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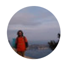

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DICHOTOMY SPECTRUM AND ALMOST TOPOLOGICAL CONJUGACY ON NONAUTONOMUS UNBOUNDED DIFFERENCE SYSTEMS

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Abstract. We construct a bijection between the solutions of a linear system of nonautonomous difference equations which is uniformly asymptotically stable and its unbounded perturbation. The key idea used to made this bijection is to consider the crossing times of the solutions with the unit sphere. This approach prompt us to introduce the concept of *almost topological conjugacy* in this nonautonomous framework. This task is carried out by simplifying both systems through a spectral approach of the notion of *almost reducibility* combined with suitable technical assumptions.

1. **Introduction.** The Hartman–Grobman Theorem shows the existence of a local and not explicit homeomorphism between the solutions of an autonomous finite dimensional (continuous or discrete) dynamical system and its linearization around an hyperbolic equilibrium. The generalization of this linearization result to a nonautonomous framework is a delicated task since there not exists a univocal definition of hyperbolicity as in the autonomous case. Nevertheless, some qualitative properties can be recovered by the properties of dichotomy, which induces a splitting between the stable and unstable directions of the solutions of a linear system.

In 1969, C. Pugh [21] studied a family of linear autonomous dynamical systems with bounded and Lipschitz nonlinear perturbations, thus allowing the construction of an explicit and global homeomorphism, making a strong difference with the Hartman–Grobman's result. This sheds light to work in the nonautonomous case. Indeed, K.J. Palmer [15] introduced the concept of topological equivalence for the continuous case and later G. Papaschinopoulos [17] studied a discrete case. This article will be focused in the discrete case and considers the nonautonomous linear system

$$x(n+1) = A(n)x(n), \tag{1}$$

and its pertubation

$$w(n+1) = A(n)w(n) + f(n, w(n)), \tag{2}$$

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where $x(n)$ and $w(n)$ are column vectors of $\mathbb{R}^d$, the matrix function $n \mapsto A(n) \in \mathbb{R}^{d\times d}$ is non singular and $f\colon \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d$ is continuous in $\mathbb{R}^d$.

Definition 1.1 ([17])**.** The systems (1) and (2) will be called topologically equivalent if there exists a map $H\colon \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d$ with the properties

(i) For each fixed $n \in \mathbb{Z}$, the map $u \mapsto H(n,u)$ is a bijection.
(ii) For any fixed $n \in \mathbb{Z}$, the maps $u \mapsto H(n,u)$ and $u \mapsto H^{-1}(n,u) = G(n,u)$ are continuous.
(iii) If $x(n)$ is a solution of (1), then $H(n,x(n))$ is a solution of (2). Similarly, if $w(n)$ is a solution of (2), then $G(n,w(n))$ is a solution of (1).

The properties of dichotomy play a fundamental role in the topological equivalence literature, being the exponential dichotomy the most relevant one.

Definition 1.2 ([9],[18],[19],[20])**.** The system (1) has an exponential dichotomy on $\mathbb{Z}$ if there exist numbers $K \geq 1$, $\rho \in (0,1)$ and a projector $P^2 = P$ such that

$$\left\{\begin{array}{rl} ||X(n)PX^{-1}(k)|| \leq K\rho^{n-k} & \text{if} \quad n \geq k \\ \\ ||X(n)(I-P)X^{-1}(k)|| \leq K\rho^{k-n} & \text{if} \quad n \leq k, \end{array}\right. \tag{3}$$

where $X(n)$ is a fundamental matrix of (1) and $X(n,k) = X(n)X^{-1}(k)$ is its corresponding transition matrix. In addition, $|\cdot|$ is a vector norm with induced matrix norm $||\cdot||$.

A classification of the topological equivalence results can be made by considering three aspects, which are strongly related between them: a) the type of dichotomy of the linear system (1), b) the properties of the nonlinear perturbation f, and c) the character either explicit or non explicit of the resulting homeomorphism H.

In [17], Papaschinopoulos assumes that (1) has an exponential dichotomy on $\mathbb{Z}$ while the perturbation is such that

$$|f(n,x)| \leq \mu \quad \text{and} \quad |f(n,x)-f(n,\tilde{x})| \leq \gamma|x-\tilde{x}| \quad \text{for any } n \in \mathbb{Z} \text{ and } x,\tilde{x} \in \mathbb{R}^d,$$

and proves that if γ is small enough, for any fixed $n \in \mathbb{Z}$ there exists a bijection

$$H(n,\xi) = \xi - \sum_{k=-\infty}^{\infty} \mathcal{G}(n,k+1)f(k,w(k,n,\xi)),$$

mapping solutions of (1) in solutions of (2), where $\mathcal{G}(n,k)$ defined by

$$\mathcal{G}(n,k) = \left\{\begin{array}{rl} X(n)PX^{-1}(k) & \text{if} \quad n \geq k, \\ X(n)(I-P)X^{-1}(k) & \text{if} \quad n \leq k \end{array}\right.$$

is the Green's function associated to (1) and $k \mapsto w(k,n,\xi)$ is the solution of (2) passing through ξ at $k = n$.

This result has been generalized in several directions, for example, the perturbation f can be considered as

$$|f(n,x)| \leq \mu(n) \text{ and } |f(n,x)-f(n,\tilde{x})| \leq \gamma(n)|x-\tilde{x}| \text{ for any } n \in \mathbb{Z} \text{ and } x,\tilde{x} \in \mathbb{R}^d,$$

where $\mu(n)$ and $\gamma(n)$ are positive and possibly unbounded sequences. Nevertheless, the map $\xi \mapsto H(n,\xi)$ still satisfies the properties of the topological equivalence provided that

$$\sum_{k=-\infty}^{+\infty} \mathcal{G}(n,k+1)\mu(k) < +\infty \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} \mathcal{G}(n,k+1)\gamma(k) < 1,$$

which can be verified when (1) has dichotomies more general than the exponential one, for example nonuniform exponential dichotomy or generalized exponential dichotomy (see *e.g.* [4, 7]).

The case when f is Lipschitz but unbounded cannot always be carried out by using the Green's function. This fact has been studied by K.J. Palmer [16] and F. Lin [14] in the continuous context by using the *crossing time* (a formal definition will be given later) of a solution with the unit sphere and its continuity properties. This fact cannot be replicated in a direct way to the discrete case due to the lack of continuity of the crossing time in this context. The first approach dealing with discrete crossing times was made by Barreira *et.al* [3], which allow to construct an homeomorphism between the solutions of two linear systems.

In order to study the topological equivalence between (1) and (2) a first step will be to consider the simpler families

$$y(n+1) = C(n)\{I + B(n)\}y(n), \tag{4}$$

and

$$w(n+1) = C(n)\{I + B(n)\}w(n) + g(n, w(n)), \tag{5}$$

where $C(n)$ is a diagonal matrix and $B(n)$ is an upper triangular matrix satisfying $||B(n)|| < \delta$, where δ can be chosen arbitrarily small.

This fact is motivated by a recent result of almost reducibility [6], which states that (1) can be transformed into (4) by a linear change of coordinates. The concept of reducibility is well known in the continuous case and we refer the reader to [8, 9] for details.

In this paper, we continue the study of the discrete crossing times initiated by Barreira *et al.* and proved that if (4) is uniformly asymptotically stable and the perturbation g satisfies some suitable properties, then the crossing times are locally constant with a possible set of discontinuities having Lebesgue measure zero, which prompt us to introduce the following definition.

