Soft Computing (2018) 22:4855–4878
https://doi.org/10.1007/s00500-018-3263-1

FOCUS

Probabilistic OWA distances applied to asset management

José M. Merigó[1,2] · Ligang Zhou[3] · Dejian Yu[4] · Nabil Alrajeh[2] · Khalid Alnowibet[5]

Published online: 30 May 2018

Abstract

Average distances are widely used in many fields for calculating the distances between two sets of elements. This paper presents several new average distances by using the ordered weighted average, the probability and the weighted average. First, the work presents the probabilistic ordered weighted averaging weighted average distance (POWAWAD) operator. POWAWAD is a new aggregation operator that uses distance measures in a unified framework between the probability, the weighted average and the ordered weighted average (OWA) operator that considers the degree of importance that each concept has in the aggregation. The POWAWAD operator includes a wide range of particular cases including the maximum distance, the minimum distance, the normalized Hamming distance, the weighted Hamming distance and the ordered weighted average distance (OWAD). The article also presents further generalizations by using generalized and quasi-arithmetic means forming the generalized probabilistic ordered weighted averaging weighted average distance (GPOWAWAD) operator and the quasi-POWAWAD operator. The study ends analysing the applicability of this new approach in the calculation of the average fixed assets. Particularly, the work focuses on measuring the average distances between the ideal percentage of fixed assets that the companies of a specific country should have versus the real percentage of fixed assets they have. The illustrative example focuses on the Asian market.

Keywords Distance measures · Aggregation operators · OWA operator · Average fixed asset · Group decision-making

1 Introduction

Aggregation operators (or functions) (Grabisch et al. 2011) are highly useful techniques for collecting information by providing summarized results (Beliakov et al. 2007, 2016; Yu 2015). Several highly popular techniques in this framework are the probability and the weighted average. These techniques give importance to the variables according to certain available subjective or objective findings. Another popular aggregation operator is the ordered weighted average (OWA) (Yager 1988; Yager et al. 2011). The OWA provides a parameterized family of aggregation operators between the minimum and the maximum, weighting the data according to the attitudinal character of the decision-maker. A key issue in the OWA operator is how to integrate it with certain other key concepts, such as the probability and the weighted average. Several authors have suggested different approaches, including the immediate probability (Engemann et al. 1996; Merigó 2010; Yager et al. 1995) and the probabilistic OWA

Communicated by C. Kahraman.

✉ José M. Merigó
jmerigo@fen.uchile.cl

Ligang Zhou
shuiqiaozlg@126.com

Dejian Yu
yudejian62@126.com

Nabil Alrajeh
nabil@ksu.edu.sa

Khalid Alnowibet
knowibet@ksu.edu.sa

1 Department of Management Control and Information Systems, School of Economics and Business, University of Chile, Av. Diagonal Paraguay 257, 8330015 Santiago, Chile

2 Biomedical Technology Department, College of Applied Medical Sciences, King Saud University, Riyadh, Saudi Arabia

3 School of Mathematical Sciences, Anhui University, 230601 Hefei, Anhui, China

4 Business School, Nanjing Audit University, 211815 Nanjing, China

5 Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia

(POWA) operator (Merigó 2012). Torra (1997) introduced the weighted OWA (WOWA) operator, Yager (1998) discussed the importance OWA, Xu and Da (2003) reported the hybrid average and Merigó (2011) proposed the OWA weighted average (OWAWA). Recently, Merigó et al. (2012) have suggested a more general approach that integrates all three concepts in the same formulation and considers the degree of importance that each concept has in the formulation. This aggregation operator is known as the probabilistic OWA weighted average (POWAWA).

Other techniques that are useful for representing information include the generalized aggregation operators. As the name indicates, these aggregation operators use the generalized and the quasi-arithmetic means in the analysis (Merigó and Gil-Lafuente 2009). Thus, these operators contain a wide range of aggregation operators that include the arithmetic mean, the geometric mean and the quadratic mean (Xian and Sun 2014). In recent years, different authors have introduced a wide range of new developments, including linguistic generalized hybrid averages (Liu et al. 2016b), generalized ordered modular averaging operator (Liu et al. 2016a), generalized compensative weighted averaging aggregation operators (Aggarwal 2015), generalized ordered weighted reference dependent utility (Gao and Liu 2017) and generalized moving averages (Merigó and Yager 2013).

Distance measures are used for measuring the differences between two elements, sets or fuzzy sets (Gil-Aluja 1999; Kaufmann 1975). Usually, when dealing with sets of elements, it is necessary to normalize the distances into an average result, such as the normalized Hamming distance (Hamming 1950) and the normalized Minkowski distance. Recently, the use of the OWA operator has also been suggested with distance measures (Merigó and Gil-Lafuente 2010; Xu and Chen 2008). In this context, it is possible to develop a wide range of new distance measures (Scherger et al. 2017) by using the available aggregation operators that have been recently introduced in the literature, including the induced OWA distance (IOWAD) (Merigó and Casanovas 2011a), the OWAWA distance (OWAWAD) (Merigó et al. 2017), the fuzzy generalized OWA distance (FGOWAD) (Zeng et al. 2012), the induced linguistic generalized OWA distance (ILGOWAD) (Zeng and Su 2012), the uncertain probabilistic OWA distance (UPOWAD) (Zeng et al. 2013), the linguistic continuous OWA distance (LCOWAD) (Zhou et al. 2014), the fuzzy linguistic induced Euclidean and Minkowski OWA distance (Xian and Sun 2014; Xian et al. 2016; Xue et al. 2017), the uncertain OWA distance (Zeng 2016), induced probabilistic OWA distance (IPOWAD) (Casanovas et al. 2016), the intuitionistic fuzzy induced OWA distance (IFIOWAD) (Zeng et al. 2017), the Bonferroni OWA distance (Blanco-Mesa et al. 2016) and cloud distances (Liu and Liu 2017).

The objective of this paper is to introduce new generalizations of the POWAWA operator by using distance measures in the analysis. First, the work introduces the probabilistic OWAWA distance (POWAWAD) operator. This operator normalizes the Hamming distance (or other distances) with the POWAWA operator. Thus, we are able to include the probability, the weighted average and the OWA operator in the same formulation with the Hamming distance and consider the degree of importance that each concept has in the analysis. Additionally, this approach can analyse the distance measures in a probabilistic way either if they are subjective or objective. The paper studies several of these technique's properties and certain particular cases, including the arithmetic probabilistic Hamming distance, the probabilistic distance, the double arithmetic OWAD operator, the OWAD operator, the normalized Hamming distance, the OWAWA distance (OWAWAD) and the probabilistic OWA distance (POWAD). The main advantage of this framework is the ability to consider a wide range of situations depending on the available information. Thus, the POWAWAD operator considers a general framework, but at the same time, it can also be reduced to the specific case needed in the analysis.

The paper also introduces further extensions by using the generalized and the quasi-arithmetic means, including the generalized probabilistic OWA weighted averaging distance (GPOWAWAD) operator and the quasi-arithmetic POWAWAD (Quasi-POWAWAD) operator. This study provides distance measures in a unified framework between the probability, the weighted average and the OWA operator. Moreover, the study uses the Minkowski and the quasi-arithmetic distance that includes the Hamming and the Euclidean distance as particular cases. Therefore, these distance aggregation operators can represent a wider range of aggregation operators as special cases and can address many complex granularities in the information (Bargiela and Pedrycz 2003; Kacprzyk and Pedrycz 2015; Zadeh 2005).

The article also studies the applicability of the POWAWAD operator, and we see that it is very broad because all of the previous studies that use average distance measures can be revised and extended with this new approach. The paper focuses on the applicability in the average distances of fixed assets. To this end, the work introduces the ordered weighted average fixed asset (OWAFA). This averaging aggregation operator provides an average of a set of fixed assets between the minimum and maximum fixed assets. The study also introduces some additional extensions including the weighted OWA fixed asset (WOWAFA) and the probabilistic weighted OWA fixed asset (PWOWAFA). These operators are very useful for calculating the average fixed asset in a set of scenarios, companies or regions. The objective of this application is to see how OWA operators can be implemented in real-world economic variables (Avilés-

Ochoa et al. 2018; Laengle et al. 2017; León-Castro et al. 2018).

The work focuses on a supranational group decision-making problem regarding the calculation of the average fixed asset under different scenarios in an industry in different Asian countries. Particularly, the work analyses the differences between the ideal percentage of fixed assets that a company should have according to the characteristics of the country where it is operating and the real percentage of fixed assets the enterprise has. To this end, the study applies the POWAWAD and GPOWAWAD operators to the fixed asset, obtaining the POWAWAD fixed asset (FAPOWAWAD) and the GPOWAWAD fixed asset (FAGPOWAWAD).

This paper is organized as follows: Section 2 briefly reviews some basic distance measures and averaging aggregation operators. Section 3 introduces the POWAWAD operator, and Sect. 4 discusses the GPOWAWAD operator. Section 5 studies the applicability of the POWAWAD operator in the calculation of the average fixed asset, and Sect. 6 develops an illustrative example. Section 7 briefly summarizes the study's primary results, findings and conclusions.

2 Preliminaries

This section briefly reviews distance measures, the OWA operator and some of its extensions by using probabilities, the generalized means and the quasi-arithmetic means.

2.1 Distance measures

The Hamming distance (Hamming 1950) is a very useful technique for calculating the differences between two elements or two sets. Recall that any distance measure should follow certain basic axioms in order to be a distance (Merigó and Casanovas 2011a; Merigó et al. 2017). In fuzzy set theory, the Hamming distance is useful for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets or hesitant fuzzy sets (Liao et al. 2015). The Hamming distance for two sets A and B is defined as follows.

Definition 1 A normalized Hamming distance of dimension n is a mapping $d_H\colon [0,1]^n \times [0,1]^n \rightarrow [0,1]$ such that:

$$d_H(A, B) = \left(\frac{1}{n}\sum_{i=1}^{n} |a_i - b_i|\right) \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B, respectively.

Occasionally, when normalizing the Hamming distance, we prefer to give different weights to each individual distance. Thus, the distance is known as the weighted Hamming distance (Merigó et al. 2017).

Note that it is possible to generalize this definition to all the real numbers by using $R^n \times R^n \rightarrow R$. See Kaufmann (1975) for further elaboration on the formulation used in fuzzy set theory.

The OWAD (or Hamming OWAD) operator is an extension of the traditional normalized Hamming distance by using the OWA operator. The main difference is the reordering of the arguments of the individual distances that is developed according to their numerical values from the highest to the lowest or vice versa. Thus, it is possible to calculate the distance between two elements, two sets or two fuzzy sets, according to the interests of the decision-maker, as explained in the following definition.

Definition 2 An OWAD operator of dimension n is a mapping OWAD: $[0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ such that:

$$\text{OWAD}(\langle \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle \mu_n, \mu_n^{(k)}\rangle) = \sum_{j=1}^{n} w_j D_j \tag{2}$$

where D_j represents the jth largest of the $|\mu_i - \mu_i^{(k)}|$, $\mu_i \in [0,1]$ for the ith characteristic of the ideal P, $\mu_i^{(k)} \in [0,1]$ for the ith characteristic of the kth alternative P_k, and $k = 1, 2, \ldots, m$.

Note that this definition can be generalized to all the real numbers R by using OWAD: $R^n \times R^n \rightarrow R$. Note also that it is possible to distinguish between ascending and descending orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the descending OWAD (DOWAD) operator and w^*_{n-j+1} the jth weight of the ascending OWAD (AOWAD) operator.

By using the generalized means (Merigó and Gil-Lafuente 2009), the OWAD operator becomes the Minkowski OWA distance (MOWAD) (Merigó and Casanovas 2011b) which is defined as follows for two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$.

Definition 3 A MOWAD operator is a mapping $f\colon R^n \times R^n \rightarrow R$ that has an associated weighting vector W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{3}$$

where b_j is the jth largest of the $|x_i - y_i|$, $|x_i - y_i|$ is the argument variable represented in the form of individual distances, and λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$.

2.2 OWA and the probabilistic OWA operator

The OWA operator (Yager 1988) is an aggregation operator that provides a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum. The aggregation operator OWA can be defined as follows.

Definition 4 An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

where b_j is the jth largest of the a_i.

The OWA operator is commutative, monotonic, bounded and idempotent. For further information on the OWA and its applications, refer to Emrouznejad and Marra (2014), He et al. (2017) and Yager et al. (2011).

By using probabilities, the OWA operator becomes the probabilistic OWA (POWA) operator (Merigó 2012). It is an aggregation operator that unifies the probability and the OWA operator in the same formulation, considering the degree of importance that each concept has in the analysis. With this approach, we can underestimate or overestimate the probabilities according to the attitudinal character of the decision-maker. POWA is defined as follows.

