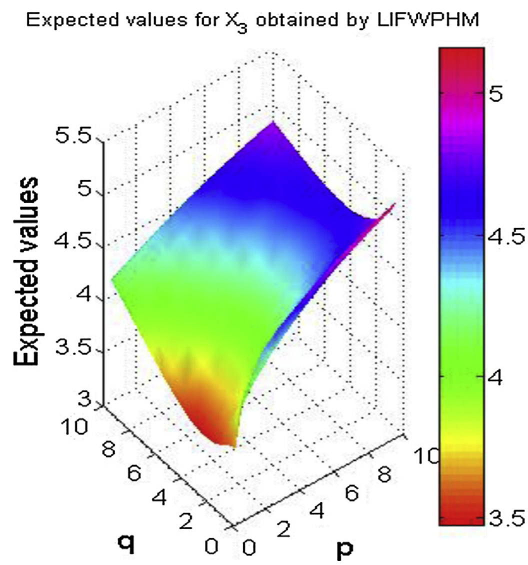

Fig. 3. Expected values of alternative X_3 when $p, q \in (0, 10)$based on *LIFWPHM* operator.

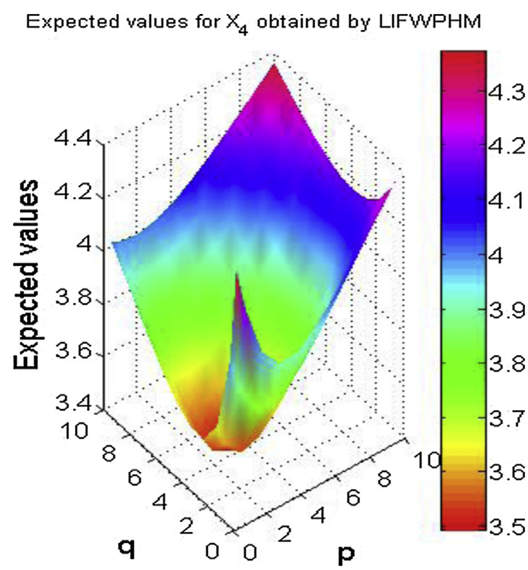

Fig. 4. Expected values of alternative$X_4$when $p, q \in (0, 10)$based on *LIFWPHM* operator.

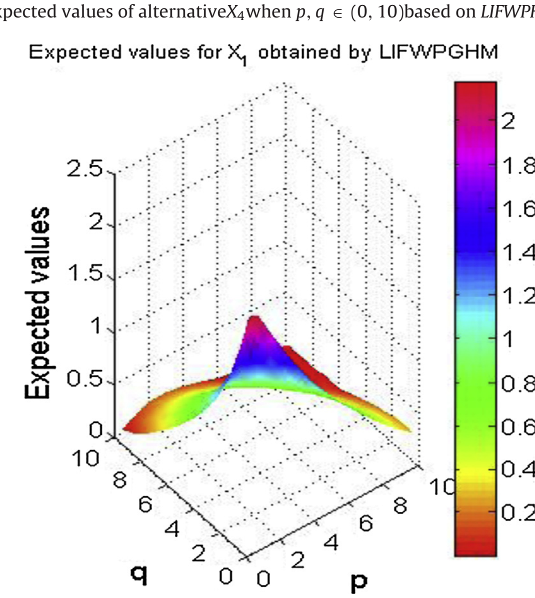

Fig. 5. Expected values of alternative X_1 when $p, q \in (0, 10)$based on *LIFWPGHM* operator.