Definition 1.3. The systems (1) and (2) will be called almost topologically equivalent if there exists a map $H\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ with the properties

(i) For each fixed $n \in \mathbb{Z}$, the map $u \mapsto H(n, u)$ is a bijection.
(ii) For any fixed $n \in \mathbb{Z}$, the maps $u \mapsto H(n, u)$ and $u \mapsto H^{-1}(n, u) = G(n, u)$ are continuous with the possible exception of a set with Lebesgue measure zero.
(iii) If $x(n)$ is a solution of (1), then $H(n, x(n))$ is a solution of (2). Similarly, if $w(n)$ is a solution of (2), then $G(n, w(n))$ is a solution of (1).

To the best of our knowledge, the Definition 1.3 seems not be introduced before in the literature. Now, when the above maps H and G are continuous in $\mathbb{R}^d$ for any $n \in \mathbb{Z}$, we recover the classical topological equivalence which has been studied in several works as [3],[17],[22].

2. Preliminaries and main results.

In this paper, we will assume that (1) and (2) verify the following properties:

(P1) $\sup\limits_{n\in\mathbb{Z}} ||A(n)|| < +\infty$ and $\sup\limits_{n\in\mathbb{Z}} ||A^{-1}(n)|| < +\infty$,

(P2) The system (1) is uniformly asymptotically stable, that is, there exist $K \geq 1$ and $0 < \rho < 1$ such that

$$||X(n)X^{-1}(m)|| \leq K\rho^{n-m} \quad \text{for any} \quad n \geq m, \tag{6}$$

(P3) The perturbation f is an element of one of the two families:

$$\mathcal{F}_1 = \left\{ f\colon \sup_{n\in\mathbb{Z}} |f(n,0)| < \infty \text{ and } \exists L \text{ s.t. } |f(n,u)-f(n,v)| \le L|u-v| \;\forall n\in\mathbb{Z} \right\},$$

$$\mathcal{F}_2 = \{ f\colon f\in\mathcal{F}_1 \quad \text{and} \quad f(n,0)=0 \quad \text{for all } n\in\mathbb{Z} \}.$$

Similarly, we will assume that

(Q1) There exists $\mathcal{K}=\theta^{-2}\in(0,1)$ such that

$$|Y(n,n_0,\xi)|^2 \le \mathcal{K}^{n-m}|Y(m,n_0,\xi)|^2 \quad \text{for any } n\ge m$$

and

$$|X(n,n_0,\xi)|^2 \le \mathcal{K}^{n-m}|X(m,n_0,\xi)|^2 \quad \text{for any } n\ge m,$$

where $Y(n,n_0,\xi)$ is the solution of (5) passing through ξ at n_0 and $X(n,n_0,\xi) = \xi\theta^{n_0-n}$ is an auxiliary comparison sequence.

A direct consequence of **(Q1)** is that $|X(n,n_0,\xi)|^2$ and $|Y(n,n_0,\xi)|^2$ are strictly decreasing with respect to n, tends to the origin when $n\to+\infty$ and tends to infinite as $n\to-\infty$. This prompt to introduce the formal definition of crossing times:

Definition 2.1. The crossing times for $n\mapsto X(n,n_0,\xi)$ and $n\mapsto Y(n,n_0,\xi)$ are the maps $M,N\colon \mathbb{Z}\times(\mathbb{R}^d\setminus\{0\})\to\mathbb{Z}$ such that, given $(n_0,\xi)\in\mathbb{Z}\times(\mathbb{R}^d\setminus\{0\})$ there exists a couple of unique integer numbers $M(n_0,\xi)$ and $N(n_0,\xi)$ satisfying

$$|\xi\theta^{n_0-\{M(n_0,\xi)+1\}}|^2 \le 1 < |\xi\theta^{n_0-M(n_0,\xi)}|^2. \tag{7}$$

$$|Y(N(n_0,\xi)+1,n_0,\xi)|^2 \le 1 < |Y(N(n_0,\xi),n_0,\xi)|^2. \tag{8}$$

Now, we state the additional property for (5)

(Q2) For any initial time $n_0\in\mathbb{Z}$ and initial condition $\xi\in\mathbb{R}^d\setminus\{0\}$, we have that

$$1 < |\xi\theta^{n_0-M(n_0,\xi)}| \le \frac{1}{\sqrt{\mathcal{K}}} \quad \text{and} \quad 1 < |Y(N(n_0,\xi),n_0,\xi)| \le \frac{1}{\sqrt{\mathcal{K}}}.$$

2.1. **Mathematical preliminaries.** In order to establish the relation between the systems (1)–(2) and (4)–(5), we need to recall some definitions from the nonautonomous dynamical systems theory.

Definition 2.2 ([11, 24])**.** Given a $\delta>0$, the linear system (1) is δ–kinematically similar to

$$y(n+1) = V(n)y(n) \tag{9}$$

if there exists a Lyapunov's transformation $F(\delta,n)$, that is

$$\sup_{n\in\mathbb{Z}} ||F(\delta,n)|| < +\infty \quad \text{and} \quad \sup_{n\in\mathbb{Z}} ||F^{-1}(\delta,n)|| < +\infty,$$

such that the change of coordinates $y(n) = F^{-1}(\delta,n)x(n)$ transforms the system (1) into (9).

Remark 1. It is straightforward to verify that δ–kinematical similarity is a particular case of topological equivalence. Indeed, the properties of Definition 1.1 are verified with $H(n,\xi) = F^{-1}(\delta,n)\xi$.

The concept of δ–kinematical similarity generalizes the well known notion of kinematical similarity and was proposed in the continuous framework by B.F. Bylov in order to introduce the concept of almost reducibility [5]. The following definition is a discrete version.

Definition 2.3 ([6])**.** The system (1) is almost reducible to

$$y(n+1) = V(n)y(n)$$

if for any $\delta > 0$, the system (1) is δ–kinematically similar to

$$y(n+1) = V(n)\{I + B(n)\}y(n), \quad \text{with} \quad ||B(n)|| \leq \delta \quad \text{for any } n \in \mathbb{Z}.$$

The almost reducibility to a diagonal system (*i.e.* V is a diagonal matrix) has been studied in the continuous context. Indeed, in [5] B.F. Bylov proved that its coefficients are real numbers and F. Lin improved this localization in [13] by using the property of exponential dichotomy and its associated spectrum. In a discrete context, there exists a version of this localization result (see [6]), which employs the same tools combined with the *diagonal significance property* introduced recently by C. Pötzsche [20].

Definition 2.4 ([1],[2],[12],[23])**.** The exponential dichotomy spectrum of (1) is the set $\Sigma(A)$ of $\lambda > 0$ such that the systems

$$x(n+1) = \lambda^{-1}A(n)x(n) \tag{10}$$

have not an exponential dichotomy on $\mathbb{Z}$.

Remark 2. The exponential dichotomy spectrum allows a deeper understanding of assumptions **(P1)**–**(P2)**. Indeed:

(P1) is equivalent (see Lemma 2.3 from [1]) to say that (1) has the property of bounded growth, namely

$$||X(n)X^{-1}(k)|| \leq C\beta^{|n-k|} \quad \textit{for any } n \textit{ and } k \in \mathbb{Z}$$

for some $C \geq 1$ and $\beta > 1$, which implies (see e.g. Theorem 2.1 from [1]) that $\Sigma(A)$ is a finite union of at most $\ell \leq d$ compact intervals

$$\Sigma(A) = \bigcup_{j=1}^{\ell}[a_j, b_j], \text{ with } a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_\ell \leq b_\ell. \tag{11}$$

(P1) also implies that (1) is δ–kinematically similar to (4). Furthermore, in [6] we proved that the diagonal terms of $C(n)$ are contained in $\Sigma(A)$.

