

A CRITERION FOR AN ABELIAN VARIETY TO BE NON-SIMPLE

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ABSTRACT. We give a criterion in terms of period matrices for an arbitrary polarized abelian variety to be non-simple. Several examples are worked out.

1. INTRODUCTION

Let (A, \mathcal{L}) be a complex abelian variety of dimension g with polarization of type $D = \text{diag}(d_1, \dots, d_g)$ defined by an ample line bundle \mathcal{L} . So $A = V/\Lambda$ where V is a complex vector space of dimension g and Λ is a lattice of maximal rank in \mathbb{C}^g such that with respect to a basis of V and a symplectic basis of Λ , A is given by a period matrix $(D \ Z)$ with Z in the Siegel upper half space of rank g . The aim of this paper is to give a set of equations in the entries of the matrix Z which characterize the fact that (A, \mathcal{L}) is non-simple. These equations are easy to work out for $g = 2$ and can be given explicitly with the help of a computer program for $g = 3$.

To be more precise, the polarization induces a bijection

$$\varphi : \text{NS}_{\mathbb{Q}}(A) \rightarrow \text{End}_{\mathbb{Q}}^s(A)$$

of the rational Néron-Severi group $\text{NS}_{\mathbb{Q}}(A) := \text{NS}(A) \otimes \mathbb{Q} = (\text{Pic}(A)/\text{Pic}^0(A)) \otimes \mathbb{Q}$ with the \mathbb{Q} -vector space $\text{End}_{\mathbb{Q}}^s(A) := \text{End}^s(A) \otimes \mathbb{Q}$ generated by the endomorphisms of A which are symmetric with respect to the Rosati involution of (A, \mathcal{L}) . Now an abelian subvariety X of A corresponds to a symmetric idempotent $\varepsilon_X \in \text{End}_{\mathbb{Q}}^s(A)$. So $\varphi^{-1}(\varepsilon_X)$ is an element of $\text{NS}_{\mathbb{Q}}(A)$. On the other hand, $\text{NS}_{\mathbb{Q}}(A)$ admits an intersection product which associates to g elements $\alpha_1, \dots, \alpha_g \in \text{NS}_{\mathbb{Q}}(A)$ a rational number $(\alpha_1 \cdots \alpha_g)$. Theorem 3.3 is a criterion for an element $\alpha \in \text{NS}_{\mathbb{Q}}(A)$ to be equal to $\varphi^{-1}(\varepsilon_X)$ for an abelian subvariety X of A in terms of the intersection numbers $(\alpha^r \cdot \mathcal{L}^{g-r})$.

Introducing coordinates of Λ as above and using the fact that

$$\text{NS}_{\mathbb{Q}}(A) = H^{1,1}(A) \cap H^2(A, \mathbb{Q})$$

we translate the criterion into terms of differential forms which finally gives the above mentioned equations in Theorem 4.1 for the matrix Z . These have been outlined in [1] in the case of a principally polarized abelian variety. For our applications we need however the generalization to an arbitrary polarized abelian variety as we will explain now.

Let G be a finite group acting faithfully on the abelian variety A . Following [3, Section 13.6], this action induces a morphism ρ from the group algebra $\mathbb{Q}[G]$ to the rational endomorphism algebra $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathbb{Q}[G]$ is a semisimple algebra, it decomposes as a product of simple algebras $Q_1 \times \cdots \times Q_r$. Each Q_i is generated by a central idempotent e_i , and these are in correspondence with the rational irreducible representations of G . By defining $A_i = \rho(me_i)$, where m is an integer such that $\rho(me_i) \in \text{End}(A)$, the so called isotypical decomposition of A is obtained. It is an isogeny $A_1 \times \cdots \times A_r \rightarrow A$ where the A_i are abelian subvarieties of A uniquely determined by the simple factors Q_i of the rational group algebra $\mathbb{Q}[G]$.

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In an analogous way the factors A_i are decomposed further up to isogeny as $A_i \sim B_i^{n_i}$. This last decomposition for each isotypical factor comes from the decomposition of Q_i as a product of minimal ideals. Therefore here B_i is an abelian subvariety of A_i , not uniquely determined, and $n_i = \frac{\deg \chi_i}{m_i}$, where χ_i is a complex irreducible representation associated to the simple factor Q_i and m_i its Schur index (see [3, Section 13.6]). The decomposition

$$(1.1) \quad A \sim B_1^{n_1} \times \cdots \times B_r^{n_r}$$

is called the *group algebra decomposition* of the G -abelian variety A . Our starting point was the question whether the abelian varieties B_i are simple. Even if A is principally polarized, the induced polarization on B_i is in general not principal. So in order to discuss the simplicity of B_i we need Theorem 4.1 also in the non-principally polarized case. We will outline several examples for this.

In Section 2 we recall and outline some more details about the relation between abelian subvarieties and symmetric idempotents of a polarized abelian variety. Section 3 contains the above criterion in terms of the Néron-Severi group and Section 4 its translation in terms of period matrices. Finally Section 5 contains the examples.

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2. ABELIAN SUBVARIETIES AND SYMMETRIC IDEMPOTENTS

According to [3, Section 5.3] there is a 1-1 correspondence between the set of abelian subvarieties of a polarized abelian variety and the set of symmetric idempotents of its endomorphism algebra. In the sequel we need however some more details of this relation which will be worked out in this section.

Let (A, \mathcal{L}) be a polarized abelian variety of dimension g of type (d_1, \dots, d_g) . Here the d_i are positive integers with $d_i | d_{i+1}$ for all i . Moreover, if $\widehat{A} := \text{Pic}^0(A)$ denotes the dual abelian variety and $\phi_{\mathcal{L}} : A \rightarrow \widehat{A}$ is the morphism $a \mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$, then

$$\ker \phi_{\mathcal{L}} \simeq (\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z})^2.$$

Now let $X \subset A$ be an abelian subvariety of dimension n . The restriction $\mathcal{L}|_X$ defines a polarization on X of some type (e_1, \dots, e_n) . We will write e_X instead of e_n and call it the *exponent* of X (in (A, \mathcal{L})). We have the commutative diagram

$$(2.1) \quad \begin{array}{ccc} A & \xrightarrow{\phi_{\mathcal{L}}} & \widehat{A} \\ \iota_X \uparrow & & \downarrow \widehat{\iota}_X \\ X & \xrightarrow{\phi_{\mathcal{L}|_X}} & \widehat{X} \end{array}$$

where ι_X is the natural inclusion. Since $\phi_{\mathcal{L}|_X}$ is an isogeny, also

$$\psi_X := e_X \phi_{\mathcal{L}|_X}^{-1} : \widehat{X} \rightarrow X$$

is an isogeny. The *norm endomorphism* of X (with respect to the polarization \mathcal{L}) is defined by the composition

$$N_X := \iota_X \psi_X \widehat{\iota}_X \phi_{\mathcal{L}} \in \text{End}(A).$$

Clearly the image of N_X is X itself. Let

$$' : \text{End}_{\mathbb{Q}}(A) \rightarrow \text{End}_{\mathbb{Q}}(A), \quad \alpha \mapsto \alpha' := \phi_{\mathcal{L}}^{-1} \widehat{\alpha} \phi_{\mathcal{L}}$$

be the Rosati involution with respect to the polarization \mathcal{L} and let $\text{End}_{\mathbb{Q}}^s(A)$ be the subspace of elements of $\text{End}_{\mathbb{Q}}(A)$ fixed under it. Clearly $N_X \in \text{End}_{\mathbb{Q}}^s(A)$ and we have according to [3, Prop. 5.2.1] that the map

$$(2.2) \quad \varphi : \text{NS}_{\mathbb{Q}}(A) \rightarrow \text{End}_{\mathbb{Q}}^s(A), \quad \alpha \mapsto \phi_{\mathcal{L}}^{-1} \phi_{\alpha}$$

is an isomorphism of \mathbb{Q} -vector spaces. We define the *numerical class of X in $\text{NS}_{\mathbb{Q}}(A)$* by

$$\delta_X := \varphi^{-1}(N_X) \in \text{NS}_{\mathbb{Q}}(A).$$

Apart from X it depends only on the polarization \mathcal{L} of A . Recall that the symmetric idempotent associated to X is defined as

$$\varepsilon_X := \frac{1}{e_X} N_X \in \text{End}_{\mathbb{Q}}^s(A).$$

Let α_X denote the class of $\text{NS}_{\mathbb{Q}}(A)$ corresponding to ε_X via the isomorphism (2.2), i.e.

$$\alpha_X = \varphi^{-1}(\varepsilon_X).$$

Lemma 2.1. *Let $[N_X^* \mathcal{L}]$ denote the class of the line bundle $N_X^* \mathcal{L}$ in $\text{NS}(A)$. Then we have*

$$\delta_X = \frac{1}{e_X} [N_X^* \mathcal{L}] \quad \text{and} \quad \alpha_X = \frac{1}{e_X^2} [N_X^* \mathcal{L}] \in \text{NS}_{\mathbb{Q}}(A).$$

