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DICHOTOMIES AND ASYMPTOTIC SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS

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1. INTRODUCTION

Consider the systems

$$y' = A(t)y + G(t, y), \qquad t \geq 0,\ y \in R^n \tag{1.1}$$

and

$$x' = A(t)x. \tag{1.2}$$

The problem of asymptotic equivalence between two systems of ordinary differential equations has been studied by many authors [1–5]. It was first used by Levinson in his theorem on asymptotic integration [6, 7]. In [1, 5], this problem is studied when system (1.2) is stable, $A(t)$ is constant and $G(t, y) = B(t)$ is integrable. In [3, theorems 5, 11], it is assumed that the linear system (1.2) has an ordinary dichotomy and the nonlinear term $G(t, y)$ is Lipschitz continuous. In [2], the previous results are extended to nonlinear perturbations $G(t, y)$ which are not Lipschitz continuous, such that $|G(t, y)| \leq r(t, y)$, where $r(t, y)$ is monotone in y. We emphasize that in all these works only the asymptotic equivalence between bounded solutions is established. Nothing is said about a possible correspondence between unbounded solutions of (1.1) and (1.2).

In this paper we suppose that the nonlinear term $G(t, y)$ satisfies some integrability conditions that we summarize in hypothesis (H2). These conditions permit one to consider systems with large nonlinearities, for example, equations where $G(t, y)$ is an oscillatory function of the variable y. No Lipschitz conditions or monotonicity properties are required of $G(t, y)$. However, we will use dichotomic properties of system (1.2).

Definition 1. Let h, k: $[0, \infty) \to (0, \infty)$ be two continuous functions. We will say that system (1.2) has an (h, k)-dichotomy if there exist a fundamental matrix $\Phi(t)$ of (1.2), a positive constant K and two projection matrices P_+ and P_- such that $P_+ + P_- = I$ (the identity matrix) and

$$|\Phi(t)P_+\Phi^{-1}(s)| \leq Kh(t)h(s)^{-1}, \qquad \text{for } t \geq s \geq 0 \tag{1.3}$$

$$|\Phi(t)P_-\Phi^{1}(s)| \leq Kk(t)^{-1}k(s), \qquad \text{for } s \geq t \geq 0. \tag{1.4}$$

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(The notation $h(t)^{-1}$ will stand for the reciprocal function of $h(t)$.)

This concept was introduced in [8]. It includes simultaneously many interesting cases including systems with exponential or ordinary dichotomies, and diagonal systems which satisfy Levinson's dichotomic conditions. This is to say that the concept of (h, k)-dichotomy has a great generality [9].

Pinto [8] has established a correspondence between both the bounded and the unbounded solutions of (1.1) and (1.2) under a global Lipschitz condition for $G(t, x)$

$$|G(t, x) - G(t, y)| \leq \lambda(t)|x - y| \qquad \text{with } \lambda \in L^1$$

when the linear system (1.2) has an (h, k)-dichotomy. We will continue this line of investigation for more general nonlinear functions $G(t, y)$.

In this work we obtain the representation of solutions of (1.1) in terms of the solutions of equation (1.2). In theorem 1, we establish the asymptotic equivalence between "h-bounded" solutions y_+ of (1.1) and x_+ of (1.2) and also between "k^{-1}-bounded" solutions y_- of (1.1) and x_- of (1.2). These are of the form

$$y_\pm(t) = x_\pm + h_\pm(t)o(1)$$

$$h_+(t) = h(t), \qquad h_-(t) = k(t)^{-1}.$$

We then present many applications of theorem 1. As a final example, we study the asymptotic integration of the second order system

$$u' = -t^{-1}u + \lambda_1(t)v + \lambda_2(t)uv, \qquad t \geq 1$$

$$v' = t^{-1}v + \mu_1(t)u + \mu_2(t)uv.$$

2. SPACES, NORMS, OPERATORS, AND HYPOTHESES

Referring to (1.3) and (1.4), we define

$$\Gamma(t, s) := \begin{cases} \Phi(t)P_+\Phi^{-1}(s) & \text{for } 0 \leq s \leq t \\ -\Phi(t)P_-\Phi^{-1}(s) & \text{for } 0 \leq t \leq s. \end{cases} \tag{2.1}$$