Definition 5 A POWA operator of dimension n is a mapping POWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\mathrm{POWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{5}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (probability) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0, 1]$, and v_j is the weight (probability) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

By choosing a different manifestation in the weighting vector, we are able to obtain a wide range of particular types of POWA operators (Merigó 2012; Yager 1993). In particular, when $\beta = 0$, we obtain the expected value, and if $\beta = 1$, the OWA operator defined earlier is obtained. Other interesting cases are found when $w_j = 1/n$ for all a_i because POWA becomes the arithmetic probability (AP), and if $v_i = 1/n$ for all a_i, it becomes the arithmetic OWA operator. Note that inside the arithmetic OWA, we obtain such values as the arithmetic maximum and the arithmetic minimum. As we can see, the use of the probability in the OWA creates new semi-boundary conditions based on the combination between the maximum and the minimum with the probability, obtaining the probabilistic maximum and the probabilistic minimum.

A further extension is the probabilistic OWA weighted average (POWAWA) operator (Merigó et al. 2012) that uses probabilities, weighted averages and OWAs in the same formulation. It unifies these three concepts by considering the degree of importance that each concept has in the aggregation, depending on the situation considered. Note that this operator can also be denoted as the probabilistic weighted OWA (PWOWA) operator. Since the literature already uses the weighted OWA (WOWA) for a different approach (Torra 1997), this article follows the names used in previous studies that follow this unification methodology (Merigó 2011). The name connects both concepts by referring to the OWA weighted average. Thus, in this case, instead of calling this approach the PWOWA, the work uses POWAWA. The POWAWA operator is defined as follows (Merigó et al. 2012).

Definition 6 A POWAWA operator of dimension n is a mapping POWAWA: $R^n \to R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{POWAWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{6}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, a probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0, 1]$, $C_1 + C_2 + C_3 = 1$, and v_j and p_j are the weights v_i and p_i ordered according to b_j, that is to say, according to the jth largest of the a_i.

2.3 Generalized and quasi-arithmetic averaging aggregation operators

Generalized averaging aggregation operators are those averaging functions that use a general framework that include a wide range of particular cases. A very common one is the generalized OWA (GOWA) operator (Yager 2004). The GOWA operator generalizes a wide range of averaging aggregation operators that include the OWA operator with its particular cases, the ordered weighted geometric (OWG) operator and the ordered weighted quadratic averaging (OWQA) operator. It is defined as follows.

Definition 7 A GOWA operator of dimension n is a mapping GOWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\text{GOWA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (7)$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$.

The GOWA operator is monotonic, bounded, commutative and idempotent. This operator also has as special cases the maximum, the minimum and the generalized mean (GM). If we look to different values of the parameter λ, we can also obtain other special cases, such as

- If $\lambda = 1$, the usual OWA operator.
- If $\lambda \rightarrow 0$, the ordered weighted geometric average (OWGA).
- If $\lambda = 2$, the ordered weighted quadratic average (OWQA).
- If $\lambda = -1$, the ordered weighted harmonic average (OWHA).

A further generalization of the GOWA operator is the quasi-arithmetic OWA (Quasi-OWA) operator, which uses the quasi-arithmetic means instead of the generalized means. Therefore, it replaces the parameter λ by a strictly continuous monotonic function g (Fodor et al. 1995; Merigó and Gil-Lafuente 2009).

Definition 8 A Quasi-OWA operator of dimension n is a mapping Quasi-OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, then:

$$\text{Quasi-OWA}(a_1, a_2, \ldots, a_n) = g^{-1} \left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right) \right) \quad (8)$$

where b_j is the jth largest of the a_i and g is a strictly continuous monotonic function.

3 Probabilistic ordered weighted averaging weighted average distance operator

The probabilistic ordered weighted averaging weighted average distance (POWAWAD) operator is a distance measure that uses the probability, the weighted average and the OWA operator in the normalization process of the Hamming distance by using the POWAWA operator. Thus, the reordering of the individual distances is developed according to the values of the individual distances formed by comparing two sets. The POWAWAD operator can be defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 9 A POWAWAD operator of dimension n is a mapping POWAWAD: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{POWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (9)$$

where b_j is the jth largest individual distance of the $|x_i - y_i|$, each argument $|x_i - y_i|$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, a probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0, 1]$, $C_1 + C_2 + C_3 = 1$, and v_j and p_j are the weights v_i and p_i ordered according to b_j, that is to say, according to the jth largest of the $|x_i - y_i|$.

Note that it is also possible to formulate the POWAWAD operator separating the part that strictly affects the OWAD operator, the weighted Hamming distance (WHD) and the part that affects the probabilistic distance. This representation is useful to see these models separated in the same formulation.

Definition 10 A POWAWAD operator is a mapping POWAWAD: $R^n \times R^n \rightarrow R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, a probabilistic vector P, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, and a weighting vector V that affects the weighted average, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\text{POWAWAD}(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \quad (10)$$

where b_j is the jth largest of the arguments $|x_i - y_i|$ and C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$.

If D is a vector corresponding to the ordered arguments b_j, we shall call this ordered argument vector, and W^{T} is the transpose of the weighting vector; then, the POWAWAD operator can be represented as follows:

$$\text{POWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = W^{\mathrm{T}} D \quad (11)$$

Observe that it is possible to distinguish between descending (DPOWAWAD) and ascending (APOWAWAD) orders.

The weights of these operators are related by $w_j = w*_{n-j+1}$, where w_j is the jth weight of the DPOWAWAD and $w*_{n-j+1}$ the jth weight of the APOWAWAD operator.

Note that if the weighting vector is not normalized, i.e. $\hat{V} = \sum_{j=1}^{n} \hat{v}_j \neq 1$, then, the POWAWAD operator can be expressed as

$$\text{POWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \frac{1}{\hat{V}} \sum_{j=1}^{n} \hat{v}_j b_j \tag{12}$$

Also note that POWAWAD $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = 0$ if and only if $x_i = y_i$ for all $i \in [1, n]$. Additionally, POWAWAD $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) =$ POWAWAD $(\langle y_1, x_1\rangle, \langle y_2, x_2\rangle, \ldots, \langle y_n, x_n\rangle)$.

The POWAWAD operator is commutative, monotonic, bounded and idempotent. It is monotonic because if $|x_i - y_i| \geq |s_i - t_i|$, for all $|x_i - y_i|$, then, POWAWAD $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \geq$ POWAWAD$(\langle s_1, t_1\rangle, \langle s_2, t_2\rangle, \ldots, \langle s_n, t_n\rangle)$. It is commutative because any permutation of the arguments has the same evaluation. It is bounded because the POWAWAD aggregation is delimited by the minimum and the maximum. That is, $\text{Min}\{|x_i - y_i|\} \leq$ POWAWAD$(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \leq \text{Max}\{|x_i - y_i|\}$. It is idempotent because if $|x_i - y_i| = |x - y|$, for all $|x_i - y_i|$, then, POWAWAD $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = |x - y|$.

Another interesting issue to consider is the attitudinal character of the POWAWAD operator $\alpha(W)$. Using a similar methodology as used for the OWA operator (Yager 1988) and following Eq. (9), we can define a new measure as follows:

$$\alpha(W) = \sum_{j=1}^{n} \hat{v}_j \left(\frac{n-j}{n-1}\right) \tag{13}$$

where n is the total number of arguments and j is the jth argument of the POWAWAD aggregation.

In this case, we could also make a distinction between descending and ascending orders. Additionally, it is also possible to use a measure that separates the OWA operator, the weighted average and the probability following the methodology of Merigó (2011) as follows:

$$\alpha(\hat{V}) = C_1 \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + C_2 \sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right) + C_3 \sum_{j=1}^{n} p_j \left(\frac{n-j}{n-1}\right) \tag{14}$$

As we can see, if $C_1 = 1$, Eq. (14) becomes the classical measure for the OWA operator (Yager 1988). Note that other measures could be discussed, such as the entropy of dispersion (Shannon 1948; Yager 1988), the divergence of W or the balance operator. The entropy of dispersion is defined as follows:

$$H(W) = -\sum_{j=1}^{n} \hat{v}_j \ln(\hat{v}_j) \tag{15}$$

In this equation, it is also possible to separate the OWA operator, the weighted average and the probability in the formulation as follows:

$$H(\hat{V}) = -\left(C_1 \sum_{j=1}^{n} w_j \ln(w_j) + C_2 \sum_{i=1}^{n} v_i \ln(v_i) + C_3 \sum_{i=1}^{n} p_i \ln(p_i)\right) \tag{16}$$

As we can see, if $C_1 = 1$, we obtain the Yager entropy (Yager 1988), and if $C_2 = 1$ or $C_3 = 1$, the result is the Shannon entropy (Shannon 1948). If $C_1 = 0$, we obtain the entropy of the PWA operator (Merigó et al. 2016), if $C_2 = 0$, we obtain the entropy of the POWA operator (Merigó 2012) and if $C_3 = 0$, we obtain the entropy of the OWAWA operator (Merigó 2011).

For the balance operator (Yager 1996), we obtain:

$$\text{BAL}(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) \hat{v}_j \tag{17}$$

And for the divergence of W (Yager 2002):

$$\text{DIV}(W) = \sum_{j=1}^{n} \hat{v}_j \left(\frac{n-j}{n-1} - \alpha(\hat{V})\right)^2 \tag{18}$$

Another interesting issue to consider is the different families of POWAWAD operators that are found in the weighting vector $\hat{V}$ and the coefficients C_1, C_2 and C_3. First, let us look into some remarkable cases that form new semi-boundary conditions for averaging aggregation operators with distance measures. If $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, the POWAWAD operator becomes the maximum probabilistic weighted average distance (Max-PWAD) which is formulated as follows:

$$\text{Max-PWAD} = C_1 \text{Max}\{b_j\} + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \tag{19}$$

If $w_n = 1$ and $w_j = 0$, for all $j \neq n$, the POWAWAD becomes the minimum probabilistic weighted average distance (Min-PWAD), which is expressed in the following way:

$$\text{Min-PWAD} = C_1 \text{Min}\{b_j\} + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \tag{20}$$

Some other particular cases of the POWAWAD operator worth noting are the following:

- The arithmetic PWAD (if $w_j = 1/n$, for all j):

$$\text{Arithmetic PWAD} = C_1 \left(\frac{1}{n} \sum_{j=1}^{n} b_j \right) + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \tag{21}$$

- The arithmetic POWAD operator (if $v_i = 1/n$, for all i):

$$\text{Arithmetic POWAD} = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - y_i| \right) + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \tag{22}$$

- The arithmetic OWAWAD operator (if $p_i = 1/n$, for all i):

$$\text{Arithmetic OWAWAD} = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - y_i| \right) \tag{23}$$

- The double arithmetic OWAD operator (if $p_i = 1/n$, for all i, and $v_i = 1/n$, for all i):

$$\text{DA-OWAD} = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - y_i| \right) + C_3 \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - y_i| \right) \tag{24}$$

- The double arithmetic WHD (if $p_i = 1/n$, for all i, and $w_j = 1/n$, for all j):

Table 1 Families of GPOWAWAD operators with the parameter λ, δ and χ

λ, δ and χ	GPOWAWAD
$\lambda = 1, \delta = 1, \chi = 1$	POWAWAD
$w_j = 1/n, \forall j$	Arithmetic PWAD
$v_i = 1/n, \forall i$	Arithmetic POWAD
$p_i = 1/n, \forall i$	Arithmetic OWAWAD
$v_i = 1/n$ and $p_i = 1/n, \forall i$	Double arithmetic OWAD
$w_j = 1/n$ and $v_i = 1/n, \forall i$ and j	Double arithmetic PAD
$w_j = 1/n$ and $p_i = 1/n, \forall i$ and j	Double arithmetic WAD
$\lambda = -1, \delta = -1, \chi = -1$	Harmonic POWAWAD
$w_j = 1/n, \forall j$	Harmonic mean PWAD
$v_i = 1/n, \forall i$	Harmonic mean POWAD
$p_i = 1/n, \forall i$	Harmonic mean OWAWAD
$v_i = 1/n$ and $p_i = 1/n, \forall i$	Double harmonic mean OWAD
$w_j = 1/n$ and $v_i = 1/n, \forall i$ and j	Double harmonic mean PAD
$w_j = 1/n$ and $p_i = 1/n, \forall i$ and j	Double harmonic mean WAD
$\lambda = 2, \delta = 2, \chi = 2$	Quadratic POWAWAD
$w_j = 1/n, \forall j$	Quadratic mean PWAD
$v_i = 1/n, \forall i$	Quadratic mean POWAD
$p_i = 1/n, \forall i$	Quadratic mean OWAWAD
$v_i = 1/n$ and $p_i = 1/n, \forall i$	Double quadratic mean OWAD
$w_j = 1/n$ and $v_i = 1/n, \forall i$ and j	Double quadratic mean PAD
$w_j = 1/n$ and $p_i = 1/n, \forall i$ and j	Double quadratic mean WAD
$\lambda \to 0, \delta \to 0, \chi \to 0$	Geometric POWAWAD
$w_j = 1/n, \forall j$	Geometric mean PWAD
$v_i = 1/n, \forall i$	Geometric mean POWAD
$p_i = 1/n, \forall i$	Geometric mean OWAWAD
$v_i = 1/n$ and $p_i = 1/n, \forall i$	Double geometric mean OWAD
$w_j = 1/n$ and $v_i = 1/n, \forall i$ and j	Double geometric mean PAD
$w_j = 1/n$ and $p_i = 1/n, \forall i$ and j	Double geometric mean WAD
$\lambda = 3, \delta = 3, \chi = 3$	Cubic POWAWAD
$w_j = 1/n, \forall j$	Cubic mean PWAD
$v_i = 1/n, \forall i$	Cubic mean POWAD
$p_i = 1/n, \forall i$	Cubic mean OWAWAD
$v_i = 1/n$ and $p_i = 1/n, \forall i$	Double cubic mean OWAD
$w_j = 1/n$ and $v_i = 1/n, \forall i$ and j	Double cubic mean PAD
$w_j = 1/n$ and $p_i = 1/n, \forall i$ and j	Double cubic mean WAD
$\lambda = -\infty, \delta = -\infty, \chi = -\infty$	Minimum distance
$\lambda = \infty, \delta = \infty, \chi = \infty$	Maximum distance
$\lambda = 1, \delta = 2, \chi = 1$	POWAD quadratic WAD
$\lambda = 1, \delta = 3, \chi = 1$	POWAD cubic WAD
$\lambda = 2, \delta = 1, \chi = 1$	PWAD quadratic OWAD
$\lambda = 2, \delta = 2, \chi = 1$	Probabilistic quadratic OWAWAD
$\lambda = 3, \delta = 1, \chi = 1$	PWAD cubic OWAD
$\lambda = 3, \delta = 3, \chi = 1$	Probabilistic cubic OWAWAD

$$\text{DA-WHD} = C_1\left(\frac{1}{n}\sum_{j=1}^{n} b_j\right) + C_2\sum_{i=1}^{n} v_i\,|x_i - y_i| + C_3\left(\frac{1}{n}\sum_{i=1}^{n}|x_i - y_i|\right) \tag{25}$$