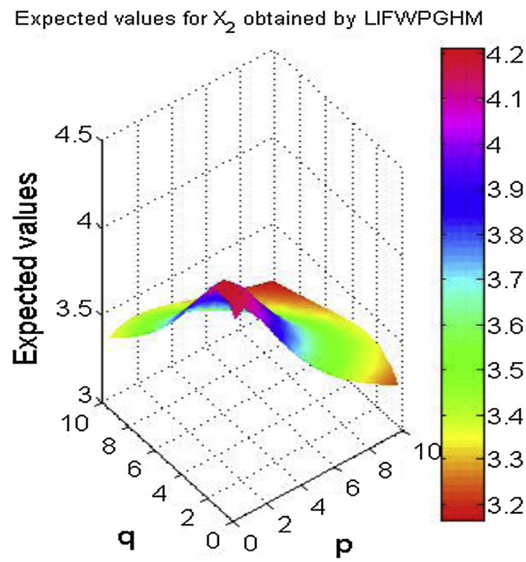

Fig. 6. Expected values of alternative$X_2$when $p, q \in (0, 10)$based on *LIFWPGHM* operator.

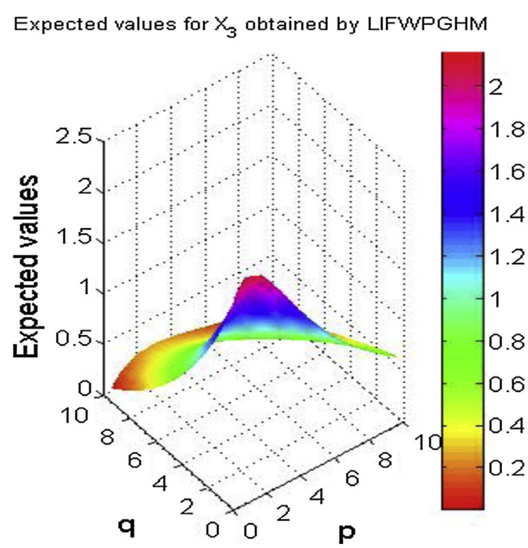

Fig. 7. Expected values of alternative X_3 when $p, q \in (0, 10)$based on*LIFWPGHM*operator.

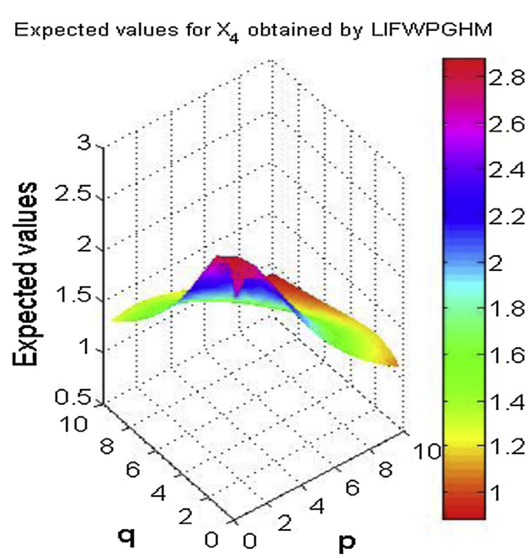

Fig. 8. Expected values of alternative X_4 when $p, q \in (0, 10)$based on *LIFWPGHM* operator.

Table 7
Uncertain linguistic decision matrix R^3 given by D_3.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_6, s_7]$	$[s_5, s_6]$	$[s_3, s_4]$	$[s_7, s_7]$	$[s_5, s_6]$
X_2	$[s_7, s_7]$	$[s_6, s_6]$	$[s_7, s_7]$	$[s_6, s_6]$	$[s_5, s_7]$
X_3	$[s_5, s_5]$	$[s_5, s_6]$	$[s_6, s_7]$	$[s_4, s_5]$	$[s_3, s_7]$
X_4	$[s_6, s_6]$	$[s_7, s_7]$	$[s_5, s_7]$	$[s_5, s_6]$	$[s_5, s_5]$

Table 8
Uncertain linguistic decision matrix R^4 given byD_4.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_5, s_5]$	$[s_4, s_4]$	$[s_7, s_7]$	$[s_5, s_7]$	$[s_4, s_6]$
X_2	$[s_6, s_7]$	$[s_7, s_7]$	$[s_6, s_7]$	$[s_5, s_6]$	$[s_6, s_7]$
X_3	$[s_5, s_6]$	$[s_3, s_4]$	$[s_6, s_6]$	$[s_3, s_5]$	$[s_5, s_6]$
X_4	$[s_4, s_5]$	$[s_5, s_7]$	$[s_4, s_6]$	$[s_6, s_6]$	$[s_5, s_6]$

Table 9
Ranking results by different methods.

