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Sums of Squares of Linear Forms

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§ 1. Introduction

Let F be a field of characteristic $\neq 2$. For any integer $n \geqq 1$ let $g_F(n)$ be the minimum of all r, such that any sum of squares of n-ary F-linear forms is a sum of r squares of n-ary F-linear forms. This number was first introduced by Mordell in [M] for $F=Q$, but the general definition was first given in [CDLR]. Mordell proved $g_Q(n)=n+3$ for all $n \geqq 1$. We extend this result in (3.1), (3.2) to local and global fields. With the aid of a new invariant $l(F)$ we can extend our investigations of $g_F(n)$ to many other fields. We define the length of F to be $l(F)=\mathrm{Min}\{r \mid \text{any totally positive quadratic form over } F \text{ of dimension } r \text{ represents } 1\}=\mathrm{Min}\{r \mid \text{any totally positive quadratic form of dimension } r \text{ represents all totally positive elements of } F\}$.

Let us recall that an element $a \in F$ is totally positive if it is positive in every ordering of F, i.e. $a \in \sum F^2=$ set of all sums of squares of F. A quadratic form $\phi=\langle a_1, \ldots, a_n\rangle$ is totally positive if all a_i are totally positive. If F is non real, i.e. $-1 \in \sum F^2$, then $l(F)$ is the usual u-invariant $u(F)=\mathrm{Max}\{\dim \phi \mid \phi \text{ an anisotropic quadratic form over } F\}=\mathrm{Min}\{r \mid \text{all forms } \phi \text{ over } F \text{ of dimension } r \text{ represent all elements of } F^*=F \setminus \{0\}\}$. In the formally real case it is interesting to relate $l(F)$ to the generalized u-invariant introduced by Elman and Lam in [E-L], i.e. $u(F)=\mathrm{Max}\{\dim \phi \mid \phi \text{ an anisotropic torsion quadratic form over } F\}$. In Sect. 2 we relate the g-invariant to the l-invariant. The main result (2.15) states that if $l(F)<\infty$, then $g_F(n)=n+l(F)-1$ for all $n \geqq l(F)-1$. We have only weaker estimates for $g_F(n)$ when $n<l(F)-1$. The result above implies that $g_F(n)$ grows asymptotically as n when $l(F)<\infty$. Conversely, if for some $n>m$ we have $g_F(n) \leqq n+m$, then $l(F) \leqq 1+m$, so that $g_F(n) \sim n$. In general we do not know the asymptotic behaviour of $g_F(n)$ when $l(F)=\infty$. We shall briefly discuss this problem in Sect. 5. In Sect. 4 we shall give some estimates for $l(F)$ in terms of $u(F)$ and other invariants of F.

To finish this introduction we shall recall some notations and definitions about quadratic forms. For further details the reader may consult [L]. If $a_1, \ldots, a_n \in F^*$ we denote by $\langle a_1, \ldots, a_n\rangle$ the quadratic form $a_1 X_1^2+\ldots+a_n X_n^2$. If

ϕ is a quadratic form over F, let $D_F(\phi)=\{\phi(x_1,\dots,x_n)|x_1,\dots,x_n\in F$, not all $0\}$ be the set of values of ϕ. The form ϕ is called isotropic if $0\in D_F(\phi)$, anisotropic otherwise. For any $r\geqq 1$, $a\in F^*$ let $r\times\langle a\rangle$ be the form $\langle a,\dots,a\rangle$ of dimension r. We denote by $W(F)$ the Witt ring of equivalence classes of quadratic forms over F and by $I(F)$ the maximal ideal of even dimensional forms. The ideal $I(F)$ is additively generated by the forms $\langle 1,a\rangle$, $a\in F^*$, and $I^n(F)$ is generated by the n-fold Pfister forms $\langle 1,a_1\rangle\otimes\dots\otimes\langle 1,a_n\rangle$, $a_1,\dots,a_n\in F^*$. Let $h(F)=2^d$ be the height of F, i.e. the smallest power of 2 with $2^d W(F)_t=0$. If F is formally real, then $h(F)=2^d$ is the smallest power of two with $2^d\geqq p(F)$, the usual Pythagoras number ([L], [CDLR]). If F is non real and $s(F)=\mathrm{Min}\{r|-1=a_1^2+\dots+a_r^2, a_i\in F\}$ is the level of F, then $h(F)=2s(F)$.