- The double arithmetic PHD (if $v_i = 1/n$, for all i, and $w_j = 1/n$, for all j):

$$\text{DA-PHD} = C_1\left(\frac{1}{n}\sum_{j=1}^{n} b_j\right) + C_2\left(\frac{1}{n}\sum_{i=1}^{n}|x_i - y_i|\right) + C_3\sum_{i=1}^{n} p_i\,|x_i - y_i| \tag{26}$$

Many other particular cases can be studied by looking at different expressions of the weighting vectors and the coefficients C_1, C_2 and C_3. For example:

- If $C_1 = 1$, we obtain the OWAD operator.
- If $C_2 = 1$, the weighted Hamming distance (WHD).
- If $C_3 = 1$, the probabilistic Hamming distance (PHD).
- If $C_1 = 0$, the probabilistic weighted averaging distance (PWAD) (Merigó and Yager 2013).
- If $C_2 = 0$, the probabilistic OWA distance (POWAD) operator (Merigó and Yager 2013).
- If $C_3 = 0$, the OWAWA distance (OWAWAD) operator (Merigó et al. 2017).
- The POWAWA operator (if one of the sets X or Y is empty).
- The normalized Hamming distance (NHD) (if $v_i = 1/n$, for all i, $p_i = 1/n$, for all i, and $w_j = 1/n$, for all j).
- The maximum arithmetic PHD ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, and $v_i = 1/n$, for all i).
- The maximum arithmetic WHD ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, and $p_i = 1/n$, for all i).
- The minimum arithmetic PHD ($w_n = 1$ and $w_j = 0$, for all $j \neq n$, and $v_i = 1/n$, for all i).
- The minimum arithmetic WHD ($w_n = 1$ and $w_j = 0$, for all $j \neq n$, and $p_i = 1/n$, for all i).
- The POWAWAD Hurwicz criteria ($w_1 = \alpha$, $w_n = 1 - \alpha$ and $w_j = 0$, for all $j \neq 1, n$).
- The step-POWAWAD ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The olympic-POWAWAD operator ($w_1 = w_n = 0$, and $w_j = 1/(n\text{-}2)$ for all others).
- The general olympic-POWAWAD operator ($w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j*} = 1/(n-2k)$, where $k < n/2$).
- The S-POWAWAD ($w_1 = (1/n)(1-(\alpha+\beta))+\alpha$, $w_n = (1/n)(1-(\alpha+\beta)+\beta)$, and $w_j = (1/n)(1-(\alpha+\beta))$ for $j = 2$ to $n-1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centred-POWAWAD (if the weighting vector W is symmetric, strongly decaying from the centre to the maximum and the minimum, and inclusive).

Note that other families of POWAWAD operators could be used following a similar methodology, as in the development of the OWA operator and its extensions (Merigó and Gil-Lafuente 2009; Merigó 2012; Yager 1993). Moreover, it is also possible to extend this analysis by using other types of distances, such as the Euclidean (or quadratic) distance, the Minkowski (or generalized) distance and the quasi-arithmetic distance.

4 Generalized POWAWAD operator

The GPOWAWAD operator is a distance aggregation operator that integrates the probability, the weighted average and the OWA operator in the same formulation by considering the degree of importance that each sub-aggregation has in the problem. Moreover, this operator uses the generalized means, providing a general framework that includes a wide range of particular cases, including quadratic and harmonic aggregations. By using the generalized means, this approach is implicitly using a generalization of the Minkowski distance (Merigó and Casanovas 2011b). This technique's primary advantage is that it can aggregate distance measures considering subjective and objective information and the attitudinal character of the decision-maker. The GPOWAWAD operator is defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 11 A GPOWAWAD operator is a mapping GPOWAWAD: $R^n \times R^n \rightarrow R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, a probabilistic vector P, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, and a weighting vector V that affects the weighted average, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\text{GPOWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = C_1\left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} + C_2\left(\sum_{i=1}^{n} v_i\,|x_i - y_i|^{\delta}\right)^{1/\delta} + C_3\left(\sum_{i=1}^{n} p_i\,|x_i - y_i|^{\chi}\right)^{1/\chi} \tag{27}$$

where b_j is the jth largest of the arguments $|x_i - y_i|$, C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$ and λ, δ and χ are parameters such that λ, δ and $\chi \in \{-\infty, \infty\} - \{0\}$.

Note that in fuzzy set theory it is very practical to simplify the mapping to $[0, 1]$. That is, $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$.

Table 2 Asian average fixed asset according to different scenarios—expert 1

Country	Population	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Afghanistan	34,656,032	0.42	0.38	0.36	0.29	0.44	0.39	0.45	0.36	0.37	0.39
Armenia	2,924,816	0.37	0.42	0.29	0.34	0.39	0.31	0.40	0.42	0.30	0.27
Azerbaijan	9,725,376	0.39	0.42	0.43	0.37	0.41	0.34	0.42	0.26	0.44	0.19
Bahrain	1,425,171	0.26	0.38	0.25	0.48	0.28	0.41	0.29	0.38	0.26	0.37
Bangladesh	162,951,560	0.48	0.24	0.36	0.42	0.50	0.43	0.51	0.39	0.37	0.42
Bhutan	797,765	0.41	0.24	0.27	0.34	0.43	0.28	0.44	0.27	0.28	0.47
Brunei	423,196	0.27	0.47	0.23	0.48	0.29	0.21	0.30	0.31	0.24	0.38
Cambodia	15,762,370	0.42	0.37	0.41	0.31	0.44	0.40	0.45	0.47	0.42	0.31
China	1,403,500,365	0.36	0.29	0.39	0.32	0.38	0.26	0.39	0.39	0.40	0.49
Cyprus	1,170,125	0.34	0.36	0.48	0.38	0.36	0.27	0.37	0.29	0.49	0.50
East Timor	1,268,671	0.31	0.41	0.36	0.30	0.33	0.34	0.34	0.48	0.37	0.38
Egypt	95,688,681	0.28	0.32	0.38	0.40	0.30	0.27	0.31	0.29	0.39	0.42
Georgia	3,925,405	0.32	0.36	0.42	0.41	0.34	0.48	0.35	0.37	0.43	0.31
India	1,324,171,354	0.47	0.38	0.36	0.50	0.49	0.36	0.50	0.28	0.37	0.47
Indonesia	261,115,456	0.45	0.28	0.26	0.38	0.47	0.23	0.48	0.35	0.27	0.38
Iran	80,277,428	0.40	0.39	0.47	0.31	0.42	0.17	0.43	0.36	0.49	0.29
Iraq	37,202,572	0.46	0.26	0.37	0.28	0.48	0.48	0.49	0.35	0.39	0.38
Israel	8,191,828	0.24	0.31	0.51	0.29	0.26	0.37	0.27	0.26	0.53	0.35
Japan	127,748,513	0.21	0.30	0.35	0.24	0.23	0.36	0.24	0.24	0.37	0.31
Jordan	9,455,802	0.34	0.27	0.21	0.34	0.36	0.35	0.37	0.27	0.23	0.35
Kazakhstan	17,987,736	0.38	0.26	0.18	0.31	0.40	0.38	0.41	0.28	0.20	0.26
Kuwait	4,052,584	0.18	0.37	0.36	0.39	0.20	0.46	0.21	0.25	0.38	0.37
Kyrgyzstan	5,955,734	0.34	0.43	0.36	0.53	0.36	0.47	0.37	0.27	0.38	0.41
Laos	6,758,353	0.36	0.26	0.36	0.49	0.38	0.37	0.39	0.24	0.38	0.36
Lebanon	6,006,668	0.35	0.37	0.28	0.43	0.37	0.28	0.38	0.48	0.30	0.33
Malaysia	31,187,265	0.38	0.51	0.38	0.42	0.40	0.24	0.41	0.37	0.40	0.39
Maldives	427,756	0.23	0.37	0.37	0.36	0.25	0.31	0.26	0.36	0.39	0.28
Mongolia	3,027,398	0.30	0.28	0.27	0.49	0.32	0.43	0.33	0.27	0.29	0.38
Myanmar	52,885,223	0.29	0.31	0.36	0.32	0.31	0.37	0.32	0.38	0.38	0.26
Nepal	28,982,771	0.46	0.35	0.23	0.45	0.48	0.21	0.49	0.31	0.25	0.30
North Korea	25,368,620	0.41	0.32	0.41	0.38	0.43	0.37	0.44	0.38	0.43	0.22
Oman	4,424,762	0.43	0.38	0.35	0.44	0.45	0.39	0.46	0.28	0.37	0.37
Pakistan	193,203,476	0.48	0.30	0.27	0.42	0.50	0.24	0.51	0.37	0.29	0.34
Palestine	4,790,705	0.42	0.27	0.36	0.49	0.44	0.32	0.45	0.43	0.38	0.39
Philippines	103,320,222	0.40	0.21	0.39	0.18	0.42	0.42	0.43	0.52	0.41	0.43
Qatar	2,569,804	0.27	0.28	0.24	0.34	0.29	0.36	0.30	0.37	0.26	0.28
Russia	143,964,513	0.39	0.37	0.23	0.21	0.41	0.31	0.42	0.28	0.25	0.45
Saudi Arabia	32,275,687	0.22	0.41	0.21	0.39	0.24	0.40	0.25	0.40	0.23	0.31
Singapore	5,622,455	0.24	0.35	0.36	0.30	0.26	0.38	0.27	0.36	0.38	0.38
South Korea	50,791,919	0.24	0.32	0.32	0.35	0.26	0.39	0.27	0.27	0.34	0.36
Sri Lanka	20,798,492	0.25	0.28	0.36	0.31	0.27	0.32	0.28	0.25	0.38	0.38
Syria	18,430,453	0.43	0.25	0.27	0.49	0.45	0.36	0.46	0.21	0.29	0.29
Tajikistan	8,734,951	0.25	0.22	0.25	0.30	0.27	0.42	0.28	0.36	0.27	0.38
Thailand	68,863,514	0.19	0.33	0.39	0.47	0.21	0.27	0.22	0.34	0.41	0.31
Turkey	79,512,426	0.36	0.32	0.43	0.31	0.38	0.19	0.39	0.38	0.45	0.40

Table 2 continued

Country	Population	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Turkmenistan	5,662,544	0.41	0.39	0.32	0.37	0.43	0.39	0.44	0.51	0.34	0.38
United Arab Emirates	9,269,612	0.34	0.45	0.44	0.47	0.36	0.27	0.37	0.42	0.46	0.27
Uzbekistan	31,446,795	0.25	0.42	0.37	0.42	0.27	0.30	0.28	0.37	0.39	0.39
Vietnam	94,569,072	0.37	0.28	0.28	0.39	0.39	0.46	0.40	0.29	0.30	0.38
Yemen	27,584,213	0.45	0.35	0.30	0.31	0.47	0.27	0.48	0.35	0.32	0.41
Asian average	4,670,858,209	0.398	0.324	0.355	0.383	0.418	0.312	0.428	0.340	0.368	0.433

Note also that it is possible to distinguish between descending GPOWAWAD and ascending GPOWAWAD operators. The weights of these operators are related by $w_j = w*_{n-j+1}$.

