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Sequential Equilibrium in Incomplete Markets with Long-Term Debt

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SEQUENTIAL EQUILIBRIUM IN INCOMPLETE MARKETS WITH LONG-TERM DEBT

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ABSTRACT. This paper proves equilibrium existence in an incomplete market sequential economy with finitely-lived debt contracts. Introducing credit constraints limiting agents' access to liquidity, we show that a competitive equilibrium *always* exists. Our results are consistent with broad forms of endogenous credit segmentation.

Keywords: General equilibrium, Incomplete markets, Financial segmentation.
JEL classification: D52, D53, C62.

1. INTRODUCTION

Two of the main challenges involved in determining equilibrium existence in infinite horizon economies with incomplete markets are discarding Ponzi schemes from agents' choice sets and ensuring that security prices are endogenously bounded from above. Conventionally, the literature on sequential economies with uncollateralized debt has imposed debt constraints limiting the growth of agents' indebtedness to solve the former (Kehoe 1989), and strong assumptions regarding preferences, asset's lifespan and/or deliveries to account for the latter (Hernández and Santos 1996; Magill and Quinzii 1994, 1996). In a seminal paper, Magill and Quinzii (1994) proved equilibrium in an economy with bounded allocations, short-lived securities and in which preferences comply with uniform impatience. In this context they established the existence of equilibria with *implicit debt constraints* (i.e. debt is restricted to the space of bounded sequences). When securities are real and long-lived, this framework is insufficient to ensure that a competitive equilibrium *always* exists. Precisely, in finite horizon economies the potential discontinuities induced by price-dependent payment matrices has lead to equilibrium results valid solely for dense subsets of economies (Hernández and Santos 1996; Magill and Quinzii 1996).

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This paper addresses the open problem of ensuring equilibrium existence with long-lived real assets. More precisely, we introduce conditions under which a competitive equilibrium *always* exists, regardless of the arbitrary election of asset delivery streams, and without requiring utility functionals consistent with uniform impatience. We model financial markets with securities in both positive and zero net supply. As recognized by Hernandez and Santos (1996), working with zero net supply securities requires additional assumptions to ensure that asset prices have endogenous upper bounds on equilibrium. To account for this issue, we adapt techniques introduced by Cea-Echenique and Torres-Martínez (2016) in the context of financial segmentation in two-period economies, and determine upper bounds on security prices based on the *super-replication* of their returns. Importantly, the bounds induced through super-replication are forward-looking in the sense that they are dependent on securities' lifespan, and thus, they are only well defined if the security lives for a finite number of periods.

To discard Ponzi schemes, we leverage on the framework of Moreno-García and Torres-Martínez (2012), who accomplish to assert equilibrium existence in an economy with long-lived real assets in positive net supply (equity contracts). Similarly to Hoelle, Pireddu, and Villanacci (2016), we introduce credit constraints limiting agent's feasible amount of borrowing to a proportion of the market value of aggregate wealth, and thus, may be understood as restrictions regarding the liquidity of the economy. Additionally, we extend the literature of financial segmentation by presenting conditions under which our equilibrium results are consistent with financial segmentation, like the one considered by Aouani and Cornet (2009) and Faias and Torres-Martínez (2017). Thus, and by allowing for segmententation in non-secured debt markets, we complement Iraola, Sepúlveda, and Torres-Martínez (2018) who introduce financial segmentation in infinitely lived collateralized markets.

The rest of the paper is organized as follows: Section 2 and 3 introduce our model, that follows closely the one proposed by Magill and Quinzii (1994, 1996). Sections 4 presents our assumptions and equilibrium results, with emphasis on the differences between them and the ones in Magill and Quinzii (1996). Section 5 extends our results as to allow for endogenous financial segmentation, and Section 6 presents a few comments regarding the difficulties of including infinitely lived debt contracts. Section 7 presents a brief conclusion. All proofs are presented in the Appendix.

2. Model

Uncertainty. Let $\mathcal{E}$ represent a discrete time, infinite horizon economy. There is a set S of states of nature characterizing uncertainty, which is homogeneous among agents and represented by a finite partition $\mathcal{F}_t$ of S at each period t. There is no information available at $t = 0$, i.e., $\mathcal{F}_0 = S$. Additionally, $\mathcal{F}_t$ is at least as fine as $\mathcal{F}_{t-1}$ at every period $t \geq 0$. Thus, there is no loss of information throughout time.

A *node* ξ is characterized by a pair (t, σ), where $t \in \mathbb{N}$ and $\sigma \in \mathcal{F}_t$. Accordingly, $t(\xi)$ and $\sigma(\xi)$ denote the date and the information set associated to ξ. Let ξ^-, and ξ^+ be, respectively, node ξ's unique predecessor and the (finite) set of all immediate successors. We also say $\mu \geq \xi$ whenever

$t(\mu) \geq t(\xi)$ and $\sigma(\mu) \subseteq \sigma(\xi)$, so $\xi^+ = \{\mu \geq \xi | t(\mu) = t(\xi) + 1\}$. Analogously, $\mu > \xi$ indicates that $\mu \geq \xi$ and $\mu \neq \xi$.

There is a unique initial node ξ_0, marking the beginning of the *event-tree* D, formed by the set of nodes in our economy. The *subtree* starting at node ξ is denoted by $D(\xi)$ and corresponds to the set $\{\mu \in D : \mu \geq \xi\}$. Moreover, define $D^t(\xi) = \{\mu \in D(\xi) | t(\mu) \leq t(\xi) + t\}$ as the branch of $D(\xi)$ spanning until date t. Similarly, $D_t(\xi) = \{\mu \in D(\xi) : t(\mu) = t + t(\xi)\}$ is the set of nodes in $D(\xi)$ whose dates coincide with $t + t(\xi)$.

Markets. There is a finite and ordered set $\mathcal{L}$ of perfectly divisible, perishable commodities, available at every $\xi \in D$. We denote as $p(\xi) = (p_l(\xi))_{l \in \mathcal{L}}$ the vector of commodity spot prices at ξ, and $p = (p(\xi))_{\xi \in D}$ as the commodity price process along D. Therefore, the set consisting of all commodities indexed across the event tree is $D \times \mathcal{L}$.

There is an ordered set $J := J_+ \cup J_0$ of financial assets, where J_+ and J_0 represent securities in positive and zero net supply respectively. Every $j \in J$ is characterized by an issuing node $\xi_j \in D$ and a payment stream consisting of a non-trivial process of commodity bundles $A_j = (A_j(\mu))_{\mu > \xi_j} \in \mathbb{R}^{\mathcal{L} \times D(\xi_j) \setminus \xi_j}$. Let $A = (A_j)_{j \in J}$ denote the security payoff process.

Securities in $\mathcal{E}$ may be finitely or infinitely lived. Precisely, asset $j \in J$ is finitely lived if there exists $T_j \in \mathbb{N}$ such that the set $\{\xi \in D | (t(\xi) > T_j) \wedge (A_j(\xi) > 0)\}$ is empty. In turn, a security $j \in J$ is infinitely lived if the latter does not hold for any $T \in \mathbb{N}$.

At ξ, there is a finite set $J(\xi) := J_+(\xi) \cup J_0(\xi)$ of assets available for trade, where $J_i(\xi) := \{j \in J_i | \exists \mu > \xi : A_j(\mu) \neq 0\}$, for $i \in \{+, 0\}$.[1] Let $D(J) = \{(\xi, j) \in D \times J : j \in J(\xi)\}$, and let the space of commodity and asset prices be $\mathbb{P} := \mathbb{R}_+^{D \times \mathcal{L}} \times \mathbb{R}_+^{D(J)}$. Define $D(J_+)$ and $D(J_0)$ analogously. Furthermore, let $\mathcal{A} := \mathbb{R}_+^{D(J) \times \mathcal{L}}$ be the security payoff process's space.