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§2. The Case $l(F)<\infty$

Let F be any field with $2\neq 0$. Our purpose in this section is to compare the following two invariants of F

$$g_F(n)=\mathrm{Min}\left\{r\,\middle|\,\begin{array}{l}\text{any sum of squares of } n\text{-ary } F\text{-linear forms}\\ \text{is a sum of } r \text{ squares of } n\text{-ary } F\text{-linear forms}\end{array}\right\}$$

$(n\geqq 1)$

$$l(F)=\mathrm{Min}\left\{r\,\middle|\,\begin{array}{l}\text{any totally positive quadratic form over } F \text{ of dimen-}\\ \text{sion } r \text{ represents all totally positive elements of } F\end{array}\right\}$$

The following fact will enable us to translate the definition of $g_F(n)$ in the language of quadratic forms.

(2.1) **Proposition.** *Let ϕ be a quadratic form over F of dimension n. Then ϕ is a sum of r squares of linear forms over F if and only if $\phi\perp\rho\cong r\times\langle 1\rangle$ for some form ρ.*

Proof. Let $\phi(X_1,\dots,X_n)$ be such that $\phi=\sum_{i=1}^{r} L_i^2$, where

$$L_i(X_1,\dots,X_n)=\sum_{j=1}^{n} a_{ji}X_j,\qquad a_{ji}\in F,\ 1\leqq i\leqq r,\ 1\leqq j\leqq n.$$

Diagonalizing ϕ, we can assume

$$\phi=a_1X_1^2+\dots+a_nX_n^2=\sum_{i=1}^{r}(a_{1i}X_1+\dots+a_{ni}X_n)^2,$$

with some $a_1,\dots,a_n\in F^*$.

Comparing coefficients we obtain $a_i=\sum_{j=1}^{r} a_{ij}^2$, $\sum_{j=1}^{r} a_{lj}a_{sj}=0$ for all $1\leqq i\leqq n$, $1\leqq l\neq s\leqq n$. This means that $r\times\langle 1\rangle$ represents the elements $a_1,\dots,a_n$ orthogonally, i.e. $\phi\perp\rho\cong r\times\langle 1\rangle$ for some form ρ. Conversely if $\phi\perp\rho\cong r\times\langle 1\rangle$ and ϕ

$=\langle a_1,\dots,a_n\rangle$, then we may reverse the above argument and we get $\phi=\sum_{j=1}^{r} L_j^2$, where $L_j(X_1,\dots,X_n)=\sum_{l=1}^{n} a_{lj}X_l$, $1\leqq j\leqq r$.

(2.3) *Notation.* We say that ϕ is a subform of ψ, in symbols $\phi\leqq\psi$, if $\phi\perp\rho\cong\psi$ for some form ρ.

(2.4) **Corollary.** *For any field F, $g_F(n)=\operatorname{Min}\{r\mid$ every n-dimensional totally positive quadratic form over F is a subform of $r\times\langle 1\rangle\}$.*

Using this description of $g_F(n)$ we deduce easily the following properties

(2.5) $$g_F(1)=p(F),\quad \text{the Pythagoras number of } F$$

(2.6) $$g_F(m)<g_F(n)\quad \text{if } m<n$$

(2.7) $$g_F(m)-m\leqq g_F(n)-n\quad \text{if } m\leqq n$$

(2.8) $$g_F(1)+n-1\leqq g_F(n)\leqq n g_F(1)\quad \text{for all } n\geqq 1$$

(2.9) $g_F(n)=n$ for some $n\geqq 1$ iff $g_F(n)=n$ for all $n\geqq 1$ iff F is Pythagorean.

(2.10) **Proposition.** *If $l=l(F)<\infty$, then for any $n\geqq 1$*

$$g_F(n)\leqq n+l-1.$$

Proof. We proceed by induction on n. For $n=1$ we have $g_F(1)=p(F)\leqq l$. Let us assume the proposition for all $m<n$. Let ϕ be a totally positive quadratic form over F of dimension n. We write $\phi=\langle a\rangle\perp\psi$, where $a\in F^*$ and ψ has dimension $n-1$. Since $g_F(n-1)\leqq n+l-2$, we get from (2.4) $\psi\leqq(n+l-2)\times\{1\}$, i.e. $\psi\perp\rho\cong(n+l-2)\times\langle 1\rangle$, where $\dim\rho=l-1$. Therefore $\psi\perp\rho\perp\langle 1\rangle\cong(n+l-1)\times\langle 1\rangle$. But $\dim(\rho\perp\langle 1\rangle)=l$ implies $a\in D_F(\rho\perp\langle 1\rangle)$, and hence $\rho\perp\langle 1\rangle\cong\langle a\rangle\perp\tau$ with some form τ. Putting the previous all together we get

$$\phi\perp\tau\cong\psi\perp\langle a\rangle\perp\tau\cong\psi\perp\rho\perp\langle 1\rangle\cong(n+l-1)\times\langle 1\rangle$$

so that $g_F(n)\leqq n+l-1$ by (2.4).