We will use the following notation

$$h_+(t) := h(t), \qquad h_-(t) := k(t)^{-1},$$

$$\|f\| := \sup_{(t_0, \infty)} |f(t)|, \qquad t_0 \geq 0,$$

$$\|y\|_\pm := \|h_\pm^{-1}y\| \qquad \text{and}$$

$$C_\pm(t_0) := \{y \in C([t_0, \infty), R^n) \colon \|h_\pm^{-1}y\| < \infty\}.$$

We will consider $C_\pm(t_0)$ as a topological vector space with respect to the family of seminorms

$$p_T(f) = \sup_{[t_0, T]} |f(t)|.$$

For a function f defined on $[0, \infty)$ we write $f \in C_\pm(t_0)$ if the restriction of f to $[t_0, \infty)$ belongs to $C_\pm(t_0)$. We denote by $C_\pm(t_0, \rho)$ the set of functions f of $C_\pm(t_0)$ with $\|f\|_\pm \le \rho$.

Taking $x_+ \in C_+(t_0)$ or $x_- \in C_-(t_0)$, we construct the two operators $T_\pm$ given by

$$T_\pm(y)(t) := x_\pm(t) + \int_{t_0}^{\infty} \Gamma(t, s)G(s, y(s))\, ds, \qquad t \ge t_0. \tag{2.2}$$

We emphasize that the operators $T_\pm$ depend on $T_\pm$ on t_0. In our paper the main results will depend on the following hypotheses:

(H1) system (1.2) has an (h, k)-dichotomy given by (1.3) and (1.4).

(H2) There exists a $\rho > 0$ such that G satisfies the inequality

$$|G(t, y)| \le r(t, y)|y|, \qquad (t, y) \in [0, \infty) \times R^n,$$

for a real function $r(t, y)$ defined for $t \ge t_0$ and $y \in \mathbf{R}^n$, where we define

$$m_\pm(t) := \sup_{|x| \le \rho} |r(t, h_\pm(t)x)| \tag{2.3}$$

such that

$$\alpha_\pm(t_0, \rho) = \int_{t_0}^{\infty} m_\pm(s)\, ds < \infty.$$

Note. In cases (see the example given in the last section) where we have the commutation

$$\Phi(t)P_\pm = P_\pm\Phi(t)$$

and

$$G(t, y) = R(t, y)y, \tag{2.4}$$

for a matrix function $R(t, y)$, in place of (2.3) it is better to define

$$m_\pm(t) := \sup\{|P_\pm R(t, h_\pm(t)x)|;\ |x| \le \rho\}. \tag{2.5}$$

We emphasize the dependence of $m_\pm$ on ρ.

(H3) $h(t)k(t)h(s)^{-1}k(s)^{-1} \le C$, for $t \ge s \ge 0$, $C \ge 1$, where C is a constant.

(H4) $\lim_{t\to\infty} h_\pm(t)^{-1}|\Phi(t)P_+| = 0$.

3. CORRESPONDENCE OF BOUNDED AND UNBOUNDED MANIFOLDS

In this section we will prove the existence of fixed points for the operators $T_\pm$.