Agents. There is a finite set H of agents participating in the economy, each one of them characterized by an utility function $U^h : \mathbb{R}_+^{D \times \mathcal{L}} \to \mathbb{R}_+ \cup \{+\infty\}$, and endowments consisting in both commodities and securities $(w^h(\xi), e^h(\xi))_{\xi \in D} \in \mathbb{R}_+^{D \times \mathcal{L}} \times \mathbb{R}_+^{D(J_+)}$. Let $\bar{e}^h(\xi)$ stand for agent h's cumulative financial endowments up to node ξ; accordingly, at ξ agent h receives aggregate endowments $W^h(\xi) = w^h(\xi) + \sum_{j \in J_+(\xi^-)} A_j(\xi) \bar{e}_j^h(\xi^-)$.[2] Assets in $J_+(\xi)$ are in *positive net supply*: $j \in J_+(\xi)$ implies that $\sum_{h \in H} \bar{e}_j^h(\xi) > 0$. In turn, securities $k \in J_0(\xi)$ are in *zero net supply*: $\sum_{h \in H} \bar{e}_k^h(\xi) = 0$. Aggregate wealth at node ξ is therefore $W(\xi) = \sum_{h \in H} W^h(\xi)$; let $W = (W(\xi))_{\xi \in D}$.

Each $h \in H$ must choose an allocation $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi)) \in \mathbb{E}_\xi := \mathbb{R}^{\mathcal{L}} \times \mathbb{R}_+^{J(\xi)} \times \mathbb{R}_+^{J(\xi)}$ for every $\xi \in D$, composed by commodity bundles and long and short positions in financial securities. Accordingly, $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in D} \in \mathbb{E} := \prod_{\xi \in D} \mathbb{E}_\xi$. Given prices $(p, q) \in \mathbb{P}$, an allocation $(x, \theta, \varphi) \in \mathbb{E}$ is said to be *budget feasible* for agent $h \in H$ if it complains with the following restriction at every node:

$$p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)\left(\theta^h(\xi) - \varphi^h(\xi) - e^h(\xi)\right) \leq \sum_{j \in J_+(\xi^-)} (p(\xi) A_j(\xi) + q_j(\xi)) \quad \left(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)\right).$$

[1]Note that we are ruling out the existence of fiat money.

[2]$A(\xi_0, j) = 0$ for every $j \in J_+(\xi_0)$.

We use $y^h(\xi) = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))$, and $y^h = (y(\xi))_{\xi \in D}$, to shorten notation.

3. Definition of choice sets and competitive equilibria

Defining the concept of a competitive equilibrium for an infinite horizon economy $\mathcal{E}$ is not straightforward due to the necessity of discarding Ponzi games from agents' budget sets. Importantly, and as highlighted by Magill and Quinzii (1994, 1996), the distinction between different definitions of a competitive equilibrium rely on particular specifications of agents' choice sets. In this paper we will discuss two types of choice set:

(1) Choice set with *debt constraints*:

$$\mathcal{C}^{\mathrm{M}}(p, q, w^h, M) = \left\{ y \in \mathbb{E} \;\middle|\; \begin{array}{c} y \text{ is budget feasible} \\ q(\xi)(\theta(\xi) - \varphi(\xi)) \geq -M, \ \forall \xi \in D \end{array} \right\},$$

where M is given exogenously.

(2) Choice set with *credit constraints*:

$$\mathcal{C}^{\kappa}(p, q, w^h, \kappa) = \left\{ y \in \mathbb{E} \;\middle|\; \begin{array}{c} y \text{ is budget feasible} \\ q(\xi)\varphi(\xi) \leq \kappa p(\xi) W(\xi), \ \forall \xi \in D \end{array} \right\},$$

where $\kappa > 0$ is given exogenously.

Choice set (1) restricts agents' net indebtedness to be no greater than M, and it appears repeatedly in macroeconomics (Kehoe 1989). Moreover, its relation with the budget set induced by transversality conditions has been widely studied (see below).

Choice set (2) is similar to the one introduced by Moreno-García and Torres-Martínez (2012) to assert equilibrium existence in an infinite horizon economy with long-lived securities in positive net supply (equity contracts). Process κ may be understood as constraining liquidity in the economy, as borrowing at node ξ is limited to a proportion κ of the market value of aggregate wealth. Choice sets (1) and (2) do not generally coincide. Nevertheless, depending on the particularization of M, and processes (κ, W, p) it may be the case that $\mathcal{C}^{\kappa}(p, q, w^h, \kappa) \subset \mathcal{C}^{\mathrm{M}}(p, q, w^h, M)$.[3]

We are now ready to introduce the different notions of a competitive equilibrium in function of the underlying budget sets.

Definition 1. A *competitive equilibrium* with credit constraints (resp., with debt constraints) for economy $\mathcal{E}$ is composed by a pair of price processes $(p, q) \in \mathbb{P}$ and a set of allocations $(y^h)_{h \in H} \in \mathbb{E}^H$ such that

(1) For every $h \in H$, y^h is maximal regarding U^h in $\mathcal{C}^{\kappa}(p, q, w^h, \kappa)$ (resp., in $\mathcal{C}^{\mathrm{M}}(p, q, w^h, M)$).

(2) Commodity and financial markets clear, i.e.,

$$\sum_{h \in H} x^h(\xi) = W(\xi), \quad \sum_{h \in H} \theta^h(\xi) = \sum_{h \in H} \varphi^h(\xi), \ \forall \xi \in D.$$

[3] This holds if $\kappa p(\xi) W(\xi) \leq M$ for all $\xi \in D$.

Magill and Quinzii (1994) established conditions under which the competitive equilibria of an economy with implicit debt constraints coincide with the competitive equilibria when using a *transversality condition.*[4] Imposing an implicit debt constraint is equivalent to restricting the portfolio value process to comply with the following:

$$q(\theta-\varphi)=(q(\xi)[\theta(\xi)-\varphi(\xi)])_{\xi\in D}\in \ell_\infty(D),$$

where $\ell_\infty(D)\subset\mathbb{R}^D$ is the subspace of all bounded sequences. As a direct corollary of this result, it follows that there exists a value for M for which the explicit debt constraint is *never binding* in equilibrium.

4. Equilibrium results

Following Magill and Quinzii (1994), we let $\ell_\infty(D\times\mathcal{L})$ equal the subset of $\mathbb{R}^{D\times\mathcal{L}}$ consisting of all bounded sequences, and $\ell_\infty^+(D\times\mathcal{L})\subset\ell_\infty(D\times\mathcal{L})$ its non-negative orthant. We will impose the following assumptions over the fundamentals of economy $\mathcal{E}$.