(2.11) **Proposition.** i) *If $g_F(n)\leqq n+m$ for some n,m, then $g_F(s)\leqq s+m$ for all $s\leqq n$. In particular, $p(F)=g_F(1)\leqq 1+m$.*

ii) *If $g_F(m)\geqq m+t$ for some m,t, then $g_F(n)\geqq n+t$ for all $n\geqq m$.*

Proof. Both results follow immediately from (2.7)

(2.12) **Proposition.** *Let F be a field with $g_F(n)\leqq n+m$ for some n,m with $n>m$. Then $l(F)\leqq m+1$.*

Proof. Because of (2.11), (i), we have $g_F(m+1)\leqq 2m+1$. Then for any totally positive quadratic form ϕ of dimension $m+1$ we have $\phi\leqq(2m+1)\times\langle 1\rangle$, i.e. $\phi\perp\psi\cong(2m+1)\times\langle 1\rangle$, where $\dim\psi=m$. Since $g_F(m)\leqq 2m$ (s. (2.11)), we have $\psi\perp\rho\cong 2m\times\langle 1\rangle$ with some form ρ. Therefore $\phi\perp\chi\cong(2m+1)\times\langle 1\rangle$

$\cong\langle 1\rangle\perp\psi\perp\rho$. Cancelling ψ we get $\phi\cong\langle 1\rangle\perp\rho$, i.e. $1\in D_F(\phi)$. This proves $l(F)\leqq m+1$.

Combining (2.12) with (2.10) we deduce

(2.13) **Corollary.** *For any field F, $g_F(n)<2n$ for some n if and only if $l(F)\leqq n$. In particular $l(F)=\infty$ implies $g_F(n)\geqq 2n$ for all $n\geqq 1$.*

(2.14) **Corollary.** *If $1<l=l(F)<\infty$, then*

$$g_F(l-1)=2l-2$$

Proof. From (2.10) we have $g_F(l-1)\leqq 2(l-1)$. If $g_F(l-1)<2l-2$, we get from (2.13) $l\leqq l-1$, a contradiction. This proves $g_F(l-1)=2l-2$.

Now we deduce immediately the main result of this section.

(2.15) **Theorem.** *Let F be a field with $l(F)<\infty$. Then*

$$g_F(n)=n+l(F)-1$$

for all $n\geqq l(F)-1$.

Proof. From (2.14) and (2.7) we get $g_F(n)\geqq n+l-1$ for all $n\geqq l-1$. Now (2.10) implies the equality.

(2.16) **Corollary.** *Let K/F be a finite extension of degree $n=[K:P]$. Then if $l(F)<\infty$,*

$$p(K)\leqq n+l(F)-1.$$

Proof. It is well known that $p(K)\leqq g_F(n)$ ([CDLR]). Thus the result follows from (2.15).

The natural question which arises from (2.15) is what values may $g_F(n)$ take for $n<l-1$. For example if $p(F)=l(F)<\infty$, then it follows from (2.10) and (2.8) that $g_F(n)=n+l-1$ for all $n\geqq 1$. This remark applies to $F=Q$ or $Q((x))$ (see Sect. 3). But in general it is not easy to determine the behaviour of $g_F(n)$ for $n<l-1$. We now give an example in this direction.

(2.17) **Proposition.** *If $p(F)=2$, $l(F)<\infty$, then $g_F(n)=2n$ for all $n\leqq l(F)-1$.* (See Sect. 5 for the case $l(F)=\infty$.)

This result follows immediatelly from (2.14) and the following (take $p=2$, $n_0=l-1$).

(2.18) **Proposition.** *Let F be a field with $g_F(n_0)=p(F)n_0$ for some $n_0\geqq 1$. Then $g_F(n)=p(F)n$ for all $n\leqq n_0$.*

Proof. We know $g_F(n)\leqq p(F)n$ for all $n\geqq 1$. Suppose $g_F(n)<p(F)n$ for some $n<n_0$. Then from (2.4) we see that $g_F(n_0)\leqq g_F(n)+g_F(n_0-n)<p(F)\,n+p(F)(n_0-n)=p(F)n_0$, which is a contradiction. This shows $g_F(n)=p(F)n$, $n\leqq n_0$.

For example let us consider $F=\mathbb{C}((t_1))((t_2))((t_3))$. It is well known that $p(F)=2$, $l(F)=8$, so that $g_F(n)=2n$ for all $n\leqq 7$ and $g_F(n)=n+7$ for all $n\geqq 7$. Next, we shall determine completely $g_F(n)$ for fields with $l(F)=4$. To this end, let us first show the following.