Proof. Using the symmetry of N_X with respect to Rosati and the fact that $N_X^2 = e_X N_X$ we have

$$\varphi \left(\frac{1}{e_X} [N_X^* \mathcal{L}] \right) = \phi_{\mathcal{L}}^{-1} \phi_{\frac{1}{e_X} N_X^* \mathcal{L}} = \frac{1}{e_X} \phi_{\mathcal{L}}^{-1} \widehat{N}_X \phi_{\mathcal{L}} N_X = \frac{1}{e_X} N_X' N_X = \frac{1}{e_X} N_X^2 = N_X.$$

This gives the first equation, since φ is an isomorphism. The second equation follows from this and $\alpha_X = \frac{1}{e_X} \delta_X$. \square

Lemma 2.2. *The map*

$$\mu_X : \text{End}_{\mathbb{Q}}(X) \rightarrow \text{End}_{\mathbb{Q}}(A), \quad \alpha \mapsto \frac{1}{e_X} \iota_X \alpha \psi_X \widehat{\iota}_X \phi_{\mathcal{L}}$$

is an injective homomorphism of \mathbb{Q} -algebras with

$$\mu_X(1_X) = \varepsilon_X.$$

Proof. The proof of the multiplicativity uses the fact that $\psi_X \widehat{\iota}_X \phi_{\mathcal{L}} \iota_X = e_X 1_X$. The other assertions are obvious. \square

Now consider a pair of complementary abelian subvarieties $\iota_{Y_i}^X : Y_i \hookrightarrow X$, $i = 1, 2$, of the polarized abelian variety $(X, \mathcal{L}|_X)$, i.e. with respect to the polarization $\mathcal{L}|_X$. Let e_{Y_i} be the exponent of the polarization $(\mathcal{L}|_X)|_{Y_i} = \mathcal{L}|_{Y_i}$ and $\psi_{Y_i} = e_{Y_i} \phi_{\mathcal{L}|_{Y_i}}^{-1}$. Let N_X and N_{Y_i} denote the norm endomorphisms of X and Y_i in $\text{End}(A)$ with respect to the polarization \mathcal{L} and ε_X and ε_{Y_i} the corresponding symmetric idempotents of $\text{End}_{\mathbb{Q}}(A)$.

Lemma 2.3. *With these notations we have*

- (a) $e_X e_{Y_2} N_{Y_1} + e_X e_{Y_1} N_{Y_2} = e_{Y_1} e_{Y_2} N_X$,
- (b) $\varepsilon_{Y_1} + \varepsilon_{Y_2} = \varepsilon_X$.

Proof. Let $N_{Y_i}^X \in \text{End}(X)$ denote the norm endomorphism of Y_i with respect to the polarization $\mathcal{L}|_X$ for $i = 1, 2$. We have for $i = 1, 2$,

$$\mu_X(N_{Y_i}^X) = \frac{1}{e_X} \iota_X N_{Y_i}^X \psi_X \widehat{\iota}_X \phi_{\mathcal{L}} = \frac{1}{e_X} \iota_X \iota_{Y_i}^X \psi_{Y_i} \widehat{\iota}_{Y_i}^X \phi_{\mathcal{L}|_X} \psi_X \widehat{\iota}_X \phi_{\mathcal{L}} = \iota_{Y_i} \psi_{Y_i} \widehat{\iota}_{Y_i} \phi_{\mathcal{L}} = N_{Y_i}$$

Now [3, page 125, equation (4)] says

$$e_{Y_2}N_{Y_1}^X + e_{Y_1}N_{Y_2}^X = e_{Y_1}e_{Y_2}1_X.$$

Applying the map μ_X we get

$$e_{Y_1}e_{Y_2}\frac{1}{e_X}N_X = e_{Y_1}e_{Y_2}\mu_X(1_X) = e_{Y_2}\mu_X(N_{Y_1}^X) + e_{Y_1}\mu_X(N_{Y_2}^X) = e_{Y_2}N_{Y_1} + e_{Y_1}N_{Y_2}.$$

This gives (a). Dividing (a) by $e_{Y_1}e_{Y_2}e_X$ gives (b). \square

Let $f : A \rightarrow B$ be an isogeny of (unpolarized) abelian varieties. So f^{-1} exists in $\text{Hom}_{\mathbb{Q}}(B, A)$ and the abelian subvarieties of A correspond bijectively to the abelian subvarieties of B under the map $X \mapsto f(X)$. Moreover the map

$$\tilde{f} : \text{End}_{\mathbb{Q}}(B) \rightarrow \text{End}_{\mathbb{Q}}(A), \quad b \mapsto f^{-1}bf$$

is an isomorphism of \mathbb{Q} -algebras. Now let

$$f : (A, \mathcal{L}_A) \rightarrow (B, \mathcal{L}_B)$$

be an isogeny of polarized abelian varieties, i.e. $f^*\mathcal{L}_B \equiv \mathcal{L}_A$. Let $\varphi_A : \text{NS}_{\mathbb{Q}}(A) \rightarrow \text{End}_{\mathbb{Q}}^s(A)$, respectively $\varphi_B : \text{NS}_{\mathbb{Q}}(B) \rightarrow \text{End}_{\mathbb{Q}}^s(B)$ be the isomorphisms associated to the polarization \mathcal{L}_A , respectively \mathcal{L}_B . Then we have,

Lemma 2.4. *\tilde{f} restricts to an isomorphism of \mathbb{Q} -vector spaces $\tilde{f} : \text{End}_{\mathbb{Q}}^s(B) \rightarrow \text{End}_{\mathbb{Q}}^s(A)$ and the following diagram commutes*

$$\begin{array}{ccc} \text{NS}_{\mathbb{Q}}(B) & \xrightarrow{\varphi_B} & \text{End}_{\mathbb{Q}}^s(B) \\ f^* \downarrow & & \downarrow \tilde{f} \\ \text{NS}_{\mathbb{Q}}(A) & \xrightarrow{\varphi_A} & \text{End}_{\mathbb{Q}}^s(A) \end{array}$$

Proof. For the first assertion it suffices to show that if $b \in \text{End}_{\mathbb{Q}}(B)$ is symmetric with respect to \mathcal{L}_B , then $\tilde{f}(b)$ is symmetric with respect to \mathcal{L}_A , since both \mathbb{Q} -vector subspaces are of the same dimension. This is a straightforward computation. For the commutativity of the diagram let $\beta \in \text{NS}_{\mathbb{Q}}(B)$. Then $\tilde{f}\varphi_B(\beta) = f^{-1}\phi_{\mathcal{L}_B}^{-1}\phi_{\beta}f = (\hat{f}\phi_{\mathcal{L}_B}f)^{-1}(\hat{f}\phi_{\beta}f) = \phi_{f^*\mathcal{L}_B}^{-1}\phi_{f^*(\beta)} = \varphi_A f^*(\beta)$. \square

Consider the direct image map $f_* : \text{NS}_{\mathbb{Q}}(A) \rightarrow \text{NS}_{\mathbb{Q}}(B)$ defined by the norm map of f .

Corollary 2.5. *Let X be an abelian subvariety of A and $Y = f(X)$ with symmetric idempotents ε_X and ε_Y . Then we have for $\alpha_X = \varphi_A^{-1}(\varepsilon_X)$ and $\alpha_Y = \varphi_B^{-1}(\varepsilon_Y)$*

$$\alpha_X = f^*\alpha_Y$$

$$f_*(\alpha_X) = \deg(f)\alpha_Y.$$

Proof. According to Lemma 2.4 we have $\alpha_X = f^*\alpha_Y$. This implies

$$f_*\alpha_X = f_*f^*\alpha_Y = \deg(f)\alpha_Y$$

where the last equation can be seen, for example, in [7, Proposition 21.10.18]. \square

Remark 2.6. The corollary can also be expressed in terms of the polarizations, since $\deg(f) = \frac{\chi(\mathcal{L}_A)}{\chi(\mathcal{L}_B)}$.

3. THE CRITERION IN TERMS OF THE NÉRON-SEVERI GROUP

Recall that for any g -tuple of line bundles \mathcal{L}_i on a g -dimensional abelian variety A an intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_g)$ is defined. It extends in the usual way to give a rational intersection number

$$(\alpha_1 \cdots \alpha_g) \in \mathbb{Q}$$

for any elements α_i in $\text{NS}_{\mathbb{Q}}(A)$. Let $\chi(\mathcal{L})$ be the Euler characteristic of \mathcal{L} ; by Riemann-Roch this is just $d_1 \cdots d_n$ if \mathcal{L} is of type (d_1, \dots, d_g) . Now let X be an abelian subvariety of dimension n of the polarized abelian variety (A, \mathcal{L}) of dimension g . Let δ_X denote the numerical class of X in $\text{NS}_{\mathbb{Q}}(A)$ and $\alpha_X = \frac{1}{e_X} \delta_X$ as in Section 2.