Observe that if some of the weighting vectors are not normalized, i.e. $W = \sum_{j=1}^{n} w_j \neq 1$, $V = \sum_{i=1}^{n} v_i \neq 1$ and $P = \sum_{i=1}^{n} p_i \neq 1$; then, we can express the GPOWAWAD operator as

$$\text{GPOWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \\ = \frac{C_1}{W}\left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} + \frac{C_2}{V}\left(\sum_{i=1}^{n} v_i |x_i - y_i|^{\delta}\right)^{1/\delta} \\ + \frac{C_3}{P}\left(\sum_{i=1}^{n} p_i |x_i - y_i|^{\chi}\right)^{1/\chi} \tag{28}$$

The GPOWAWAD operator accomplishes reflexivity when $f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = 0$ if and only if $x_i = y_i$ for all $i \in [1, n]$. In addition, it has commutativity when $f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = f(\langle y_1, x_1\rangle, \langle y_2, x_2\rangle, \ldots, \langle y_n, x_n\rangle)$. The GPOWAWAD operator is also monotonic, bounded and idempotent.

When analysing the weights, sometimes it becomes useful to characterize them. A very common technique for doing so is the entropy of dispersion, which can be defined as follows:

$$H(\hat{V}) = -\left(C_1 \sum_{j=1}^{n} w_j \ln(w_j) + C_2 \sum_{i=1}^{n} v_i \ln(v_i) + C_3 \sum_{i=1}^{n} p_i \ln(p_i)\right) \tag{29}$$

As we can see, if $C_1 = 1$, we obtain the Yager entropy (Yager 1988) and if $C_2 = 1$ or $C_3 = 1$, the Shannon entropy (Shannon 1948). If $C_1 = 0$, we obtain the entropy of the PWA operator (Merigó et al. 2016), if $C_2 = 0$, the entropy of the POWA operator (Merigó 2012) and if $C_3 = 0$, the entropy of the OWA weighted average (Merigó 2011).

Additionally, we could also extend Eq. (14) by using the generalized means, obtaining the following measure for the degree of orness:

$$\alpha(\hat{V}) = C_1\left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right)^{\lambda}\right)^{1/\lambda} \\ + C_2\left(\sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right)^{\delta}\right)^{1/\delta} \\ + C_3\left(\sum_{j=1}^{n} p_j \left(\frac{n-j}{n-1}\right)^{\chi}\right)^{1/\chi} \tag{30}$$

As we can see, if $\lambda = \delta = \chi = 1$, Eq. (30) becomes Eq. (14). In addition, if $C_1 = 1$, Eq. (14) becomes the classical measure for the OWA operator (Yager 1988). Note that $\alpha(\tilde{V}) \in [0, 1]$ and degree of andness = 1 – degree of orness.

The GPOWAWAD operator can be further generalized by using the quasi-arithmetic means (Merigó and Gil-Lafuente 2009). Thus, we obtain the quasi-arithmetic POWAWAD (Quasi-POWAWAD) operator. Its main strength is that it provides a more general approach that includes more particular cases than the GPOWAWAD operator since it includes it as a particular case. It is defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 12 A Quasi-POWAWAD operator is a mapping Quasi-POWAWAD: $R^n \times R^n \rightarrow R$ of dimension n, that has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, a probabilistic vector P, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, and a weighting vector V that affects the weighted average, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\text{Quasi-POWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \\ = C_1 f^{-1}\left(\sum_{j=1}^{n} w_j f(b_j)\right) \\ + C_2 g^{-1}\left(\sum_{i=1}^{n} v_i g(|x_i - y_i|)\right) \\ + C_3 h^{-1}\left(\sum_{i=1}^{n} p_i h(|x_i - y_i|)\right) \tag{31}$$

Table 3 Asian average fixed asset according to different scenarios—expert 2

Country	Abbreviations	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Afghanistan	AFG	0.35	0.27	0.41	0.39	0.44	0.35	0.34	0.38	0.36	0.38
Armenia	ARM	0.53	0.38	0.37	0.40	0.40	0.26	0.52	0.28	0.54	0.27
Azerbaijan	AZE	0.48	0.24	0.28	0.52	0.31	0.25	0.47	0.18	0.49	0.36
Bahrain	BAH	0.19	0.31	0.39	0.38	0.42	0.50	0.18	0.43	0.20	0.39
Bangladesh	BAN	0.38	0.46	0.33	0.34	0.36	0.32	0.37	0.47	0.39	0.42
Bhutan	BHU	0.33	0.27	0.31	0.27	0.34	0.42	0.32	0.37	0.34	0.33
Brunei	BRU	0.37	0.36	0.27	0.35	0.30	0.46	0.36	0.46	0.38	0.39
Cambodia	CAM	0.43	0.25	0.26	0.31	0.29	0.38	0.42	0.25	0.44	0.28
China	CHN	0.34	0.21	0.23	0.43	0.26	0.29	0.33	0.26	0.35	0.40
Cyprus	CYP	0.29	0.56	0.38	0.24	0.41	0.47	0.28	0.38	0.30	0.37
East Timor	ET	0.40	0.34	0.42	0.37	0.45	0.26	0.39	0.31	0.41	0.32
Egypt	EGY	0.45	0.25	0.33	0.36	0.36	0.36	0.44	0.37	0.46	0.35
Georgia	GEO	0.22	0.26	0.28	0.35	0.31	0.31	0.21	0.49	0.23	0.40
India	IND	0.39	0.34	0.36	0.19	0.39	0.46	0.38	0.27	0.40	0.32
Indonesia	INO	0.33	0.27	0.39	0.28	0.42	0.37	0.32	0.37	0.34	0.36
Iran	IRN	0.53	0.39	0.26	0.31	0.29	0.32	0.52	0.34	0.54	0.35
Iraq	IRQ	0.38	0.42	0.30	0.42	0.33	0.39	0.37	0.32	0.39	0.41
Israel	ISR	0.42	0.20	0.40	0.38	0.43	0.40	0.41	0.18	0.43	0.27
Japan	JAP	0.28	0.38	0.32	0.31	0.35	0.27	0.27	0.39	0.29	0.39
Jordan	JOR	0.26	0.37	0.36	0.27	0.39	0.30	0.25	0.44	0.27	0.35
Kazakhstan	KAZ	0.40	0.25	0.31	0.38	0.34	0.35	0.39	0.32	0.41	0.40
Kuwait	KUW	0.31	0.19	0.46	0.47	0.49	0.41	0.30	0.47	0.32	0.37
Kyrgyzstan	KYR	0.32	0.34	0.32	0.26	0.35	0.34	0.31	0.50	0.33	0.28
Laos	LAO	0.27	0.43	0.48	0.38	0.51	0.37	0.26	0.32	0.28	0.38
Lebanon	LEB	0.43	0.46	0.31	0.40	0.34	0.36	0.42	0.37	0.44	0.27
Malaysia	MLS	0.38	0.26	0.36	0.29	0.39	0.31	0.37	0.31	0.39	0.30
Maldives	MLD	0.42	0.28	0.45	0.38	0.48	0.40	0.41	0.35	0.43	0.32
Mongolia	MON	0.36	0.49	0.32	0.25	0.35	0.38	0.35	0.25	0.37	0.35
Myanmar	MYN	0.25	0.25	0.36	0.36	0.39	0.41	0.24	0.38	0.26	0.38
Nepal	NEP	0.28	0.37	0.35	0.42	0.38	0.38	0.27	0.32	0.29	0.27
North Korea	NK	0.39	0.31	0.34	0.35	0.37	0.32	0.38	0.39	0.40	0.30
Oman	OMA	0.43	0.42	0.41	0.39	0.44	0.37	0.42	0.42	0.44	0.25
Pakistan	PAK	0.19	0.39	0.38	0.28	0.41	0.38	0.18	0.36	0.20	0.32
Palestine	PAL	0.38	0.38	0.36	0.41	0.39	0.40	0.37	0.32	0.39	0.41
Philippines	PHI	0.37	0.29	0.42	0.38	0.45	0.50	0.36	0.37	0.38	0.27
Qatar	QAT	0.36	0.38	0.50	0.41	0.53	0.38	0.35	0.30	0.37	0.36
Russia	RUS	0.33	0.36	0.48	0.32	0.51	0.37	0.32	0.42	0.34	0.35
Saudi Arabia	SA	0.28	0.42	0.42	0.38	0.45	0.35	0.27	0.31	0.29	0.39
Singapore	SGP	0.40	0.28	0.32	0.39	0.35	0.31	0.39	0.32	0.41	0.38
South Korea	SK	0.22	0.31	0.44	0.40	0.47	0.39	0.21	0.38	0.23	0.40
Sri Lanka	SL	0.39	0.38	0.25	0.52	0.28	0.42	0.38	0.37	0.40	0.32
Syria	SYR	0.37	0.42	0.31	0.44	0.34	0.19	0.36	0.32	0.38	0.36
Tajikistan	TAJ	0.37	0.38	0.56	0.24	0.59	0.40	0.36	0.42	0.38	0.37
Thailand	THA	0.50	0.35	0.28	0.28	0.31	0.50	0.49	0.41	0.51	0.28
Turkey	TRK	0.29	0.38	0.49	0.15	0.52	0.38	0.28	0.28	0.30	0.39

Table 3 continued

Country	Abbreviations	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Turkmenistan	TRM	0.28	0.39	0.27	0.31	0.30	0.32	0.27	0.36	0.29	0.40
United Arab Emirates	UAE	0.38	0.31	0.38	0.42	0.41	0.47	0.37	0.35	0.39	0.36
Uzbekistan	UZB	0.43	0.34	0.18	0.36	0.21	0.28	0.42	0.40	0.44	0.32
Vietnam	VIE	0.33	0.28	0.37	0.31	0.40	0.36	0.32	0.39	0.34	0.33
Yemen	YEM	0.39	0.40	0.26	0.43	0.29	0.35	0.38	0.32	0.40	0.39
Asian average		0.354	0.302	0.324	0.318	0.354	0.370	0.344	0.310	0.364	0.357

where b_j is the jth largest of the arguments $|x_i - y_i|$, C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$ and f, g and h are strictly continuous monotonic functions.

Note that if $f = b^\lambda$, $g = |x_i - y_i|^\delta$ and $h = |x_i - y_i|^\chi$, the Quasi-POWAWAD becomes the GPOWAWAD operator. If we analyse different values of the parameter λ, δ and χ (or different functions f, g and h), we obtain a group of particular cases of the GPOWAWAD and Quasi-POWAWAD operators. For example, for the GPOWAWAD operator we may obtain the following special cases.

Remark 1 When $\lambda = \delta = \chi = 1$, the GPOWAWAD becomes the POWAWAD operator.

$$\text{GPOWAWAD} = C_1\left(\sum_{j=1}^{n} w_j b_j\right) + C_2\left(\sum_{i=1}^{n} v_i\,|x_i - y_i|\right) + C_3\left(\sum_{i=1}^{n} p_i\,|x_i - y_i|\right) \tag{32}$$

Remark 2 If $\lambda \to 0$, $\delta \to 0$, and $\chi \to 0$, we obtain the geometric POWAWAD (GPOWAWAD) operator.

$$\text{GPOWAWAD} = C_1\left(\prod_{j=1}^{n} b_j^{w_j}\right) + C_2\left(\prod_{i=1}^{n} |x_i - y_i|^{v_i}\right) + C_3\left(\prod_{i=1}^{n} |x_i - y_i|^{p_i}\right) \tag{33}$$

Remark 3 If $\lambda = \delta = \chi = -1$, the harmonic POWAWAD (HPOWAWAD) operator is obtained.

$$\text{GPOWAWAD} = C_1\frac{1}{\sum_{j=1}^{n}\frac{w_j}{b_j}} + C_2\frac{1}{\sum_{i=1}^{n}\frac{v_i}{|x_i-y_i|}} + C_3\frac{1}{\sum_{i=1}^{n}\frac{p_i}{|x_i-y_i|}} \tag{34}$$

Remark 4 If $\lambda = \delta = \chi = 2$, the quadratic POWAWAD (QPOWAWAD) operator is obtained.

$$\text{GPOWAWAD} = C_1\left(\sum_{j=1}^{n} w_j b_j^2\right)^{1/2} + C_2\left(\sum_{i=1}^{n} v_i\,|x_i - y_i|^2\right)^{1/2} + C_3\left(\sum_{i=1}^{n} p_i\,|x_i - y_i|^2\right)^{1/2} \tag{35}$$

Remark 5 If $\lambda = \delta = \chi = 3$, the cubic POWAWAD (CPOWAWAD) operator is obtained.

$$\text{GPOWAWAD} = C_1\left(\sum_{j=1}^{n} w_j b_j^3\right)^{1/3} + C_2\left(\sum_{i=1}^{n} v_i\,|x_i - y_i|^3\right)^{1/3} + C_3\left(\sum_{i=1}^{n} p_i\,|x_i - y_i|^3\right)^{1/3} \tag{36}$$

Similar families can be developed for the Quasi-POWAWAD operator. Following the previous equations described in this section, Table 1 presents an overview of some of the key special cases obtained by setting the parameter λ, δ and χ to specific values.

Note that many other families can be studied by using other values in the parameters λ, δ and χ and mixing different values of λ, δ and χ in the same aggregation. To this end, we can consider a considerably wider range of aggregation operators. Additionally, the results of Table 1 can also be obtained with the functions f, g and h of the Quasi-POWAWAD operator.