(2.19) **Proposition.** *Let F be a field with $p(F) \leqq 2^r$. If $g_F(n) < 2^r k$ for some n, k, then $g_F(n+1) \leqq 2^r k$. In particular, if $g_F(n) < 2^r$, then $g_F(n+1) \leqq 2^r$.*

Proof. Let ϕ be a totally positive quadratic form over F of dimension $n+1$. We set $\phi = \langle a \rangle \perp \psi$, with $a \in F^*$, ψ totally positive, $\dim \psi = n$. We want to show $\phi \leqq 2^r k \times \langle 1 \rangle$. We may assume $a = 1$, because if $\langle a \rangle \phi \leqq 2^r k \times \langle 1 \rangle$, then $\phi \leqq 2^r k \times \langle a \rangle$. But a is a sum of $p(F) \leqq 2^r$ squares, and since $2^r \times \langle 1 \rangle$ is round (s. [L]), it follows $\phi \leqq 2^r k \times \langle 1 \rangle$. Hence let us assume $\phi = \langle 1 \rangle \perp \psi$. Since $\dim \psi = n$ and $g_F(n) \leqq 2^r k - 1$, we have $\psi \leqq (2^r k - 1) \times \langle 1 \rangle$. Therefore $\phi \leqq 2^r k \times \langle 1 \rangle$. This proves the proposition.

(2.20) **Corollary.** *Let F be a field with $l(F) = 4$. Then $g_F(n) = n+3$ for all $n \geqq 3$ and*

$$g_F(2) = 4 \quad \text{if } p(F) = 2, 3$$

$$g_F(2) = 5 \quad \text{if } p(F) = 4.$$

Proof. Let us assume $p(F) = 4$. From (2.14) we know $g_F(3) = 6$, so that according to (2.6), (2.8) we get $1 + p(F) = 5 \leqq g_F(2) < g_F(3) = 6$, i.e. $g_F(2) = 5$. If $p(F) = 3$, then $g_F(2) \geqq 4$ ((2.6)). But on the other hand $g_F(1) = p(F) < 2^2$ implies because of (2.19), $g_F(2) \leqq 2^2$. This shows $g_F(2) = 4$. If $p(F) = 2$, then we use (2.17) to deduce $g_F(2) = 4$.

§ 3. Some Examples

i) Let F be a finite field. Then $l(F) = u(F) = 2$ ([L]), and we get $g_F(n) = n+1$ for all $n \geqq 1$.

ii) The p-adic local and global fields of number theory have length 4. This follows in the local case from $l = u$ and the well known fact that $u(F) = 4$ ([L]).

In the global case, since $l(F_p) \leqq 4$ for all completions of F, we obtain from the Hasse-Minkowski theorem, that $l(F) \leqq 4$. Using the approximation theorem, we can construct a totally positive quadratic form over F of dimension 3 which does not represent 1, and this implies $l(F) = 4$. Combining this result with (3.20) we obtain

(3.1) **Proposition.** *Let F be a p-adic field. Then $g_F(n) = n+3$ for all $n \geqq 3$. If $s(F) \leqq 2$, then $g_F(2) = 4$, and if $s(F) = 4$, then $g_F(2) = 5$.*

(3.2) **Proposition.** *Let F be a global field. Then for all $n \geqq 3$, $g_F(n) = n+3$. If $p(F) = 2$ or 3, then $g_F(2) = 4$, and if $p(F) = 4$, then $g_F(2) = 5$.*

iii) Let us consider the field $\mathbb{R}(X)$, which also satisfies a local-global principle ([K]). Let p be a prime spot of $\mathbb{R}(X)$. Then $\mathbb{R}(X)_p \cong \mathbb{R}((X))$ or $\mathbb{C}((X))$. But $l(\mathbb{R}((X)))$, $l(\mathbb{C}((X))) \leqq 2$, so that $l(\mathbb{R}(X)) = 2$.