(P2) implies that $\Sigma(A) \subset (0,1)$.

Remark 3. In Theorem 1 from [6], it was proved that if **(P1)** is satisfied, the system (1) is $\delta-$kinematically similar via $F^{-1}(\delta, n)$ to

$$y(n+1) = C(n)\{I + B(n)\}y(n), \tag{12}$$

where $C(n) = \mathrm{Diag}(C_1(n), \ldots, C_d(n))$ with $C_i(n) \in \Sigma(A)$ and $\| B(n) \| \leq \delta$.

In addition, under the same transformation, the system (2) is transformed in

$$y(n+1) = C(n)\{I + B(n)\}y(n) + F^{-1}(n+1,\delta)f(n, F(n,\delta)y(n)). \tag{13}$$

Remark 4. Some comments about **(Q1)**–**(Q2)**:

The assumption **(Q1)** is always verified when the Lipschitz constant L_g of g in the system (5) satisfies some smallness conditions which can be described in terms of $b_\ell \in \Sigma(A) = \Sigma(C[I + B])$.

The assumption **(Q2)** is a technical condition which allows to construct a bijection between the solutions of (5) and a linear diagonal autonomous system by using the crossing times described previously.

2.2. **Main results.** The principal results of this article are:

Theorem 2.5. *If $g \in \mathcal{F}_2$ and the properties **(Q1)**–**(Q2)** are verified, then (5) and*

$$z_{n+1} = \theta^{-1} I z_n \quad \text{with } \theta = \frac{1}{\sqrt{\mathcal{K}}} \tag{14}$$

are almost topologically equivalent.

The statement of the following results involve the δ–kinematical similarities between (1)–(12) and (2)–(13) via the transformation $F(n,\delta)$ with $\delta < \dfrac{1-b_\ell}{b_\ell}$.

Theorem 2.6. *If the properties **(P1)**–**(P3)** are verified with $f \in \mathcal{F}_2$ having $L = L_f$ such that*

$$L_f \leq \frac{1 - b_\ell(1+\delta)}{||F||\,||F^{-1}||}, \tag{15}$$

*then the systems (12)–(13) verify **(Q1)**. In addition if (12)–(13) also verify **(Q2)**, then (1) and (2) are almost topologically equivalent.*

In order to state our last result, we will introduce the system

$$y(n+1) = C(n)\{I + B(n)\}y(n) + F^{-1}(n+1,\delta) f_0(n, F(n,\delta)y), \tag{16}$$

where $C(n) = \mathrm{Diag}(C_1(n),\ldots,C_d(n))$ with $C_i(n) \in \Sigma(A)$, $||B(n)|| \leq \delta$ and f_0 is defined by

$$f_0(n,y) = f(n,y) - f(n,0) \quad \text{for any } n \in \mathbb{Z}. \tag{17}$$

Theorem 2.7. *If the properties **(P1)**–**(P3)** are verified with $f \in \mathcal{F}_1$ having $L = L_f = L_{f_0}$ such that*

$$L_f \leq \min\left\{\frac{1 - b_\ell(1+\delta)}{||F||\,||F^{-1}||}, \frac{1-\rho}{K}\right\}, \tag{18}$$

*then the systems (12) and (16) verify **(Q1)**. In addition if (12) and (16) also verify **(Q2)**, then (1) and (2) are almost topologically equivalent.*

3. **Some basic results.** The following proposition is a classical result of local continuity with respect to the initial conditions for difference equations.

Proposition 1. *Let us consider the difference equation*

$$x(n+1) = F(n, x(n)), \tag{19}$$

where $F \in \mathcal{F}_2$ with $L = L_F$, then for the solution $x(n,n_0,u)$ of (19) with $x(n_0,n_0,u) = u$ we have that

$$||u-v||e^{(-2L_F|n-n_0|)} \leq ||x(n+1,n_0,u) - x(n+1,n_0,v)|| \leq ||u-v||e^{(2L_F|n-n_0|)}.$$

Proof. By considering that $n > n_0$, it is easy to verify that

$$x(n+1,n_0,u) = u + \sum_{j=n_0+1}^{n+1} \big\{F(j,x(j,n_0,u)) - F(j-1,x(j-1,n_0,u))\big\},$$

which implies that

$$|\, x(n+1,n_0,u) - x(n+1,n_0,v) \,| \leq 2\Big(\,|\, u-v \,| + \sum_{j=n_0+1}^{n+1} L_F \,|\, x(j,n_0,u) - x(j,n_0,v) \,|\,\Big).$$

By using the discrete Gronwall's inequality (see *e.g.*, [10, Lemma 4.32]) we have that

$$|\, x(n+1,n_0,u) - x(n+1,n_0,v) \,| \leq |\, u-v \,| \exp\left(2L_F \,|\, n-n_0 \,|\right),$$

and the right inequality follows.

Finally, we replace u and v respectively by $x(n_0, n+1, u)$ and $x(n_0, n+1, v)$, obtaining the left inequality. $\square$

The next result is an extension to the discrete framework of [14, Proposition 5].

Proposition 2. *Assume that the system (1) has an exponential dichotomy on $\mathbb{Z}$ with $K \geq 1, 0 < \rho < 1$ and $P = I$. Let us consider the nonlinear perturbation*

$$x(n+1) = A(n)x(n) + F(n, x(n), \kappa), \tag{20}$$

where $F : \mathbb{Z} \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}^d$ and $\mathcal{B}$ is a non–empty set. Moreover, F satisfies the following conditions:

(i) *$F(n, x, \kappa)$ is bounded with respect to n, for all $x \in \mathbb{R}^d$ and $\kappa \in \mathcal{B}$ fixed,*

(ii) *there exists $L_F > 0$ such that*

$$|| F(n, x_1, \kappa) - F(n, x_2, \kappa) || \leq L_F || x_1 - x_2 || \quad \text{for any } n \in \mathbb{Z} \text{ and } \kappa \in \mathcal{B}.$$

(iii) $C_0 = \sup\limits_{n \in \mathbb{Z}, \kappa \in B} || F(n, 0, \kappa) || < +\infty.$

If $L_F K < 1 - \rho$ then for any fixed $\kappa \in \mathcal{B}$, the system (20) has a unique bounded solution $Z(n, \kappa)$ described by

$$Z(n, \kappa) = \sum_{m=-\infty}^{n-1} X(n)X^{-1}(m+1)F(m, Z(m, \kappa), \kappa), \tag{21}$$

such that $\sup\limits_{n \in \mathbb{Z}, \kappa \in \mathcal{B}} || Z(n, \kappa) || < +\infty.$

Proof. Let us consider a fixed $\kappa \in \mathcal{B}$ and construct the sequence $\{\varphi_j\}_j$ recursively defined by

$$\varphi_j(n+1, \kappa) = A(n)\varphi_{j-1}(n, \kappa) + F(n, \varphi_{j-1}(n, \kappa), \kappa),$$

where $\varphi_0(n, \kappa) \in \ell^\infty(\mathbb{Z}, \mathbb{R}^d) = \ell^\infty$, which is the Banach space of bounded sequences with norm $||\varphi||_\infty = \sup\limits_{n \in \mathbb{Z}} |\varphi(n)|$.

