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Global Bifurcation from the Eigenvalues of the p-Laplacian*

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1. INTRODUCTION

In this paper we study some bifurcation phenomena associated with the p-Laplacian operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad p>1$$

under Dirichlet boundary conditions. Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{2,\beta}$. We consider the problem

$$\text{(B)} \quad \begin{aligned} -\Delta_p u &= \lambda\,|u|^{p-2}\,u + f(x,u,\lambda) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f\colon \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies a Carathéodory condition in the first two variables and

$$f(x,s,\lambda) = o(|s|^{p-1}) \tag{1.2}$$

near $s=0$, uniformly a.e. with respect to x and uniformly with respect to λ on bounded sets.

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By a solution of (B) we understand a pair $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ satisfying (1.1) in the weak sense, i.e., such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_\Omega (\lambda |u|^{p-2} u + f(x, u, \lambda))\, v, \qquad \text{for all} \quad v \in W_0^{1,p}(\Omega). \tag{1.3}$$

We note that the pair $(\lambda, 0)$ is a solution of (B) for every $\lambda \in \mathbb{R}$. Pairs of this form will be designated as the trivial solutions of (B). We say that $P = (\bar{\lambda}, 0)$ is a bifurcation point of (B) if in any neighborhood of P in $\mathbb{R} \times W_0^{1,p}(\Omega)$ there exists a nontrivial solution of (B).

Henceforth, the function f in (1.1) will be assumed to satisfy the growth restriction

(G) There is a $1 < q < p^*$, such that

$$\lim_{|s| \to \infty} \frac{|f(x, s, \lambda)|}{|s|^{q-1}} = 0 \tag{1.4}$$

uniformly a.e. with respect to x and uniformly with respect to λ on bounded sets. Here

$$p^* = \begin{cases} NP/N - P & \text{if} \quad p < N \\ +\infty & \text{if} \quad p \geqslant N. \end{cases}$$

A rather standard compactness argument is used in Proposition 2.1 of the next section to show that a necessary condition for $(\bar{\lambda}, 0)$ to be a bifurcation point of (1.1) is that $\bar{\lambda}$ be an eigenvalue of the problem

$$(\mathrm{E}_p) \qquad \begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

Let $\lambda_1(p)$ denote the first eigenvalue of (E_p). We recall that $\lambda_1(p)$ can be variationally characterized as

$$\lambda_1(p) = \inf\left\{\int_\Omega |\nabla u|^p \,\middle|\, u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1\right\}. \tag{1.6}$$

Recently, some papers extending to a general p many of the main properties of the first eigenvalue of the usual Laplacian and its proper subspace have appeared. See, for instance, de Thelin [6], Barles [3], Bhattarcharya [4], and Anane [1]. In this last reference it is shown that $\lambda_1(p)$ is an isolated eigenvalue of (E_p). This fact will be important in the proof, which we will carry out in Section 2, of our

THEOREM 1.1. *The pair* $(\lambda_1(p), 0)$ *is a bifurcation point of* (B). *Moreover, there is a component of the set of nontrivial solution of* (B) *in* $\mathbb{R} \times W_0^{1,p}(\Omega)$ *whose closure contains* $(\lambda_1(p), 0)$ *and is either unbounded or contains a pair* $(\bar{\lambda}, 0)$ *for some* $\bar{\lambda}$, *eigenvalue of* (E_p) *with* $\bar{\lambda} \neq \lambda_1(p)$.

This result is well known in the case $p=2$ [see Rabinowitz, 10]. A key ingredient to extend the classical result to our situation is an "index formula" which is proved via a suitable homotopic deformation from a general $p>1$ to $p=2$ (see Proposition 2.2 below). We thus extend to the PDE case a result which was first derived in [7] in connection with the one-dimensional p-Laplacian.

In Section 3 we apply Theorem 1.1 to show existence of nontrivial solutions in a boundary value problem involving the p-Laplacian.

In Section 4 we restrict ourselves to the radial situation. Thus we let Ω be the unit ball in $\mathbb{R}^N$ and we deal with problem (B) where now we impose radial symmetry on f. For this Ω, the eigenvalues of (E_p), relative to the class of radial functions, are isolated and can be ordered as an increasing unbounded sequence. We show that bifurcation occurs from each of these eigenvalues and we extend the result of Theorem 1.1 to the corresponding branches. We apply these results to obtain existence of multiple nontrivial solutions for a boundary value problem under radial symmetry.

In [8] for the case $N=1$, Guedda and Veron performed a rather complete study of problem (B) for f dependent only on u and satisfying some further growth and symmetry restrictions. Their method, however, seems to extend neither to higher dimensions nor to the nonautonomous case.

We end this section by establishing some notation conventions which will be used in this paper. Thus, for $p>1$ we set $p' = p/(p-1)$, $\varphi_p(s) \equiv |s|^{p-2}s$, $W_0^{1,p} \equiv W_0^{1,p}(\Omega)$, $L^p \equiv L^p(\Omega)$, $\|u\|_p \equiv \|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p)^{1/p}$, $\|u\|_{1,p} \equiv \|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p)^{1/p}$, and we let $\| \ \|_0$ denote the sup norm. Also, for a measurable set $A \subset \mathbb{R}^n$, we denote its measure by $|A|$.

2. BIFURCATION FROM THE FIRST EIGENVALUE

We begin this section with some preliminary remarks concerning the auxiliary problem

$$(\mathrm{AP}) \qquad \begin{aligned} -\Delta_p u &= h \quad && \text{in } \Omega \\ u &= 0 \quad && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

It is well known that problem (AP) possesses a unique weak solution for each $h \in W^{-1,p'}$, i.e., the problem of finding a $u \in W_0^{1,p}$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle h, v \rangle \tag{2.2}$$

for all $v \in W_0^{1,p}$ is uniquely solvable. In (2.2), $\langle\, , \rangle$ denotes the duality pairing between $W_0^{1,p}$ and $W^{-1,p'}$. It is also known that the solution of this problem is characterized as the unique element of $W_0^{1,p}$ minimizing the coercive strictly convex functional $J_p \colon W_0^{1,p} \to \mathbb{R}$ given by

$$J_p(v) = \frac{1}{p} \int_\Omega |\nabla v|^p - \langle h, v \rangle. \tag{2.3}$$

Let us denote by $R_p(h)$ the unique weak solution of (AP). Then $R_p \colon W^{-1,p'} \to W_0^{1,p}$ is a continuous operator. Also, since $W_0^{1,p}$ embeds compactly into L^r for each $r \in (1, p^*)$ it follows that the restriction of R_p to $L^{r'}$ is a completely continuous operator. Moreover, R_p transforms weak convergence in $L^{r'}$ into strong convergence in $W_0^{1,p}$.

Now, let us reformulate problem (B). Clearly the pair (λ, u) is a solution of this problem if and only if (λ, u) satisfies

$$u = R_p(\lambda \varphi_p(u) + F(\lambda, u)). \tag{2.4}$$

In (2.4), $F(\lambda, \cdot)$ denotes the usual Nemitsky operator associated with f. From condition (G), the right hand side of (2.4) defines a completely continuous operator from $W_0^{1,p}$ into itself.

Next, let us suppose that $(\bar{\lambda}, 0)$ is a bifurcation point of problem (B). We want to show that $\bar{\lambda}$ is an eigenvalue of (E_p). Since $(\bar{\lambda}, 0)$ is a bifurcation point there is a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ of nontrivial solutions of problem (B) such that $\lambda_n \to \bar{\lambda}$ in $\mathbb{R}$ and $u_n \to 0$ in $W_0^{1,p}$. Also, since (λ_n, u_n) satisfies (2.4) we have

$$\hat{u}_n = R_p\left(\lambda_n \varphi_p(\hat{u}_n) + \frac{F(\lambda_n, u_n)}{\|u_n\|_{1,p}^{p-1}} \right), \tag{2.5}$$

where $\hat{u}_n = u_n / \|u_n\|_{1,p}$.