From (2.15) we conclude $g_{\mathbb{R}(X)}(n) = n+1$ for all $n \geqq 1$ ([P-O]).

iv) **Proposition.** *For any formally real field F*

$$l(F) = l(F((t))).$$

Proof. It is obvious that $l(F) \leqq l(F((t)))$. Let ϕ be a totally positive quadratic form over $F((t))$ of dimension n, $\phi = \langle f_1, \ldots, f_n \rangle$, $f_i \in F[\![t]\!]$. Since f_i is a sum of squares, we can alter each f_i by a square to assume that $f_i = a_i + tg_i$, $g_i \in F[\![t]\!]$, $a_i \in \sum F^2 \setminus \{0\}$. But $f_i = a_i(1 + a_i^{-1} tg_i) = a_i h_i^2$ for some $h_i \in F[\![t]\!]$, so that $\phi \cong \langle a_1, \ldots, a_n \rangle$. Therefore $l(F) \geqq l(F((t)))$, and this proves the proposition. In particular we get $l(F) = l(F((t_1))(t_2))\ldots)$.

v) Let us apply examples iii), iv) to compute $l(\mathbb{R}((X, Y)))$ where $\mathbb{R}((X, Y)) = \mathrm{Quot}(\mathbb{R}[\![X, Y]\!])$. Let ϕ be a two dimensional totally positive form over $\mathbb{R}((X, Y))$, $\phi = \langle a, b \rangle$, where $a, b \in \sum \mathbb{R}[\![X, Y]\!]^2$ without restriction. It [CDLR], Sect. 5, it has been shown that $\sum \mathbb{R}((X, Y))^2 \equiv \sum \mathbb{R}(X)[\![Y]\!]^2 (\mathrm{mod}\, \mathbb{R}((X, Y))^2)$ so that without restriction $a, b \in \sum \mathbb{R}(X)[\![Y]\!]^2$.

But by v), iii), $l(\mathbb{R}(X)[\![Y]\!]) = l(\mathbb{R}(X)) = 2$, so that ϕ represents all totally positive elements of $\mathbb{R}((X, Y))$. This shows $l(\mathbb{R}((X, Y))) \leqq 2$. But $p(\mathbb{R}((X, Y))) = 2$ ([CDLR]), so that $l(\mathbb{R}((X, Y))) = 2$. This fact has been also noticed independently by E. Hornix.

vi) It is natural to ask what values of $l(F)$ can occur. This seems to be a very difficult question. If F is non real it is well known that $l(F) = u(F)$ may take as value any power of two. If F is formally real, then it is also true that any power of two can be realized as $l(F)$ for some formally real field F. This follows from a construction of Prestel [Pr]. Let $\tilde{u}(F) = \mathrm{Min}\{n \mid \text{every totally indefinite quadratic form over } F \text{ of dimension } n+1 \text{ is isotropic}\}$. Then $l(F) \leqq \tilde{u}(F)$ because if ϕ is totally positive of dimension $\tilde{u}$, then $\phi \perp \langle -1 \rangle$ is totally indefinite of dimension $> \tilde{u}$, and hence it is isotropic. Thus ϕ represents 1, i.e. $l \leqq \tilde{u}$. Now in [Pr], Sect. 3, a chain of fields $\{K_n\}_{n \in \mathbb{N}}$ is constructed with K_n uniquely ordered, $p(K_n) = 2$ and $\tilde{u}(K_n) = 2^n$. Moreover, there is an n-fold Pfister form ρ_n defined over K_n, totally positive, anisotropic over $K_n(\sqrt{-1})$. This implies, that ρ_n', where $\rho_n = \langle 1 \rangle \perp \rho_n'$, can not represent 1 over K_n, and hence $l(K_n) > 2^n - 1$. But $l(K_n) \leqq \tilde{u}(K_n) = 2^n$, so that $l(K_n) = 2^n$.

§ 4. Some Estimates for $l(F)$

Our main purpose in this section is to compare $l(F)$ with $u(F)$ and other invariants of F. We shall assume throughout that F is formally real, because for a non real field F, $l(F) = u(F)$. When convenient, we let $h = h(F)$, $u = u(F)$, $l = l(F)$. The following fact will be used frequently, so we state it separately in the next lemma.

(4.1) Lemma. *If $\langle a_1, \ldots, a_{2n} \rangle$ is a totally positive quadratic form with $2n > u(F)$, then $\langle a_1, \ldots, a_{2n} \rangle \cong 2 \times \langle a \rangle \perp \langle b_1, \ldots, b_{2n-2} \rangle$ for some $a, b_1, \ldots, b_{2n-2} \in F^*$ totally positive.*

Proof. The quadratic form $\langle a_1, \ldots, a_n, -a_{n+1}, \ldots, -a_{2n} \rangle$ has total signature 0, so that it is a torsion form ([L]). Since $2n > u$, it is isotropic, i.e. we have $u_1^2 a_1 + \ldots u_n^2 a_n + u_{n+1}^2 a_{n+1} - \ldots - u_{2n}^2 a_{2n} = 0$ with some $u_1, \ldots, u_{2n} \in F$, not all 0. Setting $a = u_1^2 a_1 + \ldots + u_n^2 a_n$ we get $a \in D(\langle a_1, \ldots, a_n \rangle) \cap D(\langle a_{n+1}, \ldots, a_{2n} \rangle)$. This proves (4.1)