Proposition 3.1. *We have the following intersection numbers:*

$$(\delta_X^r \cdot \mathcal{L}^{g-r}) = \begin{cases} e_X^r \chi(\mathcal{L}) n! \prod_{i=n+1}^g (i-r) & \text{for } 1 \leq r \leq n, \\ 0 & \text{for } r \geq n+1. \end{cases}$$

and

$$(\alpha_X^r \cdot \mathcal{L}^{g-r}) = \begin{cases} \chi(\mathcal{L}) n! \prod_{i=n+1}^g (i-r) & \text{for } 1 \leq r \leq n, \\ 0 & \text{for } r \geq n+1. \end{cases}$$

Proof. It suffices to prove the first equation. Using Riemann-Roch and Lemma 2.1, we have that for all $t \in \mathbb{Z}$,

$$\begin{aligned} \deg(d_g(t - N_X)) &= \deg(d_g \phi_{\mathcal{L}}^{-1} \phi_{t[\mathcal{L}] - \delta_X}) \\ &= \deg(d_g \phi_{\mathcal{L}}^{-1}) \deg(\phi_{t[\mathcal{L}] - \delta_X}) \\ &= \deg(d_g \phi_{\mathcal{L}}^{-1}) \left(\frac{(t[\mathcal{L}] - \delta_X)^g}{g!} \right)^2. \end{aligned}$$

and

$$\deg(d_g \phi_{\mathcal{L}}^{-1}) = \frac{d_g^{2g}}{\chi(\mathcal{L})^2},$$

since $d_g^{2g} = \deg(d_g \cdot 1_A) = \deg(d_g \phi_{\mathcal{L}}^{-1} \phi_{\mathcal{L}}) = \deg(d_n \phi_{\mathcal{L}}^{-1}) \chi(\mathcal{L})^2$. On the other hand, according to [3, Proposition 5.1.2] we have

$$\deg(d_g(t - N_X)) = d_g^{2g} t^{2g-2n} (t - e_X)^{2n}$$

since $t^{2g-2n} (t - e_X)^{2n}$ is the characteristic polynomial of the rational representation of N_X . Together this gives, after dividing by d_g^{2g} and taking square roots (since for sufficiently large t both sides are positive)

$$(t[\mathcal{L}] - \delta_X)^g = \chi(\mathcal{L}) g! t^{g-n} (t - e_X)^n.$$

Now expanding the polynomials and comparing the coefficients gives the result. \square

Now let the notation be as in Lemma 2.3. In particular Y_1 and Y_2 are complementary abelian subvarieties of (X, \mathcal{L}_X) . For $i = 1, 2$ denote $\alpha_i := \varphi^{-1}(\varepsilon_{Y_i}) \in \text{NS}_{\mathbb{Q}}(A)$. Then we have

Proposition 3.2.

$$\sum_{j=1}^r \binom{r}{j} (\alpha_1^j \cdot \alpha_2^{r-j} \cdot \mathcal{L}^{g-r}) = \begin{cases} \chi(\mathcal{L}) n! \prod_{i=n+1}^g (i-r) & \text{for } r \leq n \\ 0 & \text{for } r \geq n+1 \end{cases}$$

Proof. Let $\mu_X : \text{End}_{\mathbb{Q}}(X) \rightarrow \text{End}_{\mathbb{Q}}(A)$ be the homomorphism of Lemma 2.2. As complementary symmetric idempotents, $\varepsilon_{Y_1}^X$ and $\varepsilon_{Y_2}^X$ commute and so do their images ε_{Y_1} and ε_{Y_2} under the map μ_X . This implies that

$$\varepsilon_X^r = (\varepsilon_{Y_1} + \varepsilon_{Y_2})^r = \sum_{j=0}^r \binom{r}{j} \varepsilon_{Y_1}^j \varepsilon_{Y_2}^{r-j}.$$

Applying the map φ^{-1} , the assertion follows from Proposition 3.1. \square

In [1, Theorem 2.6], it is shown that in the case of a principally polarized abelian variety the above intersection numbers characterize the classes that come from abelian subvarieties. The problem when \mathcal{L} is not a principal polarization, is that δ_X is not necessarily primitive. We can fix this, however, by considering symmetric idempotents instead of norm endomorphisms to get the following result.

Theorem 3.3. *Given a polarized abelian variety (A, \mathcal{L}) of dimension g . The map $X \mapsto \alpha_X$ gives a bijection between the sets of*

- (1) *abelian subvarieties of dimension n on A and*
- (2) *classes $\alpha \in \text{NS}_{\mathbb{Q}}(A)$ that satisfy*

$$(\alpha^r \cdot \mathcal{L}^{g-r}) = \begin{cases} \chi(\mathcal{L})n! \prod_{i=n+1}^g (i-r) & \text{if } 1 \leq r \leq n \\ 0 & \text{if } r \geq n+1. \end{cases}$$

Proof. Let X be an abelian subvariety of dimension n of A . According to Proposition 3.1, α_X satisfies the equations in (2). Conversely, let $\alpha \in \text{NS}_{\mathbb{Q}}(A)$ satisfy the equations in (2). Let q be a positive integer such that $q\varphi(\alpha) \in \text{End}(A)$. Then

$$\begin{aligned} \deg(t - q\varphi(\alpha)) &= \frac{1}{\chi(\mathcal{L})^2 g!^2} ((t[\mathcal{L}] - q\alpha)^g)^2 \\ &= \left(\frac{1}{\chi(\mathcal{L})g!} \sum_{r=0}^g \binom{g}{r} (-q)^r t^{g-r} (\alpha^r \cdot \mathcal{L}^{n-r}) \right)^2 \\ &= \left(\frac{1}{\chi(\mathcal{L})g!} \sum_{r=0}^n \frac{g!}{r!(g-r)!} (-q)^r t^{g-r} \chi(\mathcal{L})n!(n+1-r) \cdots (g-r) \right)^2 \\ &= \left(t^{g-n} \sum_{r=0}^n \frac{n!}{r!(n-r)!} (-q)^r t^{n-r} \right)^2 = t^{2g-2n} (t-q)^{2n} \end{aligned}$$

Since this is the characteristic polynomial of $q\varphi(\alpha)$ and $q\varphi(\alpha)$ is symmetric, then $q\varphi(\alpha)$ satisfies the equation $(q\varphi(\alpha))(q\varphi(\alpha) - q) = 0$ (see [1, Lemma 2.7]). Therefore $\varphi(\alpha)^2 = \varphi(\alpha)$, and so $\varphi(\alpha)$ is the symmetric idempotent associated to the abelian subvariety $X := \text{image}(q\varphi(\alpha))$. This completes the proof of the theorem, since clearly both maps are inverse to each other. \square

Even if one finds many elliptic curves on an abelian variety, it is not always easy to tell whether the abelian variety decomposes as the product of these curves. For example, if E and E' are isogenous elliptic curves and X is a simple abelian surface, then $E \times E' \times X$ has infinitely many elliptic curves, but does not split as the product of elliptic curves. Using the previous theorem, we can determine when this occurs. We first prove two lemmas.

Lemma 3.4. *If (A, \mathcal{L}) is a polarized abelian variety of dimension g and $E_1, \dots, E_g \subseteq A$ are elliptic curves with complementary abelian subvarieties Z_1, \dots, Z_g , then the addition map*

$$a : E_1 \times \cdots \times E_g \rightarrow A$$

is an isogeny if and only if the line bundle $\mathcal{O}_A(Z_1) \otimes \cdots \otimes \mathcal{O}_A(Z_g)$ is ample.

Proof. Set $\mathcal{S} := \mathcal{O}_A(Z_1) \otimes \cdots \otimes \mathcal{O}_A(Z_g)$, let X be the image of a and let Y be the complementary abelian subvariety of X . Clearly $Y \subseteq Z_i$ for every i and $\mathcal{S}|_Y \simeq \mathcal{O}_Y$. If a is not an isogeny, i.e. $Y \neq \{0\}$, then $\mathcal{S}|_Y$ is not ample, and therefore \mathcal{S} is not ample.