Method and adopted aggregation operators	Score values $Ls(r_i)$	Ranking
Method by Liu et al. [13] based on *GULWHM* operators	$Ls(r_1) = 1.6222$, $Ls(r_2) = 3.4953$, $Ls(r_3) = 1.1757$, $Ls(r_4) = 1.7068$	$X_2 \succ X_4 \succ X_1 \succ X_3$
Method by Xu [22] based on *ULOWA* and *ULHA*operators	$Ls(r_1) = 1.3924$, $Ls(r_2) = 3.4840$, $Ls(r_3) = 1.1683$, $Ls(r_4) = 1.9553$	$X_2 \succ X_4 \succ X_1 \succ X_3$
Our proposed method based on *LIFWPGHM* operator ($p = q = 1$)	$Ls(r_1) = 3.4568$, $Ls(r_2) = 4.6622$, $Ls(r_3) = 3.1181$, $Ls(r_4) = 3.4313$	$X_2 \succ X_4 \succ X_1 \succ X_3$

Table 10
Linguistic intuitionistic fuzzy decision matrix$R$of example 8.

	c_1	c_2	c_3	c_4	c_5
X_1	(s_5, s_3)	(s_4, s_4)	(s_3, s_4)	(s_6, s_0)	(s_3, s_5)
X_2	(s_7, s_1)	(s_5, s_2)	(s_5, s_3)	(s_6, s_1)	(s_4, s_3)
X_3	(s_4, s_4)	(s_5, s_3)	(s_6, s_0)	(s_3, s_4)	(s_4, s_4)
X_4	(s_6, s_2)	(s_4, s_3)	(s_5, s_3)	(s_7, s_1)	(s_5, s_2)

For convenience, we set $p = q = 1$ and$\omega = (0.2, 0.1, 0.3, 0.4)^T$for Xu's and Liu's method. Then the ranking results for our developed methods with the two existing methods are listed in Table 9.

From Table 9, we can find that there is the same ranking results by the methods proposed in [22], [13] and the method proposed in this paper, i.e., $X_2 \succ X_4 \succ X_1 \succ X_3$. So the methods in this paper are effective and feasible.

5.4. Further comparative analysis

In the above sub-sections, we have verified the effectiveness of our methods by comparing with the two existing methods. However, because there are the same ranking results, it is difficult to show the merits of our methods. So we further set up two examples to illustrate that our methods can be applied more extensive. Example 8 will show the advantages of new operational rules of LIFNs proposed in this paper comparing with those in [5], the Example 9 will show the advantages of HM operator of LIFNs comparing with the method based on weighted averaging and hybrid averaging aggregation operator of LIFNs proposed by [5], and the Example 10 will show the advantages of partitioned HM operator of LIFNs comparing with the generalized uncertain linguistic weighted Heronian mean (GULWHM) operator proposed by Liu et al. [13].

Example 8. Assume that a company wants to develop a new career and there are four choices$\{X_1, X_2, X_3, X_4\}$which are the real estate industry, the food industry, the education industry, the computer industry, respectively. There are five attributes (suppose their weight vector is $w = (0.2, 0.2, 0.2, 0.2, 0.2)^T$) are shown as follows: the ability to compete(C_1), the ability to grow(C_2), the influence of surrounding environment(C_3), the influence of social-politic(C_4)and the capacity of market(C_5). By investigation and analysis of the interrelationship pattern, we can assume two partition structures for the attribute sets:$P_1 = \{C_1, C_2, C_5\}$and$P_2 = \{C_3, C_4\}$.