We claim that

$$\frac{F(\lambda_n, u_n)}{\|u_n\|_{1,p}^{p-1}} \to 0 \qquad \text{in } L^{q'}, \tag{2.6}$$

where q is chosen such that condition (G) is satisfied, i.e., $q < p^*$, and, without loss of generality, such that $p < q$. We first note that

$$\frac{F(\lambda_n, u_n)}{\|u_n\|_{1,p}^{p-1}} = \frac{F(\lambda_n, u_n)}{\varphi_p(u_n)} \varphi_p(\hat{u}_n). \tag{2.7}$$

Thus from (2.7) and Hölder's inequality, we find that to prove the claim it suffices to find a real number $r > 1$ and a constant $C > 0$ so that

$$\left|\frac{F(\lambda_n, u_n)}{\varphi_p(u_n)}\right|^{q'} \to 0 \quad \text{in} \quad L^r \tag{2.8}$$

and

$$\| \, |\varphi_p(\hat{u}_n)|^{q'} \|_{L^{r'}} \leqslant C \tag{2.9}$$

for all $n \in \mathbb{N}$. To find this r, let us fix $\varepsilon > 0$ and choose positive numbers $\delta = \delta(\varepsilon)$ and $M = M(\delta)$ such that for every $x \in \Omega$ and $n \in \mathbb{N}$, the following relations hold:

$$|f(x, s, \lambda_n)| \leqslant \varepsilon \, |s|^{p-1} \quad \text{for} \quad |s| \leqslant \delta \tag{2.10}$$

and

$$|f(x, s, \lambda_n)| \leqslant M \, |s|^{q-1} \quad \text{for} \quad |s| \geqslant \delta. \tag{2.11}$$

Let r be a real number greater than 1. Then from (2.10) and (2.11) we obtain

$$\left\| \left|\frac{F(\lambda_n, u_n)}{\varphi_p(u_n)}\right|^{q'} \right\|_r^r = \int_\Omega \left|\frac{f(x, u_n, \lambda_n)}{|u_n|^{p-1}}\right|^{q'r} dx$$

$$\leqslant \varepsilon \, |\Omega| + M^{q'r} \int_\Omega |u_n|^{q'r(q-p)}.$$

From this inequality and since $u_n \to 0$ in $W_0^{1,p}$ we have that (2.8) is satisfied if

$$q'r(q-p) < p^*. \tag{2.12}$$

On the other hand, from the boundedness of $\hat{u}_n$ in L^{p^*} we see that (2.9) is satisfied if

$$q'r'(p-1) < p^*. \tag{2.13}$$

Clearly, obtaining an r satisfying (2.12) and (2.13) is equivalent to obtaining an r such that

$$\frac{q'(q-p)}{p^*} < \frac{1}{r} < \frac{p^* - (p-1)\, q'}{p^*}. \tag{2.14}$$

But this is always possible since $q < p^*$. A fixed r satisfying (2.14) will also satisfy (2.8) and (2.9), hence the claim.

Now, from (2.5), (2.6), $q < p^*$, and the compactness of R_p, we can assume, passing to a subsequence if necessary, that $\hat{u}_n \to \hat{u}$ in $W_0^{1,p}$. Taking limits as $n \to \infty$ in (2.5) we find that

$$\hat{u} = R_p(\bar{\lambda}\varphi_p(\hat{u})). \tag{2.15}$$

Since $\hat{u} \neq 0$, (2.15) implies that $\bar{\lambda}$ is an eigenvalue of (E_p). We formalize this last result in the following

PROPOSITION 2.1. *If $(\bar{\lambda}, 0)$ is a bifurcation point of problem* (B) *then $\bar{\lambda}$ is an eigenvalue of* (E_p).

Our main goal in the rest of this section is to prove Theorem 1.1. We start by quoting a result of Anane. In [1] he proved that $\lambda_1(p)$ is an isolated eigenvalue of (E_p); i.e., if we let

$$\lambda_2(p) = \inf\{\lambda > \lambda_1(p) \mid \lambda \text{ is an eigenvalue of } (\mathrm{E}_p)\} \tag{2.16}$$

then $\lambda_1(p) < \lambda_2(p)$.

Next we observe that, by definition, there is no eigenvalue of (E_p) less than $\lambda_1(p)$, thus for $\lambda < \lambda_1(p)$ or $\lambda_1(p) < \lambda < \lambda_2(p)$ the equation

$$u = R_p(\lambda\varphi_p(u)) \tag{2.17}$$

admits only the trivial solution $u \equiv 0$. If we now define the completely continuous operator $T_p^\lambda \colon W_0^{1,p} \to W_0^{1,p}$ by

$$T_p^\lambda(u) = R_p(\lambda\varphi_p(u)), \tag{2.18}$$

it is clear that for λ in the above range, the Leray–Schauder degree

$$\deg_{W_0^{1,p}}(I - T_p^\lambda, B(0, r), 0)$$

is well defined for any $r > 0$. In the next proposition we evaluate this degree. This value will be fundamental in the proof of Theorem 1.1.

PROPOSITION 2.2. *Let $r > 0$, $\bar{p} > 1$, and $\lambda \in \mathbb{R}$. Then*

$$\deg_{W_0^{1,\bar{p}}}(I - T_{\bar{p}}^\lambda, B(0, r), 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_1(\bar{p}) \\ -1 & \text{if } \lambda_1(\bar{p}) < \lambda < \lambda_2(\bar{p}). \end{cases}$$

This result is well known for $\bar{p} = 2$ and it is contained in [7] for a general $\bar{p} > 1$ and $N = 1$. The proof of Proposition 2.2 will be based on a homotopic deformation to $p = 2$ and will use some lemmas which we state and prove next.

LEMMA 2.1. *The first eigenvalue function* $\lambda_1: (1, \infty) \to \mathbb{R}$ *is continuous.*

Proof. From the variational characterization of $\lambda_1(p)$ it follows that

$$\lambda_1(p) = \sup\{\lambda > 0 \mid \lambda \, \|u\|_p^p \leqslant \|\nabla u\|_p^p, \text{ for all } u \in C_c^\infty(\Omega)\}. \tag{2.19}$$

Let $\{p_j\}_{j=1}^\infty$ be a sequence in $(1, \infty)$ convergent to $p > 1$. We will show that

$$\lim_{j \to \infty} \lambda_1(p_j) = \lambda_1(p). \tag{2.20}$$

To do this, let $u \in C_c^\infty(\Omega)$. Then, from (2.19),

$$\int_\Omega |\nabla u|^{p_j} \geqslant \lambda_1(p_j) \int_\Omega |u|^{p_j}.$$

On applying the Dominated Convergence Theorem we find

$$\int_\Omega |\nabla u|^p \geqslant \limsup_{j \to \infty} \lambda_1(p_1) \int_\Omega |u|^p. \tag{2.21}$$

Relation (2.21), the fact that u is arbitrary, and (2.19) yield

$$\limsup_{j \to \infty} \lambda_1(p_j) \leqslant \lambda_1(p). \tag{2.22}$$

Thus, to prove (2.20) it suffices to show that

$$\liminf_{j \to \infty} \lambda_1(p_j) \geqslant \lambda_1(p). \tag{2.23}$$

Let $\{p_k\}_{k=1}^\infty$ be a subsequence of $\{p_j\}_{j=1}^\infty$ such that $\lim_{k \to \infty} \lambda_1(p_k) = \liminf_{j \to \infty} \lambda_1(p_j)$.

Let us fix $\bar{\varepsilon} > 0$ so that $p - \bar{\varepsilon} > 0$ and for each $0 < \varepsilon < \bar{\varepsilon}$, $W_0^{1, p-\varepsilon}$ is compactly embedded into $L^{p+\varepsilon}$. For $k \in \mathbb{N}$, let us choose $u_k \in W_0^{1, p_k}$ such that

$$\int_\Omega |\nabla u_k|^{p_k} = 1 \tag{2.24}$$

and

$$\int_\Omega |\nabla u_k|^{p_k} = \lambda_1(p_k) \int_\Omega |u_k|^{p_k}. \tag{2.25}$$

For $0 < \varepsilon < \bar{\varepsilon}$, (2.24) and Hölder's inequality imply that

$$\|\nabla u_k\|_{p-\varepsilon} \leqslant |\Omega|^{((p_k - p) + \varepsilon)/p_k}. \tag{2.26}$$

This shows that $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in $W_0^{1,p-\varepsilon}$. Passing to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$ in $W_0^{1,p-\varepsilon}$ and hence that $u_k \to u$ in $L^{p+\varepsilon}$. Clearly $u \in L^p$ and is independent of ε. Moreover, $\|u_k\|_{p_k} \to \|u\|_p$.

We note that (2.24) and (2.25) imply that

$$[\lambda_1(p_k)]^{1/p_k} \|u_k\|_{p_k} = 1 \tag{2.27}$$

for all $k \in \mathbb{N}$. Thus letting k go to ∞ in (2.27) we find

$$\liminf_{j \to \infty} \lambda_1(p_j) \|u\|_p^p = 1. \tag{2.28}$$

On the other hand, since $u_k \rightharpoonup u$ in $W^{1,p-\varepsilon}$, from (2.26) we obtain that

$$\|\nabla u\|_{p-\varepsilon} \leqslant \liminf_{k \to \infty} \|\nabla u_k\|_{p-\varepsilon} \leqslant |\Omega|^{\varepsilon/p}.$$

Now, letting $\varepsilon \to 0^+$ and applying Fatou's Lemma we find

$$\|\nabla u\|_p \leqslant 1. \tag{2.29}$$

Hence $u \in W^{1,p}$. We claim that actually $u \in W_0^{1,p}$. Indeed, we know that $u \in W_0^{1,p-\varepsilon}$ for each $0 < \varepsilon < \bar{\varepsilon}$. For $\phi \in C_c^{\infty}(\mathbb{R}^N)$ it is easy to see that

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} \right| \leqslant \|\nabla u\|_{p-\varepsilon} \|\phi\|_{(p-\varepsilon)'}, \qquad i = 1, ..., N.$$

Then, letting $\varepsilon \to 0^+$ we obtain that

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} \right| \leqslant \|\nabla u\|_p \|\phi\|_{p'}.$$

Since ϕ is arbitrary, from Proposition IX-18 of [5] we find that $u \in W_0^{1,p}$, as desired.

Finally, combining (2.28) and (2.29) we obtain that $u \neq 0$ and

$$\liminf_{j \to \infty} \lambda_1(p_j) \|u\|_p^p \geqslant \|\nabla u\|_p^p.$$

This and the variational characterization of $\lambda_1(p)$ imply (2.23) and hence (2.20). This concludes the proof of the lemma. ∎

In the next lemma we study the joint continuity of the operator R_p with respect to p and h.