LEMMA 1. Let (σ, t_0) satisfy

$$\sigma + \rho C K \alpha_\pm(t_0, \rho) \le \rho. \tag{3.1}$$

Under hypotheses (H1)–(H3) for $x_\pm \in C_\pm(t_0, \sigma)$, it follows that

$$T_\pm\colon C_\pm(t_0, \rho) \to C_\pm(t_0, \rho).$$

Proof. From (2.2) and (3.1) we have

$$\begin{aligned}|h_\pm(t)^{-1}T_\pm(y)(t)| &\le \sigma + h_\pm(t)^{-1}\int_{t_0}^{\infty}|\Gamma(t,s)G(s,y(s))|\,\mathrm{d}s\\
&\le \sigma + Kh_\pm(t)^{-1}\left\{\int_{t_0}^{t}h(t)h(s)^{-1} + \int_t^{\infty}k(t)^{-1}k(s)\right\}r(s,y(s))|y(s)|\,\mathrm{d}s\\
&\le \sigma + K\left\{\int_{t_0}^{t}C + \int_t^{\infty}C\right\}(m_\pm(s)|h_\pm(s)^{-1}y(s)|)\,\mathrm{d}s\\
&\le \sigma + \rho CK\int_{t_0}^{\infty}m_\pm(s)\,\mathrm{d}s \le \rho.\end{aligned}$$

LEMMA 2. Under the conditions of lemma 1, the operators $T_\pm\colon C_\pm(t_0,\rho)\to C_\pm(t_0,\rho)$ are continuous in the following sense: if $(y_n)_{n=1}^{\infty}$ is a sequence converging to y_0 in $C_\pm(t_0,\rho)$, then for any interval $[t_0, T]$ the sequence $(h_\pm^{-1}Ty_n)_{n=1}^{\infty}$ converges uniformly on $[t_0, T]$.

Proof. Let $y_n \to y$ in $C_\pm(t_0,\rho)$ and let $t\in[t_0,T]$. Then because of the continuity of the function $h_\pm$ on $[t_0,T]$, for any $T>0$ we have

$$\lim_{n\to\infty}\sup_{[t_0,T]}|y_n(t)-y_0(t)| = 0. \tag{3.2}$$

Given a fixed $\varepsilon>0$ there exists a number T_0 such that for $T_0\ge T\ge t\ge t_0\colon y\in C_\pm(t_0,\rho)$ implies

$$K\int_{T_0}^{\infty}h_\pm(t)^{-1}k(t)^{-1}k(s)|G(s,y(s))|\,\mathrm{d}s\le\varepsilon. \tag{3.3}$$

Then for $t\in[0,T]$ and $y\in C_\pm(t_0,\rho)$ we obtain

$$\begin{aligned}&h_\pm(t)^{-1}|T_\pm(y_0)(t) - T_\pm(y_n)(t)|\\
&\quad\le Kh_\pm(t)^{-1}\int_{t_0}^{t}h(t)h(s)^{-1}|G(s,y_0(s)) - G(s,y_n(s))|\,\mathrm{d}s\\
&\qquad + Kh_\pm(t)^{-1}\int_t^{\infty}k(t)^{-1}k(s)|G(s,y_0(s)) - G(s,y_n(s))|\,\mathrm{d}s\\
&\quad\le Kh_\pm(t)^{-1}\int_{t_0}^{T}|G(s,y_0(s)) - G(s,y_n(s))|\,\mathrm{d}s\\
&\qquad + Kh_\pm(t)^{-1}\int_{t_0}^{T_0}k(t)^{-1}k(s)|G(s,y_0(s)) - G(s,y_n(s))|\,\mathrm{d}s\\
&\qquad + Kh_\pm(t)^{-1}\int_{T_0}^{\infty}k(t)^{-1}k(s)|G(s,y_0(s)) - G(s,y_n(s))|\,\mathrm{d}s.\end{aligned}$$

The first two integrals tend to zero as $n\to\infty$ by (3.2). The third integral is less than 2ε because of (3.3).

LEMMA 3. Under the conditions of lemma 1, each of the operators $T_\pm : C_\pm(t_0, \rho) \to C_\pm(t_0, \rho)$ has a fixed point.