The decision maker evaluates the industry$X_i (i = 1, 2, 3, 4)$with respect to the attribute $C_j (j = 1, 2, 3, 4, 5)$ by LIFNs based on the linguistic term set: $S = \{s_0$ = extremely poor, s_1 = very poor, s_2 = poor, s_3 = slightly poor, s_4 = fair, s_5 = slightly good, s_6 = good, s_7 = very good, s_8 = extremely good$\}$ and obtains following decision matrix$R = [\gamma_{ij}]_{4\times 5}$listed in Table 10, where γ_{ij} can be expressed as$(s_{a_{ij}}, s_{b_{ij}})$.

The aggregation results for the different methods are shown in Table 11.

As we all know that in practical applications, it is often unavoidable that some subscripts in the non-membership degrees of LIFNs are zero. In this situation, the some existing methods are not able to get an effective or even the wrong ranking results. From Table 11, we know there are the different ranking results and two best choices X_2 or X_3 for the different methods. For traditional operations of LIFNs from [5], as we have known, it is difficult to get the right aggregation result when the subscript in non-membership degree of any one LIFN is zero. In this example, because there are zero in the subscripts of non-membership degrees of a_{14} and a_{33} from the choicesX_1 and X_3,

Table 11
Ranking results by different methods of example 8.

Method and adopted aggregation Operators	Score values $Ls(r_i)$	Ranking
Method by Chen et.al. [5] based on *LIFWA* and *LIFHA* operators	$Ls(\tilde{r}_1)=4.4056$, $Ls(\tilde{r}_2)=3.8652$, $Ls(\tilde{r}_3)=4.5625$, $Ls(\tilde{r}_4)=3.6002$	$X_3 \succ X_1 \succ X_2 \succ X_4$
Our proposed method based on *LIFWPGHM* operator	$Ls(\tilde{r}_1)=2.6470$, $Ls(\tilde{r}_2)=3.5162$, $Ls(\tilde{r}_3)=3.1397$, $Ls(\tilde{r}_4)=3.4414$	$X_2 \succ X_4 \succ X_3 \succ X_1$

Table 12
Linguistic intuitionistic fuzzy matrix R' of example 9.

	c_1	c_2	c_3	c_4	c_5
X_1	(s_5, s_2)	(s_4, s_3)	(s_3, s_4)	(s_6, s_1)	(s_3, s_5)
X_2	(s_5, s_1)	(s_5, s_2)	(s_4, s_3)	(s_4, s_1)	(s_4, s_3)
X_3	(s_3, s_4)	(s_5, s_2)	(s_4, s_2)	(s_3, s_4)	(s_4, s_3)
X_4	(s_5, s_2)	(s_4, s_3)	(s_4, s_3)	(s_5, s_1)	(s_5, s_2)

Table 13
Uncertain linguistic decision matrix $R'$of example 9.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_5, s_6]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_6, s_7]$	$[s_3, s_4]$
X_2	$[s_5, s_7]$	$[s_5, s_6]$	$[s_4, s_5]$	$[s_4, s_7]$	$[s_4, s_5]$
X_3	$[s_3, s_4]$	$[s_5, s_6]$	$[s_4, s_6]$	$[s_3, s_4]$	$[s_4, s_5]$
X_4	$[s_5, s_6]$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_5, s_7]$	$[s_5, s_6]$

Table 14
Ranking results by different methods of example 9.