We will prove by induction that $\varphi_j \in \ell^\infty$ for any $j \in \mathbb{N}$. Indeed, if $\varphi_j \in \ell^\infty$, it follows that $F(n, \varphi_j(n), \kappa)$ is bounded and **(P2)** implies that

$$\varphi_{j+1}(n, \kappa) = \sum_{k=-\infty}^{n-1} X(n)X^{-1}(k+1)F(k, \varphi_j(k, \kappa), \kappa)$$

is the unique sequence in ℓ^∞ satisfying the recursivity stated above (we refer the reader to Lemma 2 from [7] for details).

On the other hand, by using $KL_F < 1 - \rho$, it is easy to see that $\{\varphi_j\}$ is a Cauchy sequence, convergent to the fixed point $Z(n, \kappa)$ defined by (21).

As we are considering a fixed $\kappa \in \mathcal{B}$ we have that $\sup\limits_{n \in \mathbb{Z}} | Z(n, \kappa) | < C(\kappa)$. That is, $Z(\cdot, \kappa) \in \ell^\infty$ but its bound $C(\kappa)$ could be dependent of $\kappa \in \mathcal{B}$. Nevertheless, we will prove that $C(\kappa)$ has an upper bound independently of κ. Indeed, by combining properties (ii), (iii) with the exponential dichotomy of (1), we can deduce that

$$\begin{aligned} | Z(n, \kappa) | &\leq C_0 \sum_{m=-\infty}^{n-1} K\rho^{n-(m+1)} + KL_F \sum_{m=-\infty}^{n-1} K\rho^{n-(m+1)} | Z(m, \kappa) | \\ &\leq \frac{C_0 K}{1-\rho} + \frac{KL_F C(\kappa)}{1-\rho}. \end{aligned}$$

Taking supremum over n, we have that

$$C(\kappa) \leq \frac{C_0 K}{1-\rho}\left(1 - \frac{KL_F}{1-\rho}\right)^{-1}.$$

□

4. **Proof of Theorem 2.5.** In order to construct the bijection relating (5) and the linear autonomous system (14), we will introduce some notation and definitions. The solution of (14) passing through ξ at $n = n_0$ will be denoted by $Z(n, n_0, \xi) = \xi\theta^{n_0 - n}$. In addition, we can verify that

$$|Z(n, n_0, \xi)| = \theta^{m-n}|Z(m, n_0, \xi)| \quad \text{with } n \geq m \quad \text{and} \quad \theta = \frac{1}{\sqrt{\mathcal{K}}} > 1. \tag{22}$$

Remark 5. The maps N and M stated in Definition 2.1 verify the identities:

$$N(n_0, \xi) = N(n, Y(n, n_0, \xi)) \quad \text{for any } n \in \mathbb{Z} \tag{23}$$

and

$$M(n, \xi\theta^{n_0 - n}) = M(n_0, \xi). \tag{24}$$

Remark 6. The crossing times N and M are inspired in an idea carried out by Barreira *et al.* in [3], which use them in order to construct a bijection between two linear systems.

Now, we will verify that the systems (5) and (14) are almost topologically equivalent through the maps $H\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ and $G\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$H(k, \xi) = \begin{cases} Y(N(k, \xi), k, \xi)\theta^{N(k,\xi)-k} & \text{if} \quad \xi \neq 0 \\ 0 & \text{if} \quad \xi = 0, \end{cases}$$

and

$$G(k, \xi) = \begin{cases} Y(k, M(k, \xi), \xi\theta^{k - M(k,\xi)}) & \text{if} \quad \xi \neq 0 \\ 0 & \text{if} \quad \xi = 0, \end{cases}$$

and the proof will be decomposed in several lemmas and remarks.

Lemma 4.1. *For any $(k, \xi) \in \mathbb{Z} \times (\mathbb{R}^d \setminus \{0\})$, the maps H and G satisfy the identities*

$$H(N(k, \xi), Y(N(k, \xi), k, \xi)) = Y(N(k, \xi), k, \xi) \tag{25}$$

and

$$G(M(k, \xi), \xi\theta^{k - M(k,\xi)}) = \xi\theta^{k - M(k,\xi)}. \tag{26}$$

Proof. The proof follows by using the identities (23) and (24) from Remark 5. □

The above result shows the maps H and G have a fixed points in the solutions of (5) and (14) at their respective crossing times.

Lemma 4.2. *If $n \mapsto Y(n, k, \xi)$ is solution of (5), then $n \mapsto H(n, Y(n, k, \xi))$ is solution of (14). Similarly, if $n \mapsto \xi\theta^{k-n}$ is solution of (14) then $n \mapsto G(n, \xi\theta^{k-n})$ is solution of (5).*

Proof. The identity (23) implies

$$H(n, Y(n, k, \xi)) = Y(N(k, \xi), k, \xi)\theta^{N(k,\xi)-n}$$

and the reader can easily verify that is solution of (14). Analogously, by using (24), it follows that

$$\begin{aligned} G(n, \xi\theta^{k-n}) &= Y(n, M(n, \xi\theta^{k-n}), \xi\theta^{k - M(n, \xi\theta^{k-n})}) \\ &= Y(n, M(k, \xi), \xi\theta^{k - M(k,\xi)}), \end{aligned}$$

which is clearly a solution of (5). $\square$

Lemma 4.3. *For any $(k,\xi)\in\mathbb{Z}\times(\mathbb{R}^d\setminus\{0\})$, the following crossing times identities are verified*

$$M(k,H(k,\xi))=N(k,\xi), \tag{27}$$

and

$$N(k,G(k,\xi))=M(k,\xi). \tag{28}$$

Proof. By the previous result, we know that the solution of (5) passing through ξ at $n=k$ is mapped into a solution of (14) passing through $H(k,\xi)$ at $n=k$. This fact combined with (25) and **(Q2)** imply that

$$1<|Z(N(k,\xi),k,H(k,\xi))|^2=|Y(N(k,\xi),k,\xi)|^2\leq\theta$$

and the identity (27) is a consequence of the uniqueness of crossing times.

In order to prove (28), we know by Definition 2.1 that $N(k,G(k,\xi))$ is the unique integer satisfying the inequalities

$$|Y(N(k,G(k,\xi))+1,k,G(k,\xi))|^2\leq 1<|Y(N(k,G(k,\xi)),k,G(k,\xi))|^2,$$

which are equivalent to

$$|Y(N(k,G(k,\xi))+1,M(k,\xi),\xi\theta^{k-M(k,\xi)})|^2\leq 1,$$

and

$$1<|Y(N(k,G(k,\xi)),M(k,\xi),\xi\theta^{k-M(k,\xi)})|^2.$$

On the other hand, $M(k,\xi)$ is the unique integer verifying

$$|\xi\theta^{k-\{M(k,\xi)+1\}}|^2\leq 1<|\xi\theta^{k-M(k,\xi)}|^2=|Y(M(k,\xi),M(k,\xi),\xi\theta^{k-M(k,\xi)})|^2,$$

which implies that $N(k,G(k,\xi))\geq M(k,\xi)$.

We will verify that $N(k,G(k,\xi))=M(k,\xi)$. Indeed, if $N(k,G(k,\xi))>M(k,\xi)$, we have that $\mathcal{K}^{N(k,G(k,\xi))-M(k,\xi)}\leq\mathcal{K}$. Moreover, **(Q1)** combined with the above inequalities, the identities $\theta^{-2}=\mathcal{K}$ and $Y(M(k,\xi),M(k,\xi),\xi\theta^{k-M(k,\xi)})=\xi\theta^{k-M(k,\xi)}$ imply that

$$\begin{aligned}1<|Y(N(k,G(k,\xi)),M(k,\xi),\xi\theta^{k-M(k,\xi)})|^2 &\leq \mathcal{K}|\xi\theta^{k-M(k,\xi)}|^2\\ &= |\xi\theta^{k-\{M(k,\xi)+1\}}|^2\leq 1,\end{aligned}$$

obtaining a contradiction and the identity (28) follows. $\square$

Remark 7. The assumption **(Q2)** was essential to prove (27) but it was not necessary to deduce (28). This follows from the fact that any solution of (14) with non–null initial condition passes for all the annuli $\{z\in\mathbb{R}^d\colon\theta^j<|z|\leq\theta^{j+1}\}_{j\in\mathbb{Z}}$ in forward and backward time while this is not always true for the solutions of (5).