Assumption A1 (Utility)**.** *$U^h(x)=\sum_{\xi\in D}u^h(\xi,x(\xi))$ for each $h\in H$, where $u^h(\xi,.):\mathbb{R}_+^{\mathcal{L}}\to\mathbb{R}_+$ is continuous, concave, strictly increasing and unbounded. Moreover, it holds that $U^h(0)=0$.*

Assumption A2 (Endowments)**.** *There exist $\omega,\bar\omega\in\mathbb{R}_{++}^{\mathcal{L}}$ such that $w^h(\xi)\geq\omega$ and $W(\xi)\leq\bar\omega$ for all (ξ,h) in $D\times H$. Furthermore, it holds that $U^h(W)<+\infty$ for every $h\in H$.*

Assumption A3 (Lifespan)**.** *Assets in J_0 are finitely-lived.*

Assumption A4 (Deliveries)**.** *The asset payment process A belongs to $\hat{\mathcal{A}}\subset\mathcal{A}$, where $A\in\hat{\mathcal{A}}$ implies that $A_j\in\ell_\infty^+(D(\xi_j)\times\mathcal{L})$ for every $j\in J$.*

Node-by-node separable utility functions are a standard assumption in the literature, necessary to obtain equilibrium existence in the infinite horizon case through asymptotic techniques relying on the existence of competitive equilibria for the finite horizon case. Unboundedness of $(u^h(\xi,.))_{(\xi,h)\in D\times H}$ implies that agents are impatient, in the sense that at any node they may compensate reductions in future consumption with large enough bundles of present consumption . It is necessary to induce both upper and lower bounds on asset prices. This property is not a requirement in Magill and Quinzii (1994, 1996), who accomplish the latter by imposing uniform impatience on preference relations. Importantly, preferences complying with uniform impatience do not necessarily have an unbounded utility representation; analogously, not all utility functions considered in A1 are consistent with uniform impatience.[5]

[4] When financial markets are incomplete, the absence of an objective (market based) vector of present value prices implies that the formulation of a transversality condition is not straightforward. See Hernández and Santos (1996) for a discussion about the relevance of choosing between different sets of deflators.

[5] Indeed, preferences complying with uniform impatience may be represented by utility functions of the sort

$$U^h(x)=\sum_{\xi\in D}\rho(\xi)\delta^{t(\xi)}u^h(x(\xi)),$$

Assumption A2 imposes that endowments are bounded away from zero, and that there is an upper bound on aggregate wealth throughout D. In turn, Assumption A4 restricts the deliveries of financial securities to be bounded; both are present in Magill and Quinzii (1996). Assumption A3 is particular to our model.

When financial securities are real, and long-lived, the potential discontinuities caused by price-dependent payments matrices imply that endogenous bounds on portfolio holdings become indeterminate. Thus, for finite horizon economies, a competitive equilibrium may fail to exist. If this is the case, equilibrium for the infinite horizon economy can no longer be approximated as the accumulation point of equilibria in finite horizon economies. This problem persists when the budget set is defined in function of an implicit debt constraint or a tranversality condition. This is why the literature relying on any of these mechanisms to rule out Ponzi schemes has results that are only valid for subsets of $\hat{\mathcal{A}}$.[6]

Theorem (Theorem 5.5., Magill and Quinzii 1996)**.** *Assume the following hold.*

(1) Agent's consumption space is restricted to $\ell_\infty^+(D \times \mathcal{L})$. Moreover, utility functions $(U^h)_{h\in H}$ are consistent with uniform impatience.

(2) At every ξ there exists a short-lived security $j' \in J(\xi)$ paying one unit of a numeraire security $l' \in \mathcal{L}$ at each successor.

Then, under Assumptions A1-A4, there exists a dense subset $A^* \subset \hat{\mathcal{A}}$ such that, if $A \in A^*$, there exists M for which $\mathcal{E}$ has a competitive equilibrium with debt constraints.

As shown by Magill and Quinzii, A^* is dense, but not generic. Therefore, no assertions can be made regarding the measure of economies for which a competitive equilibrium with debt constraint actually exists. In contrast, the presence of credit constraints allows to assert equilibrium existence *regardless* of the election of $A \in \hat{\mathcal{A}}$.

Theorem 1. *Assume the following hold.*

(1) At every ξ there exists a risk-free security $k \in J_+(\xi)$ paying one unit of each $l \in \mathcal{L}$ at every $\mu \in \xi^+$.

Then, under Assumptions A1-A4, economy $\mathcal{E}$ has a competitive equilibrium with credit constraints.

The presence of borrowing constraints induces endogenous Radner bounds on short-sale portfolios regardless of the particularization of assets' payment streams, and thus, solves the indeterminacy problem that appears when choice sets are defined in terms of debt constraints or transversality conditions. Nevertheless, and as noted by Hernández and Santos (1996) and Moreno-García and Torres-Martínez (2012), the presence of assets in zero net supply is problematic as non-arbitrage

where $\rho(\xi)$ is a probability induced by a probability measure over S, $\delta \in (0,1)$ is a discount factor and $u^h : \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ is continuous, increasing, and concave function with $u^h(0) = 0$. Note that this does not allows for hyperbolic discounting, which Assumption A1 does allow (Moreno-García and Torres-Martínez 2012).

[6] See Magill and Quinzii (1996) for a detailed explanation.

conditions may be incompatible with finite prices in equilibrium (see Hernández and Santos 1996, example 3.9, p.118). Thus, we require an additional mechanism to ensure that prices of assets in J_0 have endogenous upper bounds. To achieve this, we leverage from Cea-Echenique and Torres-Martínez (2016), who introduced a *super-replication* property and used it to assert equilibrium existence in the context of a two-period economy with financial segmentation. Super-replication of an asset requires the existence of a portfolio of financial securities whose returns are greater than those of the former, at every successor. Intuitively, the equilibrium price of an asset should be smaller than the price of a portfolio that yields greater returns at every state of nature. Therefore, if the price of the super-replicating portfolio is correctly defined, this property allows to assert that the prices of the super-replicated securities are bounded in equilibrium.

When securities are short-lived, their returns are captured solely by the market value of their deliveries. In contrast, when considering a longer lifespan, returns may involve potential re-selling the security, and thus, super-replication of long-lived assets requires accounting for future prices. Thus, we obtain upper bounds at node ξ which are determined as a function of potential prices at nodes in ξ^+. The recursive nature of the endogenous bounds determined through super-replication implies that bounds on asset prices are dependent on each asset's terminal payments. Hence, we show that when securities in J_0 are finitely-lived, the presence of a risk-free security in J_+ ensures that a super-replicating portfolio always exists, and thus, we determine endogenous upper bounds of assets in J_0. In contrast, we are unable of incorporating infinitely lived debt contracts, as the absence of terminal payments implies that the aforementioned bounds are undetermined.

Remark 1. Magill and Quinzii (1994, 1996) need to restricts agents' consumption space to $\ell_\infty^+(D\times\mathcal{L})$ to prove equilibrium results using Separating Hyperplane results. This restriction is unnatural as it does not originate from any budgetary or individual rationality considerations, and is not required in our equilibrium results.

Remark 2. The proof of Theorem 1 actually requires substantially weaker requirements in both endowments and the deliveries of securities. Indeed, and as shown in the following section, Assumption A2 may be replaced by the interiority of endowments, whereas Assumption A3 may be totally dismissed.

5. Credit segmentation

Unequal access to financial markets may emerge as a consequence of informational frictions, market imperfections, or due to institutional considerations. Nonetheless, a broad range of restrictions are observed in financial markets, such as income-based access to funding, differential investment opportunities, and collateral requirements. Moreover, financial segmentation may be relevant to understand the prevalence of a wide range of phenomena in financial markets, such as negative equity loans (Iraola and Torres-Martínez 2014), asset pricing puzzles (Guvenen 2009; Gromb and Vayanos 2017) and may even play an important role in determining the impact of macroprudential policies (Vayanos and Vila 2009; Chen, Cúrdia, and Ferrero 2012; He and Krishnamurthy 2013).

There is a growing body of theoretical literature studying financial segmentation in two-period/finite horizon economies with incomplete markets. Indeed, the presence of trading constraints curtailing agents' participation in financial markets poses several methodological challenges, which in turn require additional techniques to determine endogenous bounds on asset prices. Cea-Echenique and Torres-Martínez (2016) accomplish the latter through super-replicatio of segmented securities' deliveries, whereas Seghir and Torres-Martínez (2011) and Faias and Torres-Martínez (2017) do so by imposing impatience properties and super-modularity on utility functionals, respectively.