LEMMA 2.2. *Let* $p_0 > 1$ *and* $1 < q < p_0^*$. *Then the operator* $R: [p_0, +\infty) \times L^{q'} \to L^q$ *defined by* $R(p, h) \equiv R_p(h)$ *is completely continuous.*

Proof. We show first the compactness of R. Thus let $\{(p_n, h_n)\}_{n=1}^{\infty}$ be a bounded sequence in $[p_0, \infty) \times L^{q'}$. We will show that $u_n \equiv R(p_n, h_n)$ has a convergent subsequence in L^q. We have that u_n, $n \in \mathbb{N}$, satisfies

$$\int_\Omega |\nabla u_n|^{p_n} = \int_\Omega h_n u_n \leqslant \|h_n\|_{q'} \|u_n\|_q \leqslant C \|\nabla u_n\|_{p_0}$$

for some $C > 0$ independent on n. Since $p_n \geqslant p_0$, we obtain that

$$\int_\Omega |\nabla u_n|^{p_0} \leqslant C'$$

for some $C' > 0$ independent on n. Hence $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in W_0^{1,p_0} and since $q < p_0^*$, it possesses a convergent subsequence in L^q, hence R is compact.

Next we show the continuity of R. Let $h_n \to h$ in $L^{q'}$, $p_n \to p$ in $[p_0, \infty)$. Let $u_n \equiv R(p_n, h_n)$, $n \in \mathbb{N}$, and $u \equiv R(p, h)$. We have to prove that $u_n \to u$ in L^q. Since R is compact we only have to show that any accumulation point of the sequence $\{u_n\}_{n=1}^{\infty}$ is equal to u. Thus let $\{u_k\}_{k=1}^{\infty}$ be a subsequence of $\{u_n\}_{n=1}^{\infty}$ converging to v in L^q.

We recall that u is characterized as the unique element of $W_0^{1,p}$ satisfying

$$\frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega hu \leqslant \frac{1}{p}\int_\Omega |\nabla z|^p - \int_\Omega hz \tag{2.30}$$

for every $z \in W_0^{1,p}$. We will show that $v \in W_0^{1,p}$ and satisfies (2.30) for every $z \in W_0^{1,p}$ and hence $v = u$.

We have that

$$\frac{1}{p_k}\int_\Omega |\nabla u_k|^{p_k} - \int_\Omega h_k u_k \leqslant \frac{1}{p_k}\int_\Omega |\nabla z|^{p_k} - \int_\Omega h_k z. \tag{2.31}$$

Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that $1 < p - \varepsilon < p_k$ for all $k \geqslant k_0$. Hölder's inequality implies that

$$\left(\int_\Omega |\nabla u_k|^{p-\varepsilon}\right)^{p_k/(p-\varepsilon)} |\Omega|^{-(p_k - p + \varepsilon)/p_k} \leqslant \int_\Omega |\nabla u_k|^{p_k} \tag{2.32}$$

and hence the sequence $\{\|\nabla u_k\|_{p-\varepsilon}\}_{k=1}^{\infty}$ is bounded in $W_0^{1,p-\varepsilon}$. Passing to a subsequence if necessary we can assume that $u_k \rightharpoonup v$ in $W_0^{1,p-\varepsilon}$ and thus

$$\int_\Omega |\nabla v|^{p-\varepsilon} \leqslant \liminf_{k \to \infty} \int_\Omega |\nabla u_k|^{p-\varepsilon}. \tag{2.33}$$

Now, letting $k \to \infty$ in (2.31), (2.33) and using (2.32) we obtain

$$\frac{1}{p} \int_{\Omega} |\nabla v|^{p-\varepsilon} |\Omega|^{-\varepsilon/p} - \int_{\Omega} hv \leqslant \frac{1}{p} \int_{\Omega} |\nabla z|^{p} - \int_{\Omega} hz. \tag{2.34}$$

Taking the limit as $\varepsilon \to 0^+$ in (2.34) and applying Fatou's Lemma we find that $|\nabla v| \in L^p$ and that

$$\frac{1}{p} \int_{\Omega} |\nabla v|^{p} - \int_{\Omega} hv \leqslant \frac{1}{p} \int_{\Omega} |\nabla z|^{p} - \int_{\Omega} hz \tag{2.35}$$

for every $z \in C_c^\infty(\Omega)$. Clearly, by density, (2.35) is also valid for all $z \in W_0^{1,p}$. It remains to show that $v \in W_0^{1,p}$. This follows directly from $v \in W^{1,p-\varepsilon}$, for all $\varepsilon > 0$, $|\nabla v| \in L^p$, and the use of Proposition IX-18 of [5] as in Lemma 2.1. This concludes the proof of the lemma. ∎

In the following lemma we give a modified version of Anane's result, $\lambda_1(p) < \lambda_2(p)$, in a form suitable to our purpose. This is done by following his proof in [1] and using our previous lemmas.

LEMMA 2.3. *For every interval* $[a, b] \subset (1, +\infty)$ *there is a* $\delta > 0$ *such that for all* $p \in [a, b]$ *there is no eigenvalue of* (E_p) *in* $(\lambda_1(p), \lambda_1(p) + \delta]$.

Proof. Suppose that the assertion of the lemma is not true. Then there are sequences $\{p_n\}_{n=1}^\infty$ in $(1, +\infty)$, $\{\lambda_n\}_{n=1}^\infty$ in $\mathbb{R}^+$, and $\{u_n\}_{n=1}^\infty$ in $W_0^{1,p}\setminus\{0\}$ such that $\lim_{n\to\infty} p_n = \bar{p} \in (1, +\infty)$, $\lambda_n > \lambda_1(p_n)$, $\lim_{n\to\infty} (\lambda_n - \lambda_1(p_n)) = 0$, and

$$u_n = R_{p_n}(\lambda_n \varphi_{p_n}(u_n)), \qquad n \in \mathbb{N}. \tag{2.36}$$

We note that, from Lemma 2.1, $\lambda_n \to \lambda_1(\bar{p})$. Let us choose $\varepsilon > 0$ such that $1 < \bar{p} - \varepsilon < \bar{p} + \varepsilon < (\bar{p} - \varepsilon)^*$ and call $p_0 = \bar{p} - \varepsilon$, $q = \bar{p} + \varepsilon$. Also assume $\|u_n\|_q = 1$. Then $\{\varphi_{p_n}(u_n)\}_{n=1}^\infty$ is a bounded sequence in $L^{q'}$. Hence from Lemma 2.2 we can assume $u_n \to u$ in L^q and correspondingly $\varphi_{p_n}(u_n) \to \varphi_{\bar{p}}(u)$ in $L^{q'}$. Taking limits as $n \to \infty$ in (2.36) and using Lemma 2.2 we obtain

$$u = R_{\bar{p}}(\lambda_1(\bar{p})\, \varphi_{\bar{p}}(u)); \tag{2.37}$$

that is, u is an eigenfunction associated to $\lambda_1(\bar{p})$.

Now, it is known that $u \in C^{1,\alpha}(\bar{\Omega})$ and u is one-signed in Ω, say positive in Ω. On the other hand, if we call $\Omega_n^- = \{x \in \Omega \mid u_n(x) < 0\}$, Proposition 2 of [1] implies that

$$|\Omega_n^-| \geqslant C \tag{2.38}$$

for some positive C independent of n. Since $u_n \to u$ in L^q, it is easily seen that (2.38) implies that u must change its sign in Ω, which is impossible. This contradiction completes the proof. ▮

We conclude the preliminaries to the proof of Proposition 2.2 with an abstract Leray–Schauder degree property.

LEMMA 2.4. *Let X, Y be Banach spaces with respective norms $\| \ \|_X$ and $\| \ \|_Y$. Assume that $Y \subset X$ and that the inclusion $i\colon Y \to X$ is continuous.*

Let Ω_X, Ω_Y be open bounded sets in X and Y, respectively, both containing 0, *and $T\colon X \to Y$ a completely continuous operator such that*

$$x - Tx \neq 0 \qquad \text{for every} \quad x \in X, \qquad x \neq 0. \tag{2.39}$$

Then

$$\deg_X(I - i \circ T, \Omega_X, 0) = \deg_Y(I - T \circ i, \Omega_Y, 0). \tag{2.40}$$

Proof. From (2.39) and the continuity of the inclusion $Y \subset X$, without loss of generality we can assume $\bar{\Omega}_Y \subset \Omega_X$. Let $\varepsilon > 0$. It is well known that we can find a finite dimensional vector space $E_\varepsilon \subset Y$ and a completely continuous operator $T_\varepsilon\colon X \to Y$ whose range is included in E_ε and such that

$$\sup_{x \in \bar{\Omega}_X} \|T_\varepsilon x - Tx\|_Y < \varepsilon. \tag{2.41}$$

From (2.39) and (2.41) we observe that, choosing ε small enough,

$$x - T_\varepsilon x \neq 0 \qquad \text{if} \quad x \in \bar{\Omega}_X \backslash \Omega_Y. \tag{2.42}$$

The definition of the Leray–Schauder degree [see, for example, 11], then yields

$$\deg_X(I - i \circ T, \Omega_X, 0) = \deg_{E_\varepsilon}((I - T_\varepsilon)|_{E_\varepsilon}, \Omega_X \cap E_\varepsilon, 0) \tag{2.43}$$

and

$$\deg_Y(I - T \circ i, \Omega_Y, 0) = \deg_{E_\varepsilon}((I - T_\varepsilon)|_{E_\varepsilon}, \Omega_Y \cap E_\varepsilon, 0), \tag{2.44}$$

for ε small. On the other hand, (2.42) implies

$$\deg_{E_\varepsilon}((I - T_\varepsilon)|_{E_\varepsilon}, \Omega_X \cap E_\varepsilon \backslash \overline{\Omega_Y \cap E_\varepsilon}, 0) = 0.$$

The additivity and excision properties of the degree imply that the right hand sides of (2.43) and (2.44) coincide, thus (2.40) follows. ▮

Proof of Proposition 2.2. Suppose that $\lambda_1(\bar{p}) < \lambda < \lambda_2(\bar{p})$. The continuity of $\lambda_1(p)$ and Lemma 2.3 imply the existence of a continuous function $\nu: (1, \infty) \to \mathbb{R}$ such that $\lambda_1(p) < \nu(p) < \lambda_2(p)$ for every $p > 1$ and $\nu(\bar{p}) = \lambda$.