Finally, let us look into a further generalization of the GPOWAWAD and Quasi-POWAWAD by using the generalized and the quasi-arithmetic means in the coefficients C.

Table 4 Asian average fixed asset according to different scenarios—expert 3

Country	Weights	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Afghanistan	0.007420	0.36	0.36	0.35	0.37	0.34	0.38	0.40	0.38	0.37	0.35
Armenia	0.000626	0.27	0.32	0.28	0.39	0.27	0.27	0.31	0.32	0.30	0.39
Azerbaijan	0.002082	0.38	0.38	0.41	0.42	0.40	0.40	0.42	0.39	0.43	0.32
Bahrain	0.000305	0.30	0.27	0.33	0.35	0.32	0.32	0.34	0.40	0.35	0.36
Bangladesh	0.034887	0.21	0.36	0.39	0.32	0.38	0.31	0.25	0.38	0.41	0.42
Bhutan	0.000171	0.38	0.32	0.37	0.33	0.36	0.35	0.42	0.28	0.39	0.32
Brunei	0.000091	0.42	0.41	0.33	0.31	0.32	0.34	0.46	0.40	0.35	0.38
Cambodia	0.003375	0.39	0.27	0.32	0.40	0.31	0.37	0.43	0.35	0.34	0.37
China	0.300480	0.32	0.36	0.36	0.38	0.35	0.35	0.36	0.32	0.38	0.35
Cyprus	0.000251	0.35	0.40	0.40	0.35	0.39	0.39	0.39	0.40	0.42	0.31
East Timor	0.000272	0.37	0.32	0.30	0.32	0.29	0.40	0.41	0.38	0.32	0.28
Egypt	0.020486	0.32	0.38	0.27	0.39	0.26	0.43	0.36	0.32	0.29	0.38
Georgia	0.000840	0.37	0.42	0.41	0.40	0.40	0.27	0.41	0.39	0.43	0.37
India	0.283496	0.37	0.37	0.38	0.38	0.37	0.38	0.41	0.40	0.40	0.38
Indonesia	0.055903	0.43	0.35	0.36	0.40	0.35	0.42	0.47	0.32	0.38	0.42
Iran	0.017187	0.27	0.32	0.38	0.38	0.37	0.38	0.31	0.38	0.40	0.36
Iraq	0.007965	0.38	0.28	0.33	0.32	0.32	0.42	0.42	0.40	0.35	0.28
Israel	0.001754	0.31	0.40	0.34	0.36	0.33	0.27	0.35	0.36	0.36	0.25
Japan	0.027350	0.32	0.37	0.41	0.38	0.40	0.39	0.36	0.42	0.43	0.38
Jordan	0.002024	0.42	0.33	0.27	0.36	0.26	0.40	0.46	0.31	0.29	0.32
Kazakhstan	0.003851	0.37	0.32	0.33	0.39	0.32	0.36	0.41	0.38	0.35	0.26
Kuwait	0.000868	0.36	0.37	0.31	0.42	0.30	0.27	0.40	0.32	0.33	0.39
Kyrgyzstan	0.001275	0.39	0.35	0.44	0.27	0.43	0.38	0.43	0.40	0.46	0.27
Laos	0.001447	0.27	0.39	0.28	0.39	0.27	0.42	0.31	0.35	0.30	0.39
Lebanon	0.001286	0.36	0.29	0.39	0.42	0.38	0.36	0.40	0.32	0.41	0.19
Malaysia	0.006677	0.41	0.42	0.41	0.35	0.40	0.38	0.45	0.39	0.43	0.32
Maldives	0.000092	0.27	0.37	0.44	0.39	0.43	0.27	0.31	0.40	0.46	0.27
Mongolia	0.000648	0.36	0.35	0.28	0.34	0.27	0.39	0.40	0.42	0.30	0.25
Myanmar	0.011322	0.38	0.39	0.37	0.32	0.36	0.34	0.42	0.36	0.39	0.32
Nepal	0.006205	0.39	0.27	0.32	0.40	0.31	0.42	0.43	0.32	0.34	0.34
North Korea	0.005431	0.40	0.41	0.39	0.42	0.38	0.31	0.44	0.34	0.41	0.38
Oman	0.000947	0.42	0.29	0.47	0.35	0.46	0.27	0.46	0.33	0.49	0.42
Pakistan	0.041364	0.36	0.39	0.43	0.32	0.42	0.28	0.40	0.31	0.45	0.37
Palestine	0.001026	0.31	0.43	0.38	0.38	0.37	0.39	0.35	0.38	0.40	0.33
Philippines	0.022120	0.36	0.47	0.32	0.32	0.31	0.42	0.40	0.36	0.34	0.31
Qatar	0.000550	0.32	0.28	0.31	0.42	0.30	0.44	0.36	0.42	0.33	0.27
Russia	0.030822	0.35	0.39	0.48	0.36	0.47	0.36	0.39	0.27	0.50	0.38
Saudi Arabia	0.006910	0.39	0.40	0.33	0.34	0.32	0.38	0.43	0.29	0.35	0.29
Singapore	0.001204	0.40	0.32	0.36	0.31	0.35	0.42	0.44	0.24	0.38	0.36
South Korea	0.010874	0.50	0.51	0.39	0.39	0.38	0.38	0.54	0.30	0.41	0.38
Sri Lanka	0.004453	0.48	0.27	0.42	0.40	0.41	0.39	0.52	0.32	0.44	0.42
Syria	0.003946	0.42	0.36	0.39	0.28	0.38	0.34	0.46	0.36	0.41	0.41
Tajikistan	0.001870	0.32	0.40	0.37	0.36	0.36	0.32	0.36	0.27	0.39	0.38
Thailand	0.014743	0.27	0.38	0.33	0.34	0.32	0.28	0.31	0.29	0.35	0.28
Turkey	0.017023	0.38	0.35	0.40	0.42	0.39	0.40	0.42	0.22	0.42	0.36

Table 4 continued

Country	Weights	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
		I	R	I	R	I	R	I	R	I	R
Turkmenistan	0.001212	0.36	0.32	0.38	0.38	0.37	0.39	0.40	0.31	0.40	0.39
United Arab Emirates	0.001985	0.42	0.41	0.37	0.39	0.36	0.32	0.46	0.24	0.39	0.30
Uzbekistan	0.006733	0.32	0.38	0.41	0.40	0.40	0.40	0.36	0.31	0.43	0.32
Vietnam	0.020247	0.37	0.36	0.39	0.32	0.38	0.35	0.41	0.37	0.41	0.27
Yemen	0.005906	0.39	0.35	0.33	0.35	0.32	0.32	0.43	0.29	0.35	0.35
Asian average	1	0.349	0.368	0.374	0.372	0.364	0.365	0.389	0.349	0.394	0.363

Note that this is possible because the coefficients C can be seen as a weighted average that can be generalized by using the weighted generalized mean and the weighted quasi-arithmetic mean. Following Eq. (27), the formulation with the weighted generalized mean is as follows:

$$\begin{aligned} &\text{GPOWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \\ &= \left(C_1 \left(\left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \right)^{\varepsilon} \right. \\ &\quad + C_2 \left(\left(\sum_{i=1}^{n} v_i \left| x_i - y_i \right|^{\delta} \right)^{1/\delta} \right)^{\varepsilon} \\ &\quad \left. + C_3 \left(\left(\sum_{i=1}^{n} p_i \left| x_i - y_i \right|^{\chi} \right)^{1/\chi} \right)^{\varepsilon} \right)^{1/\varepsilon} \end{aligned} \quad (37)$$

where ε is a parameter such that $\varepsilon \in \{-\infty, \infty\} - \{0\}$.

With the quasi-arithmetic means, using a strictly continuous monotonic function z yields:

$$\begin{aligned} &\text{Quasi-POWAWAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \\ &= z^{-1} \left(C_1 z \left(f^{-1} \left(\sum_{j=1}^{n} w_j f(b_j) \right) \right) \right. \\ &\quad + C_2 z \left(g^{-1} \left(\sum_{i=1}^{n} v_i g\left(|x_i - y_i| \right) \right) \right) \\ &\quad \left. + C_3 z \left(h^{-1} \left(\sum_{i=1}^{n} p_i h\left(|x_i - y_i| \right) \right) \right) \right) \end{aligned} \quad (38)$$

where z is a strictly continuous monotonic function.

Observe that these generalizations can also be implemented with the OWA operator, that is, by reordering the final results of the OWAD, WHD and PHD in descending or ascending order and using the coefficients C as the OWA weights. The formulation would be as follows:

$$\begin{aligned} &\text{GPOWAWAD}_{\text{OWA}}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \\ &= \sum_{k=1}^{q} C_k m_k \end{aligned} \quad (39)$$

where m_k is the kth largest of $\left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}$, $\left(\sum_{i=1}^{n} v_i \left|x_i - y_i\right|^{\delta}\right)^{1/\delta}$ and $\left(\sum_{i=1}^{n} p_i \left|x_i - y_i\right|^{\chi}\right)^{1/\chi}$.

In addition, with the generalized OWA:

$$\begin{aligned} &\text{GPOWAWAD}_{\text{GOWA}}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \\ &= \left(\sum_{k=1}^{q} C_k m_k^{\pi} \right)^{1/\pi} \end{aligned} \quad (40)$$

where π is a parameter such that $\pi \in (-\infty, \infty) - \{0\}$, and m_k is the kth largest of $\left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}$, $\left(\sum_{i=1}^{n} v_i \left|x_i - y_i\right|^{\delta}\right)^{1/\delta}$ and $\left(\sum_{i=1}^{n} p_i \left|x_i - y_i\right|^{\chi}\right)^{1/\chi}$.

This formulation is also interesting because many times it is not clear which of the three aggregations to use. Therefore, a decision-maker may weight each of them according to the results they provide following the OWA philosophy. Note that here it is also possible to use the OWAWA operator and the POWAWA operator by, for example, mixing Eq. (27) with Eq. (39). Additionally, following the aggregation operator literature (Beliakov et al. 2007, 2016) it is possible to unify the OWA, the weighted average and the probability with the coefficients C by using many other types of averages, such as the Bonferroni mean (Blanco-Mesa et al. 2016), logarithmic aggregations (Alfaro-García et al. 2018; Zhou et al. 2015) and Choquet integrals (Belles-Sampera et al. 2014).

5 Applicability of the POWAWAD operator

This section analyses the applicability of the POWAWAD operator and its extensions with a main focus on information aggregation and decision-making. First, the work analyses some ideas about multi-person aggregation. Next, the paper presents an application in the calculation of the average

Table 5 Asian average fixed asset according to different scenarios—collective results

Country	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
	I	R	I	R	I	R	I	R	I	R
Afghanistan	0.37	0.32	0.38	0.36	0.42	0.37	0.39	0.37	0.37	0.38
Armenia	0.43	0.38	0.33	0.38	0.37	0.28	0.44	0.33	0.42	0.29
Azerbaijan	0.43	0.32	0.35	0.46	0.36	0.31	0.45	0.25	0.46	0.30
Bahrain	0.23	0.32	0.34	0.40	0.36	0.44	0.25	0.41	0.25	0.38
Bangladesh	0.38	0.37	0.35	0.36	0.41	0.35	0.39	0.43	0.39	0.42
Bhutan	0.36	0.27	0.31	0.30	0.37	0.36	0.38	0.32	0.33	0.37
Brunei	0.35	0.40	0.27	0.38	0.30	0.36	0.36	0.40	0.33	0.39
Cambodia	0.42	0.29	0.32	0.33	0.34	0.38	0.43	0.34	0.41	0.31
China	0.34	0.26	0.30	0.39	0.31	0.29	0.35	0.31	0.37	0.42
Cyprus	0.32	0.47	0.41	0.30	0.39	0.39	0.33	0.36	0.38	0.40
East Timor	0.37	0.36	0.38	0.34	0.38	0.31	0.38	0.38	0.38	0.33
Egypt	0.37	0.30	0.33	0.38	0.32	0.35	0.39	0.34	0.41	0.38
Georgia	0.28	0.32	0.35	0.38	0.34	0.35	0.29	0.43	0.33	0.37
India	0.41	0.36	0.36	0.32	0.42	0.41	0.42	0.30	0.39	0.38
Indonesia	0.39	0.29	0.35	0.33	0.42	0.34	0.40	0.35	0.33	0.38
Iran	0.44	0.38	0.35	0.32	0.35	0.29	0.45	0.35	0.50	0.33
Iraq	0.40	0.34	0.33	0.36	0.37	0.42	0.42	0.35	0.38	0.38
Israel	0.34	0.27	0.42	0.35	0.36	0.37	0.36	0.24	0.45	0.29
Japan	0.27	0.35	0.35	0.30	0.32	0.32	0.28	0.35	0.34	0.36
Jordan	0.32	0.33	0.30	0.31	0.36	0.34	0.33	0.36	0.26	0.34
Kazakhstan	0.39	0.27	0.28	0.36	0.35	0.36	0.40	0.32	0.34	0.33
Kuwait	0.28	0.28	0.40	0.44	0.37	0.40	0.29	0.37	0.34	0.37
Kyrgyzstan	0.34	0.37	0.36	0.34	0.37	0.39	0.35	0.41	0.37	0.32
Laos	0.30	0.37	0.40	0.42	0.42	0.38	0.31	0.30	0.31	0.38
Lebanon	0.39	0.40	0.32	0.41	0.36	0.34	0.40	0.39	0.39	0.27
Malaysia	0.39	0.37	0.38	0.34	0.40	0.30	0.40	0.34	0.40	0.33
Maldives	0.33	0.33	0.42	0.38	0.40	0.35	0.35	0.36	0.42	0.30
Mongolia	0.34	0.40	0.30	0.34	0.33	0.40	0.35	0.29	0.33	0.34
Myanmar	0.29	0.30	0.36	0.34	0.36	0.38	0.30	0.38	0.32	0.33
Nepal	0.36	0.34	0.31	0.43	0.40	0.34	0.37	0.32	0.29	0.29
North Korea	0.40	0.33	0.37	0.37	0.39	0.33	0.41	0.38	0.41	0.29
Oman	0.43	0.38	0.40	0.40	0.45	0.36	0.44	0.36	0.43	0.32
Pakistan	0.31	0.36	0.36	0.33	0.44	0.32	0.32	0.35	0.28	0.34
Palestine	0.38	0.36	0.36	0.43	0.40	0.37	0.39	0.37	0.39	0.39
Philippines	0.38	0.30	0.39	0.31	0.41	0.46	0.39	0.41	0.38	0.33
Qatar	0.33	0.33	0.38	0.39	0.41	0.39	0.34	0.35	0.33	0.32
Russia	0.35	0.37	0.41	0.30	0.47	0.35	0.36	0.35	0.35	0.39
Saudi Arabia	0.28	0.41	0.34	0.38	0.36	0.37	0.30	0.33	0.28	0.35
Singapore	0.35	0.31	0.34	0.35	0.32	0.35	0.36	0.32	0.40	0.38
South Korea	0.28	0.35	0.39	0.38	0.39	0.39	0.29	0.33	0.30	0.38
Sri Lanka	0.37	0.33	0.32	0.43	0.30	0.38	0.38	0.32	0.40	0.36
Syria	0.40	0.36	0.31	0.42	0.38	0.27	0.41	0.30	0.36	0.35
Tajikistan	0.32	0.34	0.43	0.28	0.45	0.39	0.34	0.37	0.35	0.38
Thailand	0.36	0.35	0.32	0.35	0.28	0.39	0.37	0.37	0.45	0.29
Turkey	0.33	0.36	0.45	0.25	0.45	0.33	0.34	0.30	0.37	0.39