In the infinite horizon setting, Iraola, Sepúlveda, and Torres-Martínez (2018) modeled general secured debt markets where agents where subject to exogenous trading constraints. By incorporating general formulations of collateral requirements, coupons and prepayment costs, they obtain equilibrium results in which the default decision may be heterogeneous among agents, as well as underwater borrowers.

We further extend the literature of financial segmentation by introducing endogenous credit segmentation to a sequential economy with non-secured debt contracts. We follow Cea-Echenique and Torres-Martínez (2016) and introduce financial segmentation by assuming that agents are subject to trading constraints $(\Phi^h)_{h\in H}$ limiting their access to financial markets. More precisely, each h faces trading constraints of the form $\Phi^h = \prod_{\xi\in D}\Phi^h_\xi$, where $\Phi^h_\xi : \mathbb{P} \twoheadrightarrow \mathbb{E}_\xi$ is a set-valued function determining the set of available allocations at node ξ. In consequence, we update the definition of agents' choice set to:

$$\widehat{\mathcal{C}}^\kappa(p,q,w^h,\kappa) = \left\{ y \in \Phi^h(p,q) \;\middle|\; \begin{array}{c} y \text{ is budget feasible} \\ q(\xi)\varphi(\xi) \leq \kappa p(\xi)W(\xi),\ \forall \xi \in D \end{array} \right\}.$$

We impose the following assumptions over the fundamentals of economy $\mathcal{E}$.

Assumption B1 (Utility). *$U^h(x) = \sum_{\xi\in D} u^h(\xi, x(\xi))$ for each $h \in H$, where $u^h(\xi,.) : \mathbb{R}^{\mathcal{L}}_+ \to \mathbb{R}_+$ is continuous, concave, strictly increasing and unbounded. Moreover, it holds that $U^h(0) = 0$.*

Assumption B2 (Endowments). *For every $(\xi,h) \in D\times H, w^h(\xi) \gg 0$. Also, $U^h(W) < +\infty$.*

Assumption B3 (Assets' lifespan). *Assets in J_0 are finitely-lived.*

Assumption B4 (Trading constraints). *For every pair $(h,\xi) \in H \times D$, Φ^h_ξ complies with the following properties:*

a. Φ^h_ξ is lower hemicontinuous, has a closed graph, convex values and contains 0.

b. Agents may always sell their financial endowments $(e^h(\xi))_{\xi\in D}$. Also,

$$\Phi^h_\xi(p,q) + \left(\mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{J(\xi)}_+ \times \{0\}\right) \subseteq \Phi^h_\xi(p,q).$$

c. If $(x(\xi),\theta(\xi),\varphi(\xi)) \in \Phi^h_\xi(p,q)$, then $(x(\xi),\theta(\xi),\varphi(\xi)) - (0,\bar\theta(\xi),0) \in \Phi^h_\xi(p,q)$, for any $\bar\theta(\xi)$ such that $\bar\theta_j(\xi) \in [0,\theta_j(\xi)]$ if $j \in J_0(\xi)$, and zero otherwise.

Assumption B1 and B3 are identical to Assumptions A1 and A3 respectively, and we repeat them for the sake of clarity. Assumption A4 is not required, whereas Assumption B2 is weaker than its counterpart in Section 4, as it imposes only the interiority of endowments. Thus, we depart from Magill and Quinzii (1996) as we may consider economies with unrestricted economic growth.

Assumption B3 characterized the scope of the financial segmentation considered in our results. Closed graph of trading constraints implies that for every convergent sequence of prices and feasible allocations there is a feasible limit allocation. Convexity of Φ^h_ξ ensures that linear combinations of feasible allocations are feasible as well. Introducing financial market segmentation without incorporating any *financial survival* assumptions may lead to empty interiors of choice sets if there is no access to credit and no physical wealth or agents are prevented from consuming their endowments due to binding portfolio constraints (Seghir and Torres-Martínez 2011). An empty interior, in turn, menaces the choice set's continuity, a requisite for determining equilibrium existence. Assumption B3.a, plus the interiority of commodity endowments solve this issue. Assumption B3.b simply states agents are not obliged to keep any financial endowments they receive, and that both commodity consumption and investment may always increase independently of the allocation.

In order to induce endogenous upper bonds for zero net supply securities through super-replication, we build a non-arbitrage argument that relies on the capacity of agents of reducing their investments in an asset in $J_0(\xi)$, and use those resources to purchase its respective super-replicating portfolio. Importantly, this arguments is not valid if agents are using investments in zero net supply securities to obtain access to credit, as reducing their positions in those assets could compromise the feasibility of their chosen allocations. Thus, and in a manner akin to Cea-Echenique and Torres-Martínez (2016), we need to rule out the possibility that zero net supply assets may be used as financial collateral. Thus, Assumption B3.c simply states that long positions in assets in $J_0(\xi)$ may be always reduced without compromising the feasibility of the allocation. We are now ready to state the main result of this section.

Theorem 2. *Assume that at every ξ there exists a risk-free security $k \in J_+(\xi)$ paying one unit of each $l \in \mathcal{L}$ at every $\mu \in \xi^+$. Then, under Assumptions B1-B4, economy $\mathcal{E}$ has a competitive equilibrium with borrowing constraints.*

The following examples illustrate the nature of the trading constraints considered in our setup.

Example 1. The following constraints are examples of restrictions that are included in our framework:

$$(x,\theta,\varphi) \in \Phi_h(p,q) \Rightarrow \begin{cases} \varphi_k(\xi) = 0, \text{ for any } k \in \bar{J}(\xi). \\ q(\xi)\varphi_j(\xi) \geq \alpha_\xi q(\mu), \text{ for some } (\mu, j) \in D \times J(\xi) \setminus \bar{J}(\xi). \\ \varphi_s(\xi) \leq \max\{\theta_1(\xi), \theta_2(\xi)\}, \text{ for } s \in J(\xi) \setminus \bar{J}(\xi). \end{cases}$$

Note that we do not require to impose any *financial survival* assumptions; that is, agents may not have access to credit throughout event-tree D (Aouani and Cornet 2009). Moreover, access to credit may depend on past or future prices, as illustrated by the second restriction. Finally, the third constraint represents a typical case of financial collateral.

Example 2. The following constraints are examples of restrictions that are ruled out in our framework:

$$(x,\theta,\varphi)\in\Phi_h(p,q)\Rightarrow\begin{cases} p(\xi)x(\xi)+q(\xi)\theta(\xi)\leq A, \text{ for } A>0.\\ \theta_j(\xi)\in[1,3], \text{ for some } j\in J(\xi).\\ \varphi(\xi)\geq\alpha\varphi(\xi^-), \text{ for } \alpha>0.\end{cases}$$

The first restriction violates Assumptions B3.c, as it curtails hypothetical agents from freely consuming commodities and investing in financial securities. The second constraint obliges agents to maintain a positive and bounded investment in security j; this scenario is explicitly ruled out by Assumption AB.c as well. Lastly, the third restriction forces long positions in financial markets to be dependent on those at the previous node, which contravenes the fact that Φ^h_ξ is solely dependent on prices.

6. Comments on infinitely lived debt contracts

As already discussed, the techniques used to induce endogenous upper bounds on prices of securities in J_0 in the proofs of Theorems 1-2 are inapplicable if these securities are infinitely lived. Indeed, super-replication of returns implies accounting for future prices, which in turn may depend on future prices as well, a terminal condition for this recursion may be only determined if securities live a finite amount of periods. In this section, we provide an additional result for equilibrium existence in finite horizon economies, which gives a role to the abundance of liquidity, captured by the value of κ. More precisely, we show that for a given time horizon there exists a level of liquidity that guarantees equilibrium existence in the finite horizon economy. Indeed, and for any $T\in\mathbb{N}$, let $\mathcal{E}^T$ be the finite-horizon version of economy $\mathcal{E}$ up to time T, without financial segmentation.[7] We are able to prove the following equilibrium result.