The result will follow by showing that the integer-valued function

$$d(p) \equiv \deg_{W_0^{1,p}}(I - T_p^{\nu(p)}, B(0, r), 0) \tag{2.45}$$

is locally constant in $(1, \infty)$ and hence constant. Recall that $d(2) = -1$.

Let $p_0 > 1$ and choose $\varepsilon > 0$ such that $1 < p_0 - \varepsilon < p_0 + \varepsilon < (p_0 - \varepsilon)^*$. Let us take $p \in [p_0 - \varepsilon, p_0 + \varepsilon]$ and set $q = p_0 + \varepsilon$. It is easy to see that the operator $S_p: L^q \to W_0^{1,p}$ defined by $S_p(u) = R_p(\nu(p))\, \varphi_p(u))$ is completely continuous. Also we have that $T_p^{\nu(p)} = S_p \circ i$ where $i: W_0^{1,p} \to L^q$ is the usual inclusion. Hence, from Lemma 2.4 with $T = S_p$, $X = L^q$, $Y = W_0^{1,p}$, we obtain

$$d(p) = \deg_{L^q}(I - i \circ S_p, A, 0) \qquad \text{for} \quad p \in [p_0 - \varepsilon, p_0 + \varepsilon]. \tag{2.46}$$

Here A is any open bounded set in L^q containing 0. Now, it is easily verified that the operator $\varphi: [p_0 - \varepsilon, p_0 + \varepsilon] \times L^q \to L^{q'}$ defined by $\varphi(p, u) = \varphi_p(u)$ is continuous and maps bounded sets into bounded sets. These facts, the continuity of $\nu(p)$, and Lemma 2.2 allow the conclusion that the homotopy

$$[p_0 - \varepsilon, p_0 + \varepsilon] \times L^q \to L^q$$

$$(p, u) \mapsto R_p(\nu(p)\, \varphi_p(u)) = (i \circ S_p)(u)$$

is completely continuous. The invariance of the Leray–Schauder degree under a compact homotopy and (2.46) then yield $d(p) \equiv$ *constant* for $p \in [p_0 - \varepsilon, p_0 + \varepsilon]$. Thus $d(p)$ is locally constant and hence constant on $(1, \infty)$. In particular, $d(\bar{p}) = d(2) = -1$, as desired.

The same proof applies if $\lambda < \lambda_1(\bar{p})$; however, a simpler argument can be used in this case. Clearly the degree $\deg_{W_0^{1,\bar{p}}}(I - R_{\bar{p}}(s\lambda\varphi_{\bar{p}}(\cdot)), B(0, r), 0)$ is well defined for $s \in [0, 1]$. Hence, from the invariance of the degree under homotopies, this degree equals 1, its value at $s = 0$. ∎

Proof of Theorem 1.1. Let us set

$$H_\lambda(u) = R_p(\lambda\varphi_p(u) + F(\lambda, u)).$$

Suppose that $(\lambda_1(p), 0)$ is not a bifurcation point of problem (B). Then there exist $\varepsilon > 0$, $\delta_0 > 0$ such that for $|\lambda - \lambda_1(p)| \leqslant \varepsilon$ and $\delta < \delta_0$ there is no nontrivial solution of the equation

$$u - H_\lambda(u) = 0$$

with $\|u\|_{1,p}=\delta$. From the invariance of the degree under a compact homotopy we obtain that

$$\deg_{W_0^{1,p}}(I-H_\lambda, B(0,\delta), 0) \equiv constant, \qquad \text{for} \quad \lambda \in [\lambda_1(p)-\varepsilon, \lambda_1(p)+\varepsilon]. \tag{2.47}$$

By taking ε smaller if necessary, we can assume that there is no eigenvalue of (E_p) in $(\lambda_1(p), \lambda_1(p)+\varepsilon]$. Fix now $\lambda \in (\lambda_1(p), \lambda_1(p)+\varepsilon]$. It is easy to see that if we choose δ sufficiently small then the equation

$$u - R_p(\lambda\varphi(u) + sF(\lambda, u)) = 0$$

has no solution u with $\|u\|_{1,p}=\delta$ for every $s \in [0, 1]$. Indeed, assuming the contrary and reasoning as in the proof of Proposition 2.1, we would find that λ is an eigenvalue of (E_p). From the invariance of the degree under homotopies and Proposition 2.2 we then obtain

$$\deg_{W_0^{1,p}}(I-H_\lambda, B(0,\delta), 0) = \deg_{W_0^{1,p}}(I-T_p^\lambda, B(0,\delta), 0) = -1. \tag{2.48}$$

Similarly, for $\lambda \in [\lambda_1(p)-\varepsilon, \lambda_1(p))$ we find that

$$\deg_{W_0^{1,p}}(I-H_\lambda, B(0,\delta), 0) = 1. \tag{2.49}$$

Relations (2.48) and (2.49) contradict (2.47) and hence $(\lambda_1(p), 0)$ is a bifurcation point of problem (B).

The rest of the proof is entirely similar to that of the Rabinowitz Global Bifurcation Theorem [see 10], so we omit it here. ∎

3. An Existence Result

In this section we will apply Theorem 1.1 to prove an existence result for the nonlinear problem

$$(\mathrm{D}) \qquad \begin{aligned} -\Delta_p u &= g(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $g: \mathbb{R} \to \mathbb{R}$ is continuous and $g(0)=0$. We note that $u \equiv 0$ is a trivial solution of problem (D).

In the next theorem we will be concerned with existence of nontrivial solutions to this problem.

THEOREM 3.1. *Assume that $g(s)/\varphi_p(s)$ is bounded and*

$$\underline{\lambda} \equiv \lim_{s \to 0} \frac{g(s)}{\varphi_p(s)} < \lambda_1(p) < \liminf_{|s| \to \infty} \frac{g(s)}{\varphi_p(s)}. \tag{3.2}$$

Then problem (D) *possesses at least one nontrivial solution $u \in C^{1,\alpha}(\bar{\Omega})$ which does not vanish in Ω.*

The proof of this theorem will use some properties, which we discuss next, of the component predicted by Theorem 1.1 for a bifurcation problem associated with (D). Thus let us assume that g satisfies the hypotheses of Theorem 3.1, then from (3.2) we can write g as

$$g(s) = \underline{\lambda}\varphi_p(s) + f(s), \tag{3.3}$$

where $\underline{\lambda} < \lambda_1(p)$ and $f(s) = o(|s|^{p-1})$ near $s = 0$. Now let us consider the bifurcation problem

$$(\mathrm{B}_1) \qquad \begin{aligned} -\Delta_p u &= \lambda\varphi_p(u) + f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

From Theorem 1.1, we know that there is a component $\mathscr{S} \subset \mathbb{R} \times W_0^{1,p}$ of the set of nontrivial solutions of (B_1) such that its closure $\bar{\mathscr{S}}$ contains $(\lambda_1(p), 0)$ and is either unbounded or contains a point $(\bar{\lambda}, 0)$ where $\bar{\lambda} > \lambda_1(p)$ is an eigenvalue of (E_p).

In the following lemmas we study some further properties of the component $\mathscr{S}$ from which Theorem 3.1 will follow immediately. Let us set

$$\mathscr{P} = \{v \in C^{1,\alpha}(\bar{\Omega}) \mid v(x) \neq 0, \quad \text{for all} \quad x \in \Omega\}.$$

LEMMA 3.1. *We have*

$$\bar{\mathscr{S}} \subset \mathscr{C} \equiv \{\lambda_1(p), 0)\} \cup (\mathbb{R} \times \mathscr{P})$$

and $\mathscr{S}$ is unbounded in $\mathbb{R} \times W_0^{1,p}$.