Assume now that a is an isogeny. We see that for all $I = \{i_1, \dots, i_r\} \subsetneq \{1, \dots, g\}$ and all $j \notin \{i_1, \dots, i_r\}$, Z_j intersects $Z_I := Z_{i_1} \cap \cdots \cap Z_{i_r}$ properly. Therefore,

$$(\mathcal{S}^r \cdot \mathcal{L}^{n-r}) = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \#I=r}} (\mathcal{L}^{n-r}|_{Z_I}),$$

which is positive since \mathcal{L} is ample. By the Nakai-Moishezon Criterion for abelian varieties ([3, Corollary 4.3.3]), we get that \mathcal{S} is ample. \square

Lemma 3.5. *If E is an elliptic curve on A with complementary abelian subvariety Z , then α_E is a positive multiple of $[\mathcal{O}_A(Z)]$, where $[\mathcal{O}_A(Z)]$ denotes the numerical class of the line bundle $\mathcal{O}_A(Z)$,*

Proof. By Lemma 2.1 it is clearly sufficient to prove that $N_E^* \mathcal{L}$ is algebraically equivalent to a positive multiple of $\mathcal{O}_A(Z)$. Now according to [3, Proposition 12.1.3] and since an elliptic curve is self dual, we can identify $A/Z = \widehat{E} = E$ and the norm map N_E factors as

$$N_E : A \xrightarrow{p} E \xrightarrow{\iota} A.$$

This gives $N_E^* \mathcal{L} = p^*(\mathcal{L}|_E)$. Since E is an elliptic curve, we have $\mathcal{L}|_E \equiv m\mathcal{O}_{\widehat{E}}([0])$ for some positive integer m , where $[0]$ denotes the origin of E . Therefore,

$$N_E^* \mathcal{L} \equiv p^* \mathcal{L}|_E \equiv mp^* \mathcal{O}_E([0]) \simeq m\mathcal{O}_A(Z),$$

since Z is the kernel of the projection p . \square

Corollary 3.6. *A polarized abelian variety (A, \mathcal{L}) of dimension g splits isogenously as the product of elliptic curves if and only if there exist $\alpha_1, \dots, \alpha_g \in \text{NS}_{\mathbb{Q}}(A)$ such that*

$$(\alpha_i^r \cdot \mathcal{L}^{g-r}) = \begin{cases} \chi(\mathcal{L}) \prod_{i=1}^g (i-r) & \text{if } r=1 \\ 0 & \text{if } r \geq 2. \end{cases}$$

and such that $\alpha_1 + \cdots + \alpha_g$ is an ample \mathbb{Q} -class.

Proof. If A splits up to isogeny as the product of sub elliptic curves E_1, \dots, E_g , then, by Theorem 3.3 and Lemma 3.4, we obtain classes $\alpha_1, \dots, \alpha_g$ that have the intersection numbers above. According to Lemma 3.5 we have

$$r_i \alpha_i = [\mathcal{O}_A(Z_i)]$$

where Z_i is the complementary abelian subvariety of E_i in A and $r_i = \frac{1}{m_i}$ for some positive integer m_i . Define $r := \max\{r_1, \dots, r_g\}$. Since $\delta = r_1 \alpha_1 + \cdots + r_g \alpha_g$ is ample by Lemma 3.4, then

$$r(\alpha_1 + \cdots + \alpha_g) = \delta + (r - r_1) \alpha_1 + \cdots + (r - r_g) \alpha_g$$

is ample, being the sum of an ample class and a nef class. Therefore $\alpha_1 + \cdots + \alpha_g$ is ample.

Reciprocally, if we have $\alpha_1, \dots, \alpha_g \in \text{NS}_{\mathbb{Q}}(A)$ with the intersection numbers above, then Theorem 3.3 implies that these come from elliptic curves E_1, \dots, E_g on A . By Lemma 3.5 there exists $r_i \in \mathbb{Z}_{>0}$ such that $r_i \alpha_i$ is the numerical cycle class of the complementary abelian subvariety of E_i . If $\alpha_1 + \cdots + \alpha_g$ is ample, then $r_1 \alpha_1 + \cdots + r_g \alpha_g$ is ample (it lies within the ample cone), and by Lemma 3.4, A splits as a product of the E_i . \square

To determine when a \mathbb{Q} -class is ample, we use the Nakai-Moishezon Criterion for abelian varieties (see [3, Corollary 4.3.3]). This theorem states that if \mathcal{L} is an ample line bundle on an abelian variety of dimension g and α is a \mathbb{Q} -divisor class, then α is ample if and only if $(\alpha^r \cdot \mathcal{L}^{g-r}) > 0$ for all $r = 1, \dots, n$. This, along with Corollary 3.6, gives us a numerical criterion for when an abelian variety splits as a product of elliptic curves.

4. THE CRITERION IN TERMS OF THE PERIOD MATRIX

Let (A, \mathcal{L}) be an abelian variety of dimension g with polarization of type (d_1, \dots, d_g) . The Néron-Severi group of $A = V/\Lambda$ can be seen as a subgroup of $H^2(A, \mathbb{Z}) \simeq \wedge^2 \mathbb{Z}^{2g}$. To be more precise, given a basis of V , let $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ be a symplectic basis for (A, \mathcal{L}) such that with respect to these bases, A is given by the period matrix $(D \ Z)$. If x_1, \dots, x_{2g} are the real coordinate functions of $\Lambda \otimes \mathbb{R}$ associated to the basis of Λ and z_i the complex coordinate functions with respect to the basis of V , these functions are related by the equation

$$(4.1) \quad \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} = (D \ Z) \begin{pmatrix} x_1 \\ \vdots \\ x_{2g} \end{pmatrix}$$

where $D = \text{diag}(d_1, \dots, d_g)$. The global 1-forms dz_1, \dots, dz_g form a \mathbb{C} -basis of the cotangent space of A at 0. Since the forms $dz_i \wedge d\bar{z}_j$, $i, j = 1, \dots, g$ generate the \mathbb{C} -vector space $H^{1,1}(A)$, this implies that

$$H^{1,1}(A) = \{\omega \in H^2(A, \mathbb{C}) \mid \omega \wedge dz_1 \wedge \dots \wedge dz_g = 0\}$$

Now the first Chern class induces an isomorphism

$$\text{NS}_{\mathbb{Q}}(A) \simeq H^2(A, \mathbb{Q}) \cap H^{1,1}(A).$$

Moreover, $H^2(A, \mathbb{Q}) \simeq \wedge^2 H^1(A, \mathbb{Q})$. So we can identify $H^2(A, \mathbb{Q})$ with respect to the above real basis as

$$H^2(A, \mathbb{Q}) = \wedge^2 \mathbb{Q}^{2g}.$$

In other words, we consider $\{dx_i \wedge dx_j \mid 1 \leq i < j \leq g\}$ as the canonical basis of $\wedge^2 \mathbb{Q}^{2g}$. So together we obtain the following identification

$$(4.2) \quad \text{NS}_{\mathbb{Q}}(A) = \{\omega \in \wedge^2 \mathbb{Q}^{2n} : \omega \wedge dz_1 \wedge \dots \wedge dz_g = 0\},$$

With these identifications Theorem 3.3 translates into the following theorem.

Theorem 4.1. *Given a polarized abelian variety (A, \mathcal{L}) of dimension g and type (d_1, \dots, d_g) . The above identifications induce a bijection between the sets of*

- (1) *abelian subvarieties of A of dimension n and*
 - (2) *differential forms $\omega \in \wedge^2 \mathbb{Q}^{2g}$ such that*
 - (a) $\omega \wedge dz_1 \wedge \dots \wedge dz_g = 0$,
 - (b) $\omega^{\wedge r} \wedge \theta^{\wedge (g-r)} = \begin{cases} \chi(\mathcal{L})n! \prod_{i=n+1}^g (i-r) \cdot \omega_0 & 1 \leq r \leq n \\ 0 & r \geq n+1 \end{cases}$
- where θ is the first Chern class of \mathcal{L} and $\omega_0 = (-1)^g dx_1 \wedge dx_{g+1} \wedge \dots \wedge dx_g \wedge dx_{2g}$.*

Proof. According to [3, Exercise 2.6,(2b)], the first Chern class of \mathcal{L} corresponds to the differential form

$$(4.3) \quad \theta := - \sum_{i=1}^n d_i dx_i \wedge dx_{i+n}.$$

Moreover, in these terms the intersection product in $\text{NS}_{\mathbb{Q}}(A)$ corresponds to the wedge product of differential forms. So the assertion is a translation of Theorem 3.3. \square

Example 4.2. In [4, Theorem 8] a particular set of period matrices is shown that represents all principally polarized abelian varieties of dimension g that contain an abelian subvariety of dimension n and of type (d_1, \dots, d_n) . Indeed, Borówka obtains the family of matrices $Z = (z_{ij})_{i,j}$ where

$$\begin{aligned} z_{ij} &= d_i z_{(g-n+i)j} & i, j &= 1, \dots, n \\ z_{ij} &= 0 & i &= n+1, \dots, g-n, j = 1, \dots, n \end{aligned}$$

If Z_n represents the principal $n \times n$ submatrix of Z , then putting $X = \mathbb{C}^n / (Z_n \mathbb{Z}^n + D \mathbb{Z}^n)$ with $D = \text{diag}(d_1, \dots, d_n)$ and $A = \mathbb{C}^g / (Z \mathbb{Z}^g + \mathbb{Z}^g)$, we have an inclusion $X \hookrightarrow A$ whose analytic representation acts as

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0, \frac{1}{d_1} x_1, \dots, \frac{1}{d_n} x_n).$$