Proposition 1. *Under Assumptions A1, A2 and A4, there exists κ^T such that economy $\mathcal{E}^T$ has a competitive equilibrium whenever $\kappa>\kappa^T$.*

Abundance of liquidity allows to determine endogenous upper bounds on security prices. Indeed, Assumption A1 implies the existence of bundles $(a(\xi))_{\xi\in D}\in\mathbb{R}^{\mathcal{L}\times D}_{++}$ providing greater instant utility at every node than the one feasible through aggregate wealth across the entire event-tree D. Moreover, the liquidity level induced by κ^T is sufficiently large as to allow agents to borrow enough resources to purchase bundle $a(\xi)$ at every ξ in $\mathcal{E}^T$. Thus, Assumptions A1-A4 induce endogenous upper bounds on asset prices at every ξ, as no agent should be able to purchase $a(\xi)$ by short-selling a positive amount of any security $j\in J(\xi)$ and honor her commitments with her future endowment streams. Importantly, the bounds induced by A1 do not rely on hypothetical returns, and thus, are independent of securities' lifespan. Hence, we would be able to incorporate infinitely lived securities provided that there existed a value of κ for which a competitive equilibrium existed for all finite horizon economies.

[7]For a more thorough definition of finite horizon economies, see the Appendix.

Proposition 2. *If there exists $\hat{\kappa}$ for which the sequence of economies $\{\mathcal{E}^T\}_{T\in\mathbb{N}}$ all have a competitive equilibrium whenever $\kappa > \hat{\kappa}$, then $\mathcal{E}$ has a competitive equilibrium whenever $\kappa > \hat{\kappa}$.*

We are unable to assert that the sequence formed by the liquidity thresholds, $(\kappa^T)_{T\in\mathbb{N}}$, is bounded, and hence, we are unable to assert equilibrium existence for $\mathcal{E}$. Nevertheless, if such a limiting value for κ did exist, we would retrieve equilibrium existence for the original economy.

7. Conclusion

This paper extends the literature of general equilibrium in incomplete market sequential economies by presenting conditions under which a competitive equilibrium always exists in an economy with long-lived debt contracts. More precisely, we show that replacing debt constraints by credit constraints allows to find endogenous Radner bounds on portfolios independently of the particularization of assets' payment process, a necessary step in the determination of equilibrium existence.

The previous literature on non-secured financial markets had relied on strong assumptions on preference relations (Magill and Quinzii 1994, 1996) or in the positive net supply of securities (Moreno-García and Torres-Martínez 2012) to determine endogenous upper bounds on asset prices. We are able to incorporate long, but finitely lived, debt contracts by adapting a technique developed by Cea-Echenique and Torres-Martínez (2016), which relies on the presence of a risk-free security with positive deliveries at every node. Importantly, the bounds determined in our proof are dependent on securities' terminal payments, and hence, we are not able to incorporate infinitely lived debt contracts.

We additionally provide conditions under which our results are compatible with broad forms of endogenous financial segmentation, and thus, complement Iraola, Sepúlveda, and Torres-Martínez (2018) by expanding the literature on financial segmentation to infinite horizon unsecured debt contracts.

8. Appendix

Proof of Theorem 1

Truncated finite horizon economies. Let $\mathcal{E}_n^T$ be the finite horizon version of economy $\mathcal{E}$ up to time $T \in \mathbb{N}$ where, additionally, the net supply of financial securities has been increased by an amount proportional to $\frac{1}{n} > 0$, for $n \in \mathbb{N}$ given. In particular, $\mathcal{E}_n^T$ starts at node ξ_0 and is circumscribed to event-tree $D^T(\xi_0)$. At every node $\xi \in D^{T-1}(\xi_0)$ there is a set $J^T(\xi) = \{j \in J(\xi) | \exists \mu > \xi : t(\mu) < T, A_j(\mu) \neq 0\}$ of assets available for trade; $J^T(\xi) = \emptyset$ for all $\xi \in D_T(\xi)$, by assumption. Importantly, given $\xi \in D^{T-1}(\xi_0)$, $J^T(\xi) = J(\xi)$ for T large enough. Let $D^T(J) = \{(\xi, j) \in D^T(\xi_0) \times J^T(\xi) : j \in J^T(\xi)\}$.

We consider prices (p, q) belonging to space

$$\mathbb{P}^T = \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^{\mathcal{L}} \times \mathbb{R}_+^{J^T(\xi)} \right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^{\mathcal{L}},$$

where $\Delta_+^{\mathcal{L}} := \{p \in \mathbb{R}_+^{\mathcal{L}} : \|p\|_\Sigma = 1\}$.

Agents' problem is reformulated to fit event-tree $D^T(\xi_0)$. In particular, agent $h \in H$ is characterized by a modified utility functional over consumption streams, $U^{h,T} = \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$, and commodity and financial endowments $(w^h(\xi), e^h(\xi))_{\xi \in D^T(\xi_0)} \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{D^T(J)}$. More precisely, we assume that each h receives a financial endowment of $\frac{1}{n} > 0$ at node ξ_k, for every asset $k \in J_0$ available in economy $\mathcal{E}_n^T$:

$$e_k^h(\xi) = \begin{cases} \frac{1}{n} & \xi = \xi_k. \\ 0 & \xi \neq \xi_k. \end{cases}$$

The deliveries of assets in J_0^T, scaled by $\frac{1}{n}$, should now be considered as part of each agent's endowments, and thus, accounted for in aggregate wealth at ξ:

$$W_n(\xi) = W(\xi) + \sum_{h \in H} \left(\frac{1}{n} \sum_{k \in J^T(\xi^-)} A_k(\xi) \right).$$

Let $W_n^T = (W_n(\xi))_{\xi \in D^T(\xi_0)}$ stand for economy $\mathcal{E}_n^T$'s aggregate wealth throughout event-tree $D^T(\xi_0)$.

Each agent must choose an allocation $(y^h(\xi))_{\xi \in D^T(\xi_0)} = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in D^T(\xi_0)}$ belonging to space $\mathbb{E}^T := \mathbb{R}_+^{D^T(\xi_0) \times \mathcal{L}} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)}$. We denote $y^h = (y^h(\xi))_{\xi \in D^T(\xi_0)}$. For prices $(p, q) \in \mathbb{P}^T$, agent h's truncated choice set correspondence $C^{h,T}(p, q)$ considers allocations $y^h \in \mathbb{E}^T$ complying with constraints:

$$g^{h,T}(\xi, y^h(\xi), y^h(\xi^-); p, q) \leq 0, \quad \forall \xi \in D^T(\xi_0),$$
$$q(\xi)\varphi^h(\xi) \leq \kappa(\xi) p(\xi) W(\xi),$$

where $(\theta^h(\xi_0^-), \varphi^h(\xi_0^-)) = (0, 0)$ and for every $\xi \in D^T(\xi_0)$ the function $g^{h,T}(\xi, .)$ is given by:

$$\begin{aligned} g^{h,T}(\xi, y(\xi), y(\xi^-); p, q) := & \; p(\xi) \left(x^h(\xi) - w^h(\xi) \right) + q(\xi)(\theta^h(\xi) - \varphi^h(\xi) - e^h(\xi)) \\ & - \sum_{j \in J^T(\xi^-)} (p(\xi) A_j(\xi) + q_j(\xi))(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)). \end{aligned}$$

Definition 2. A *competitive equilibrium* for economy $\mathcal{E}_n^T$ is composed by a price process $(p,q) \in \mathbb{P}^T$ and allocations $(y^h)_{h\in H} \in (\mathbb{E}^T)^H$ such that

(1) For every $h \in H$, $y^h \in \operatorname{argmax}_{y\in C^{h,T}(p,q)} U^{h,T}(x)$.