Proof. We first observe that the boundedness of $f(s)/\varphi_p(s)$ and Lemma 1 of [1] imply that any $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}$ solution of (B_1) is such that $u \in C^{1,\alpha}(\bar{\Omega})$. Hence $\mathscr{S} \subset C^{1,\alpha}(\bar{\Omega})$. We claim now that there is a neighborhood $\mathscr{N}$ of $(\lambda_1(p), 0)$ in $\mathbb{R} \times W_0^{1,p}$ such that $\mathscr{S} \cap \mathscr{N} \subset \mathbb{R} \times \mathscr{P}$. Otherwise, there would exist a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ of nontrivial solutions of (B_1) such that $(\lambda_n, u_n) \to (\lambda_1(p), 0)$ in $\mathbb{R} \times W_0^{1,p}$ and u_n changes sign in Ω, for each $n \in \mathbb{N}$. Since $\lambda_n + f(u_n)/\varphi_p(u_n)$ is uniformly bounded in

Ω, it follows that $\mu = 1$ is an eigenvalue, different from the first eigenvalue, of the problem

$$\begin{aligned} -\Delta_p u &= \mu\left(\lambda_n + \frac{f(u_n)}{\varphi(u_n)}\right)\varphi_p(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.5}$$

Proposition 2 of [1] then implies the existence of a $C > 0$, independent on n, such that

$$|\{x \in \Omega \mid u_n(x) < 0\}| \geqslant C \qquad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

On the other hand, a compactness argument shows that $\hat{u}_n \equiv u_n/\|u_n\|_{1,p}$ converges to some $\hat{u}$ in $W_0^{1,p} \setminus \{0\}$, and this $\hat{u}$ is an eigenfunction associated to $\lambda_1(p)$ in (E_p). Thus $\hat{u} \in C^{1,\alpha}(\bar{\Omega})$ and does not vanish in Ω, say $\hat{u} > 0$ in Ω.

After a simple measure argument, it follows that (3.6) is not compatible with the fact that $\hat{u}_n \to \hat{u}$ in $W_0^{1,p}$, and the claim follows.

Suppose now the assertion of the lemma is not true. Then $\bar{\mathscr{S}}$ leaves $\mathscr{C}$ at some point $(\bar{\lambda}, \bar{u}) \neq (\lambda_1(p), 0)$. Necessarily $\bar{u} \neq 0$, for otherwise $\bar{\lambda}$ would be an eigenvalue of (E_p) different to $\lambda_1(p)$ and an argument similar to the one just employed would lead to a contradiction.

A continuity argument shows that $(\bar{\lambda}, \bar{u})$ weakly satisfies

$$\begin{aligned} -\Delta_p \bar{u} &= \left(\bar{\lambda} + \frac{f(\bar{u})}{\varphi_p(\bar{u})}\right)\varphi_p(\bar{u}) && \text{in } \Omega \\ \bar{u} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.7}$$

Thus $\bar{u} \in C^{1,\alpha}(\bar{\Omega})$ and does not change sign in Ω, say $\bar{u} \geqslant 0$ in Ω.

Lemma 1 of [1] then yields $\bar{u} > 0$ on Ω. An argument similar to that employed in the first part of this proof implies that we cannot approximate $(\bar{\lambda}, \bar{u})$ by elements of $\mathscr{S}$ from without $\mathscr{C}$. This contradicts the definition of $(\bar{\lambda}, \bar{u})$ and proves the first part of the lemma. That $\mathscr{S}$ is unbounded is an immediate consequence of Theorem 1.1. ∎

LEMMA 3.2. *There is a $C > 0$ such that $(\lambda, u) \in \mathscr{S}$ implies $\lambda \leqslant C$.*

Proof. Denote $\bar{m} = \sup_{s \in \mathbb{R}} |f(s)/\varphi_p(s)|$ and let $(\lambda, u) \in \mathscr{S}$. Then, from Lemma 3.1, $u > 0$ on Ω. This implies that $\mu = 1$ is the first eigenvalue of the problem

$$\begin{aligned} -\Delta_p v &= \mu(m(x) + \lambda)\,\varphi(v) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

Here, $m(x) = f(u(x))/\varphi_p(u(x))$. It follows from the variational characterization of the first eigenvalue that

$$\lambda = \inf\left\{\int_\Omega |\nabla v|^p - \int_\Omega m\,|v|^p \,\Big|\, v \in W_0^{1,p}, \int_\Omega |v|^p = 1\right\}$$

$$\leqslant \inf\left\{\int_\Omega |\nabla v|^p \,\Big|\, v \in W_0^{1,p}, \int_\Omega |v|^p = 1\right\} + \bar{m}$$

$$= \lambda_1(p) + \bar{m}.$$

We obtain the result by defining $C = \lambda_1(p) + \bar{m}$. ∎

LEMMA 3.3. *There exists an $M > 0$ such that for every $\lambda \in [\underline{\lambda}, \infty)$ we have that $(\lambda, u) \in \mathcal{S}$ implies $\|u\|_{1,p} \leqslant M$.*

Proof. Suppose not; then there exist sequences $\{\lambda_n\}_{n=1}^\infty$ in $[\underline{\lambda}, C]$ and $\{u_n\}_{n=1}^\infty$ in $W_0^{1,p}$ such that $(\lambda_n, u_n) \in \mathcal{S}$, $\lambda_n \to \lambda_0$ and $\|u_n\|_{1,p} \to \infty$. Here C is as in the last lemma.

Now, defining $\hat{u}_n = u_n/\|u_n\|_{1,p}$ and calling on the definition of R_p we have

$$\hat{u}_n = R_p\left(\lambda_n \varphi_p(\hat{u}_n) + \frac{f(u_n)}{\varphi_p(u_n)}\,\varphi_p(\hat{u}_n)\right). \tag{3.9}$$

Here for simplicity f also denotes the corresponding Nemitsky operator. Now, since the argument of R_p in (3.9) is bounded in $L^{p'}$, we can assume that $\hat{u}_n \to \hat{u}$ in $W_0^{1,p}$. Also, from the boundedness of $g(s)/\varphi_p(s)$, we obtain that the sequence $\{f(u_n)/\varphi_p(u_n)\}_{n=1}^\infty$ is uniformly bounded and hence $f(u_n)/\varphi_p(u_n) \to h \in L^\infty$, weakly in any L^r, $r > 1$. Taking limits in (3.9) we obtain

$$\begin{aligned} -\Delta_p \hat{u} &= (\lambda_0 + h(x))\,\varphi_p(\hat{u}) && \text{in } \Omega \\ \hat{u} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3.10}$$

and thus $\hat{u} \in C^{1,\alpha}(\bar{\Omega})$. From Lemma 3.1 we can assume $u_n > 0$ in Ω, $n \in \mathbb{N}$. A similar argument works if $u_n < 0$ in Ω, $n \in \mathbb{N}$.

Since $\hat{u}_n \to \hat{u}$ a.e. in Ω, it follows that $\hat{u} \geqslant 0$ in Ω. But $\hat{u} \not\equiv 0$, hence from Lemma 1 of [1] we must have $\hat{u} > 0$ in Ω. Now let us take $\bar{\lambda}$ such that $\lambda_1(p) < \bar{\lambda} < \liminf_{|s| \to +\infty} g(s)/\varphi_p(s)$. We claim that $h \geqslant \bar{\lambda} - \underline{\lambda}$ a.e. in Ω. Otherwise, there is a set $A \subset \Omega$ with $|A| > 0$ such that $h(x) < \bar{\lambda} - \underline{\lambda}$, for all $x \in A$. Using the facts that $\hat{u}_n \to \hat{u}$, $\hat{u} > 0$, and Egorov's theorem, we obtain the existence of a set $\tilde{\Omega} \subset \Omega$ with $|\Omega \backslash \tilde{\Omega}| < |A|$ such that $u_n(x) \to +\infty$ uniformly on $\tilde{\Omega}$. Then the inequality $f(u_n)/\varphi(u_n) \leqslant \bar{\lambda} - \underline{\lambda}$ holds in $\tilde{\Omega}$ for $n \geqslant n_0$ and so $h \geqslant \bar{\lambda} - \underline{\lambda}$ a.e. in $\tilde{\Omega}$. Hence, we obtain the contradiction $|A| \leqslant |\Omega \backslash \tilde{\Omega}| < |A|$, which proves our claim.

Thus, we have that $\lambda_0 + h \geqslant \lambda_0 + \bar{\lambda} - \underline{\lambda} > \lambda_1(p)$ a.e. in Ω. On the other hand, $\hat{u} > 0$ satisfies (3.10) and thus $\mu_1 = 1$ is the first eigenvalue of the problem

$$\begin{aligned} -\Delta_p v &= \mu(\lambda_0 + h(x))\, \varphi_p(v) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.11}$$

Thus from the variational characterization of μ_1 it directly follows that

$$\lambda_0 + (\bar{\lambda} - \underline{\lambda}) \leqslant \lambda_1(p),$$

which is a contradiction. Hence the lemma follows. ∎

Proof of Theorem 3.1. Lemma 3.3 shows that $\mathscr{S} \cap [\underline{\lambda}, \infty) \times W_0^{1,p}$ is bounded. Since $\mathscr{S}$ is unbounded and connected it must cross the axis $\lambda = \underline{\lambda}$ at some point $(\underline{\lambda}, \underline{u})$. Furthermore from Lemma 1 of [1], $\underline{u} \in C^{1,\alpha}(\bar{\Omega})$ and does not vanish in Ω. Thus $\underline{u}$ is a nontrivial solution of (3.1). ∎

4. The Radial Case

In this section we consider the bifurcation problem (B) in the presence of radial symmetry.