The rational representation of the norm endomorphism associated to X in A (with respect to the given polarization) is found to be

$$N = d_n \begin{pmatrix} R & 0 \\ 0 & R^t \end{pmatrix}$$

where

$$R = \begin{pmatrix} I_n & 0 \\ 0 & 0 \\ D^{-1} & 0 \end{pmatrix}$$

(with the zero matrices having the appropriate size) and $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$. By [1, Prop. 3.2], the matrix of the alternating form associated to X (with respect to the given symplectic basis) is

$$M = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} N = d_n \begin{pmatrix} 0 & -R^t \\ R & 0 \end{pmatrix}.$$

Therefore, the differential form associated to the symmetric idempotent of X is

$$\omega_{d_1, \dots, d_n} := - \sum_{i=1}^n (dx_i \wedge dx_{g+i} + \frac{1}{d_i} dx_i \wedge dx_{g-n+i})$$

and the equations above on the period matrices can be simply represented by the equation $\omega_{d_1, \dots, d_n} \wedge dz_1 \wedge \dots \wedge dz_g = 0$. We then have that the moduli space of all principally polarized abelian varieties that contain an abelian subvariety of dimension n and type (d_1, \dots, d_n) is

$$\mathcal{A}_g(\omega_{d_1, \dots, d_n}) := \pi(\{Z \in \mathcal{H}_g : \omega_{d_1, \dots, d_n} \wedge dz_1 \wedge \dots \wedge dz_g = 0\}),$$

where $\pi : \mathcal{H}_g \rightarrow \mathcal{A}_g$ is the natural projection. This shows that our general approach to studying non-simple abelian varieties gives the same equations as those found in [4] when using a particular differential form. \square

Now we endow \mathbb{Z}^{2g} with the canonical symplectic form and extend this form to the complex vector space \mathbb{C}^{2g} . Let x_1, \dots, x_{2g} denote the coordinate functions with respect to the canonical basis of \mathbb{C}^{2g} . We identify the vector space $\wedge^2 \mathbb{C}^{2g}$ with the \mathbb{C} -vector space of differential forms $\sum_{i < j} a_{ij} dx_i \wedge dx_j$ with constants $a_{ij} \in \mathbb{C}$.

Let (A, \mathcal{L}) be a polarized abelian variety of dimension g of type (d_1, \dots, d_g) and suppose $A = V/\Lambda$. Given a basis of V and a symplectic basis of Λ , we can identify this basis with the canonical basis of $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$, such that the restrictions to the reals of the above coordinate functions x_i (also denoted by x_i) are real coordinates of the complex vector space V . Let (D, Z) be the period matrix of A with respect to the bases. We introduce complex coordinates z_1, \dots, z_g of V by means of equation (4.1).

The matrix Z is symmetric and positive definite and thus an element of the Siegel upper half space \mathcal{H}_g of rank g , which is a complex manifold of dimension $\binom{g+1}{2}$. With respect to the real coordinates x_i the first Chern class θ of \mathcal{L} is given by (4.3). With these notations we define for every positive integer $n < g$ a complex subvariety of $\wedge^2 \mathbb{C}^{2g} \times \mathcal{H}_g$ by

$$(4.4) \quad \mathcal{E}(g, n) := \{(\omega, Z) \in \wedge^2 \mathbb{C}^{2g} \times \mathcal{H}_g \mid \omega \text{ satisfies equations (a) and (b) of Theorem 4.1}\}.$$

If we denote for any $Z \in \mathcal{H}_g$ by A_Z the polarized abelian variety defined by the period matrix $(D Z)$, then Theorem 4.1 translates into the following corollary.

Corollary 4.3. *The abelian variety A_Z admits an abelian subvariety of dimension n if and only if there exists an $\omega \in \wedge^2 \mathbb{Q}^{2g} \subset \wedge^2 \mathbb{C}^{2g}$ such that $(\omega, Z) \in \mathcal{E}(g, n)$. In other words,*

$$\{Z \in \mathcal{H}_g \mid A_Z \text{ admits an abelian subvariety of dimension } n\} = \mathcal{E}(g, n) \cap (\wedge^2 \mathbb{Q}^{2g} \times \mathcal{H}_g).$$

The following proposition gives explicit equations for $g = 2$. For the principally polarized case see [1, Thm. 3.1], or [8, Lemma 5.3]. Since any polarization is a multiple of a primitive polarization, we may assume that the polarization of A is of type $(1, d_2)$ with some positive integer d_2 .

Proposition 4.4. *Let (A, \mathcal{L}) be a polarized abelian surface of type $(1, d_2)$ with period matrix $Z = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$. Then A admits a sub elliptic curve if and only if there exists a vector $(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) \in \mathbb{Q}^6$ satisfying*

$$\begin{aligned} -d_2 &= d_2 a_{13} + a_{24}, \\ 0 &= (\tau_1 \tau_3 - \tau_2^2) a_{12} - d_2 a_{14} \tau_1 + d_2 a_{13} \tau_2 - a_{24} \tau_2 + a_{23} \tau_3 + d_2 a_{34} \text{ and} \\ 0 &= a_{14} a_{23} - a_{13} a_{24} + a_{12} a_{34}. \end{aligned}$$

Proof. This comes from writing out the previous conditions. The first equation is (a) and the second and third equation (b) for $r = 1$ and 2 of Theorem 4.1. \square

Similarly the equations for $g = 3$ can be given explicitly. Again we assume that the polarization is primitive, i.e. $d_1 = 1$.

Proposition 4.5. *Let (A, \mathcal{L}) be a principally polarized abelian threefold of type $(1, d_2, d_3)$ with period matrix $Z = (\tau_{ij})_{ij}$. Then A admits a sub elliptic curve if and only if there exists a vector $(a_{ij})_{1 \leq i < j \leq 6} \in \mathbb{Q}^{15}$ such that*

$$\begin{aligned}
0 &= -a_{16}d_2d_3\tau_{11} + a_{14}d_2d_3\tau_{13} + a_{46}d_2d_3 - a_{26}d_3\tau_{12} - a_{36}d_2\tau_{13} + a_{24}d_3\tau_{23} + \\
&\quad a_{34}d_2\tau_{33} - (d_3\tau_{12}\tau_{13} - d_3\tau_{11}\tau_{23})a_{12} - (d_2\tau_{13}^2 - d_2\tau_{11}\tau_{33})a_{13} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})a_{23} \\
0 &= -a_{16}d_2d_3\tau_{12} + a_{15}d_2d_3\tau_{13} + a_{56}d_2d_3 - a_{26}d_3\tau_{22} - a_{36}d_2\tau_{23} + \\
&\quad a_{25}d_3\tau_{23} + a_{35}d_2\tau_{33} - (d_3\tau_{13}\tau_{22} - d_3\tau_{12}\tau_{23})a_{12} - (d_2\tau_{13}\tau_{23} - d_2\tau_{12}\tau_{33})a_{13} - \\
&\quad (\tau_{23}^2 - \tau_{22}\tau_{33})a_{23} \\
0 &= a_{56}d_2\tau_{13} - a_{46}d_2\tau_{23} + a_{45}d_2\tau_{33} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + \\
&\quad (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})a_{12} - (d_2\tau_{13}\tau_{23} - d_2\tau_{12}\tau_{33})a_{14} + (d_2\tau_{13}^2 - d_2\tau_{11}\tau_{33})a_{15} - \\
&\quad (d_2\tau_{12}\tau_{13} - d_2\tau_{11}\tau_{23})a_{16} - (\tau_{23}^2 - \tau_{22}\tau_{33})a_{24} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})a_{25} - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})a_{26} \\
0 &= -a_{15}d_2d_3\tau_{11} + a_{14}d_2d_3\tau_{12} + a_{45}d_2d_3 - a_{25}d_3\tau_{12} - a_{35}d_2\tau_{13} + a_{24}d_3\tau_{22} + \\
&\quad a_{34}d_2\tau_{23} - (d_3\tau_{12}^2 - d_3\tau_{11}\tau_{22})a_{12} - (d_2\tau_{12}\tau_{13} - d_2\tau_{11}\tau_{23})a_{13} - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})a_{23} \\
0 &= a_{56}d_2d_3\tau_{11} - a_{46}d_2d_3\tau_{12} + a_{45}d_2d_3\tau_{13} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + \\
&\quad (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})a_{23} + (d_3\tau_{13}\tau_{22} - d_3\tau_{12}\tau_{23})a_{24} - (d_3\tau_{12}\tau_{13} - d_3\tau_{11}\tau_{23})a_{25} + \\
&\quad (d_3\tau_{12}^2 - d_3\tau_{11}\tau_{22})a_{26} + (d_2\tau_{13}\tau_{23} - d_2\tau_{12}\tau_{33})a_{34} - (d_2\tau_{13}^2 - d_2\tau_{11}\tau_{33})a_{35} + \\
&\quad (d_2\tau_{12}\tau_{13} - d_2\tau_{11}\tau_{23})a_{36} \\
0 &= -a_{56}d_3\tau_{12} + a_{46}d_3\tau_{22} - a_{45}d_3\tau_{23} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + \\
&\quad (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})a_{13} + (d_3\tau_{13}\tau_{22} - d_3\tau_{12}\tau_{23})a_{14} - (d_3\tau_{12}\tau_{13} - d_3\tau_{11}\tau_{23})a_{15} + \\
&\quad (d_3\tau_{12}^2 - d_3\tau_{11}\tau_{22})a_{16} - (\tau_{23}^2 - \tau_{22}\tau_{33})a_{34} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})a_{35} - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})a_{36} \\
d_2d_3 &= -a_{14}d_2d_3 - a_{36}d_3 - a_{25}d_3 \\
0 &= -2a_{16}a_{34}d_2 - 2a_{13}a_{46}d_2 - 2a_{15}a_{24}d_3 - 2a_{12}a_{45}d_3 + (a_{36}d_2 + a_{25}d_3)a_{14} + \\
&\quad (a_{14}d_3 + a_{36})a_{25} - 2a_{26}a_{35} + (a_{14}d_2 + a_{25})a_{36} - 2a_{23}a_{56} \\
0 &= (a_{36}a_{45} - a_{35}a_{46} + a_{34}a_{56})a_{12} - (a_{26}a_{45} - a_{25}a_{46} + a_{24}a_{56})a_{13} + \\
&\quad (a_{26}a_{35} - a_{25}a_{36} + a_{23}a_{56})a_{14} - (a_{26}a_{34} - a_{24}a_{36} + a_{23}a_{46})a_{15} + \\
&\quad (a_{25}a_{34} - a_{24}a_{35} + a_{23}a_{45})a_{16} + (a_{16}a_{45} - a_{15}a_{46} + a_{14}a_{56})a_{23} - \\
&\quad (a_{16}a_{35} - a_{15}a_{36} + a_{13}a_{56})a_{24} + (a_{16}a_{34} - a_{14}a_{36} + a_{13}a_{46})a_{25} - \\
&\quad (a_{15}a_{34} - a_{14}a_{35} + a_{13}a_{45})a_{26} + (a_{16}a_{25} - a_{15}a_{26} + a_{12}a_{56})a_{34} - \\
&\quad (a_{16}a_{24} - a_{14}a_{26} + a_{12}a_{46})a_{35} + (a_{15}a_{24} - a_{14}a_{25} + a_{12}a_{45})a_{36} + \\
&\quad (a_{16}a_{23} - a_{13}a_{26} + a_{12}a_{36})a_{45} - (a_{15}a_{23} - a_{13}a_{25} + a_{12}a_{35})a_{46} + \\
&\quad (a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34})a_{56}
\end{aligned}$$