(4.2) **Lemma.** *Let $a \in \sum F^2$. Suppose for a given integer $m \geqq 1$, every totally positive quadratic form ϕ over F of dimension $>m$ contains a subform $\langle b \rangle \langle 1, a \rangle$ for some $b \in \sum F^2$. Let ϕ be a totally positive quadratic form over F. If $\dim \phi \geqq m(2^t - 1) + 1$, $t \geqq 1$, then ϕ contains a subform $\langle d \rangle \langle 1, a \rangle^t = \langle d \rangle 2^{t-1} \times \langle 1, a \rangle$ for some $d \in \sum F^2$.*

Proof. The proof is by induction on t. The case $t=1$ is obvious. Now assume $t \geqq 2$. Let $\dim \phi = m(2^t - 1) + 1$. Using the hypothesis repeatedly, we can write ϕ as $\phi = \langle 1, a \rangle \psi \perp \rho$, where $\dim \psi = m(2^{t-1} - 1) + 1$, $\dim \rho = m - 1$. By induction, ψ contains $\langle d \rangle \langle 1, a \rangle^{t-1}$ as a subform, and therefore ϕ contains $\langle d \rangle \langle 1, a \rangle^t$ as a subform, where $d \in \sum F^2$.

(4.3) *Remark.* Assuming the hypothesis of (4.2), we deduce from (4.2) that if ϕ is a totally positive quadratic form of dimension $\geqq m(2^t - 1) + 2^t + 1$, $t \geqq 1$, then ϕ contains a subform

$$\langle 1, a \rangle^t \langle b \rangle \langle 1, c \rangle \cong 2^{t-1} \times \langle 1, a \rangle \langle b \rangle \langle 1, c \rangle \quad \text{for some } b, c \in \textstyle\sum F^2.$$

Lemma (4.2) enables us to quickly obtain several estimates relating $l(F)$ and $u(F)$.

(4.4) **Proposition.** *Let F be a formally real field and assume $u(F) < \infty$. Then $l(F) \leqq (h(F) - 1)u(F) + 1$.*

Proof. If $h=1$, then $p(F)=1$ and therefore $l=1$. Now assume $h \geqq 2$. Because of (4.1), we can apply (4.2) which $m=u+1$, $a=1$. Let $h=2^t$, $t \geqq 1$. If ϕ is a totally positive quadratic form with $\dim \phi \geqq (u+1)(h-1)+1$, then ϕ contains a subform $\langle d \rangle \langle 1, 1 \rangle^t \cong h \times \langle 1 \rangle$. If ϕ is a totally positive quadratic form with $\dim \phi \geqq (h-1)u + 1$, then $\phi \perp (h-1) \times \langle 1 \rangle$ contains $h \times \langle 1 \rangle$ as a subform and therefore $1 \in D_F(\phi)$ by cancellation. This shows $l(F) \leqq u(h-1)+1$.

(4.5) **Corollary.** *If $u(F) < \infty$, then $l(F) < \infty$.*

Proof. Use (4.4) and the fact $h \leqq u$.

The converse of (4.5) is not true. Example iv) of Sect. 3 shows $l(F) = l(F((t_1))((t_2))\ldots)$ when F is formally real. Taking $K = Q((t_1))((t_2))\ldots$, we see $l(K) = l(Q) = 4$, but it is well known that $u(K) = \infty$.

(4.6) **Proposition.** *Let F be a formally real field and suppose $I^{n+1}(F)_t = 0$ for some $n \geqq 0$. Then $l(F) = 1$ if $n=0$ and $l(F) \leqq 2^{n-1} + 1 + (2^{n-1} - 1)u(F)$ if $n \geqq 1$.*

Proof. This is clear for $n=0$ since F is Pythagorean in this case. Let ϕ be a totally positive form of dimension $2^{n-1} + 1 + (2^{n-1} - 1)(u+1)$, $n \geqq 1$. Because of (4.1) we may apply (4.3) with $m = u+1$, $t = n-1$, $a=1$. Then ϕ contains $\langle 1, 1 \rangle^{n-1} \langle b \rangle \langle 1, c \rangle \cong 2^{n-1} \times \langle b \rangle \langle 1, c \rangle$ as a subform for some $b, c \in \sum F^2$. But $2^{n-1} \times \langle b \rangle \langle 1, c \rangle \cong 2^{n-1} \times \langle 1, c \rangle$ since $I^{n+1}(F)_t = 0$. Therefore if ϕ is totally positive of dimension $2^{n-1} + 1 + (2^{n-1} - 1)u$, then $\phi \perp (2^{n-1} - 1) \times \langle 1 \rangle$ contains a subform $2^{n-1} \times \langle 1, c \rangle$. Cancellation shows $1 \in D_F(\phi)$ and the result is proved.