Method and adopted aggregation operators	Score values $Ls(r_i)$	Ranking
Method by Chen et.al. [5] based on*LIFWA* and *LIFHA*operators	$Ls(r_1)=1.9141$, $Ls(r_2)=2.6522$, $Ls(r_3)=1.0091$, $Ls(r_4)=2.5865$	$X_2 \succ X_4 \succ X_1 \succ X_3$
Method by Liu et al. [13] based on *GULWHM* operators	$Ls(r_1)=1.7368$, $Ls(r_2)=2.4696$, $Ls(r_3)=1.2450$, $Ls(r_4)=2.5485$	$X_4 \succ X_2 \succ X_1 \succ X_3$
Our proposed method based on *LIFWPGHM* operator $(p=q=1)$	$Ls(r_1)=2.0724$, $Ls(r_2)=2.9560$, $Ls(r_3)=1.7262$, $Ls(r_4)=2.9989$	$X_4 \succ X_2 \succ X_1 \succ X_3$

we can get the aggregated results of X_1 and X_3 are$(s_{4.4056}, s_0)$and$(s_{4.45625}, s_0)$. Because the subscripts in the aggregated non-membership degrees of X_1 and X_3 are also zero, obviously, they are an unreasonable result, and make the ranking positions of X_1 andX_3 unreasonable (they are becoming better because the score functions of X_1 and X_3 are becoming bigger). However, because the method based on the proposed *LIFWPGHM* operator in this paper adopts the new operational laws, and they can overcome the weaknesses existing in [5]. So the ranking results and the best choice by the proposed method are more reasonable than [5].

Example 9. In practical applications, there are many decision making problems with interrelationships between attributes arguments. In order to consider this situation, we use another decision matrix R' in Example 8 and show in Table 12. To begin, as mentioned in remark 2 of Section 2.3, we can transform the uncertain linguistic information into the LIFN form (see Remark 2). Therefore, we can compare our methods with Liu et al. [13] by transforming the LIFNs into ULNs. The transformation results are listed in Table 13. Then we can get the ranking results in Table 14.

From Table 14, we can find that there are the same ranking results from the our method and Liu et al. [13]' method, and the ranking result based on the method in [5] is different from our proposed method and the method proposed in [13]. The reason can be explained as follows. The method in [5] used the simple weighted averaging operator and hybrid averaging operator, and these operators do not consider the interrelationship of the attributes and only provide the simple weighted function. On the contrary, the method in [13] and our methods are based on the Heronian mean operator, they successfully consider the interrelationship between two attributes. Obviously, in this example, the attributes have the interrelationships, so the ranking results produced by the method in [13] and our methods are more reasonable than that produced by [13].

Example 10. In some practical decisions, the decision making problem is with interrelationships for the attributes only in same a partition and with no interrelationships in different partitions. In order to consider this situation, we replace γ_{41} in Table 12 with the value (s_4, s_1), and the changed decision matrix R'' is shown in Table 15, then we can compare our methods with Liu et al. [13] by transforming the LIFNs into ULNs (see Remark 2). The transformation results are listed in Table 16. Then we can get the ranking results in Table 17.

In Table 15, we should notice that the attribute values for the choices X_2 and X_4 are same, and only have a simple permutation for different attributes. From Table 17, we can see that the ranking result based on the method in Liu et al. [13] does not rank the choices X_2 and X_4. It is obvious that the method in Liu et al. [13] used the Heronian mean operator and considered the interrelationship between any two parameters. But in this example, there are interrelationships between C_1, C_2 and C_5, and between C_3 and C_4, and there is no interrelationship between$P_1 = \left\{C_1, C_2, C_5\right\}andP_2 = \left\{C_3, C_4\right\}$. The method in [13] does not consider this situation, so the result got by the

Table 15
Linguistic intuitionistic fuzzy matrix R'' of example 10.

	c_1	c_2	c_3	c_4	c_5
X_1	(s_5, s_2)	(s_4, s_3)	(s_3, s_4)	(s_6, s_1)	(s_3, s_5)
X_2	(s_5, s_1)	(s_5, s_2)	(s_4, s_3)	(s_4, s_1)	(s_4, s_3)
X_3	(s_3, s_4)	(s_5, s_2)	(s_4, s_2)	(s_3, s_4)	(s_4, s_3)
X_4	(s_4, s_1)	(s_4, s_3)	(s_4, s_3)	(s_5, s_1)	(s_5, s_2)