(2) Physical and financial markets clear, i.e.,

$$\sum_{h\in H} x^h(\xi) = W_n(\xi) \ \ \forall \xi \in D^T(\xi_0), \ \text{ and } \ \sum_{h\in H} \theta^h(\xi) = \sum_{h\in H} (\varphi^h(\xi) + \bar{e}^h(\xi)) \, \forall \xi \in D^{T-1}(\xi_0).$$

Finite horizon economy $\mathcal{E}^T$ is defined equivalently to $\mathcal{E}_n^T$, with the exception that there are no perturbations in the net supply of financial securities: $e^h(\xi) = 0$, for all $(h,\xi) \in H \times D^{T-1}(\xi_0)$. Analogously, infinite horizon economy $\mathcal{E}_n$ is equivalent to $\mathcal{E}$ with the modified net supply of financial assets.

Equilibrium in truncated economies. Assumptions A1, A4 an A5 ensure that, for every agent $h \in H$, $U^h(W_n^T) < +\infty$. Thus, and as all assets are in positive net supply, equilibrium existence for economy $\mathcal{E}_n^T$ follows directly from Lemma 2 of Moreno-García and Torres-Martínez (2012). Now, consider any competitive equilibrium $\left[(\bar{p}_n, \bar{q}_n); (\bar{y}_n^h)_{h\in H}\right]$ of $\mathcal{E}_n^T$. Because equilibrium allocations are bounded by aggregate wealth, Moreno García and Torres-Martínez (2012) showed that asset prices at any $\xi \in D^{T-1}(\xi_0)$ comply with:

$$\bar{q}_{n,j}(\xi) \le \frac{\#H\|a_n(\xi)\|}{\sum_{h\in H} \bar{e}_j(\xi)}, \text{ for } j \in J_+^T(\xi), \text{ and } \bar{q}_{n,k}(\xi) \le n\|a_n(\xi)\| \text{ for } k \in J_0^T(\xi).$$

Bundles $(a_n(\xi))_{\xi\in D^{T-1}(\xi_0)} \in \mathbb{R}_+^{\mathcal{L}\times D^{T-1}(\xi_0)}$ retrieve greater utility at every ξ that the one attainable trough aggregate wealth in the infinite horizon economy $\mathcal{E}_n$:

$$u^h(\xi, a_n(\xi)) > U^h(W_n), \ \forall h \in H.$$

Analogously, and for any $T \in \mathbb{N}$, Assumptions A1 and A2 ensure we may define bundles $(a(\xi))_{\xi\in D^{T-1}(\xi_0)} \in \mathbb{R}_+^{\mathcal{L}\times D^{T-1}(\xi_0)}$ such that:

$$u^h(\xi, a(\xi)) > U^h(W), \ \forall h \in H,$$

where W stands for aggregate wealth at original economy $\mathcal{E}$. Furthermore, Assumption A4 plus the continuity of utility functions allows us to assert that there exists $\overline{N} \in \mathbb{N}$ such that:

$$u^h(\xi, a(\xi)) > U^h(W_n), \ \forall h \in H,$$

for any $n \ge \overline{N}$; we assume this holds from now on.

Due to Moreno-García and Torres-Martínez (2012) we know that at any $\xi \in D^{T-1}(\xi_0)$ the price of the risk-free security $k \in J_+(\xi)$ is bounded by

$$\bar{q}_k(\xi) \le R_k(\xi) := \frac{\#H\|a(\xi)\|}{\sum_{h\in H} \bar{e}_k(\xi)}.$$

Importantly, this bound does not depend of the time horizon $T \in \mathbb{N}$ nor the truncation value of the net supply of securities in J_0, given that $n \ge \overline{N}$.

Now, choose any $(\xi, j) \in D^{T-1}(\xi_0) \times J_0^T(\xi)$ and define the following recursion:

$$M_j(\xi) = \begin{cases} R_k(\xi) \max_{\mu \in \xi^+}\{\|A_j(\mu)\| + M_j(\mu)\} & \xi \notin \bar{\xi}_j, \\ R_k(\xi) \max_{\mu \in \xi^+}\{\|A_j(\mu)\|\} & \xi \in \bar{\xi}_j, \end{cases}$$

where $\bar{\xi}_j$ is the set of *terminal nodes* for j: $\xi \in \bar{\xi}_j \Leftrightarrow (j \in J(\xi)) \wedge (j \notin J(\mu),\ \forall \mu > \xi^+)$. Let $M_j = (M_j(\xi))_{\xi \geq \xi_j}$, and note that $M_j(\mu) = 0$ for any $\mu > \xi$ with $\xi \in \bar{\xi}_j$. Note that M_j does not depend on $n > \overline{N}$, and because assets in J_0 are finitely lived, there exists $T_j \in \mathbb{N}$ such that M_j does not depend on $T > T_j$ as well.

Lemma 8.1. *At every $\xi \in D^{T-1}(\xi_0)$ there exists $M(\xi)$, independent of $n \in \mathbb{N}$, such that $\bar{q}_{n,j}(\xi) \leq M(\xi)$, for all $j \in J_0^T(\xi)$.*

Proof. Fix any $j \in J_0^T(\xi)$ and assume that $\bar{q}_{n,j}(\xi) > M_j(\xi)$. Recall that, as asset j is in positive net supply, there is at least one agent $h \in H$ who purchased $\varepsilon > 0$ units of j at ξ. If $\xi \notin \bar{\xi}_j$, agent h could follow the following strategy:

- Decrease her position in j by ε units, and use those resources to purchase $\varepsilon\alpha(\xi)$ units of security k, where:

$$\alpha(\xi) = \begin{cases} \max_{\mu \in \xi^+}\left(\|A_j(\mu)\| + M_j(\mu)\right) & \xi \notin \bar{\xi}_j, \\ \max_{\mu \in \xi^+}\|A_j(\mu)\| & \xi \in \bar{\xi}_j, \end{cases}.$$

- Increase her consumption at ξ by a total value of $\varepsilon\left(\bar{q}_{n,j}(\xi) - \bar{q}_{n,k}(\xi)\alpha(\xi)\right) > 0$.

By pursuing this strategy, agent h would be increasing her consumption at ξ and receiving enough resources at any $\mu \in \xi^+$ as to maintain the rest of her consumption stream unaltered (this may involve repurchasing j at some μ). This alternative strategy provides greater utility than the one pursued by h in equilibrium, and thus, it contradicts the optimality of h's behavior. Hence, we learn that $\bar{q}_{n,j}(\xi) \leq M_j(\xi)$. Defining $M(\xi) = \max_{j \in J_0^T(\xi)}\{M_j(\xi)\}$ ends the proof. $\square$

As the election of $\xi \in D^{T-1}(\xi_0)$ was arbitrary, we learn that prices are bounded by $M := (M(\xi))_{\xi \in D^{T-1}(\xi_0)}$ in any competitive equilibrium of an economy $\mathcal{E}_n^T$. Importantly, bounds M are independent from the perturbation on the net supply of financial securities, provided $n > \overline{N}$. Moreover, bound $M_j(\xi)$ are independent of the truncation horizon, provided that $J^T(\xi) = J(\xi)$ (which holds for T large enough).

