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THE EXCESS UTILITY FUNCTIONS AND THE WELFARE ADJUSTMENT PROCESS

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This paper introduces the excess utility functions which are shown to have all the properties the excess demand functions have. Then, these functions are used to simplify the proofs of existing results about the stability of the welfare adjustment process.

1. Introduction

In the second section of this brief paper I discuss the welfare maximization problem (WMP) in the particular case in which the welfare function is linear in the individual utility functions. A condition is stated for the weights of the welfare function so that the solution of the WMP will be a competitive equilibrium.

In section 3 I introduce the excess utility functions. These functions have, at least, all the properties the excess demand functions have and allow us to characterize the competitive equilibrium in a similar way to the excess demand functions.

The similarities between both types of functions are used to study the stability of the welfare adjustment process. The results are the same as those already established by Mantel (1971); the originality of my approach lies in the treatment.

Finally, the last section presents a proof of the existence of a solution to the general equilibrium problem. This proof was originally done by Negishi (1960). Now I present a simpler version that uses the excess utility function concept.

2. The welfare maximization problem

Assume an exchange economy with n consumers and m commodities, where $\omega^i = (\omega_1^i, \dots, \omega_m^i)$ represents the initial endowment of the ith consumer, and $x^i = (x_1^i, \dots, x_m^i)$ his consumption bundle. Assume also that the consumer's preferences can be represented by utility functions u^i, $i = 1, \dots, n$ which are strictly concave and locally non-saturating. Finally, assume that the social welfare can be represented by a linear function in the individual utility functions. Let a_i denote the weight given to the ith consumer, then the mathematical formulation of the WMP is

$$\max \quad \sum_{i=1}^{n} a_i u^i(x^i), \tag{1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x^i \leqslant \sum_{i=1}^{n} \omega^i, \tag{2}$$

$$x^i \geqslant 0, \qquad i = 1, \dots, n. \tag{3}$$

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Since the utility functions are concave and the feasible set is compact, a solution to the WMP does exist. We will denote it $(x^1(a),\dots, x^n(a), p(a))$, where $p(a)^t = (p_1(a),\dots, p_m(a))$ are the Lagrange multipliers (as usual a superindex t will denote the transpose).

The Kuhn-Tucker conditions satisfied by the optimal solution are

$$a_i Du^i(x^i(a)) - p(a) \leqslant 0, \qquad i = 1,\dots, n, \tag{4}$$

$$x^i(a)\{a_i Du^i(x^i(a)) - p(a)\} = 0, \qquad i = 1,\dots, n, \tag{5}$$

$$\sum_{i=1}^{n} \omega^i - \sum_{i=1}^{n} x^i(a) \geqslant 0, \tag{6}$$

$$\left(\sum_{i=1}^{n} \omega^i - \sum_{i=1}^{n} x^i(a)\right) p(a) = 0, \tag{7}$$

where

$$Du^i = \left(\frac{\partial u^i}{\partial x_1},\dots,\frac{\partial u^i}{\partial x_m}\right)^t.$$

For each commodity, call it j, we have that

$$p_j(a) \geqslant a_i \frac{\partial u^i}{\partial x_j}(x^i(a)), \qquad i = 1,\dots, n. \tag{8}$$

Hence assuming that $\Sigma_i \omega_j^i > 0$,

$$p_j(a) = \max_i \left\{ a_i \frac{\partial u^i}{\partial x_j}(x^i(a)) \right\}. \tag{9}$$

For any vector of weights $a = (a_1,\dots, a_n)$ the solution $(x^1(a), x^2(a),\dots, x^n(a), p(a))$ is Pareto optimal, but in general it does not represent a competitive equilibrium.

Theorem 1. If the solution $(x^1(a),\dots, x^n(a), p(a))$ of the WMP is such that

$$\omega^i p(a) \geqslant x^i(a) p(a), \qquad i = 1,\dots, n, \tag{10}$$

then it is a competitive equilibrium.

Proof. Assume that the solution does not represent a competitive equilibrium; therefore, there is at least one consumer, say i, for which exists a bundle $\bar{x}^i$ such that

$$u^i(x^i(a)) < u^i(\bar{x}^i), \quad \text{and} \tag{11}$$

$$\bar{x}^i p(a) \leqslant \omega^i p(a). \tag{12}$$

From (7) and (10) it follows that

$$\omega^i p(a) = x^i(a)p(a), \tag{13}$$

therefore

$$\bar{x}^i p(a) \leqslant x^i(a)p(a). \tag{14}$$

But (11) and (14) contradict the fact that $(x^1(a), x^2(a), \ldots, x^n(a), p(a))$ maximizes the expression

$$\sum_{i=1}^{n} a_i u^i(x^i) \tag{15}$$

subject to restrictions (2) and (3).

3. The excess utility functions

Given a solution to the WMP, the expression $(\omega^i - x^i(a))p(a)$ represents the ith consumer's budget surplus. Then, recalling eq. (4) it seems natural to define the excess utility functions as follows:

$$g_i(a) = (1/a_i)(\omega^i - x^i(a))p(a). \tag{16}$$

Lemma 1. If the utility functions, u^i, $i = 1, \ldots, n$ are continuously differentiable, then the excess utility functions are continuous in the positive orthant.