Table 16
Uncertain linguistic decision matrix R'' of example 10.

	c_1	c_2	c_3	c_4	c_5
X_1	$[s_5, s_6]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_6, s_7]$	$[s_3, s_4]$
X_2	$[s_5, s_7]$	$[s_5, s_6]$	$[s_4, s_5]$	$[s_4, s_7]$	$[s_4, s_5]$
X_3	$[s_3, s_4]$	$[s_5, s_6]$	$[s_4, s_6]$	$[s_3, s_4]$	$[s_4, s_5]$
X_4	$[s_4, s_7]$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_5, s_7]$	$[s_5, s_6]$

Table 17
Ranking results by different methods of example 10.

Method and adopted aggregation operators	Score values $Ls(r_i)$	Ranking
Method by Liu et al. [13] based on *GULWHM* operators	$Ls(r_1) = 1.7702, Ls(r_2) = 2.4690,$ $Ls(r_3) = 1.2918, Ls(r_4) = 2.4690$	$X_2 = X_4 \succ X_1 \succ X_3$
Our proposed method based on *LIFWPGHM* operator ($p = q = 1$)	$Ls(r_1) = 2.0724, Ls(r_2) = 2.9560,$ $Ls(r_3) = 1.7262, Ls(r_4) = 2.8954$	$X_2 \succ X_4 \succ X_1 \succ X_3$

Table 18
The characteristic comparisons of different methods.

Methods	information by linguistic intuitionistic fuzzy number	whether consider interrelationships between aggregating parameters	whether consider the interactions between membership function and the non-membership function	whether consider the partition of the input arguments
Chen [5]		No	No	No
Xu [16]	No	No	No	No
Liu et al. [13]	No	Yes	No	No
Our proposed methods	Yes	Yes	Yes	Yes

method in [13] is unreasonable. In the contrast, our method based on partitioned Heronian mean operator successfully solves this problem, which avoids the influence of the irrelevant attributes and only considers the interrelationship for relevant attributes. Of course, if there is only one partition, the proposed method in this paper will be reduced to that in Heronian mean operator like the method in [13]. Thus our method is more general and effective than that in [13].

As discussed above, we simply summarize the differences between our methods and the existing methods which are listed in Table 18.

In a word, from Table 18, we can know the proposed methods in this paper are based on linguistic intuitionistic fuzzy numbers and partitioned Heronian mean operator with the parameters p, q. So these methods are more general and more flexible than some existing methods, at the same time, they not only consider the interrelationship of the attributes and the interactions between membership function and non-membership function, but also can partition the input arguments into several parts to connect the expressed interrelationship structure among the attributes to the corresponding input arguments in MAGDM problems.

6. Conclusion

In this paper, we firstly proposed the partitioned Heronian mean (PHM) operator, extended it to the LIFNs, and proposed some linguistic intuitionistic fuzzy PHM operators, such as LIFPHM operator, LIFWPHM operator, LIFPGHM operator and LIFWPGHM operator, further, we studied some desirable properties and some special cases of them, and developed two methods for the MAGDM based on the proposed operators. Comparing with the existing methods, the methods we proposed in this paper are more general, and they considered the interrelationships between the attitudes and partitioned them into some different parts. In addition, we used the new operational rules in these PHM operators to solve some situation when the subscript of the non-membership is zero. So it is more flexible to deal with the MAGDM problems based on the linguistic intuitionistic fuzzy environment.

In the future research, we will study how to take the applications of theses operators to deal with the decision making problems in real world. In addition, considering the advantages of PHM operator, we should also study some extensions to hesitant fuzzy sets (HFSs), neutrosophic set (NS), and so on.

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