Consider the sequence consisting of equilibrium prices and allocations given by

$$\left[(\bar{p}_n, \bar{q}_n); (\bar{y}_n^h)_{h \in H}\right]_{n \geq \overline{N}}.$$

The results exposed by Moreno-García and Torres-Martínez (2012, Lemma 2) ensure that the latter is node-by-node bounded. Hence, we may use Tychonoff's theorem to ensure it has a convergent subsequence $(n_k)_{k \in \mathbb{N}}$ such that:

$$\lim_{n_k \to +\infty}\left[(\bar{p}_{n_k}, \bar{q}_{n_k}); (\bar{y}_{n_k}^h)_{h \in H}\right] = \left[(\bar{p}, \bar{q}); (\bar{y}^h)_{h \in H}\right].$$

We treat $\left[(\bar{p},\bar{q});(\bar{y}^h)_{h\in H}\right]$ as our candidate for a competitive equilibrium for economy $\mathcal{E}^T$. As equilibrium allocations comply with market clearing and belong to agent's budget sets for every $n\in\mathbb{N}$, the same holds for the limit allocations. Thus, we must solely show that the limit allocation is optimal for agents regarding the limit's prices.

By contradiction, assume that for some agent h there exists $\tilde{y}=(\tilde{x},\tilde{\theta},\tilde{\varphi})\in C^{h,T}(\bar{p},\bar{q})$ such that $U^{h,T}(\tilde{x})>U^{h,T}(\bar{x})$. The continuity of $C^{h,T}(.)$ allows to assert that there exists $\{\tilde{y}_{n_k}\}_{k\in\mathbb{N}}$ complying with $\lim_{k\to+\infty}\tilde{y}_{n_k}=\tilde{y}$ and $\tilde{y}_{n_k}\in C^{h,T}(\bar{p}_{n_k},\bar{q}_{n_k})$ for all n_k. The continuity of $U^{h,T}$ then implies that there exists n_k^* large enough such that $U^{h,T}(\tilde{x}_{n_k})>U^{h,T}(\bar{x}_{n_k})$ for all $n_k\geq n_k^*$. This leads to a contradiction, as we already know $\bar{x}_{n_k}$ is optimal regarding h's budget set. Thus, we have proved the existence of a competitive equilibrium for economy $\mathcal{E}^T$.

As the election of $T\in\mathbb{N}$ was arbitrary, we know a competitive equilibrium exists for any finite horizon truncated economy. Equilibrium existence for the infinite horizon case then follows directly from Moreno-García and Torres-Martínez (2012, *Asymptotic equilibria*, p.141).

Proof of Theorem 2

Consider a truncated, finite horizon economy $\mathcal{E}^T$ where additionally every agent in H is subject to trading constraints $\Phi^{h,T}(p,q):\mathbb{P}^T\twoheadrightarrow\mathbb{E}^T$, where $\Phi^{h,T}$ is defined as the *projection* of Φ^h onto subtree $D^T(\xi_0)$. That is, for prices $(p,q)\in\mathbb{P}^T$, $\Phi^{h,T}$ is defined as all allocations $y\in\mathbb{E}^T$ for which we may find a pair $(\tilde{y},(\tilde{p},\tilde{q}))\in\mathbb{E}$ such that:

$$\Big(\tilde{y}\in\Phi^h(\tilde{p},\tilde{q})\Big)\wedge\Big(\big(y(\xi),(p(\xi),q(\xi))\big)_{\xi\in D^{T-1}(\xi_0)}=\big(\tilde{y}(\xi),(\tilde{p}(\xi),\tilde{q}(\xi))\big)_{\xi\in D^{T-1}(\xi_0)}\Big).$$

Thus, for prices $(p,q)\in\mathbb{P}^T$, agent h's truncated choice set correspondence $\hat{C}^{h,T}(p,q)$ considers allocations $y^h\in\mathbb{E}^T$ complying with constraints:

$$g^{h,T}(\xi,y^h(\xi),y^h(\xi^-);p,q)\leq 0,\quad\forall\xi\in D^T(\xi_0)$$
$$(y^h(\xi))_{\xi\in D^T(\xi_0)}\in\Phi^{h,T}(p,q),$$

In contrast with Theorem 1, it is not obvious that the results exposed in Moreno-García and Torres-Martínez (2012) suffice to ensure that our finite horizon economy has a competitive equilibrium if we truncate the net supply of securities in J_0, because financial segmentation may menace the continuity of agents' choice sets. Hence, we now show that agents' choice set in a truncated generalized game setup does comply with the former property, a necessary step in order to determine equilibrium existence in $\mathcal{E}^T$.

Lemma 8.2. *Consider the following compact set $\mathcal{K}(\mathcal{X},\Theta,\Psi)$, where $\mathcal{K}(\mathcal{X},\Theta,\Psi):=[0,\mathcal{X}]\times[0,\Theta]\times[0,\Psi]$, for $(\mathcal{X},\Theta,\Psi,M)\in\mathbb{E}^T\times\mathbb{R}^{D(J)}_{++}$. Then, the correspondence $\widehat{C}^{h,T}\cap\mathcal{K}(\mathcal{X},\Theta,\Psi)$ is continuous.*

Proof. Lower hemicontinuity. We prove first the lower hemicontinuity of the trading constraints correspondence $\Phi^{h,T}$. Fix $(y^T,(p^T,q^T))\in\mathbb{E}^T\times\mathbb{P}^T$ such that $y^T\in\Phi^{h,T}(p^T,q^T)$ and let there be a sequence $\{(p^T_n,q^T_n)\}_{n\in\mathbb{N}}\subset\mathbb{P}^T$ whose limit is (p^T,q^T). The definition of $\Phi^{h,T}$ implies there exist

$(y,(p,q)) \in \mathbb{E}\times\mathbb{P}$ complying with both $y \in \Phi^h(p,q)$ and $(y^T,(p^T,q^T)) = (y(\xi),(p(\xi),q(\xi)))_{\xi\in D^{T-1}(\xi_0)}$. Consider the following sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}} \subset \mathbb{P}$:

$$(p_n(\xi),q_n(\xi)) = \begin{cases} (p_n^T(\xi),q_n^T(\xi)) & \text{if } \xi \in D^{T-1}(\xi_0) \\ (p(\xi),q(\xi)) & \text{if } \xi \notin D^{T-1}(\xi_0) \end{cases},$$

Clearly, $(p_n(\xi),q_n(\xi)) \to (p,q)$. Therefore, Assumption B3.a and the sequential characterization of lower hemicontinuity ensure there exists $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{E} : \left(y_n \in \Phi^h(p_n,q_n)\ \forall n \in \mathbb{N}\right) \wedge (y_n \to y)$. Moreover, the construction of $\{y_n\}_{n\in\mathbb{N}}$ allows us to assert that the sequence $\{y_n^T\}_{n\in\mathbb{N}} \subset \mathbb{E}^T$ defined as

$$y_n^T(\xi) = \begin{cases} y_n(\xi) & \text{for } \ \xi \in D^{T-1}(\xi_0) \\ y^T(\xi) & \text{for } \ \xi \in D^T(\xi_0) \end{cases} \quad \text{for every } n \in \mathbb{N},$$

complies with both $y_n^T \in \Phi^{h,T}(p_n^T,q_n^T)\ \forall n \in \mathbb{N}$ and $y_n^T \to y^T$, which implies that $\Phi^{h,T}$ is lower hemicontinuous.