Proof. Since the utility functions are strictly concave $x^i(a)$ is a continuous function of a. Furthermore, because the utility functions are continuously differentiable, $p(a)$ is also a continuous function of a, and consequently the excess utility functions are continuous in the positive orthant.

It is easy to prove that the excess utility functions are homogeneous of degree zero and that they satisfy a condition analogous to the Walras' law, i.e.,

$$\sum_{i=1}^{n} a_i g_i(a) = 0. \tag{17}$$

The conditions that characterize the excess demand functions are homogeneity of degree zero and the Walras' law. In addition, if the utility functions that represent the consumers' preferences are strictly concave, then the excess demand functions are continuous. Hence we have the following result.

Theorem 2. The excess utility functions have, at least, all the properties that characterize the excess demand functions.

4. The welfare adjustment process

A competitive equilibrium is characterized by the conditions

$$g_i(a) \geqslant 0, \qquad i = 1, \ldots, n. \tag{18}$$

Now, if in any given solution of the WMP, a consumer has a budget deficit, i.e., $g_i(a) < 0$ for some i, then it is possible to lower his deficit by reducing his relative weight a_i in the social welfare function. This result suggests the following adjustment mechanism:

$$\frac{\mathrm{d}a_i}{\mathrm{d}t} = g_i(a). \tag{19}$$

Hence, if a consumer has a budget deficit his weight is reduced, but on the contrary, if he has a budget surplus his weight is raised. This is the so-called welfare adjustment process.

Since the excess utility functions have all the properties characterizing the excess demand functions and the welfare adjustment process is analogous to the price adjustment process, all the results demonstrated for the latter process also apply to the former. For instance, we can state without need of proof the following result [see Arrow, Block and Hurwicz (1959)]:

Theorem 3. Assume that

$$\frac{\partial g_i(a)}{\partial a_j} \leqslant 0, \qquad i \neq j, \quad i,\ j = 1,\ldots,n. \tag{20}$$

Then the welfare adjustment mechanism is stable.

Condition (20), which is not always fulfilled, has some intuitive appeal. If the weight of any consumer in the social welfare function is increased, it seems plausible to expect a reduction in the consumption level of the remaining consumers, reducing in this way their budget surpluses. For instance, if all consumers have identical homothetic indifference curves, condition (20) is readily satisfied.

5. Existence of a solution to the general equilibrium problem

In this section, I will provide a simpler version of Negishi's (1960) proof of the existence of a solution to the general equilibrium problem. In order to probe the existence of a competitive equilibrium I find a vector a^* such that $g_i(a^*) \geqslant 0$, $i = 1,\ldots,n$. Let S_n denote the simplex

$$S_n = \left\{ a \mid \sum_{i=1}^{n} a_i = 1, \qquad a_i \geqslant 0 \right\}, \tag{21}$$

and $g(a)$ the row vector $(g_1(a), g_2(a),\ldots, g_n(a))$. Following Arrow and Hahn (1971) I will introduce the following functions:

$$M_i(a) = \max\{0,\ g_i(a)\}, \qquad i = 1,\ldots,n, \tag{22}$$

$$\alpha(a) = \beta\left(\sum_{i=1}^{n} g_i(a) \right), \tag{23}$$

and

$$\mathrm{N}(a) = [1 - \alpha(a)]\,M(a) + \alpha(a)e, \tag{24}$$

where e is the row vector whose components are all one and $\beta(\ )$ is a real continuous function such that $0 \leqslant \beta(x) \leqslant 1$, all x, $\beta(x) = 0$, all $x \leqslant 0$, $\beta(x) = 1$, all $x \geqslant x_0 > 0$.

Then the function

$$T(a) = \frac{a + \mathrm{N}(a)}{1 + \mathrm{N}(a)e^t} \tag{25}$$

is a continuous mapping of the simplex S_n into itself, and, therefore, has a fixed point a^*. Hence

$$a^* = \frac{a^* + \mathrm{N}(a^*)}{1 + \mathrm{N}(a^*)e^t}, \tag{26}$$

and

$$(\mathrm{N}(a^*)e^t)a^* = \mathrm{N}(a^*). \tag{27}$$

Postmultiplying eq. (27) by $g(a^*)^t$ and recalling that $a^*g(a^*)^t = 0$ it follows that $\mathrm{N}(a^*)g(a^*)^t = 0$, i.e.,

$$[1 - \alpha(a^*)]\,M(a^*)g(a^*)^t + \alpha(a^*)g(a^*)e^t = 0. \tag{28}$$

By construction $M(a^*)g(a^*)^t \geqslant 0$. Now $\alpha(a^*) > 0$ would imply that $M(a^*)g(a^*)^t < 0$, hence $\alpha(a^*)$ must be zero. Therefore $M(a^*)g(a^*)^t = 0$, which in turn implies that $g(a^*) \leqslant 0$.

Now, because $\alpha(a^*) = 0$, every component of a^* must be greater than zero. Then recalling that $a^*g(a^*)^t = 0$ it immediately follows that $g(a^*) = 0$. Hence the existence of a competitive equilibrium has been demonstrated.

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