Table 5 continued

Country	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
	I	R	I	R	I	R	I	R	I	R
Turkmenistan	0.34	0.38	0.31	0.34	0.35	0.36	0.35	0.40	0.33	0.39
United Arab Emirates	0.38	0.37	0.40	0.43	0.39	0.38	0.39	0.35	0.41	0.32
Uzbekistan	0.35	0.37	0.28	0.39	0.27	0.31	0.37	0.37	0.42	0.34
Vietnam	0.35	0.30	0.35	0.34	0.39	0.39	0.36	0.36	0.34	0.33
Yemen	0.41	0.38	0.29	0.38	0.35	0.32	0.42	0.32	0.37	0.39
Asian average	0.366	0.321	0.341	0.349	0.375	0.350	0.377	0.327	0.371	0.383

fixed assets. The section ends by analysing how to aggregate information with this new framework in real-world problems associated with asset management.

5.1 Introduction

The POWAWAD operator generalizes a wide range of distance aggregation operators. Moreover, if one of the sets X and Y is empty, it becomes the POWAWA operator, which also includes a wide range of averaging aggregation operators. Therefore, all of the previous studies that use average distances, OWA operators, weighted averages or probabilistic aggregations can be revised and extended with this new approach. Note that in simple problems, the POWAWAD operator can be reduced to some of its special cases. However, the main advantage of the POWAWAD is the ability to better represent more complex frameworks that address subjective and objective information and with the attitudinal character of the decision-maker.

In group decision-making problems (Blanco-Mesa et al. 2017; Figueira et al. 2005; Kahraman et al. 2015), the POWAWAD operator is useful as a technique to assess the information. However, in this instance, it is necessary to extend the POWAWAD to a multi-person framework. To this end, following Merigó (2012), we can introduce the multi-person POWAWAD (MP-POWAWAD) operator. This approach uses an additional weighted average aggregation process in the aggregation of the POWAWAD that summarizes the information provided by different persons or experts.

Definition 13 An MP-POWAWAD operator is a mapping MP-POWAWAD: $R^n \times R^n \rightarrow R$ that has a weighting vector Z of dimension t with $\sum_{k=1}^{t} z_k = 1$ and $z_k \in [0, 1]$, a weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, a probabilistic vector P, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, and a weighting vector V that affects the weighted average, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\text{MP-POWAWAD}\left(\{\langle x_1, y_1\rangle^1, \ldots, \langle x_1, y_1\rangle^t\}, \ldots, \{\langle x_n, y_n\rangle^1, \ldots, \langle x_n, y_n\rangle^t\}\right) =$$

$$= C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i d_i + C_3 \sum_{i=1}^{n} p_i d_i \quad (41)$$

where b_j is the jth largest of the d_i, $d_i = \sum_{k=1}^{t} z_k d_i^k$, d_i^k is the individual distance $|x_i - y_i|^k$ between the sets $X = \{x_1, \ldots, x_n\}^k$ and $Y = \{y_1, \ldots, y_n\}^k$ provided by the kth person (or expert), and C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$.

Note that following Eq. (41), we can also develop similar formulations for all the definitions and formulas presented in the previous sections, including the multi-person GPOWAWAD and the multi-person quasi-arithmetic POWAWAD. Additionally, observe that all the particular cases mentioned in Sects. 3 and 4 are also available in Eq. (41). By using Definition 13, we can build a group decision-making process (Merigó 2012).

5.2 Calculation of the average fixed assets with OWA operators

In this paper, let us focus on the calculation of the average fixed asset. The average fixed asset addresses a set of fixed assets providing a numerical value that summarizes the information of the set. Usually, researchers and economists use the arithmetic mean or the weighted average in the analysis of the average fixed assets. However, it is also possible to use many other averaging aggregation operators, including the OWA and the WOWA operator. The use of the OWA operator produces the ordered weighted average fixed asset (OWAFA). The OWAFA operator is an aggregation operator that analyses a set of fixed assets providing a parameterized family of aggregation operators between the minimum and maximum fixed assets. It is very useful to analyse the information of a set of assets under complex and uncertain environments, where it is possible to under- or overestimate the data according to the attitudinal character of the decision-maker. Following Eq. (5), the OWAFA operator is defined as follows for a set of fixed assets $A = \{a_1, a_2, \ldots, a_n\}$:

Table 6 Individual distances between the ideal percentage of fixed assets and the actual ones

Country	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Afghanistan	0.052	0.027	0.052	0.011	0.012
Armenia	0.05	0.052	0.094	0.112	0.126
Azerbaijan	0.111	0.104	0.051	0.199	0.162
Bahrain	0.09	0.068	0.079	0.164	0.13
Bangladesh	0.002	0.009	0.055	0.04	0.032
Bhutan	0.093	0.007	0.007	0.054	0.038
Brunei	0.053	0.111	0.06	0.041	0.053
Cambodia	0.129	0.011	0.045	0.095	0.107
China	0.078	0.083	0.021	0.043	0.046
Cyprus	0.151	0.11	0.003	0.028	0.016
East Timor	0.01	0.039	0.07	0.004	0.05
Egypt	0.076	0.045	0.025	0.049	0.028
Georgia	0.042	0.03	0.016	0.142	0.037
India	0.052	0.043	0.002	0.123	0.014
Indonesia	0.097	0.011	0.083	0.044	0.051
Iran	0.063	0.023	0.058	0.097	0.163
Iraq	0.06	0.031	0.05	0.071	0.007
Israel	0.071	0.072	0.006	0.116	0.156
Japan	0.087	0.044	0.003	0.072	0.022
Jordan	0.016	0.012	0.02	0.035	0.082
Kazakhstan	0.121	0.086	0.007	0.08	0.005
Kuwait	0.001	0.036	0.032	0.081	0.034
Kyrgyzstan	0.029	0.013	0.018	0.059	0.054
Laos	0.074	0.011	0.043	0.007	0.062
Lebanon	0.007	0.096	0.021	0.011	0.12
Malaysia	0.019	0.035	0.092	0.054	0.07
Maldives	0.008	0.048	0.054	0.018	0.126
Mongolia	0.057	0.043	0.072	0.064	0.007
Myanmar	0.008	0.022	0.024	0.076	0.01
Nepal	0.012	0.117	0.059	0.051	0.005
North Korea	0.065	0.002	0.057	0.033	0.119
Oman	0.046	0.007	0.091	0.08	0.109
Pakistan	0.052	0.027	0.121	0.03	0.059
Palestine	0.021	0.064	0.027	0.025	0.001
Philippines	0.075	0.083	0.047	0.024	0.055
Qatar	0.005	0.007	0.026	0.008	0.011
Russia	0.017	0.11	0.122	0.016	0.041
Saudi Arabia	0.129	0.036	0.01	0.037	0.062
Singapore	0.043	0.007	0.03	0.048	0.019
South Korea	0,071	0,011	0,001	0,037	0,085
Sri Lanka	0,038	0,116	0,081	0,054	0,044
Syria	0,041	0,109	0,11	0,115	0,01
Tajikistan	0,012	0,147	0,058	0,036	0,026
Thailand	0,011	0,026	0,105	0,008	0,159
Turkey	0,027	0,202	0,125	0,043	0,018
Turkmenistan	0,041	0,035	0,002	0,048	0,065
United Arab Emirates	0,004	0,033	0,005	0,039	0,09

Table 6 continued

Country	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Uzbekistan	0,018	0,103	0,044	0,007	0,082
Vietnam	0,054	0,011	0,005	0,006	0,009
Yemen	0,033	0,092	0,03	0,097	0,022
Asian average	0,060	0,057	0,033	0,066	0,039

$$\text{OWAFA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{42}$$

where b_j is the jth largest of the a_i.

Note that the OWAFA operator assumes that there is no information regarding the importance of each of the fixed assets. However, in real-world problems, it is common to have some information concerning the importance of the fixed assets in the whole set. Therefore, a better approach for aggregating the information is by using a technique that combines the OWA operator with the weighted average, such as the WOWA operator (Torra 1997), the hybrid average (Xu and Da 2003), the importance OWA (Yager 1998) and the OWAWA operator (Merigó 2011). In this work, let us use the OWAWA operator. However, in order to produce a better terminology, we will use the WOWA denomination. Therefore, under this framework, we obtain the weighted OWA fixed asset (WOWAFA), which is defined in the following way for a set of fixed assets $A = \{a_1, a_2, \ldots, a_n\}$:

$$\text{WOWAFA}(a_1, a_2, \ldots, a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i \tag{43}$$

where b_j is the jth largest of the a_i, $\beta \in [0, 1]$, V is a weighting vector with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$ and W is a weighting vector with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$.

Another extension consists in using probabilities in the analysis to address objective and subjective information and the attitudinal character of the decision-maker. For doing so, let us use the POWAWA operator. Following Eq. (6), we would obtain the probabilistic ordered weighted averaging weighted average fixed asset (POWAWAFA) as follows:

$$\text{POWAWAFA}(a_1, a_2, \ldots, a_n) = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i a_i + C_3 \sum_{i=1}^{n} p_i a_i \tag{44}$$

where b_j is the jth largest of the a_i, v_i is the ith weight of the weighted average with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, p_i is the ith probability with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, w_j is the jth OWA weight with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, C_1, C_2 and $C_3 \in [0, 1]$, and $C_1 + C_2 + C_3 = 1$.

Equations (42)–(44) can be extended by using the generalized and the quasi-arithmetic means. With the generalized means, Eq. (44) would become the generalized POWAWAFA (GPOWAWAFA) as follows:

$$\text{GPOWAWAFA}(a_1, a_2, \ldots, a_n) = C_1 \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} + C_2 \left(\sum_{i=1}^{n} v_i a_i^{\delta} \right)^{1/\delta} + C_3 \left(\sum_{i=1}^{n} p_i a_i^{\chi} \right)^{1/\chi} \tag{45}$$

where λ, δ and χ are parameters such that λ, δ and $\chi \in \{-\infty, \infty\} - \{0\}$. Note that with the quasi-arithmetic means, we should replace λ, δ and χ by strictly continuous monotonic functions. Observe that if $\lambda = \delta = \chi = 1$, Eq. (45) becomes the POWAWAFA operator. If $\lambda = \delta = \chi = 2$, we obtain the quadratic POWAWAFA operator. Similarly, we can also analyse a wide range of other particular cases following the results of Sect. 4.