Lemma 4.4. *For any fixed $k\in\mathbb{Z}$, the map H is bijective being G its inverse.*

Proof. By using (28), we have that

$$\begin{aligned}H(k,G(k,\xi)) &= Y(N(k,G(k,\xi)),k,G(k,\xi))\theta^{N(k,G(k,\xi))-k}\\ &= Y(M(k,\xi),k,G(k,\xi))\theta^{M(k,\xi))-k}\\ &= Y(M(k,\xi),k,Y(k,M(k,\xi),\xi\theta^{k-M(k,\xi)}))\theta^{M(k,\xi)-k}\\ &= \xi,\end{aligned}$$

and the identity $G(k,H(k,\xi))=\xi$ can proved in a similar way by using (27). $\square$

Lemma 4.5. *The sets*

$$\{\xi\in\mathbb{R}^d\colon |\xi|<1\},\quad \{\xi\in\mathbb{R}^d\colon |\xi|=1\}\quad \text{and}\quad \{\xi\in\mathbb{R}^d\colon |\xi|>1\}$$

are invariant under the maps H and G for any $k\in\mathbb{Z}$.

Proof. If $|\xi|>1$, we have $N(k,\xi)\geq k$, as $\theta>1$ and $|Y(N(k,\xi),k,\xi)|>1$, the inequality $|H(k,\xi)|>1$ follows from our definition of H. Similarly, if $|\xi|>1$, it follows that $M(k,\xi)\geq k$. As the crossing time definition implies that $1<|\xi\theta^{k-M(k,\xi)}|$, the inequality $|G(k,\xi)|>1$ can be deduced from our definition of G.

If $|\xi|=1$, it follows that $N(k,\xi)=k-1$ and by **(Q1)**, we know that

$$|\xi|^2=|Y(k,k,\xi)|^2\leq\mathcal{K}|Y(k-1,k,\xi)|^2=|Y(k-1,k,\xi)\theta^{-1}|^2,$$

that is $1=|\xi|^2\leq|H(k,\xi)|^2$. Nevertheless, the case $|H(k,\xi)|>1$ is not possible due to the bijectivity of H combined with the invariance of $\{\xi\in\mathbb{R}^d\colon|\xi|>1\}$ under G.

If $|\xi|<1$, we can deduce that $|G(k,\xi)|<1$. Indeed, otherwise if $|G(k,\xi)|\geq 1$, the invariance of the set $\{\xi\in\mathbb{R}^d\colon|\xi|\geq 1\}$ under H will implies that $|\xi|\geq 1$, obtaining a contradiction.

Now, if $|\xi|<1$ we have that $N(k,\xi)\leq k-1$ and $\theta^{N(k,\xi)-k}\leq\theta^{-1}$. On the other hand, by **(Q2)** it follows that

$$1\leq|Y(N(k,\xi),k,\xi)|<\theta.$$

We couple the above inequalities and obtain

$$\theta^{N(k,\xi)-k}\leq|Y(N(k,\xi),k,\xi)|\theta^{N(k,\xi)-k}=|H(k,\xi)|<\theta\,\theta^{N(k,\xi)-1}\leq 1,$$

then $|H(k,\xi)|<1$ and the invariance of $\{\xi\in\mathbb{R}^d\colon|\xi|<1\}$ under H and G follows.

Finally, if $|\xi|=1$, it follows that $M(k,\xi)=k-1$ and by **(Q1)**, we know that

$$|G(k,\xi)|^2=|Y(k,k-1,\xi\theta)|^2\leq\mathcal{K}|Y(k-1,k-1,\xi\theta)|^2=|\xi|^2=1,$$

that is $1=|\xi|^2\geq|G(k,\xi)|^2$. Nevertheless, the case $|G(k,\xi)|<1$ is not possible due to bijectivity of G combined with the invariance of $\{\xi\in\mathbb{R}^d\colon|\xi|<1\}$ under H. $\square$

The next results are devoted to study the continuity properties of the maps G and H.

Lemma 4.6. *For any fixed $k\in\mathbb{Z}$, the map $\xi\mapsto G(k,\xi)$ is continuous with the possible exception of a set of initial conditions with Lebesgue measure zero.*

Proof. We will focus our attention in the map $M(k,\xi)$. We consider the sets

$$\Gamma_k=\left\{\xi\in\mathbb{R}^d\setminus\{0\}\colon|\xi|\theta^{k-(M(k,\xi)+1)}=1\right\}.$$

It is easy to see that $\xi\in\Gamma_k$ if only if $M(k,\xi)$ is written as follows

$$M(k,\xi)=k-1-\frac{\ln(1/|\xi|)}{\ln(\theta)}.$$

The above identity prompt us to construct the step map

$$\xi\to\left\lceil k-1-\frac{\ln(1/|\xi|)}{\ln(\theta)}\right\rceil \tag{29}$$

for any fixed k, where $\lceil\cdot\rceil$ denotes the ceiling function, whose discontinuities are in the level sets

$$\mathcal{D}_j=\left\{\xi\in\mathbb{R}^d\setminus\{0\}\colon|\xi|=\theta^j\right\}\quad\text{for any } j\in\mathbb{Z},$$

and we know that for any $\xi \in \mathcal{D}_j$ it follows that $M(k,\xi) = k-1+j$. Moreover, it is easy to see that $M(k,\xi)$ is constant in the annuli Λ_j:

$$\Lambda_j = \left\{\xi \in \mathbb{R}^d \setminus \{0\} \colon \theta^{j-1} < |\xi| \leq \theta^j \right\}.$$

An important consequence of the above property is that for any fixed $k \in \mathbb{Z}$ the map G is Lipschitz on the annuli Λ_j. In fact, by using Proposition 1 and recalling that the Lipschitz constant of the system (5) is $L = ||C||(1+||B||)+\delta$, for any couple $\xi, \tilde{\xi}$ in the annulus $\operatorname{Int}\Lambda_j$ we have that

$$\begin{aligned} |G(k,\xi) - G(k,\tilde{\xi})| &= |Y(k,M(k,\xi),\xi\theta^{k-M(k,\xi)}) - Y(k,M(k,\xi),\tilde{\xi}\theta^{k-M(k,\xi)})| \\ &\leq \theta^{k-M(k,\xi)}e^{2L|k-M(k,\xi)|}|\xi - \tilde{\xi}|, \end{aligned}$$

and we conclude that for any fixed k, the map $\xi \mapsto G(k,\xi)$ is continuous with the possible exception of the sets $\mathcal{D}_j$, which have Lebesgue measure zero. □

Lemma 4.7. *For any fixed $k \in \mathbb{Z}$, the map $\xi \mapsto H(k,\xi)$ is continuous with the possible exception of a set of initial conditions with Lebesgue measure zero.*

Proof. It will be useful to consider the sets of initial conditions

$$\mathcal{I}_k = \left\{\xi \in \mathbb{R}^d \setminus \{0\} \colon |Y(N(k,\xi)+1,k,\xi)^2| = 1\right\}$$

and the sets

$$G(k,E) = \left\{G(k,\xi) \in \mathbb{R}^d \setminus \{0\} \colon \xi \in E\right\},$$

where E is any subset of $\mathbb{R}^d$. The proof will be decomposed in several steps.