Proof. For the computation we used the open source computer algebra system SAGE. The first 6 equations correspond to the coefficient of $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_6$ (where dx_i is omitted) of equation (a) of Theorem 4.1. The last 3 equations correspond to $r = 1, 2, 3$ in this order of equation (b) of Theorem 4.1. \square

5. EXAMPLES

As pointed out in the introduction, we want to apply Propositions 4.4 and 4.5 to factors of group algebra decompositions of some principally polarized abelian varieties in order to check whether they decompose further or not.

5.1. The family \mathcal{F}_5 . In [10] a 3-dimensional family \mathcal{F}_5 of principally polarized abelian varieties of dimension 5 admitting an action of the dihedral group D_5 of order 10 was investigated. It is

given by the Riemann matrices

$$Z = \begin{pmatrix} z_1 & z_2 & z_3 & z_3 & z_2 \\ & z_1 & z_2 & z_3 & z_3 \\ & & z_1 & z_2 & z_3 \\ & & & z_1 & z_2 \\ & & & & z_1 \end{pmatrix} \in \mathcal{H}_5.$$

Let $A = A_Z = V/\Lambda_Z$ denote the abelian variety corresponding to Z . With respect to a basis of V and a symplectic basis $\{\lambda_1, \dots, \lambda_5, \mu_1, \dots, \mu_5\}$ of Λ_Z the period matrix of A is

$$\Pi_Z = (I_5 Z).$$

According to [10] the group decomposition of A_Z is

$$A_Z \sim E \times B^2.$$

with an elliptic curve E and an abelian surface B . We want to see whether B decomposes further. Clearly the abelian subvariety B of A is not uniquely determined, but since the decomposability does not depend on the chosen subvariety in the equivalence class, we may assume according to [10, Proposition 4.4] that the sublattice of Λ defining B has the basis

$$\{\lambda_1 - \lambda_5, \lambda_2 - \lambda_4, \mu_1 - \mu_5, \mu_2 - \mu_4\}$$

and moreover with respect to this basis, from the proof of [10, Prop. 4.4] it is deduced that the period matrix of B with respect to the induced polarization of A is given by

$$\Pi_B = \begin{pmatrix} 2 & 0 & 2(z_1 - z_2) & 2(z_2 - z_3) \\ 0 & 2 & 2(z_2 - z_3) & 2(z_1 - z_3) \end{pmatrix}.$$

Proposition 5.1. (a) *For a general $A \in \mathcal{F}_5$ the abelian subvariety B is irreducible.*

(b) *There is a union of surfaces in \mathcal{F}_5 whose members split isogenously as the product of elliptic curves.*

Proof. The abelian surface B with period matrix Π_B splits if and only if the isogenous principally polarized abelian surface \tilde{B} with period matrix

$$\frac{1}{2}\Pi_B = \begin{pmatrix} z_1 - z_2 & z_2 - z_3 \\ z_2 - z_3 & z_1 - z_3 \end{pmatrix}.$$

splits. According to Proposition 4.4, \tilde{B} splits if and only if there exists a vector $(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) \in \mathbb{Q}^6$ such that

$$\begin{aligned} -1 &= a_{13} + a_{24} \\ (5.1) \quad 0 &= [(z_1 - z_2)(z_1 - z_3) - (z_2 - z_3)^2]a_{12} + \\ &\quad + z_1(a_{23} - a_{14}) + z_2(a_{13} + a_{14} - a_{24}) + z_3(a_{24} - a_{13} - a_{23}) + a_{34} \\ 0 &= a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34} \end{aligned}$$

Suppose that for a general $Z \in \mathcal{H}_5$ the abelian surface \tilde{B}_Z splits. This implies that all coefficients of the quadratic polynomial in the z_i vanish. So we get $a_{12} = a_{34} = 0$, $a_{23} = a_{14}$ and the system of equations

$$a_{24} = -a_{13} - 1, \quad a_{24} = a_{13} + a_{14}, \quad a_{14}^2 = a_{13}a_{24}.$$

Eliminating a_{14} and a_{23} we remain with the equation $5a_{13}^2 + 5a_{13} + 1 = 0$ which does not have a rational solution. This completes the proof of (a).

As for (b), consider the integers $a_{12} = a_{13} = a_{23} = a_{34} = 0$, $a_{24} = -1$ and $a_{14} := a$ an arbitrary rational number. Then the first and the last equation of (5.1) are satisfied and the second equation defines the surface

$$S_a := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_3 = (a + 1)z_2 - az_1\}$$

We will show that there exist infinitely many $a \in \mathbb{Q}$ for which S_a gives a non-empty family in \mathcal{H}_5 . In fact, denote by y_i the imaginary part of z_i . Then the eigenvalues of the imaginary part of the matrix Z with $(z_1, z_2, z_3) \in S_a$ are:

$$2ay_2 + y_1 + 4y_2 - 2ay_1, \quad \left(1 + \frac{a}{2} + \frac{\sqrt{5}a}{2}\right)(y_1 - y_2),$$

$$\left(1 + \frac{a}{2} - \frac{\sqrt{5}a}{2}\right)(y_1 - y_2), \quad \left(1 + \frac{a}{2} + \frac{\sqrt{5}a}{2}\right)(y_1 - y_2), \quad \left(1 + \frac{a}{2} - \frac{\sqrt{5}a}{2}\right)(y_1 - y_2).$$

The following conditions imply that these numbers are positive.

$$\frac{-2}{\sqrt{5} + 1} < a < \frac{2}{\sqrt{5} - 1}, \quad y_1 > y_2.$$

So for all rational numbers a in that interval, S_a defines a surface in \mathcal{F}_5 all of whose abelian surfaces split up to isogeny. \square

Remark 5.2. According to the proof of Proposition 5.1 there are infinitely many rational numbers a for which S_a defines a surface in \mathcal{F}_5 . We do not know how many of them are pairwise isomorphic. So we do not know how many such surfaces there are.