Henceforth Ω will denote the unit ball in $\mathbb{R}^N$. We will consider the problem

$$(\tilde{\mathrm{B}}) \quad \begin{aligned} -\Delta_p u &= \lambda \varphi_p(u) + f(|x|, u, \lambda) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $f\colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(r, s, \lambda) = o(|s|^{p-1})$ near $s = 0$, uniformly for (r, λ) on bounded subsets of $[0, 1] \times \mathbb{R}$.

We are interested in classical radial solutions of $(\tilde{\mathrm{B}})$, that is, in pairs (λ, u) with $u(x) = v(|x|)$ and $v \in C^1[0, 1]$ satisfying the ODE.

$$(\tilde{\mathrm{B}}') \quad \begin{aligned} &-(r^{N-1}\varphi_p(v'))' = r^{N-1}(\lambda\varphi_p(v) + f(r, v, \lambda)), \qquad r \in (0, 1) \\ &v'(0) = 0, \qquad v(1) = 0. \end{aligned} \tag{4.2}$$

The results of this section will rest upon some properties of the solutions of the radial eigenvalue problem for $-\Delta_p$, that is of the problem

$$(\tilde{\mathrm{E}}_p) \quad \begin{aligned} &-(r^{N-1}\varphi_p(v'))' = \mu r^{N-1}\varphi_p(v), \qquad r \in (0, 1) \\ &v'(0) = 0, \qquad v(1) = 0. \end{aligned} \tag{4.3}$$

We begin by the following proposition

PROPOSITION 4.1. *The set of scalars μ such that* $(\tilde{\mathrm{E}}_p)$ *admits a nontrivial solution consists of an unbounded increasing sequence*

$$0<\mu_1(p)<\mu_2(p)<\cdots<\mu_k(p)<\cdots.$$

Moreover, the set of solutions of $(\tilde{\mathrm{E}}_p)$ *for* $\mu=\mu_k(p)$ *is the one-dimensional space spanned by a solution* ϕ_k *of* $(\tilde{\mathrm{E}}_p)$ *with exactly* $k-1$ *zeros in* $(0, 1)$, *all of them simple. Furthermore,* μ_k *as a function of* $p\in(1, +\infty)$ *is continuous for each* $k\in\mathbb{N}$.

Basically, this result is known [see 2, Theorem 4.1]. For convenience of the reader we sketch a proof in the Appendix, different from that in [2].

Remark. Regularity implies that all the radial eigenfunctions of $-\Delta_p$ are of class $C^1(\bar{\Omega})$. From this fact it can be shown that they satisfy the ODE (4.3). Also, from [6], we have that $\mu_1(p)=\lambda_1(p)$.

The following result extends Theorem 1.1 to the situation of radial symmetry.

THEOREM 4.1. *For each* $k\in\mathbb{N}$ *there is a component* $\mathscr{S}_k\subseteq\mathbb{R}\times C[0, 1]$ *of the set of nontrivial solutions of* $(\tilde{\mathrm{B}}')$ *whose closure* $\bar{\mathscr{S}}_k$ *contains* $(\mu_k(p), 0)$. *Moreover,* $\mathscr{S}_k$ *is unbounded in* $\mathbb{R}\times C[0, 1]$, *and* $(\lambda, v)\in\mathscr{S}_k$ *implies that* v *possesses exactly* $k-1$ *zeros in* $(0, 1)$.

Before proving Theorem 4.1 we need some preliminary facts. First, we observe that for a given $h\in C[0, 1]$ the unique solution of

$$\begin{aligned} -\Delta_p u&=h(|x|) &&\text{in } \Omega\\ u&=0 &&\text{on } \partial\Omega \end{aligned} \tag{4.4}$$

is of class C^1 and radial. This follows from the fact that the radial version of (4.4), i.e.,

$$\begin{aligned} -(r^{N-1}\varphi_p(v'))'&=r^{N-1}h(r), \qquad r\in(0, 1)\\ v'(0)=0, &\qquad v(1)=0 \end{aligned} \tag{4.4$'$}$$

has a unique solution, namely,

$$v(r)=-\int_0^1 \varphi_{p'}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{h-1}h(\tau)\,d\tau\right)ds \tag{4.5}$$

and hence $u(x)=v(|x|)$ is the solution of (4.4).

Denote by $\bar{R}(p, h)$ the unique solution of (4.4)$'$. From its integral representation (4.5) and the Ascoli–Arzéla Theorem, we find that $\bar{R}$ defines a continuous operator from $(1, \infty)\times C[0, 1]$ into $C^1[0, 1]$ and a completely continuous operator from $[p_0, \infty) \times C[0, 1]$ into $C[0, 1]$, for each $p_0 > 1$.

Let $\mu_1(p) < \mu_2(p) < \cdots$ be the sequence of eigenvalues of $(\tilde{\mathrm{E}}_p)$. We can easily obtain the following analogue to Proposition 2.2.

PROPOSITION 4.2. *If λ is not an eigenvalue of $(\tilde{\mathrm{E}}_p)$, $r > 0$, $\bar{p} > 1$, and we set $\bar{T}_{\bar{p}}^{\lambda}(u) \equiv \bar{R}(\bar{p}, \lambda\varphi_{\bar{p}}(u))$, then*

$$\deg_{C[0,1]}(I - \bar{T}_{\bar{p}}^{\lambda}, B(0, r), 0) = \begin{cases} 1 & \text{if } \lambda < \mu_1(\bar{p}) \\ (-1)^k & \text{if } \mu_k(\bar{p}) < \lambda < \mu_{k+1}(\bar{p}). \end{cases} \tag{4.6}$$

Proof. Assume first that $\mu_k(\bar{p}) < \lambda < \mu_{k+1}(\bar{p})$. Since for $j \in \mathbb{N}$, $\mu_j(p)$ is a continuous function of p, we can find a continuous function $v: (1, \infty) \to \mathbb{R}$ with $\mu_k(p) < v(p) < \mu_{k+1}(p)$ and $v(\bar{p}) = \lambda$. The invariance of the Leray–Schauder degree under compact homotopies yields

$$d(p) \equiv \deg_{C[0,1]}(I - T_p^{v(p)}, B(0, r), 0) = \text{constant}$$

for $p \in (1, \infty)$. In particular $d(\bar{p}) = d(2) = (-1)^k$ and the result follows.

The case $\lambda < \mu_k(p)$ is analogous. ∎

Proof of Theorem 4.1. As in Theorem 1.1, following Rabinowitz [10], we obtain the existence of a component $\mathscr{S}_k \subset \mathbb{R} \times C[0, 1]$ whose closure $\bar{\mathscr{S}}_k$ contains $(\mu_k(p), 0)$ and is either unbounded or contains a point $(\mu_j(p), 0)$ for some $j \neq k$.

Let us prove first that $(\lambda, v) \in \mathscr{S}_k$ implies that v possesses exactly $k-1$ zeros in $(0, 1)$, all of them simple. We claim that there is a neighborhood $\mathscr{N}$ of $(\mu_k(p), 0)$ such that for every $(\lambda, v) \in N \cap S_k$, v has the above property. Otherwise there would be sequences $\lambda_n \to \mu_k(p)$, $v_n \to 0$ in $C[0, 1]$ with $(\lambda_n, v_n) \in \mathscr{S}_k$ and v_n not having $k-1$ zeros. Using the compactness of $\bar{R}$ we can assume that the sequence $\{\hat{v}_n\}_{n=1}^{\infty}$, $\hat{v}_n = v_n/\|v_n\|_0 \to \hat{v}$ in $C[0, 1]$, where $\hat{v}$ is an eigenfunction associated with $\lambda_k(p)$. But $\hat{v}$ has exactly $k-1$ zeros, all of them simple. It follows that v_n must have the same property for large n which is a contradiction. This proves our claim.

Now suppose that there is a $(\bar{\lambda}, \bar{v}) \in \bar{\mathscr{S}}_k$, $\bar{\lambda} \neq \lambda_k(p)$ that can be approximated by elements $(\lambda, v) \in S_k$ with v having exactly $k-1$ zeros in $(0, 1)$ and by elements $(\lambda, v) \in \mathscr{S}_k$ without this property. We have that $\bar{v} \neq 0$. In fact if this were not so we would have $\bar{\lambda} = \mu_j(p)$, some $j \neq k$. This, the fact that an eigenfunction associated with $\mu_j(p)$ has exactly $(j-1)$ simple zeros in $(0, 1)$, and an argument similar to the one just employed would lead to a contradiction. Hence $\bar{v} \neq 0$, and also $(\bar{\lambda}, \bar{v}) \in S_k$.

From Lemma 5.1 in the Appendix, $\bar{v}$ can have only simple zeros in $[0, 1]$. Hence, there is a neighborhood of $\bar{v}$ in $C[0, 1]$ such that every element in this neighborhood intersected with $\mathscr{S}_k$ has the same number of zeros, contrary to the definition of $\bar{v}$. Therefore, each element (λ, v) of S_k is such that v has exactly $k-1$ zeros. Also $\mathscr{S}_k$ is unbounded, since it cannot contain a point $(\mu_j(p), 0)$ with $j \neq k$. This completes the proof. ∎

In the rest of this section wewill apply Theorem (4.1) to the problem

$$(\tilde{\mathrm{D}}) \quad \begin{aligned} -\Delta_p u &= g(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.7}$$

with $g: \mathbb{R} \to \mathbb{R}$ continuous and $g(0) = 0$.