To focus more specifically on the contributions of this paper, let us implement the POWAWAD operator on the average fixed assets. In this instance, the assumption is that the initial information is assessed with two sets of arguments instead of one. To perform the aggregation, first there is a comparative process between the two sets with a distance measure. With the individual distances, the process is treated in a similar way as with one set, where the data are aggregated with an average. As mentioned previously, it is common to use the arithmetic mean and the weighted average, which in this case would imply the use of the normalized Hamming distance and the weighted Hamming distance. However, it is also possible to use other formulations, such as the OWAD and the POWAWAD. By using the OWAD operator, we obtain the OWAD fixed asset (OWADFA), which is formulated as follows, with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$:

$$\text{OWADFA}(\langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j b_j \tag{46}$$

where b_j is the jth largest of the $|x_i - y_i|$, and w_j is the jth weight of a weighting vector W with $w_j \in [0, 1]$

Table 7 Asian average fixed asset—aggregated results 1

Country	NHDFA	WHDFA	PHDFA	OWADFA	OWAWADFA	POWAWADFA
Afghanistan	0,031	0,028	0,027	0,023	0,026	0,027
Armenia	0,087	0,084	0,093	0,073	0,080	0,085
Azerbaijan	0,125	0,127	0,134	0,105	0,118	0,125
Bahrain	0,106	0,102	0,114	0,092	0,098	0,104
Bangladesh	0,028	0,022	0,031	0,019	0,021	0,025
Bhutan	0,040	0,038	0,036	0,027	0,034	0,035
Brunei	0,064	0,070	0,062	0,056	0,064	0,064
Cambodia	0,077	0,075	0,074	0,059	0,069	0,072
China	0,054	0,061	0,051	0,045	0,054	0,053
Cyprus	0,062	0,071	0,049	0,037	0,058	0,056
East Timor	0,035	0,036	0,034	0,024	0,031	0,033
Egypt	0,045	0,045	0,042	0,037	0,042	0,042
Georgia	0,053	0,044	0,063	0,040	0,042	0,051
India	0,047	0,040	0,054	0,031	0,036	0,044
Indonesia	0,057	0,051	0,052	0,045	0,048	0,050
Iran	0,081	0,084	0,084	0,063	0,076	0,080
Iraq	0,044	0,036	0,045	0,035	0,035	0,039
Israel	0,084	0,095	0,089	0,065	0,083	0,086
Japan	0,046	0,045	0,044	0,032	0,040	0,042
Jordan	0,033	0,037	0,035	0,024	0,032	0,034
Kazakhstan	0,060	0,060	0,056	0,040	0,052	0,054
Kuwait	0,037	0,033	0,045	0,028	0,031	0,037
Kyrgyzstan	0,035	0,034	0,038	0,026	0,031	0,034
Laos	0,039	0,042	0,033	0,028	0,036	0,035
Lebanon	0,051	0,069	0,051	0,031	0,054	0,055
Malaysia	0,054	0,050	0,058	0,043	0,047	0,052
Maldives	0,051	0,061	0,052	0,035	0,051	0,052
Mongolia	0,049	0,040	0,049	0,040	0,040	0,044
Myanmar	0,028	0,021	0,035	0,020	0,021	0,026
Nepal	0,049	0,050	0,053	0,033	0,043	0,048
North Korea	0,055	0,058	0,052	0,040	0,051	0,052
Oman	0,067	0,061	0,070	0,052	0,057	0,063
Pakistan	0,058	0,051	0,056	0,046	0,049	0,052
Palestine	0,028	0,029	0,028	0,021	0,026	0,027
Philippines	0,057	0,064	0,052	0,048	0,057	0,056
Qatar	0,011	0,010	0,012	0,009	0,009	0,010
Russia	0,061	0,063	0,061	0,041	0,054	0,058
Saudi Arabia	0,055	0,060	0,046	0,040	0,052	0,050
Singapore	0,029	0,024	0,030	0,023	0,024	0,026
South Korea	0,041	0,047	0,038	0,027	0,039	0,039
Sri Lanka	0,067	0,069	0,068	0,055	0,064	0,066
Syria	0,077	0,066	0,084	0,060	0,064	0,072
Tajikistan	0,056	0,064	0,058	0,039	0,054	0,057
Thailand	0,062	0,069	0,062	0,037	0,056	0,060
Turkey	0,083	0,088	0,085	0,055	0,075	0,080
Turkmenistan	0,038	0,043	0,039	0,031	0,038	0,039

Table 7 continued

Country	NHDFA	WHDFA	PHDFA	OWADFA	OWAWADFA	POWAWADFA
United Arab Emirates	0,034	0,042	0,038	0,022	0,034	0,036
Uzbekistan	0,051	0,064	0,050	0,035	0,052	0,053
Vietnam	0,017	0,018	0,012	0,012	0,015	0,014
Yemen	0,055	0,054	0,061	0,041	0,049	0,054
Asian average	0,051	0,051	0,052	0,038	0,046	0,049

and $\sum_{j=1}^{n} w_j = 1$. Note that if $w_j = 1/n$ for all j, the OWADFA becomes the normalized Hamming distance fixed asset (NHDFA).

By using the POWAWAD operator under this framework, we would obtain the POWAWAD fixed asset (POWAWADFA). Note that the main advantage of this approach is the possibility of using distance measures in the calculation of the fixed assets in a unified framework between the probability, the weighted average and the OWA operator. Following Eq. (44), the POWAWADFA operator is defined as follows:

$$\text{POWAWADFA}(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i |x_i - y_i| + C_3 \sum_{i=1}^{n} p_i |x_i - y_i| \tag{47}$$

where b_j is the jth largest of the arguments $|x_i - y_i|$ and C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$.

Finally, note that it is also possible to generalize the POWAWADFA by using the generalized and the quasi-arithmetic means, obtaining the generalized POWAWADFA (GPOWAWADFA) and the Quasi-POWAWADFA operator. Observe that the GPOWAWADFA uses parameters λ, δ and χ such that λ, δ and $\chi \in \{-\infty, \infty\} - \{0\}$ and the Quasi-POWAWADFA strictly continuous monotonic functions f, g and h, as in Eq. (31). From this, it is straightforward to analyse a wide range of particular cases in a similar way to the development in Sect. 4.

5.3 Aggregation systems and asset management with the POWAWADFA operator

To use these averaging aggregation operators in a real-world problem, let us briefly explain the steps to follow in the aggregation process. Note that here the work focuses on the POWAWADFA and GPOWAWADFA, although it is straightforward to use any of the other cases mentioned in the paper. Let us analyse the aggregation process at the country and supranational levels. Observe that we implicitly assume a previous aggregation step where the fixed assets of each of the companies involved in the country are aggregated with an averaging aggregation operator. To address distance measures, let us design a problem where the experts compare the ideal percentage of fixed assets that the companies of a country have versus the actual percentage of fixed assets that they have. Note that each country may have a different ideal percentage due to the specific characteristics of the country.

Step 1 The experts $E = \{e_1, e_2, \ldots, e_t\}$ of the problem define their data regarding the ideal and actual percentage of fixed assets of each country under different scenarios that may occur in future $S = \{s_1, s_2, \ldots, s_n\}$, building two sets of fixed assets, where the ideal percentage of fixed assets is $X_k = \{x_1^k, x_2^k, \ldots, x_n^k\}$ and the actual percentage is $Y_k = \{y_1^k, y_2^k, \ldots, y_n^k\}$. Note that the percentage is a real number between 0 and 1.
Step 2 Produce the collective results for the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ by aggregating the data given by the group of experts using a weighted average $Z = (z_1, z_2, \ldots, z_t)$. Observe that it is also possible to use other averaging aggregation operators instead of the weighted average.
Step 3 Define all the weights of the POWAWADFA (or the GPOWAWADFA) for producing the aggregation process. Accordingly, following Eq. (47), define the weighting vector W, V and P.
Step 4 Calculate the individual distances between the two sets X and Y forming one single set $D = \{d_1, d_2, \ldots, d_n\}$.
Step 5 Aggregate the individual distances of the collective results with the weighting vectors studied in Step 3. Consider the results with the POWAWADFA and the GPOWAWADFA from the perspective of the different particular cases mentioned in Sects. 3 and 4.
Step 6 Analyse the results to obtain some general conclusions concerning the available data. Verify whether all the averaging aggregation operators provide similar results that lead to the same decisions or not.

6 Numerical example

To understand numerically the applicability of the POWAWADFA and GPOWAWADFA explained in the pre-

Table 8 Asian average fixed asset—Aggregated results 2

Rank	NHDFA	WHDFA	PHDFA	OWADFA	OWAWADFA	POWAWADFA
1	QAT	QAT	QAT	QAT	QAT	QAT
2	VIE	VIE	VIE	VIE	VIE	VIE
3	BAN	MYN	AFG	BAN	BAN	BAN
4	MYN	BAN	PAL	MYN	MYN	MYN
5	PAL	SGP	SGP	PAL	SGP	SGP
6	SGP	AFG	BAN	UAE	AFG	AFG
7	AFG	PAL	LAO	AFG	PAL	PAL
8	JOR	KUW	ET	SGP	ET	ET
9	UAE	KYR	JOR	ET	KUW	JOR
10	ET	ET	MYN	JOR	KYR	KYR
11	KYR	IRQ	BHU	KYR	JOR	BHU
12	KUW	JOR	KYR	BHU	BHU	LAO
13	TRM	BHU	SK	SK	UAE	UAE
14	LAO	IND	UAE	KUW	IRQ	KUW
15	BHU	MON	TRM	LAO	IND	IRQ
16	SK	LAO	EGY	IND	LAO	SK
17	IRQ	UAE	JAP	LEB	TRM	TRM
18	EGY	TRM	IRQ	TRM	SK	EGY
19	JAP	GEO	KUW	JAP	JAP	JAP
20	IND	EGY	SA	NEP	MON	IND
21	MON	JAP	CYP	IRQ	EGY	MON
22	NEP	SK	MON	MLD	GEO	NEP
23	LEB	MLS	UZB	UZB	NEP	INO
24	MLD	NEP	CHN	CYP	MLS	SA
25	UZB	INO	LEB	EGY	INO	GEO
26	GEO	PAK	INO	THA	PAK	MLS
27	CHN	YEM	MLD	TAJ	YEM	MLD
28	MLS	NK	NK	GEO	MLD	NK
29	NK	KAZ	PHI	KAZ	NK	PAK
30	SA	SA	NEP	MON	KAZ	CHN
31	YEM	CHN	IND	NK	SA	UZB
32	TAJ	MLD	KAZ	SA	UZB	KAZ
33	INO	OMA	PAK	RUS	CHN	YEM
34	PHI	RUS	MLS	YEM	LEB	LEB
35	PAK	PHI	TAJ	MLS	RUS	CYP
36	KAZ	TAJ	RUS	CHN	TAJ	PHI
37	RUS	UZB	YEM	INO	THA	TAJ
38	CYP	SYR	BRU	PAK	OMA	RUS
39	THA	LEB	THA	PHI	PHI	THA
40	BRU	SL	GEO	OMA	CYP	OMA
41	OMA	THA	SL	SL	BRU	BRU
42	SL	BRU	OMA	TRK	SL	SL
43	CAM	CYP	CAM	BRU	SYR	CAM
44	SYR	CAM	IRN	CAM	CAM	SYR
45	IRN	ARM	SYR	SYR	TRK	IRN
46	TRK	IRN	TRK	IRN	IRN	TRK

Table 8 continued

Rank	NHDFA	WHDFA	PHDFA	OWADFA	OWAWADFA	POWAWADFA
47	ISR	TRK	ISR	ISR	ARM	ARM
48	ARM	ISR	ARM	ARM	ISR	ISR
49	BAH	BAH	BAH	BAH	BAH	BAH
50	AZE	AZE	AZE	AZE	AZE	AZE

vious section, let us develop an illustrative example. This study focuses on an application regarding the calculation of the average distances between the ideal and the actual percentage of fixed assets of companies in countries of the Asian region. To assess the problem correctly, a group of three experts analyses the information. The steps to follow in the aggregation process can be described as follows.

Step 1 Assume three experts analyse the average fixed assets of the companies of a specific industry of countries in Asia. Consider that five possible scenarios may occur in future depending on the evolution of the economic environment in the region. Tables 2, 3 and 4 present the results.

Step 2 With the information of Tables 2, 3 and 4, the experts unify their opinions in order to build a collective result that considers all the information. To do so, the experts reach an agreement assuming the following degrees of importance for each of the three experts: $Z = (0.3, 0.5, 0.2)$. Table 5 presents the results.

Step 3 The experts define the weights to use in the aggregation process. They assume that it is necessary to consider subjective and objective information and an attitudinal character that underestimates the results. The weighting vectors are as follows:

- OWA: $W = (0{,}1; 0{,}1; 0{,}2; 0{,}3; 0{,}3)$.
- Weighted average: $V = (0{,}2; 0{,}3; 0{,}1; 0{,}1; 0{,}3)$.
- Probability: $P = (0{,}1; 0{,}2; 0{,}2; 0{,}3; 0{,}2)$.
- OWAWA: $\beta = 0{,}4$.
- POWAWA: $C_1 = 0{,}2$; $C_2 = 0{,}4$; $C_3 = 0{,}4$.

Step 4 Calculate the individual distances between the ideal and actual percentage of fixed assets of each country and for each scenario. Table 6 shows the results.