5.2. The family \mathcal{F}_7 . In [10] a 4-dimensional family \mathcal{F}_7 of principally polarized abelian varieties of dimension 7 admitting an action of the dihedral group D_7 of order 14 was investigated. It is given by the Riemann matrices

$$Z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_4 & z_3 & z_2 \\ & z_1 & z_2 & z_3 & z_4 & z_4 & z_3 \\ & & z_1 & z_2 & z_3 & z_4 & z_4 \\ & & & z_1 & z_2 & z_3 & z_4 \\ & & & & z_1 & z_2 & z_3 \\ & & & & & z_1 & z_2 \\ & & & & & & z_1 \end{pmatrix} \in \mathcal{H}_7,$$

Let $A = A_Z = V/\Lambda_Z$ denote the abelian variety corresponding to Z . With respect to a basis of V and a symplectic basis $\{\lambda_1, \dots, \lambda_7, \mu_1, \dots, \mu_7\}$ of Λ_Z the period matrix of A is

$$\Pi_Z = (I_7 \ Z).$$

According to [10], the group algebra decomposition of A_Z is

$$A_Z \sim E \times B^2.$$

with an elliptic curve E and an abelian threefold B . We want to see whether B decomposes further. As before the abelian subvariety B of A is not uniquely determined, but since the decomposability does not depend on the chosen subvariety in the equivalence class, we may assume according to [10, Proposition 4.4] that the sublattice of Λ defining B has the basis

$$\{\lambda_1 - \lambda_7, \lambda_2 - \lambda_6, \lambda_3 - \lambda_5, \mu_1 - \mu_7, \mu_2 - \mu_6, \mu_3 - \mu_5\}$$

and moreover by the proof of [10, Prop. 4.4], it is deduced that with respect to this basis the period matrix of B with respect to the induced polarization of A is given by

$$\Pi_B = 2 \begin{pmatrix} 1 & 0 & 0 & z_1 - z_2 & z_2 - z_3 & z_3 - z_4 \\ 0 & 1 & 0 & z_2 - z_3 & z_1 - z_4 & z_2 - z_4 \\ 0 & 0 & 1 & z_3 - z_4 & z_2 - z_4 & z_1 - z_3 \end{pmatrix}.$$

Proposition 5.3. *For a general principally polarized abelian variety in \mathcal{F}_7 , B is simple.*

Proof. We need to show that a general B contains no elliptic curves. Let

$$\omega = \sum_{i < j} a_{ij} dx_i \wedge dx_j \in \wedge^2 \mathbb{Q}^6.$$

We see that if we divide the period matrix of B by 2, we obtain an isomorphic abelian variety which is principally polarized. After writing out the first six equations from Proposition 4.5 for ω and the modified period matrix for B , we obtain polynomials in z_1, z_2, z_3 and z_4 with rational coefficients. If we take these to be algebraically independent, then each coefficient of the polynomials must be equal to 0, and we obtain the following relations:

$$\begin{aligned} a_{12} &= a_{13} = a_{23} = a_{45} = a_{46} = a_{56} = 0 \\ a_{14} &= -a_{15} + a_{25} - a_{34} \\ a_{16} &= a_{34} \\ a_{24} &= a_{15} \\ a_{26} &= a_{35} = a_{15} + a_{34} \\ a_{36} &= a_{25} - a_{34} \end{aligned}$$

The parameters a_{15}, a_{25} and a_{34} are free, and the rest can be written in terms of these. If we now ask for $\omega \wedge \theta \wedge \theta = -dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_5 \wedge dx_3 \wedge dx_6$ (which is the 7th equation in 4.5 with $d_2 = d_3 = 1$), we obtain the equation

$$a_{15} = -\frac{1}{2} + 3a_{25} - 2a_{34}.$$

The equation $\omega \wedge \omega \wedge \theta = 0$ leads to

$$42a_{25}^2 - 42a_{25}a_{34} + 14a_{34}^2 - 14a_{25} + 7a_{34} + 1 = 0.$$

Solving for a_{25} in terms of a_{34} , we get that

$$a_{25} = \frac{42a_{34} + 14 \pm \sqrt{(42a_{34} + 14)^2 - 168(14a_{34}^2 + 7a_{34} + 1)}}{84}.$$

Now $(42a_{34} + 14)^2 - 168(14a_{34}^2 + 7a_{34} + 1) = -588a_{34}^2 + 28$. We see that $a_{34} = 1/7$ makes this number a square, and using the stereographic projection, we get that this expression is the square of a rational number if and only if

$$a_{34} = -\frac{168 + 8t}{588 + t^2} + \frac{1}{7}$$

for $t \in \mathbb{Q}$. In this case, we obtain the solutions

$$\begin{aligned} a_{25} &= -\frac{2t^2}{21(t^2 + 588)} - \frac{6t}{t^2 + 588} - \frac{84}{t^2 + 588} + \frac{2}{7} \\ a_{25} &= \frac{2t^2}{21(t^2 + 588)} - \frac{2t}{t^2 + 588} - \frac{84}{t^2 + 588} + \frac{4}{21}. \end{aligned}$$

If we now ask for the equation $\omega \wedge \omega \wedge \omega = 0$, we obtain the two equations

$$t^6 - 756t^5 + 139356t^4 + 1481760t^3 - 52898832t^2 - 261382464t + 5489031744 = 0$$

$$3t^6 - 1092t^5 + 88788t^4 + 2140320t^3 - 29618736t^2 - 377552448t + 3817474752 = 0.$$

A quick check in SAGE shows us that these two polynomials have no rational roots, and so the equations of Proposition 4.5 have no rational solutions. \square

Remark 5.4. We note that the proof of Proposition 5.3 shows that the Picard number of a very general B in the above family is 3 (since the parameters a_{15} , a_{25} and a_{34} are free), and the Picard number of any B is at least 3.

Remark 5.5. Not every B is simple. For example, setting $z_2 = z_3 = z_4$ we obtain a one dimensional family of threefolds that are isomorphic (as varieties) to the product of elliptic curves. We see that the differential forms

$$\eta = \frac{1}{2}dx_1 \wedge dx_4 \quad \text{and} \quad \mu = \frac{1}{2}dx_2 \wedge dx_5$$

satisfy the equations of Proposition 4.5 for this family, for example. By fixing η and using the equations from Proposition 4.5, we see that the Néron-Severi group of a B contains η (and μ) if and only if

$$z_2 = z_3 = z_4.$$

This gives a surface in \mathcal{F}_7 whose members split as the product of three elliptic curves and two abelian surfaces.

5.3. The family \mathcal{G} . In this subsection we study the factors of the group algebra decomposition of a 3-dimensional family of principally polarized abelian varieties of dimension 6 with an action of the dihedral group of order 24

$$G := \langle r, s : r^{12} = s^2 = (rs)^2 = 1 \rangle.$$

First, recall some known results from [2], and the references given there. Let K be a finite group, an action of K on a Riemann surface X is an injective homomorphism from K to the group of holomorphic automorphisms of X . K acts with signature $(\gamma; m_1, \dots, m_t)$ if the quotient surface X/K is of genus γ and the covering $\pi : X \rightarrow X/K$ is ramified over t points over which π is locally $m_i : 1$. Notice that the Riemann-Hurwitz equation

$$g = |K|(\gamma - 1) + 1 + \frac{|K|}{2} \sum_{i=1}^t \left(1 - \frac{1}{m_i}\right)$$

must be satisfied, where g is the genus of X . This imposes restrictions on γ and the m_i 's for given K and g . A signature for K satisfying this numerical condition is called an admissible signature. Moreover, given an admissible signature $(0; m_1, \dots, m_t)$ for a group K , a covering where K acts with that signature can be constructed by a *generating vector*, which is a t -tuple (g_1, \dots, g_t) of elements of K generating K and satisfying $g_1 \cdots g_t = 1$ such that g_i is of order m_i for all i .

In fact, denote by Γ the fundamental group of \mathbb{P}^1 with t points removed. Choose a set of generators $\alpha_1, \dots, \alpha_t$ of Γ satisfying $\alpha_1 \cdots \alpha_t = 1$. Then a generating vector induces an exact sequence

$$1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow K \rightarrow 1$$

and the normal subgroup Γ' of Γ defines the Galois covering $\pi : X \rightarrow \mathbb{P}^1$.

Moreover, in [2] there is an algorithm to compute a symplectic basis of $H_1(X, \mathbb{Z})$ together with the induced action of K on this basis, given a generating vector for the action (under the assumption of having \mathbb{P}^1 as total quotient). The algorithm starts by finding a basis that reflects the action, which is turned into a symplectic basis by the Frobenius algorithm, giving in this

way a symplectic representation for the group K . This algorithm is implemented in a computer program, which is available at <http://www.geometry.uchile.cl>.

For the group studied in this subsection we have that the tuple (r^3, r^4, r^5s, s) is a generating vector for the signature $(0; 4, 3, 2, 2)$. By the Riemann-Hurwitz formula, the corresponding curve is of genus 6. Therefore there is a one-dimensional family of Jacobians with the action of G coming from its action on the corresponding Riemann surface. The parameter of this family comes from the fact that the covering ramifies over four points on \mathbb{P}^1 , therefore fixing three of them and allowing one to move we obtain the family.