(4.7) **Corollary.** *If $2^n \leqq u(F) < 2^{n+1}$ then $l(F) \leqq 2^{n-1} + 1 + (2^{n-1} - 1)u(F)$. In particular $l \leqq \frac{u^2}{2} - \frac{u}{2} + 1$.*

Proof. Since $u(F)<2^{n+1}$, the theorem of Arason-Pfister ([E-L]) implies that $I^n(F)_t=0$, and hence by (4.6) $l(F)\leqq 2^{n-1}+1+(2^{n-1}-1)u$. Since $2^{n-1}\leqq\frac{u}{2}$, we obtain $l\leqq\frac{u^2}{2}-\frac{u}{2}+1$.

(4.8) **Corollary.** *Let F be a formally real field with $2h(F)>u(F)$. Then $l\leqq\frac{h}{2}+1+\left(\frac{h}{2}-1\right)u$.*

Proof. Let $2^n\leqq u<2^{n+1}$. Then $u<2h$ implies $2^{n+1}\leqq 2h$. The result now follows from (4.7).

Finally let us relate $l(F)$ with the values of the u-invariant of quadratic extensions of F.

(4.9) **Proposition.** *Let F be a formally real field and let $a\in\sum F^2$.*

i) *If $u(F(\sqrt{-a}))\leqq 2^n$, $n\geqq 0$, then $l(F)\leqq 1+(2^n-1)^2$*

ii) *If $u(F(\sqrt{-a}))\leqq 4$, then $l(F)\leqq 6$.*

Proof. i) If $n=0$, $u(F(\sqrt{-a}))=1$. This implies F is Euclidean ([Be]) and $l(F)=1$. Now assume $n\geqq 1$. Because of (3.7), Chap. 7 in [L], we can apply Lemma (4.2) with $m=2^n$, $t=n$. Let ϕ be a totally positive quadratic form of dimension $1+(2^n-1)^2$. Then $\phi'=\phi\perp(2^{n-1}-1)\times\langle 1\rangle\perp 2^{n-1}\times\langle a\rangle$ has dimension $(2^n-1)2^n+1$. Therefore (4.2) implies ϕ' contains a subform $\langle d\rangle 2^{n-1}\times\langle 1,a\rangle\cong 2^{n-1}\times\langle 1,a\rangle$, since $u(F(\sqrt{-a}))\leqq 2^n$ implies $I^{n+1}(F)_t=0$ ([E]). Cancellation yields that ϕ contains $\langle 1\rangle$ as a subform. This shows $l(F)\leqq 1+(2^n-1)^2$.

ii) The assumption $u(F(\sqrt{-a}))\leqq 4$ implies $I^3(F)_t=0$, and hence every form in I^2F represents all totally positive elements. Let us first consider a totally positive quadratic form ϕ with $\dim\phi=7$. Since $\phi\otimes F(\sqrt{-a})$ contains two hyperbolic planes, it follows $\phi\cong\langle b,c\rangle\langle 1,a\rangle\perp\phi_1$, so that by the above remark, $\langle 1,bc\rangle\langle 1,a\rangle\cong\langle b,c\rangle\langle 1,a\rangle$ is a subform of ϕ. Therefore, every totally positive form of dimension 7 contains $\langle 1,a\rangle$ as a subform. Now let ϕ be any totally positive form with $\dim\phi=6$. Then $\phi\perp\langle a\rangle$ contains $\langle 1,a\rangle$ as a subform and cancellation shows $1\in D_F(\phi)$. This proves ii).

(4.10) **Corollary.** *Let R be a real closed field and F/R a formally real extension with $tr(F/R)=n$. Then $l(F)\leqq 1+(2^n-1)^2$. If $n=2$, then $l(F)\leqq 6$ and moreover $l(F)\neq 5$.*

Proof. By the theorem of Tsen-Lang we know $u(F(\sqrt{-1}))\leqq 2^n$ ([G]). Therefore $l(F)\leqq 1+(2^n-1)^2$. If $n=2$, we have from (4.9), ii) that $l(F)\leqq 6$. Since in this case $I^3(F)_t=0$, we deduce $l(F)\neq 5$ from the following result (4.11), which we state separately.