Step 1. for any fixed $k \in \mathbb{Z}$, we prove the following identity

$$\mathcal{I}_k = \bigcup_{r\in\mathbb{Z}} G(k,\mathcal{D}_r) \quad \text{where } G(k,\mathcal{D}_j) \text{ and } G(k,\mathcal{D}_i) \text{ pairwise disjoint.} \tag{30}$$

If $\xi \in \bigcup_{r\in\mathbb{Z}} G(k,\mathcal{D}_r)$, there exists a unique $\eta \in \mathcal{D}_j$ such that $\xi = G(k,\eta)$.

As $\eta \in \mathcal{D}_j$ it follows that $i \mapsto \eta\theta^{k-i}$ is a solution of (14) passing through $\eta\,\theta^{k-(M(k,\eta)+1)} = \eta_0 \in S^1$ at $i = M(k,\eta)+1$. Now, by Lemma 4.5 combined with (24) we can deduce that

$$G(M(k,\eta)+1,\eta_0) = Y(M(k,\eta)+1,M(k,\eta),\eta_0\theta) \in S^1.$$

On the other hand, Lemmas 4.2 and 4.3 imply that $i \mapsto Y(i,k,G(k,\eta))$ is a solution of (5) passing through $G(k,\eta)$ at $i=k$ satisfying

$$\begin{aligned} G(M(k,\eta)+1,\eta_0) &= Y(M(k,\eta)+1,k,G(k,\eta)) \in S^1 \\ &= Y(N(k,G(k,\eta))+1,k,G(k,\eta)) \in S^1 \end{aligned}$$

and we conclude that $G(k,\eta) = \xi \in \mathcal{I}_k$, then $G(k,\mathcal{D}_j) \subset \mathcal{I}_k$.

Analogously, if $i \mapsto Y(i,k,\xi)$ is a solution of (5) with $\xi \in \mathcal{I}_k$. We can also see that the Lemma 4.2 combined with (23) and the fact that S^1 is invariant by H imply that $i \mapsto H(k,\xi)\theta^{k-i}$ is a solution of (14) with

$$|H(k,\xi)\theta^{k-(M(k,H(k,\xi))+1)}| = |H(k,\xi)\theta^{k-(N(k,\xi)+1)}| = 1.$$

In consequence, it follows that if $\xi \in \mathcal{I}_k$, then $H(k,\xi) \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$ or equivalently $\xi \in G(k,\mathcal{D}_j)$.

Step 2. We will prove that $N(k,\eta)$ is constant for any $\eta \in G(k,\Lambda_j)$.

Indeed $\eta \in G(k,\Lambda_j)$ if and only if $\eta = G(k,\xi)$ with $\xi \in \Lambda_j$. This fact combined with Lemma 4.3 imply that

$$M(k,\xi) = N(k,G(k,\xi)) = N(k,\eta),$$

and as $M(k,\xi)$ is locally constant in Λ_j, the property follows.

Step 3. We will prove that $\xi \mapsto H(k,\xi)$ is continuous with the possible exception of the set $\mathcal{I}_k$.

In fact, by using again Proposition 1 with $\eta, \tilde{\eta} \in G(k, \operatorname{Int}\Lambda_j)$, we can deduce that

$$\begin{aligned} |H(k,\eta) - H(k,\tilde{\eta})| &= |Y(N(k,\eta),k,\eta) - Y(N(k,\eta),k,\tilde{\eta})|\theta^{N(k,\eta)-k} \\ &\leq |\eta - \tilde{\eta}|(\theta e^{2L})^{|N(k,\eta)-k|}, \end{aligned}$$

thus the map $\eta \mapsto H(k,\eta)$ is Lipschitz on $G(k,\operatorname{Int}\Lambda_j)$.

As $G(k,\Lambda_j) = G(k,\operatorname{Int}\Lambda_j) \cup G(k,\mathcal{D}_j)$, we can deduce that the $\eta \mapsto H(k,\eta)$ is continuous with the possible exception of the set $\mathcal{I}_k$.

Step 4. We will show that $\mathcal{I}_k$ has Lebesgue measure zero.

By using (30), we only need to verify that $G(k,\mathcal{D}_j)$ has Lebesgue measure zero for any $j \in \mathbb{Z}$.

Notice that $\mathcal{D}_j$ can be seen as the limit of the sequence of sets $\mathcal{D}_j^n$ defined by

$$\mathcal{D}_j^n = \left\{\xi \in \mathbb{R}^d \setminus \{0\} \colon |\xi| = \theta^j r_n, \quad \text{where} \quad 0 < r_n < r_{n+1} < 1 \quad \text{and} \quad r_n \to 1\right\}.$$

As $\xi \mapsto G(k,\xi)$ is Lipschitz in the interior of Λ_j and $m(\mathcal{D}_j^n) = 0$ then the measure of $G(k,\mathcal{D}_j^n)$ is zero. (See Theorem 3.33 from [25] and its details).

Now, note that $G(k,\mathcal{D}_j)$ can be seen that as the limit of the sequence of sets $E_{n+1} \subset E_n$ defined by

$$E_n = \bigcup_{i=n}^{\infty} G(k,\mathcal{D}_j^i)$$

and $m(E_n) = 0$ follows from the fact that the sets $G(k,\mathcal{D}_j^i)$ are pairwise disjoints. □

Remark 8. A careful inspection of the map G shows that it can be seen as the composition of a continuous function with a piecewise constant map. This fact prompted us to consider the almost topological equivalence.

By considering $g(n,y(n)) = 0$ and following the lines of the above proof, we can deduce the following result.

Corollary 1. *The linear systems (4) and (14) are almost topologically equivalent.*

5. **Proof of Theorem 2.6.** In order to verify that the systems (12) and (13) verify assumption **(Q1)**, we have the following Lemma

Lemma 5.1. *Let us consider the system*

$$y(n+1) = C(n)\{I + B(n)\}y(n) + g(n,y(n)) \tag{31}$$

*where $C(n), B(n)$ are the same of the system (12) with $\delta < \frac{b_\ell - 1}{b_\ell}$. If **(P1)–(P3)** are satisfied with $g \in \mathcal{F}_2$ having Lipschitz constant L_g such that*

$$0 < L_g < 1 - b_\ell(1+\delta), \tag{32}$$

then there exist $0 < \mathcal{K}(L_g) < 1$ such that the solutions $Y(n, n_0, \xi)$ of (31) verifies

$$| Y(n, n_0, \xi) |^2 \leq \mathcal{K}(L_g)^{n-m} \, | Y(m, n_0, \xi) |^2 \text{ for any } n > m. \tag{33}$$

Proof. Let $w(n) = | Y(n, n_0, \xi) |^2$, and recall (see Remark 3) that

$$|| \, C(n) \, || < b_\ell \quad \text{and} \quad || \, I + B(n) \, || < 1 + \delta.$$

Now, by using the Cauchy–Schwarz inequality combined of properties of g it is easy to deduce that

$$w(n+1) \leq \mathcal{K}(L_g) w(n) \quad \text{with} \quad \mathcal{K}(L_g) = b_\ell^2(1+\delta)^2 + 2b_\ell(1+\delta)L_g + L_g^2.$$

We can see that $0 < \mathcal{K}(L_g) < 1$ when δ verifies (32) and we can prove by an inductive approach that

$$w(n) \leq \mathcal{K}(L_g)^{n-m} w(m) \quad \text{for any} \quad n > m,$$

and the results follows. □

Remark 9. An interesting feature of the above result is that the uniform asymptotic stability of (12) is preserved for any $\mathcal{F}_2$–perturbation whose Lipschitz constant has an upper bound which is dependent of $\Sigma(C[I+B]) = \Sigma(A)$.