Using the method described in [2] we compute the corresponding symplectic representation of G , and we compute the Riemann matrices in \mathcal{H}_6 fixed by this action. We obtain a 3-dimensional family \mathcal{G} of principally polarized abelian varieties of dimension 6 with Riemann matrices

$$Z = \begin{pmatrix} -2x - 2y + 2z & x - \frac{3}{2}z & 0 & x + 2y & 2x + y - 2z & -x - y + z \\ & 2y + 2z & x & -x - 2y - \frac{1}{2}z & -2x + 2y + 2z & x - y - z \\ & & -2y + z & y - \frac{1}{2}z & -x + y - \frac{1}{2}z & x \\ & & & z & -3y & y \\ & & & & 3z & -\frac{3}{2}z \\ & & & & & z \end{pmatrix}$$

The three parameters are complex numbers such that the corresponding matrix has positive definite imaginary part. The period matrix of a generic element A in \mathcal{G} is $\Pi = (I_6 \ Z)$, after fixing a symplectic basis

$$\Gamma = \{\alpha_1, \dots, \alpha_6, \beta_1, \dots, \beta_6\}$$

for the lattice of A_Z . A generic element $A \in \mathcal{G}$ decomposes as $A \sim S \times F$, where S is an abelian surface and F is a 4-dimensional abelian variety. We note that for the parameters $x = y = 0$ and $z = 1$, the matrix

$$Z_0 = \begin{pmatrix} 2 & -\frac{3}{2} & 0 & 0 & -2 & 1 \\ & 2 & 0 & -\frac{1}{2} & 2 & -1 \\ & & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ & & & 1 & 0 & 0 \\ & & & & 3 & -\frac{3}{2} \\ & & & & & 1 \end{pmatrix}$$

is positive definite and $\tau Z_0 \in \mathcal{G}$ for any $\tau \in \mathbb{H}$. The family \mathcal{G} is therefore non-empty and thus three-dimensional.

Following the method in [11, Table 4], we determine that the induced polarization on S is of type $(2, 2)$, and on F is of type $(1, 1, 2, 2)$. Besides, the lattices for S and F are generated by the following symplectic bases:

$$\Gamma_S = \{\alpha_2 + \alpha_5 - \alpha_6, \alpha_1 - \alpha_2 + \alpha_4 - \alpha_5, \beta_1 + \beta_2 - \beta_6, \beta_4 - \beta_5 - \beta_6\},$$

$$\Gamma_F = \{\alpha_3, -\alpha_1 + \alpha_2, 2\alpha_1 - \alpha_2 - \alpha_5 + \alpha_6, -\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5, \beta_3, \beta_2 - \beta_5, \beta_1 + \beta_2 + \beta_6, \beta_4 + \beta_5 + \beta_6\}.$$

S and F are isotypical factors; this is, each one is the image of a central idempotent in the group algebra $\mathbb{Q}[G]$. Therefore they have the action of G . Using the group algebra decomposition, it is possible to decompose both factors further:

$$S \sim E^2, \text{ and } F \sim S_1 \times S_2,$$

where E is an elliptic curve with induced polarization of type (4), and S_i is an abelian surface with induced polarization of type (2, 4). S_i corresponds to a primitive idempotent in $\mathbb{Q}[G]$, therefore using the group action we cannot decompose further either of the surfaces decomposing F .

From Γ_F , replacing each β_i by the combination of α_j given by the Riemann matrix of A , we get the following period matrix for $F = V_F/L_F$. Recall that the basis for V_F is

$$\left\{ \alpha_3, -\alpha_1 + \alpha_2, \frac{2\alpha_1 - \alpha_2 - \alpha_5 + \alpha_6}{2}, \frac{-\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5}{2} \right\},$$

in order to consider F with its induced polarization. Under this consideration the period matrix of L_F is

$$P_F = \begin{pmatrix} 1 & 0 & 0 & 0 & -2y + z & 2x - y + \frac{z}{2} & 2x & 2y - z \\ 0 & 1 & 0 & 0 & 2x - y + \frac{z}{2} & 4x - 2y + z & 2x - 2y + z & -2x + 2y - z \\ 0 & 0 & 2 & 0 & 2x & 2x - 2y + z & -4y + 2z & 2y - z \\ 0 & 0 & 0 & 2 & 2y - z & -2x + 2y - z & 2y - z & -4y - 2z \end{pmatrix}.$$

Moreover, we explore the primitive idempotents in $\mathbb{Q}[G]$ giving varieties decomposing F , and we find 12 of them, all defining abelian surfaces with the same type of polarization (2, 4). We choose two of these abelian surfaces; the advantage of the chosen ones is that they define an isomorphism from $S_1 \times S_2$ to F .

Let Γ_i be a symplectic basis for the lattice L_i of $S_i = V_i/L_i$ for $i = 1, 2$. Then we have

$$\Gamma_1 = \{ \alpha_1 - \alpha_2, -\alpha_2 - 2\alpha_3 + 2\alpha_4 + 3\alpha_5 - \alpha_6, \beta_1 - \beta_2 + \beta_3 - \beta_4 + \beta_5, \beta_4 + \beta_5 + \beta_6 \},$$

$$\Gamma_2 = \{ -\alpha_3 + \alpha_4 + \alpha_5, \alpha_1 - \alpha_2 + \alpha_4 + \alpha_5, \beta_2 - \beta_3 + \beta_4 + \beta_6, \beta_1 - \beta_2 + 2\beta_3 + 2\beta_5 + \beta_6 \}.$$

Note that $\langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle = \langle \Gamma_F \rangle$, therefore there is an isomorphism defined by the addition

$$S_1 \times S_2 \rightarrow F, \quad (s_1, s_2) \mapsto s_1 + s_2.$$

Considering that the induced polarization on S_i is of type (2, 4), and replacing the curves β_j by the corresponding combination of curves α_k given by the Riemann matrix of A , we get the following period matrix for the lattice of S_1 and S_2

$$P = \begin{pmatrix} 2 & 0 & -4x - 6y + 3z & 4x + 4y - 2z \\ 0 & 4 & 4x + 4y - 2z & -4y + 2z \end{pmatrix},$$

We obtain the same period matrix for both subvarieties, therefore they are isomorphic.

Proposition 5.6. *There are polynomials $Q, R \in \mathbb{C}[x, y, z, a_{12}, a_{13}, a_{14}, a_{23}, a_{34}]$ such that in the family of surfaces*

$$f : \text{Spec } \mathbb{C}[x, y, z, a_{12}, a_{13}, a_{14}, a_{23}, a_{34}]/(Q, R) \rightarrow \text{Spec } \mathbb{C}[a_{12}, a_{13}, a_{14}, a_{23}, a_{34}]/(R),$$

each fiber over a rational point of the base scheme corresponds to a (possibly empty) family of principally polarized abelian varieties in \mathcal{G} that split isogenously as the product of elliptic curves. Explicitly, if (x, y, z) belongs to a fiber of f , we replace these in the matrix Z ; if this matrix lies in the Siegel half space it corresponds to a completely decomposable principally polarized abelian variety in \mathcal{G} .

Proof. In order for a member of \mathcal{G} to split as the product of elliptic curves, it is only necessary to make S_1 (and therefore S_2) split. Since S_1 with the restricted polarization is of type $(2, 4)$, we can divide this polarization by 2 to obtain a polarization of type $(1, 2)$. Writing the equations from Proposition 4.4 for when an abelian surface splits as the product of elliptic curves in this scenario and replacing $a_{24} = 2(1 - a_{23})$, the second equation becomes

$$Q = a_{12}(-4x^2 + 2y^2 + \frac{1}{2}z^2 - 4xy + 2xz - 2yz) + 4a_{13}(2x + 2y - z) + a_{14}(4x + 6y - 3z) \\ + a_{23}(-2y + z) + 2a_{34} + 4x + 4y - 2z.$$

and the third equation becomes

$$R = 2a_{13}^2 + 2a_{13} + a_{14}a_{23} + a_{12}a_{34}.$$

These polynomials give us the family of surfaces above. \square

Corollary 5.7. *If we put $a_{12} = 0$, we obtain linear equations that give us families of surfaces in \mathcal{G} that consist of completely decomposable principally polarized abelian varieties.*

For example, the rational point $(0, 0, 0, 1, 0)$ gives us the surface $4x + 2y - z = 0$, and for $\tau \in \mathcal{H}_1$, the matrix

$$\tau \cdot \begin{pmatrix} 6 & -5 & 0 & 1 & -6 & 3 \\ & 8 & 1 & -3 & 6 & -3 \\ & & 4 & -2 & -3 & 1 \\ & & & 4 & 0 & 0 \\ & & & & 12 & -6 \\ & & & & & 4 \end{pmatrix}$$

has positive definite imaginary part and corresponds to a point of this fiber.

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