(4.11) **Lemma.** *Let F be a field satisfying $I^3(F)_t=0$. If $1<l(F)<\infty$, then $l(F)\not\equiv 1 \pmod 4$.*

Proof. Assume $l=1+4k$, $k\geqq 1$, and let ϕ be a totally positive form of dimension $4k$. Since $d=\det(\phi)$ is totally positive, $\phi\perp\langle d\rangle\cong\langle 1\rangle\perp\psi$, with $\dim\psi=4k$,

$\det(\psi)=1$. Then $\psi \in I^2(F)$, since $d(\psi)=1$. But $I_t^3=0$ implies ψ represents all totally positive elements, and so we have $\psi \cong \langle d \rangle \perp \rho$. This implies $\phi \cong \langle 1 \rangle \perp \rho$, and this shows $l(F) \leqq 4k$, a contradiction.

§ 5. The Case $l(F)=\infty$

When $l(F)<\infty$, the behaviour of $g_F(n)$ as a function of n is fairly well described by Theorem (2.15). If $l(F)=\infty$, we saw in (2.8) and (2.13) that $2n \leqq g_F(n) \leqq g_F(1)n$ for all $n \geqq 1$. Therefore, $l(F)<\infty$ if and only if $g_F(n) \sim n$. When $l(F)=\infty$ we treat below the case $p(F)=2$ and the non real case.

(5.1) **Proposition.** *If $p(F)=2$ and $l(F)=\infty$, then $g_F(n)=2n$ for all $n \geqq 1$.*

Proof. This follows from (2.13) and (2.8).

(5.2) **Proposition.** *Let F be a field with $l(F)=\infty$ and level $s=s(F)<\infty$. For any n let $r \in \{0, 1, \ldots, s-1\}$ be determined by $n+r \equiv 0 \pmod{s}$. Then $2n \leqq g_F(n) \leqq 2n+r$ and these bounds are best possible. In particular $g_F(n) \sim 2n$.*

Proof. Let ϕ be a n-dimensional quadratic form over F. Assume first $n \equiv 0 \pmod{s}$. Therefore $n \times \langle 1, -1 \rangle \cong n \times \langle 1, 1 \rangle = 2n \times \langle 1 \rangle$, and hence $\phi \perp -\phi \cong n \times \langle 1, -1 \rangle \cong 2n \times \langle 1 \rangle$ implies $g_F(n) \leqq 2n$. How from (2.13) we obtain $g_F(n) = 2n$. Let us now assume $n \not\equiv 0$ (*mod s*) and take $r \in \{0, 1, \ldots, s-1\}$ with $n+r \equiv 0 \pmod{s}$. We see from above that $\phi \perp r \times \langle 1 \rangle \leqq 2(n+r) \times \langle 1 \rangle$, and cancel ling we get $\phi \leqq (2n+r) \times \langle 1 \rangle$, i.e. $g_F(n) \leqq 2n+r$. Together with (2.13) we have $2n \leqq g_F(n) \leqq 2n+r$. The following example shows that these bounds can be realized. Let k be a field of level $s<\infty$, and define $F=k((t_1))((t_2))\ldots$. It is clear that $s(F)=s$, $l(F)=\infty$. Hence $2n \leqq g_F(n) \leqq 2n+n$ with r as above. We shall prove $g_F(n)=2n+r$. From the proof above we may assume $n \not\equiv 0 \pmod{s}$. Let us consider the quadratic form $\phi_n = \langle t_1, \ldots, t_n \rangle$. We have $\phi_n \leqq (2n+r) \times \langle 1 \rangle$. If $\phi_n \leqq (2n+r-1) \times \langle 1 \rangle$, then $\phi_n \perp \psi \cong (2n+r-1) \times \langle 1 \rangle$ with $\dim \psi = n+r-1$. Therefore $\phi_n \perp \psi \perp (r+1) \times \langle 1 \rangle \cong 2(n+r) \times \langle 1 \rangle \cong (n+r) \times \langle 1, -1 \rangle$, because $s \mid n+r$. But $(n+r) \times \langle 1, -1 \rangle \cong \langle 1, -1 \rangle \perp \psi \perp -\psi$, so that cancelling ψ, we get $\phi_n \perp (r-1) \times \langle 1 \rangle \cong \langle 1, -1 \rangle \perp -\psi$. Therefore $\phi_n \perp (r+1) \times \langle 1 \rangle$ is isotropic over F. Since $r<s$, it is easy to see, that this is impossible. Therefore $g_F(n)=2n+r$. Also (5.1) shows that the lower bound can be realized.

(5.3) *Remark.* For all examples calculated in the case $l(F)=\infty$, $g_F(n) \sim 2n$. We know no example when this fails.

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