It is easy to see that the systems (12)–(13) satisfy assumption **(Q1)** with the constant $\mathcal{K} = b_\ell^2(1+\delta)^2 + 2b_\ell(1+\delta)L_f||F|| \, ||F^{-1}|| + (L_f||F|| \, ||F^{-1}||)^2$ and (15) implies that $0 < \mathcal{K} < 1$.

The end of proof is now clear, indeed is a trivial consequence of Theorem 2.5 combined with the fact that (almost) topological equivalence is an equivalence relation:

- (i) As the systems (12) and (13) satisfies **(Q1)–(Q2)**, Theorem 2.5 says that are almost topologically equivalent.
- (ii) By Theorem 1 from [6], the systems (1) and (12) are δ–kinematically similar by a Lypaunov transformation F.
- (iii) By using the same transformation F, the system (2) and (13) are topologically equivalent.
- (iv) By (ii) and (iii) we can deduce that the couples (1)–(12) and (2)-(13) are topologically equivalent since the map $\xi \mapsto F^{-1}(n,\delta)\xi$ fulfils the properties of topological equivalence.

Corollary 2. *If the properties **(P1)–(P2)** and there exists a $d \times d$ matrix $A_0(n)$ such that*

$$||A_0|| \leq \frac{1 - b_\ell(1+\delta)}{||F|| \, ||F^{-1}||} \quad \text{with} \quad \delta < \frac{1-b_\ell}{b_\ell}, \tag{34}$$

then the systems (12) and

$$y(n+1) = C(n)[I + B(n)]y(n) + F^{-1}(n+1,\delta)A_0(n)F(n,\delta)y(n) \tag{35}$$

*verify **(Q1)**. In addition if (12) and (35) also verify **(Q2)**, then (1) and*

$$y(n+1) = A(n)y(n) + A_0(n)y(n)$$

are almost topologically equivalent.

6. **Proof of Theorem 2.7.** Firstly, by (17) we have that $f \in \mathcal{F}_1$ implies $f_0 \in \mathcal{F}_2$. As f and f_0 have the same Lipschitz constant, the inequality (18) combined with Theorem 2.6 imply that (12) and (16) verify **(Q1)**.

As **(Q2)** is satisfied by Hypothesis, Theorem 2.6 says us that (1) and

$$u(n+1) = A(n)u(n) + f_0(n, u(n)) \tag{36}$$

are almost topologically equivalent. In consequence, the Theorem follows if we prove that (36) is topologically equivalent to (2).

Let $U(n, n_0, \xi)$ be the unique solution of (36) passing through ξ at $n = n_0$ (resp. $z(n, n_0, \xi)$ be the unique solution of (2) passing through ξ at $n = n_0$). Now, let $\mathcal{B} = \mathbb{Z} \times \mathbb{R}^d$ and define the functions $P, Q \colon \mathbb{Z} \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}^d$ as

$$\begin{aligned} P(n, z, (n_0, \xi)) &= f(n, z + U(n, n_0, \xi)) - f_0(n, U(n, n_0, \xi)) \\ &= f(n, z + U(n, n_0, \xi)) - f(n, U(n, n_0, \xi)) + f(n, 0). \end{aligned} \tag{37}$$

and

$$\begin{aligned} Q(n, v, (n_0, \xi)) &= f_0(n, v + z(n, n_0, \xi)) - f(n, z(n, n_0, \xi)) \\ &= f(n, v + z(n, n_0, \xi)) - f(n, 0) - f(n, z(n, n_0, \xi)). \end{aligned} \tag{38}$$

It is easy to verify that

$$\begin{cases} |P(n, z, (n_0, \xi))| \leq L_f |z| + D, \\ |Q(n, v, (n_0, \xi))| \leq L_f |v| + D, \\ |P(n, z, (n_0, \xi)) - P(n, \tilde{z}, (n_0, \xi))| \leq L_f |z - \tilde{z}| \\ |Q(n, z, (n_0, \xi)) - Q(n, \tilde{z}, (n_0, \xi))| \leq L_f |z - \tilde{z}|, \end{cases} \tag{39}$$

where $D = \sup\{n \in \mathbb{Z} \colon |f(n, 0)|\}$.

Notice that P and Q verifies the assumptions of Proposition 2, which implies that the system

$$r(n+1) = A(n)r(n) + P(n, r(n), (n_0, \xi)) \tag{40}$$

has a unique bounded solution Z defined by

$$Z(n, (n_0, \xi)) = \sum_{k=-\infty}^{n-1} X(n, m+1)\{f(k, Z(k, (n_0, \xi)) + U(k, n_0, \xi)) - f_0(k, U(k, n_0, \xi))\}.$$

Similarly the system

$$r(n+1) = A(n)r(n) + Q(n, r(n), (n_0, \xi)) \tag{41}$$

has a unique bounded solution $\tilde{Z}$ defined by

$$\widetilde{Z}(n, (n_0, \xi)) = \sum_{k=-\infty}^{n-1} X(n, m+1)\{f_0(k, \widetilde{Z}(k, (n_0, \xi)) + z(k, n_0, \xi)) - f(k, z(k, n_0, \xi))\}.$$

Now, let us construct the maps $H, G \colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ as:

$$H(n, \xi) = \xi + Z(n, (n, \xi)) \quad \text{and} \quad G(n, \xi) = \xi + \widetilde{Z}(n, (n, \xi)). \tag{42}$$

Lemma 6.1. *For any $(j, n) \in \mathbb{Z} \times \mathbb{Z}$ and $(n_0, \xi) \in \mathbb{Z} \times \mathbb{R}^d$ it follows that*

$$Z(j, (n, U(n, n_0, \xi))) = Z(j, (n_0, \xi)). \tag{43}$$

Proof. Firstly, notice that

$$
\begin{aligned}
&Z(j,(n,U(n,n_0,\xi))) = \\
&= \sum_{k=-\infty}^{j-1} X(j,m+1)f(k,Z(k,(n,U(n,n_0,\xi))) + U(k,n,U(n,n_0,\xi))) \\
&- \sum_{k=-\infty}^{j-1} X(j,m+1)f_0(k,U(k,n_0,\xi)) \\
&= \sum_{k=-\infty}^{j-1} X(j,m+1)f(k,Z(k,(n,U(n,n_0,\xi)) + U(k,n_0,\xi)) \\
&- \sum_{k=-\infty}^{j-1} X(j,m+1)f_0(k,U(k,n_0,\xi))
\end{aligned}
$$

and

$$
Z(j,(n_0,\xi)) = \sum_{k=-\infty}^{j-1} X(j,m+1)\{f(k,Z(k,(n_0,\xi))+U(k,n_0,\xi)) - f_0(k,U(k,n_0,\xi)).
$$

Secondly, we have that

$$
\begin{aligned}
&|Z(j,(n,U(n,n_0,\xi))) - Z(j,(n_0,\xi)) \leq \\
&\leq \sum_{k=-\infty}^{j-1} K\rho^{j-(m+1)}L_f|Z(k,(n,U(n,n_0,\xi))) - Z(k,(n_0,\xi))| \\
&\leq \frac{KL_f}{1-\rho}\sup_{k\in\mathbb{Z}}|Z(k,(n,U(n,n_0,\xi))) - Z(k,(n_0,\xi))|
\end{aligned}
$$

and the Lemma follows. □

Lemma 6.2. *If $n \mapsto U(n,n_0,\xi)$ is solution of (36) passing throguh ξ at $n=n_0$, then $n \mapsto H(n,U(n,n_0,\xi))$ is solution of (2).*

Proof. By using (42) combined with (43), we have that

$$
H(n,U(n,n_0,\xi)) = U(n,n_0,\xi) + Z(n,(n_0,\xi))
$$

and the rest of the proof follows by a direct computation. □

Lemma 6.3. *The map $\xi \mapsto H(n,\xi)$ is continuous for any fixed $n \in \mathbb{Z}$.*

Proof. By (42), we only need to prove that the map $\xi \mapsto Z(n,(n,\xi))$ is continuous for any fixed n.