Now, consider the correspondence defined as $\dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$, where $\dot{C}^{h,T}(p,q)$ corresponds to all allocations $y \in \widehat{C}^{h,T}(p,q)$ complying with budget constraints with strict inequalities. Assumptions B2 and B3 ensure that $\dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ has non-empty values, as commodity endowments are strictly positive, commodity spot prices always belong to the unitary simplex and there are no trading constraints curtailing h from consuming a fraction of her endowments. Moreover, given a pair $(y,(p,q)) \in \mathbb{E}^T \times \mathbb{P}^T$ such that $y \in \dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ and a sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}}$ converging to (p,q), the lower hemicontinuity of $\Phi^{h,T}$ allows us to assert that there exists $\{y_n\}_{n\in\mathbb{N}}$ complying with $y_n \in \Phi^{h,T}(p_n,q_n)\ \forall n \in \mathbb{N}$ and converging to y. Thus, and for $n \in \mathbb{N}$ sufficiently large, $y_n \in \dot{C}^{h,T}(p_n,q_n) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ as well; $\dot{C}^{h,T} \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ is lower hemicontinuous.

Since $\dot{C}^{h,T}$ has non-empty and convex values, its closure coincides with $\widehat{C}^{h,T}$, and so we learn that $\widehat{C}^{h,T} \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ is lower hemicontinuous.

Upper hemicontinuity. As $\mathcal{K}(\mathcal{X},\Theta,\Psi)$ is a closed, convex and compact set containing 0, it follows from Assumptions B2 and B3.a that $\widehat{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ has non-empty, convex and compact values and a closed graph, which are sufficient conditions for upper hemicontinuity. □

Moreover, note that Assumption B3 implies that agents may invest and consume freely, and that long positions in securities in J_0 may be always reduced without compromising the feasibility of allocations. Thus, equilibrium for any economy $\mathcal{E}_n^T$ follows directly from Moreno-García and Torres-Martínez (2012), whereas equilibrium for the limit economies $\mathcal{E}^T$ and $\mathcal{E}$ follow from Theorem 1.

Proofs of Propositions 1 and 2

Proposition 1. Assumptions A1, A4 an A5 ensure that, for every agent $h \in H$, and for any $n \in \mathbb{N}$, $U^h(W_n^T) < +\infty$. Thus, and as all assets are in positive net supply, equilibrium existence for economy $\mathcal{E}_n^T$ follows directly from Lemma 2 of Moreno-García and Torres-Martínez (2012). Now, consider any competitive equilibrium $\left[(\bar{p}_n,\bar{q}_n);(\bar{y}_n^h)_{h\in H}\right]$ of $\mathcal{E}_n^T$.

For large values of n the utility provided by equilibrium allocations is smaller than the one determined by bundles $(a(\xi))_{\xi\in D^T(\xi_0)}$ as well. Hence, and for truncated economy $\mathcal{E}_n^T$, there cannot be any feasible allocation in agents' budget set that involves purchasing bundle $a(\xi)$ at any node ξ. This induces, jointly with a sufficient level of liquidity throughout $D^T(\xi_0)$, a set of upper bounds on asset that are independent of $n \geq \overline{N}$. The following lemma formalizes this result.

Lemma 8.3. *Fix $\xi \in D^{T-1}(\xi_0)$ and assume that κ is large enough. Then, there exists $M_\delta(\xi)$, independent of n such that $\bar{q}_{n,k}(\xi) < M_\delta$, for all $k \in J^T(\xi)$.*

Proof. Fix any $k \in J^T(\xi)$, and recall that Assumption A5 implies that at node ξ there is one agent h_k that may sustain a debt consisting of δ_k units of asset k, by disposing of the market value of her future endowments.

Now, assume that $\kappa \geq \hat{\kappa}(\xi) := \frac{\|a(\xi)\|}{\underline{\delta}\underline{w}(\xi)}$, where $\underline{\delta} := \min_{j\in J^T(\xi)}\{\delta_j\}$ and $\underline{w}(\xi) := \min_{(h,l)\in H\times\mathcal{L}}\{w_l^h(\xi)\}$. The definition of $\hat{\kappa}(\xi)$ implies that all agents where able to receive enough resources through debt at ξ as to buy $a(\xi)$, and thus, agents did not pursue this strategy because they were not able to finance any debt portfolio that involved disposing of that amount of resources.

In particular, this implies that the price of asset k is bounded by:

$$\bar{q}_{n,k}(\xi) < \frac{\|a(\xi)\|}{\delta_k}.$$

Otherwise, agent h_k would short-sell δ_k units of the security, purchase and consume $a(\xi)$, and honor their debt with the deliveries of her future endowments at any $\mu > \xi$. Hence, we learn that at node ξ prices of securities in $J^T(\xi)$ are smaller than $M_\delta(\xi) := \frac{\|a(\xi)\|}{\underline{\delta}}$. ☐

As the election of $\xi \in D^{T-1}(\xi_0)$ was arbitrary, we learn that asset prices are bounded by $M_\delta := (M_\delta(\xi))_{\xi\in D^{T-1}(\xi_0)}$ in any competitive equilibrium of an economy $\mathcal{E}_n^T$ such that $\kappa \geq \hat{\kappa} := (\hat{\kappa}(\xi))_{\xi\in D^{T-1}(\xi_0)}$; we assume the latter holds from now on. Importantly, bounds M_δ are independent from the perturbation on the net supply of financial securities.

Consider the sequence consisting of equilibrium prices and allocations given by

$$\left[(\bar{p}_n, \bar{q}_n); (\bar{y}_n^h)_{h\in H}\right]_{n\geq\overline{N}}.$$

The results exposed by Moreno-García and Torres-Martínez (2012, Lemma 2) ensure that the latter is node-by-node bounded. Hence, we may use Tychonoff's theorem to ensure it has a convergent subsequence $(n_k)_{k\in\mathbb{N}}$ such that:

$$\lim_{n_k\to+\infty}\left[(\bar{p}_{n_k}, \bar{q}_{n_k}); (\bar{y}_{n_k}^h)_{h\in H}\right] = \left[(\bar{p}, \bar{q}); (\bar{y}^h)_{h\in H}\right].$$

We treat $\left[(\bar{p}, \bar{q}); (\bar{y}^h)_{h\in H}\right]$ as our candidate for a competitive equilibrium for economy $\mathcal{E}^T$. As equilibrium allocations comply with market clearing and belong to agent's budget sets for every $n \in \mathbb{N}$, the same holds for the limit allocations. Thus, we must solely show that the limit allocation is optimal for agents regarding the limit's prices.

By contradiction, assume that for some agent h there exists $\tilde{y} = (\tilde{x}, \tilde{\theta}, \tilde{\varphi}) \in C^{h,T}(\bar{p}, \bar{q})$ such that $U^{h,T}(\tilde{x}) > U^{h,T}(\bar{x})$. The continuity of $C^{h,T}(.)$ allows to assert that there exists $\{\tilde{y}_{n_k}\}_{k\in\mathbb{N}}$ complying with $\lim_{k\to+\infty} \tilde{y}_{n_k} = \tilde{y}$ and $\tilde{y}_{n_k} \in C^{h,T}(\bar{p}_{n_k}, \bar{q}_{n_k})$ for all n_k. The continuity of $U^{h,T}$ then implies that there exists n_k^* large enough such that $U^{h,T}(\tilde{x}_{n_k}) > U^{h,T}(\bar{x}_{n_k})$ for all $n_k \geq n_k^*$. This leads to a contradiction, as we already know $\bar{x}_{n_k}$ is optimal regarding h's budget set. Thus, we have proved the existence of a competitive equilibrium for economy $\mathcal{E}^T$.

As the election of $T \in \mathbb{N}$ was arbitrary, we know a competitive equilibrium exists for any finite horizon truncated economy.

Proposition 2. Equilibrium existence for the infinite horizon case follows directly from Moreno-García and Torres-Martínez (2012, *Asymptotic equilibria*, p.141).

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