The following result extends Theorem 3.1.

THEOREM 4.2. *Assume*

$$\sup_{s \in \mathbb{R}} \left| \frac{g(s)}{\varphi_p(s)} \right| < +\infty \tag{4.8}$$

and that for some natural numbers k, n with $k \leqslant n$

$$\lim_{s \to 0} \frac{g(s)}{\varphi_p(s)} < \mu_k(p) \leqslant \mu_n(p) < \liminf_{|s| \to \infty} \frac{g(s)}{\varphi_p(s)}. \tag{4.9}$$

Then (4.7) *possesses at least $n-k+1$ nontrivial radial solutions. More precisely, for each $k \leqslant j \leqslant n$, there is a radial solution of problem* ($\tilde{\mathrm{D}}$) *with exactly $j-1$ nodes in Ω.*

The idea of the proof of Theorem 4.2 is similar to that of Theorem 3.1. We will need the following Sturm Comparison Lemma whose proof we defer to the Appendix.

LEMMA 4.1. *Let $a, b \in L^\infty(0, 1)$ with $a \leqslant b$ a.e. on $(0, 1)$. Assume that $u, v \in C^1[0, 1] \backslash \{0\}$, $u'(0) = v'(0) = 0$, $r^{N-1}\varphi_p(u')$, and $r^{N-1}\varphi_p(v')$ are absolutely continuous functions on $[0, 1]$ and respectively satisfy*

$$(r^{N-1}\varphi_p(u'))' + r^{N-1}a(r)\,\varphi_p(u) = 0 \qquad \textit{a.e. on } (0, 1) \tag{4.10}$$

$$(r^{N-1}\varphi_p(v'))' + r^{N-1}b(r)\,\varphi_p(v) = 0 \qquad \textit{a.e. on } (0, 1). \tag{4.11}$$

Then:

(i) *If u has a zero in $(0, 1)$ then v does too. The first zero of v is then less than or equal to the first zero of u.*

(ii) *If $(r_0, r_1) \subset [0, 1]$, $u(r_0) = u(r_1) = 0$, $u(r) \neq 0$, $r \in (r_0, r_1)$, and $a < b$ in some subset of (r_0, r_1) of positive measure, then v has at least one zero in (r_0, r_1).*

Proof of Theorem 4.2. Clearly $g(s)$ can be written as $g(s) = \underline{\lambda}\varphi(s) + f(s)$

where $\underline{\lambda}=\lim_{s\to 0} g(s)/\varphi_p(s)$ and $f(s)=0(|s|^{p-1})$ near $s=0$. Consider the bifurcation problem

$$\begin{gathered} -(r^{N-1}\varphi_p(v'))'=r^{N-1}(\lambda\varphi_p(v)+f(r)), \qquad r\in(0,1) \\ v'(0)=0, \qquad v(1)=0. \end{gathered} \tag{4.12}$$

Let $k\leqslant j\leqslant n$, and S_j be the component of the set of nontrivial solutions of (4.12) predicted by Theorem 4.1.

Let $(\lambda, v)\in S_j$ and $C=\mu_j(p)+\sup_{s\in\mathbb{R}}|g(s)/\varphi_p(s)|$. We claim that $\lambda\leqslant C$. To prove this claim, we first observe that the pair (λ, v) satisfies

$$\begin{gathered} -(r^{N-1}\varphi_p(v'))'=b(r)\,r^{N-1}\varphi_p(v), \qquad r\in(0,1) \\ v'(0)=0, \qquad v(1)=0, \end{gathered} \tag{4.13}$$

where $b(r)=\lambda+g(v(r))/\varphi_p(v(r))$. Suppose $\lambda>C$. Then $b(r)>\mu_j(p)$, $r\in(0,1)$. It follows from Lemma 4.1, parts (i) and (ii), that v has at least j zeros is $(0,1)$. From Theorem 4.1 this is impossible, hence $\lambda\leqslant C$.

Next we claim that if $\lambda\in[\tilde{\lambda}, C]$ then there is an $M>0$ such that $(\lambda, v)\in S_j$ implies $\|v\|_0\leqslant M$. Otherwise, reasoning in a similar way to the proof of Lemma 3.3, we obtain the existence of a sequence $\{v_n\}_{n=1}^{\infty}$ in $C[0,1]$, v_n having exactly $j-1$ simple zeros in $(0,1)$ and $\|v_n\|_0=1$, convergent in $C[0,1]$ to a v satisfying the equation

$$\begin{gathered} -(r^{N-1}\varphi_p(v'))'=r^{N-1}h(r)\,\varphi_p(v), \qquad r\in(0,1) \\ v'(0)=0, \qquad v(1)=0, \end{gathered} \tag{4.14}$$

where $h\in L^{\infty}(0,1)$ and $h>\mu_j(p)$ a.e. on $(0,1)$. Lemma 4.2, then implies that v has at least j simple zeros in $(0,1)$ and hence the same is true for v_n with n sufficiently large. This contradiction shows the claim.

From these considerations and reasoning as in Theorem 3.1 we find that S_j must cross the axis $\lambda=\underline{\lambda}$. Thus the result of the theorem follows. ∎

5. Appendix

In this Appendix we will prove some results used in Section 4. We begin with a preliminary lemma.

Lemma 5.1. *Let* $a\in L^{\infty}(0,\alpha)$ *and* $u\in C^1[0,\alpha]$ *with* $r^{N-1}\varphi_p(u')$ *absolutely continuous on* $[0,1]$ *satisfying*

$$\begin{gathered} (r^{N-1}\varphi_p(u'))'+r^{N-1}a(r)\,\varphi_p(u)=0 \qquad \text{a.e. in } (0,\alpha) \\ u(r_0)=0, \qquad u'(r_0)=0. \end{gathered} \tag{5.1}$$

for some $r_0\in[0,\alpha]$. *Then* $u\equiv 0$.

Proof. Clearly u satisfies the integral equation

$$u(r)=\int_{r_0}^{r} \varphi_{p'}\left(\frac{1}{s^{N-1}}\int_{r_0}^{s} \tau^{N-1}a(\tau)\,\varphi_p(u(\tau))\,d\tau\right)ds. \tag{5.2}$$

From (5.2) and assuming first $r_0\in(0,\alpha)$ it follows that

$$|u(r)|\leqslant \frac{\delta^{p'}}{N^{p'-1}}\,\|a\|_{L^\infty}^{p'-1} \sup_{\tau\in[r_0-\delta,\,r_0+\delta]} |u(\tau)| \qquad \text{if} \quad r\in(r_0-\delta,r_0+\delta)$$

and hence $u\equiv 0$ on $(r_0-\delta, r_0+\delta)$ for δ sufficiently small. A standard argument then shows that actually $u\equiv 0$ on $(0,\alpha)$. The cases $r_0=0$ and $r_0=\alpha$ are treated similarly. ▮

Proof of Lemma 4.1. (i) Assume that u does have a zero in $(0,1)$ and let t_1 be the first zero of u. Suppose that v has no zeros on $(0,t_1)$. We will show that in this case v must vanish at t_1. Define $w_a(r)=r^{N-1}\varphi_p((u'/u)(r))$, $w_b(r)=r^{N-1}\varphi_p((v'/v)(r))$, for $r\in(0,t_1)$.

It is easy to verify that w_a and w_b respectively satisfy

$$w_a'+(p-1)\,\frac{|w_a|^{p'}}{r^{(p'-1)(N-1)}}+r^{N-1}a(r)=0 \qquad \text{a.e. on } (0,t_1) \tag{5.3}$$

$$w_b'+(p-1)\,\frac{|w_b|^{p'}}{r^{(p'-1)(N-1)}}+r^{N-1}b(r)=0 \qquad \text{a.e. on } (0,t_1). \tag{5.4}$$

Also, from Lemma 5.1 and since $u'(0)=0=v'(0)$, it follows that $w_a(0)=0=w_b(0)$. Subtracting (5.4) from (5.3) we obtain

$$(w_a-w_b)'+m(r)(w_a-w_b)\geqslant 0 \tag{5.5}$$

a.e. on $(0,t_1)$, where

$$m(r)=p\int_0^1 \varphi_{p'}\left(\frac{w_b(r)+s(w_a-w_b)(r)}{r^{N-1}}\right)ds. \tag{5.6}$$

Since $(w_a-w_b)(0)=0$, (5.5) easily yields

$$(w_a-w_b)(r)\geqslant 0 \qquad \text{for all} \quad r\in(0,t_1). \tag{5.7}$$

Since $w_a(r)\to-\infty$ as $r\to t_1^-$, from (5.7) we obtain that $w_b(r)\to-\infty$ as $r\to t_1^-$. Hence $v(t_1)=0$ and the result follows.

(ii) Assume that v has no zeros in (r_0,r_1), and define w_a, w_b on (r_0,r_1) as in (i). We will show that $w_a=w_b$ on (r_0,r_1). This, (5.3), and (5.4) will in turn imply that $a=b$ a.e. on (r_0,r_1) which is a contradiction.