We will follow the lines of the proof of Theorem 1 from [7]. Indeed, let us recall $n \mapsto Z(n,(n_0,\xi))$ is the unique bounded solution of (40), which was constructed by successive approximations in Proposition 2. That is

$$
\lim_{j\to+\infty} Z_j(n,(n_0,\xi)) = Z(n,(n_0,\xi)),
$$

where

$$Z_{j+1}(n,(n_0,\xi)) = \sum_{k=-\infty}^{n-1} X(n)X^{-1}(k+1)F(k, Z_j(k,(n_0,\xi)),(n_0,\xi)).$$

In addition, it is important to emphasize that for any $\varepsilon > 0$, there exists a positive integer $\kappa \geq \ln(\rho_\varepsilon)/\ln(\rho)$ with

$$\rho_\varepsilon = \frac{\varepsilon(1-\rho)\rho}{12K(L_f M_0)+D} < 1 \quad \text{and} \quad M_0 = \sup\{n \in \mathbb{Z}\,, (k,\xi) \in \mathbb{Z}\times\mathbb{R}^d \colon |Z(k,(n,\xi))|\},$$

where M_0 is well defined by Proposition 2. Now, we have

$$\begin{aligned} |Z_{j+1}(n,(n,\xi))| &\leq \sum_{k=-\infty}^{n-\kappa} |X(n)X^{-1}(k+1)F(k,Z_j(k,(n,\xi)),(n,\xi))| \\ &\leq \sum_{k=-\infty}^{n-\kappa} K^{n-(k+1)}|L_F M_0 + D| < \frac{\varepsilon}{12}. \end{aligned}$$

In addition, we know that for any $\varepsilon > 0$, there exists $J(\varepsilon) > 0$ such that for any $j > J$, it follows that

$$\begin{aligned} |Z(n,(n,\xi)) - Z(n,(n,\xi'))| &\leq |Z(n,(n,\xi)) - Z_j(n,(n,\xi))| \\ &\quad +|Z_j(n,(n,\xi)) - Z_j(n,(n,\xi'))| \\ &\quad +|Z_j(n,(n,\xi')) - Z(n,(n,\xi))| \\ &< \frac{2}{3}\varepsilon + |Z_j(n,(n,\xi)) - Z_j(n,(n,\xi'))| \end{aligned}$$

We will prove by induction that for any $j \in \mathbb{N}$, there exists $\delta_j > 0$ such that

$$|Z_j(n,(n,\xi)) - Z_j(n,(n,\xi'))| \leq \frac{\varepsilon}{3} \quad \text{if} \quad |\xi - \xi'| < \delta_j. \tag{44}$$

Indeed, let us consider an initial term

$$Z_0(n,(n,\xi)) = Z_0(n,(n,\xi)) = \phi_0 \in \ell^\infty$$

and suppose that (44) is verified for some j as inductive hipothesis. Now, we have that

$$|Z_{j+1}(n,(n,\xi)) - Z_{j+1}(n,(n,\xi))| \leq \Delta_1 + \Delta_2$$

where

$$\Delta_1 = \sum_{k=-\infty}^{n-\kappa} X(n,k+1)\left\{F(k,Z_j(k,(n,\xi)),(n,\xi)) - F(k,Z_j(k,(n,\xi)),(n,\xi'))\right\}$$

and

$$\begin{aligned} \Delta_2 &= \sum_{k=n-\kappa+1}^{n-1} X(n,k+1)f(k,Z_j(k,(n,\xi)) + U(k,n,\xi)) \\ &\quad - \sum_{k=n-\kappa+1}^{n-1} X(n,k+1)f(k,Z_j(k,(n,\xi')) + U(k,n,\xi')) \\ &\quad + \sum_{k=n-\kappa+1}^{n-1} X(n,k+1)\left\{f_0(k,U(k,n,\xi)) - f_0(k,U(k,n,\xi'))\right\}. \end{aligned}$$

The inequality $\rho_\varepsilon < 1$ implies that $|\Delta_1| < \varepsilon/6$. On the other hand, by using Gronwall's discrete Lemma combined with inductive hypothesis we can deduce that

$$\begin{aligned}
|\Delta_2| \leq & \sum_{k=n-\kappa+1}^{n-1} K\rho^{n-(k+1)} L_f |Z_j(k,(n,\xi)) - Z_j(k,(n,\xi')| \\
& + \sum_{k=n-\kappa+1}^{n-1} K\rho^{n-(k+1)} (L_f + L_{f_0}) |U(k,n,\xi)) - U(k,n,\xi'))| \\
\leq & \frac{\varepsilon}{3} \sum_{k=n-\kappa+1}^{n-1} K\rho^{n-(k+1)} L_f \\
& + \sum_{k=n-\kappa+1}^{n-1} K\rho^{n-(k+1)} (L_f + L_{f_0}) |\xi - \xi'| \exp\left(\sum_{i=k}^{n-1} (||A(i) - I|| + L_{f_0})\right) \\
\leq & \frac{\varepsilon}{3} K L_f \frac{1-\rho^\kappa}{1-\rho} + |\xi - \xi'| \Gamma(n,\kappa),
\end{aligned}$$

where

$$\Gamma(n,\kappa) = \max\left\{ [n-\kappa+1, n-1] \cap \mathbb{Z} \colon \exp\left(\sum_{i=k}^{n-1} (||A(i) - I|| + L_{f_0})\right)\right\},$$

and (44) is verified for $j+1$ when choosing $\delta_{j+1} = \min\left\{\delta_j, \left(\frac{1}{2} - \frac{KL_F}{1-\rho}\right)\frac{\varepsilon}{6\Gamma(n,\kappa)}\right\}$ and the continuity of $\xi \mapsto Z(n,(n,\xi))$ follows. Finally, H is continuous for any fixed n. $\square$

The proof of the following results is similar to the previous ones,

Lemma 6.4. *For any $(j,n) \in \mathbb{Z} \times \mathbb{Z}$ and $(n_0,\xi) \in \mathbb{Z} \times \mathbb{R}^d$ it follows that*

$$\widetilde{Z}(j,(n,z(n,n_0,\xi))) = \widetilde{Z}(j,(n_0,\xi)).$$

Lemma 6.5. *If $n \mapsto z(n,n_0,\xi)$ is solution of (2) passing throguh ξ at $n = n_0$, then $n \mapsto G(n,z(n,n_0,\xi))$ is solution of (36).*

Lemma 6.6. *The map $\xi \mapsto G(n,\xi)$ is continuous for any fixed $n \in \mathbb{Z}$.*

By summarizing all this Lemmas, we easily conclude that the systems (2) and (36) are topologically equivalents. As (1) and (36) are almost topologically equivalent by Theorem 2.6, the result follows.

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