Step 5 Aggregate the individual distances for each country by using the three weighting vectors explained in Step 3. Present the results of the average fixed assets for the normalized Hamming distance (NHD), weighted Hamming distance (WHD), probabilistic Hamming distance (PHD), OWAD, OWAWAD and POWAWAD. Table 7 presents the aggregated results.

Step 6 Analyse the results in order to draw some conclusions. Rank the countries from the lowest to the highest distance according to each method of Step 5. Table 8 presents the results with the abbreviated names of the countries, which are available in Table 3. Additionally, calculate the average distance results for all the countries forming the Asian average distance between the ideal and actual percentage of fixed assets. To this end, use a weighted average that is formed according to the population of each country and six decimals. Normalize the weighting vector such that the sum of the weights is equal to one. Thus, divide the population of each country by the total population of Asia to define the weight of the country. The data regarding the population of each country are shown in Table 2, and the weights are shown in Table 4.

7 Conclusions

This work introduces new averaging aggregation operators by using distance measures, probabilities, weighted averages, OWA operators, the generalized means and the quasi-arithmetic means. First, the article analyses the POWAWAD operator. This operator is an averaging aggregation operator that uses distance measures in a unified framework between subjective and objective information and the attitudinal character of the decision-maker. The POWAWAD operator includes a wide range of distance aggregation operators, including the normalized Hamming distance, the weighted Hamming distance, the OWAD, the OWAWAD and the POWAD operators. The main advantage of this approach is that it provides a wider framework that includes a wide range of classical average distances. Thus, the model can represent a wide range of issues in the same formulation and can adapt to the specific needs of the problem.

Next, the paper studies the GPOWAWAD and Quasi-POWAWAD operators. They are averaging distance aggregation operators that permit studying distance measures in a more complete and flexible way because they can address different sources of information in the analysis, including the probability, the weighted average and the OWA operator. The use of the generalized means (Minkowski distance) and the quasi-arithmetic means has shown the possibility of including the Hamming and the Euclidean distance in the same formulation. A number of other particular cases have also been considered, including generalizations in the unification process of the probability, the weighted average and the OWA operator.

The paper also analyses the applicability of the POWAWAD and GPOWAWAD operators, which is notably broad because all the previous studies that use average distances can be revised and extended with this new approach. The work focuses on the calculation of the average fixed assets, introducing the ordered weighted average fixed asset (OWAFA) and the weighted ordered weighted average fixed asset (WOWAFA). Particularly, the article studies the use of distance measures in the analysis of fixed assets by measuring the distance between the ideal percentage of fixed assets and the actual percentage of fixed assets that the companies of a country have. The work also develops a numerical example regarding the calculation of the average fixed assets in Asia by using a multi-person analysis and distance measures with ideal and actual percentages of fixed assets. This approach provides a general picture of different potential results that can occur according to different events between the minimum and maximum results.

Future research in this direction should consider further extensions and generalizations by using a wide range of methodologies, including induced aggregation operators, moving averages, norms and other types of distances. Other problems in the calculation of the average fixed asset can be considered and in other supranational regions. Additionally, it is possible to develop more complex aggregation structures where the average fixed asset is calculated not only at the country and supranational level but also considering, for example, the provincial level and all the different companies that intervene in the analysis. Finally, note that many other applications could be considered in a wide range of areas, including business, economics and engineering (Maldonado et al. 2018; Zadeh et al. 2014).

Acknowledgements We would like to thank the anonymous reviewers for valuable comments that have improved the quality of the paper. Support from the Distinguished Scientist Fellowship Programme from King Saud University (Saudi Arabia) and the Chilean Government through the Fondecyt Regular Programme (Project Number 1160286) is gratefully acknowledged.

Compliance with ethical standards

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

Conflict of interest The authors declare that they have no conflict of interests.

References

Alfaro-García VG, Merigó JM, Gil-Lafuente AM, Kacprzyk J (2018) Logarithmic aggregation operators and distance measures. Int J Intell Syst 33:1488–1506

Aggarwal M (2015) Generalized compensative weighted averaging aggregation operators. Comput Ind Eng 87:81–90

Avilés-Ochoa E, León-Castro E, Pérez-Arellano LA, Merigó JM (2018) Government transparency measurement through prioritized distance operators. J Intell Fuzzy Syst 34:2783–2794

Bargiela A, Pedrycz W (2003) Granular computing: an introduction. Kluwer Academic Publishers, Dordrecht

Beliakov G, Pradera A, Calvo T (2007) Aggregation functions: a guide for practitioners. Springer, Berlin

Beliakov G, Bustince-Sola H, Calvo-Sánchez T (2016) A practical guide to averaging functions. Springer, Heidelberg

Belles-Sampera J, Merigó JM, Guillén M, Santolino M (2014) Indicators for the characterization of discrete Choquet integrals. Inf Sci 267:201–216

Blanco-Mesa F, Merigó JM, Kacprzyk J (2016) Bonferroni means with distance measures and the adequacy coefficient in entrepreneurial group theory. Knowl Based Syst 111:217–227

Blanco-Mesa F, Merigó JM, Gil-Lafuente AM (2017) Fuzzy decision making: a bibliometric-based review. J Intell Fuzzy Syst 32:2033–2050

Casanovas M, Torres-Martínez A, Merigó JM (2016) Decision making in reinsurance with induced OWA operators and Minkowski distances. Cybern Syst 47:460–477

Emrouznejad A, Marra M (2014) Ordered weighted averaging operators 1988–2014. A citation based literature survey. Int J Intell Syst 29:994–1014

Engemann KJ, Filev DP, Yager RR (1996) Modelling decision making using immediate probabilities. Int J Gen Syst 24:281–294

Figueira J, Greco S, Ehrgott M (2005) Multiple criteria decision analysis: state of the art surveys. Springer, Boston

Fodor J, Marichal JL, Roubens M (1995) Characterization of the ordered weighted averaging operators. IEEE Trans Fuzzy Syst 3(2):236–240

Gao JW, Liu HH (2017) Generalized ordered weighted reference dependent utility aggregation operators and their applications to group decision-making. Group Decis Negot 26:1173–1207

Gil-Aluja J (1999) Elements for a theory of decision under uncertainty. Kluwer Academic Publishers, Dordrecht

Grabisch M, Marichal JL, Mesiar R, Pap E (2011) Aggregation functions: means. Inf Sci 181:1–22

Hamming RW (1950) Error-detecting and error-correcting codes. Bell Syst Tech J 29:147–160

He XR, Wu YY, Yu D, Merigó JM (2017) Exploring the ordered weighted averaging operator knowledge domain: a bibliometric analysis. Int J Intell Syst 32:1151–1166

Kacprzyk J, Pedrycz W (2015) Handbook of computational intelligence. Springer, Berlin

Kahraman C, Onar SC, Oztaysi B (2015) Fuzzy multicriteria decision-making: a literature review. Int J Comput Intell Syst 8:637–666

Kaufmann A (1975) Introduction to the theory of fuzzy subsets. Academic Press, New York

Laengle S, Loyola G, Merigó JM (2017) Mean-variance portfolio selection with the ordered weighted average. IEEE Trans Fuzzy Syst 25:350–362

León-Castro E, Avilés-Ochoa E, Merigó JM, Gil-Lafuente AM (2018) Heavy moving averages and their application in econometric forecasting. Cybern Syst 49:26–43

Liao HC, Xu ZS, Zeng XJ, Merigó JM (2015) Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. Knowl Based Syst 76:127–138
Liu PD, Liu X (2017) Multi-attribute group decision-making method based on cloud distance operators with linguistic information. Int J Fuzzy Syst 19:1011–1024
Liu JP, Chen HY, Xu Q, Zhou LG, Tao ZF (2016a) Generalized ordered modular averaging operator and its application to group decision making. Fuzzy Sets Syst 299:1–25
Liu PD, He L, Yu XC (2016b) Generalized hybrid aggregation operators based on the 2-dimension uncertain linguistic information for multiple attribute group decision making. Group Decis Negot 25:103–126
Maldonado S, Merigó JM, Miranda J (2018) Redefining support vector machines with the ordered weighted average. Knowl Based Syst 148:41–46
Merigó JM (2010) Fuzzy decision making with immediate probabilities. Comput Ind Eng 58:651–657
Merigó JM (2011) A unified model between the weighted average and the induced OWA operator. Expert Syst Appl 38:11560–11572
Merigó JM (2012) Probabilities in the OWA operator. Expert Syst Appl 39(13):11456–11467
Merigó JM, Casanovas M (2011a) Decision making with distance measures and induced aggregation operators. Comput Ind Eng 60:66–76
Merigó JM, Casanovas M (2011b) A new Minkowski distance based on induced aggregation operators. Int J Comput Intell Syst 4(2):123–133
Merigó JM, Gil-Lafuente AM (2009) The induced generalized OWA operator. Inf Sci 179:729–741
Merigó JM, Gil-Lafuente AM (2010) New decision-making techniques and their application in the selection of financial products. Inf Sci 180:2085–2094
Merigó JM, Yager RR (2013) Generalized moving averages, distance measures and OWA operators. Int J Uncertain Fuzziness Knowl Based Syst 21:533–559
Merigó JM, Lobato-Carral C, Carrilero-Castillo A (2012) Decision making in the European Union under risk and uncertainty. Eur J Int Manag 6:590–609
Merigó JM, Palacios-Marqués D, Zeng SZ (2016) Subjective and objective information in linguistic multi-criteria group decision making. Eur J Oper Res 248:522–531
Merigó JM, Palacios-Marqués D, Soto-Acosta P (2017) Distance measures, weighted averages, OWA operators and Bonferroni means. Appl Soft Comput 50:356–366
Scherger V, Terceño A, Vigier H (2017) The OWA distance operator and its application in business failure. Kybernetes 46:114–130
Shannon CE (1948) A mathematical theory of communication. Bell Syst Tech J 27:379–423
Torra V (1997) The weighted OWA operator. Int J Intell Syst 12:153–166
Xian SD, Sun WJ (2014) Fuzzy linguistic Euclidean OWA distance operator and its application in group linguistic decision making. Int J Intell Syst 29:478–491
Xian SD, Sun WJ, Xu SH, Gao YY (2016) Fuzzy linguistic induced OWA Minkowski distance operator and its application in group decision making. Pattern Anal Appl 19:325–335
Xu ZS, Chen J (2008) Ordered weighted distance measure. J Syst Sci Syst Eng 17:432–445
Xu ZS, Da QL (2003) An overview of operators for aggregating information. Int J Intell Syst 18:953–969
Xue WT, Xian SD, Dong YF (2017) A novel intuitionistic fuzzy induced ordered weighted Euclidean distance operator and its application for group decision making. Int J Intell Syst 32:739–753
Yager RR (1988) On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans Syst Man Cybern B 18:183–190
Yager RR (1993) Families of OWA operators. Fuzzy Sets Syst 59:125–148
Yager RR (1996) Constrained OWA aggregation. Fuzzy Sets Syst 81:89–101
Yager RR (1998) Including importances in OWA aggregation using fuzzy systems modelling. IEEE Trans Fuzzy Syst 6:286–294
Yager RR (2002) Heavy OWA operators. Fuzzy Optim Decis Mak 1:379–397
Yager RR (2004) Generalized OWA aggregation operators. Fuzzy Optim Decis Mak 3:93–107
Yager RR, Engemann KJ, Filev DP (1995) On the concept of immediate probabilities. Int J Intell Syst 10:373–397
Yager RR, Kacprzyk J, Beliakov G (2011) Recent developments on the ordered weighted averaging operators: theory and practice. Springer, Berlin
Yu D (2015) A scientometrics review on aggregation operator research. Scientometrics 105:115–133
Zadeh LA (2005) Toward a generalized theory of uncertainty (GTU): an outline. Inf Sci 172:1–40
Zadeh LA, Abbasov AM, Yager RR, Shahbazova SN, Reformat MZ (2014) Recent developments and new directions in soft computing. Springer, Switzerland
Zeng SZ (2016) An extension of OWAD operator and its application to uncertain multiple-attribute group decision-making. Cybern Syst 47:363–375
Zeng SZ, Su W (2012b) Linguistic induced generalized aggregation distance operators and their application to decision making. Econ Comput Econ Cybern Stud Res 46:155–172
Zeng SZ, Su WH, Le A (2012a) Fuzzy generalized ordered weighted averaging distance operator and its application to decision making. Int J Fuzzy Syst 14:402–412
Zeng SZ, Merigó JM, Su WH (2013) The uncertain probabilistic OWA distance operator and its application in group decision making. Appl Math Model 37:6266–6275
Zeng SZ, Merigó JM, Palacios-Marqués D, Jin HH, Gu FJ (2017) Intuitionistic fuzzy induced ordered weighted averaging distance operator and its application to decision making. J Intell Fuzzy Syst 32:11–22
Zhou LG, Wu JX, Chen HY (2014) Linguistic continuous ordered weighted distance measure and its application to multiple attributes group decision making. Appl Soft Comput 25:266–276
Zhou LG, Tao ZF, Chen HY, Liu JP (2015) Generalized ordered weighted logarithmic harmonic averaging operators and their applications to group decision making. Soft Comput 19:715–730