Let $\tilde{r} \in (r_0, r_1)$. Assume first that $(w_a - w_b)(\tilde{r}) \leqslant 0$. Since w_a, w_b satisfy the differential inequality (5.5) a.e. on (r_0, r_1), we obtain that

$$(w_a - w_b)(r) \leqslant 0 \qquad \text{for} \quad r \in (r_0, \tilde{r}). \tag{5.8}$$

Now from $u(r_0) = 0$, we have that $w_a(r) \to +\infty$ as $r \to r_0^+$. This, and (5.8) imply that $w_b(r) \to +\infty$ as $r \to 0^+$ and then $v(r_0) = 0$.

On the other hand, the definitions of w_a and w_b and (5.8) imply that

$$m(r) \geqslant p\varphi_{p'}\left(\frac{w_a(r)}{r^{N-1}}\right) = p\frac{u'}{u}(r) \tag{5.9}$$

for all $r \in (r_0, \tilde{r})$ and where $m(r)$ is given by (5.6). Multiplying the inequality (5.5) by $|u(r)|^p$ and on using (5.9) we find that

$$(|u|^p (w_a - w_b))' \geqslant 0 \qquad \text{a.e. on } (r_0, \tilde{r}). \tag{5.10}$$

A direct verification yields that $|u|^p (w_a - w_b)(r) \to 0$ as $r \to r_0^+$, and hence from (5.10)

$$(w_a - w_0)(r) \geqslant 0 \qquad \text{on } (r_0, \tilde{r}). \tag{5.11}$$

Relations (5.8) and (5.11) then imply that $w_a = w_b$ on $(r_0, \tilde{r})$.

A symmetric argument on $(\tilde{r}, r_1)$ shows that if we originally had supposed $(w_a - w_b)(\tilde{r}) \geqslant 0$, then $w_a = w_b$ on $(\tilde{r}, r_1)$. Combining these facts we obtain that, in either case, $w_a = w_b$ on (r_0, r_1) as desired. This concludes the proof. ▮

Next we will briefly study existence, uniqueness, extendibility, and oscillation of the solution to the initial value problem

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' = r^{N-1}\varphi_p(u), \qquad r > 0. \\ u(0) = 1, \qquad u'(0) = 0. \end{gathered} \tag{5.12}$$

After doing this we will prove Proposition 4.1. Existence of a local solution of the problem can be established by writing (5.12) as the equivalent integral equation

$$u(r) = 1 - \int_0^r \varphi_{p'}\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1}\varphi_p(u(\tau))\,d\tau\right) ds \equiv A_p(u)(s). \tag{5.13}$$

For $\delta > 0$ the operator $A_p\colon C[0, \delta] \to C[0, \delta]$ satisfies

$$\|A_p(u)\|_{C[0,\delta]} \leqslant \delta^{p'} \|u\|_{C[0,\delta]} + C, \tag{5.14}$$

for some $C > 0$. Schauder's Fixed Point Theorem then implies that, for

$\delta < 1$, (5.13) possesses a solution in $C[0, \delta]$ and hence (5.12) has a local solution. Now, let us show uniqueness of this local solution. Let u_1, u_2 be two local solutions of (5.12) and note that

$$\lim_{s \to 0} \frac{1}{s^N} \int_0^s \tau^{N-1} \varphi_p(u_i(\tau))\, d\tau = \frac{1}{N}, \qquad i = 1, 2.$$

Then since $\varphi_{p'}$ is C^1 near $1/N$ we obtain from (5.13) the existence of $k > 0$ such that

$$|u_1(r) - u_2(r)| \leqslant k \int_0^r s^{p'-1} |u_1(s) - u_2(s)|\, ds \tag{5.15}$$

for $r > 0$ sufficiently small. Relation (5.15) clearly implies that $u_1 \equiv u_2$ near $r = 0$. Thus we have shown existence and uniqueness of a local solution to (5.12).

We note that similar arguments show that for every $r_0 > 0$, $\alpha, \beta \in \mathbb{R}$ the IVP problem

$$\begin{aligned} -(r^{N-1} \varphi_p(u'))' &= r^{N-1} \varphi_p(u) \\ u(r_0) = 0, \qquad & u'(r_0) = \beta \end{aligned} \tag{5.16}$$

has a unique local solution.

On the other hand, via Hölder's inequality one can show that a solution u to problem (5.12) defined on $[0, a)$ satisfies the Gronwall's inequality

$$|u'(r)|^p + |u(r)|^p \leqslant K\left(1 + \int_0^r (|u'(s)|^p + |u(s)|^p)\, ds\right), \qquad r \in [0, a), \tag{5.17}$$

for some constant K dependent on a. From (5.17)

$$\int_0^r (|u'(s)|^p + |u(s)|^p)\, ds \leqslant e^{Kr}, \qquad r \in [0, a). \tag{5.18}$$

Relation (5.18) and a standard argument imply that the local solution of (5.12) can be extended to $[0, +\infty)$.

Hence, we have the validity of the following lemma

Lemma 5.2. *The IVP (5.12) has a unique solution $\phi(r)$ defined on $[0, \infty)$.*

Next we will show that the solution of (5.12) is oscillatory.

LEMMA 5.3. *ϕ, the solution of* (5.12), *is oscillatory; that is, given any $r>0$, there is a $\tau>r$ such that $\phi(\tau)=0$.*

Proof. We use an idea of Hartman's [9]. Suppose ϕ is not oscillatory; that is for some $r_0>0$ ϕ does not vanish on $[r_0, \infty)$. Define

$$w(r)=r^{N-1}\varphi_p\left(\frac{\phi'(r)}{\phi(r)}\right), \qquad r\in[r_0,\infty).$$

Then w satisfies

$$w'+(p-1)\frac{|w|^{p'}}{r^{(p'-1)(N-1)}}+r^{N-1}=0, \qquad \text{on } [r_0,\infty). \tag{5.19}$$

Integrating (5.19) from r_0 to $t>r_0$ we get

$$w(t)+(p-1)\int_{r_0}^{t}\frac{|w|^{p'}}{r^{(p'-1)(N-1)}}=\frac{-t^N}{N}+w(r_0). \tag{5.20}$$

In particular, we see that

$$|w(t)|=-w(t)\geqslant Ct^N \tag{5.21}$$

for some $C>0$ and t large. Define

$$k(t)=\int_{r_0}^{t}\frac{|w|^{p'}}{r^{(p'-1)(N-1)}}\,dr. \tag{5.22}$$

From (5.21) and (5.22) we obtain that

$$k(t)\geqslant\tilde{c}t^{p'+N} \qquad \text{for } t \text{ large and some } \tilde{c}>0. \tag{5.23}$$

On the other hand from (5.20)

$$(p-1)\,k(t)\leqslant|w(t)|$$

or

$$(p-1)^{p'}k(t)^{p'}\leqslant t^{(p'-1)(k-1)}k'(t), \qquad \text{for } t \text{ large}.$$

The latter inequality implies

$$A\left(\frac{1}{k(t)^{p'-1}}-\frac{1}{k(s)^{p'-1}}\right)\geqslant\frac{1}{t^{(p'-1)(N-1)-1}}-\frac{1}{s^{(p'-1)(N-1)-1}} \tag{5.24}$$

for some $A>0$ and t, s large with $t<s$. Letting $s\to+\infty$ in (5.24) and noting that $k(s)\to+\infty$, we find

$$k(t)\leqslant A^{1/(p'-1)}t^{N-1-1/(p'-1)}. \tag{5.25}$$

However (5.23) and (5.25) are not compatible. This contradiction shows that ϕ must be oscillatory. ▮

With these preliminaries we are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Let $0 < v_1(p) < v_2(p) < \cdots < v_k(p) < \cdots$ be the zeros of $\phi(r)$, the solution of (5.12). Lemma 5.1 implies that these zeros are simple. Lemma 5.3 tells us that $v_k(p) \to +\infty$ as $k \to \infty$. Next define $\mu_k(p) \equiv (v_k(p))^p$. Clearly $\lambda = \mu_k(p)$ is an eigenvalue of $(\tilde{\text{E}}_p)$, with $\phi_k(r) = \phi(v_k(p)\,r)$, $r \in [0, 1]$, being a corresponding eigenfunction with $k-1$ zeros in $(0, 1)$. We claim that there are no eigenvalues of $(\tilde{\text{E}}_p)$ other than these μ_k's.

Let μ be an eigenvalue of $(\tilde{\text{E}}_p)$. Clearly $\mu > 0$. Let $u(r)$ be a nontrivial solution of $(\tilde{\text{E}}_p)$ for $\lambda = \mu$. The uniqueness of the solution of (5.12) then implies that $u(r) = u(0)\,\phi(\lambda^{1/p} r)$. Moreover, since $u(1) = 0$ we have that $\lambda = v_k(p)^p$ for some $k \in \mathbb{N}$, and $u = u(0)\,\phi_k$.

Finally, since ϕ satisfies the integral equations (5.13) we have that ϕ is continuous in p in the sense of uniform convergence on bounded intervals. This and the fact that the zeros of ϕ are simple imply that the v_k's and hence the μ_k's define continuous functions of p. This completes the proof. ▮

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