Proof. We will verify the validity of the hypotheses of the Schauder–Tijonov fixed point theorem as given in [3]. Lemma 2 says that each of the operators $T_\pm$ is continuous. It remains to prove that the sequence $\{h_\pm^{-1} T_\pm(y_n) : y_n \in C_\pm(t_0, \rho)\}_{n=1}^\infty$ is equicontinuous at each point $t \in [t_0, \infty)$, that is, for $\varepsilon > 0$ and s fixed, there exists $\delta(\varepsilon, s) > 0$ such that $|t - s| < \delta$ implies

$$|h_\pm(t)^{-1} T_\pm(y_n)(t) - h_\pm(s)^{-1} T_\pm(y_n)(s)| < \varepsilon \qquad \text{for all } n.$$

For $s < t < \delta$ we have

$$\begin{aligned} |h_\pm(t)^{-1} T_\pm(y_n)(t) - h_\pm(s)^{-1} T_\pm(y_n)(s)| &\le I_1 + I_2, \\ I_1 &:= h_\pm(t)^{-1} |T_\pm(y_n)(t) - T_\pm(y_n)(s)|, \\ I_2 &:= |h_\pm(t)^{-1} - h_\pm(s)^{-1}|\,|T_\pm(y_n)(s)|. \end{aligned}$$

From the continuity of $h_\pm$ at t we have

$$I_2 \le \rho |1 - h_\pm(s) h_\pm(t)^{-1}| < \varepsilon$$

for small δ. In order to estimate I_1, we have

$$\begin{aligned} &T_\pm(y_n)(t) - T_\pm(y_n)(s) \\ &\quad = \left[\int_{t_0}^{s} (\Phi(t) - \Phi(s)) P_+ \Phi^{-1}(u) - \int_s^\infty (\Phi(t) - \Phi(s)) P_- \Phi^{-1}(u) \right. \\ &\qquad \left. + \int_s^t \Phi(t) P_+ \Phi^{-1}(u) + \int_s^t \Phi(t) P_- \Phi^{-1}(u) \right] G(u, y_n(u))\, du \\ &\quad = \left[\int_{t_0}^{s} (\Phi(t)\Phi^{-1}(s) - I)\Phi(s) P_+ \Phi^{-1}(u) - \int_s^\infty (\Phi(t)\Phi^{-1}(s) - I)\Phi(s) P_- \Phi^{-1}(u) \right. \\ &\qquad \left. + \int_s^t \Phi(t)\Phi^{-1}(u) \right] G(u, y_n(u))\, du. \end{aligned}$$

If δ is small then $|\Phi(t)\Phi^{-1}(s) - I| \le \varepsilon$. Then we obtain

$$\begin{aligned} |I_1| &\le \varepsilon K h_\pm(t)^{-1} \left[\int_{t_0}^{s} h(s) h(u)^{-1} + \int_s^\infty k(s)^{-1} k(u) \right] \cdot m_\pm(u) |y_n(u)|\, du \\ &\quad + \left[\int_s^t |\Phi(t)\Phi^{-1}(u)| \right] h_\pm(u) m_\pm(u) |h_\pm(u)^{-1} y_n(u)|\, du \\ &\le \varepsilon \rho C K h_\pm(t)^{-1} [h_+(s) + h_-(s)] \|m_\pm\|_{L^1} + \rho h_\pm(t)^{-1} \int_s^t |\Phi(t)\Phi^{-1}(u)| h_\pm(u) m_\pm(u)\, du, \end{aligned}$$

from which lemma 3 follows.

According to (2.1), the fixed points $y_\pm$ found in lemma 3 can be expressed in the form

$$y_\pm(t) = x_\pm(t) + \Phi(t)P_+u_\pm(t) + \Phi(t)P_-v_\pm(t),$$

$$u_\pm(t) := \int_{t_0}^{t} P_+\Phi^{-1}(s)g(s, y_\pm(s))\,\mathrm{d}s,$$

$$v_\pm(t) := \int_{t_0}^{\infty} P_-\Phi^{-1}(s)g(s, y_\pm(s))\,\mathrm{d}s.$$

LEMMA 4. If (H1)–(H3) are fulfilled then

$$\Phi(t)P_-v_\pm = h_\pm(t)o(1), \qquad o(1) = 0\left(\int_t^\infty m_\pm(s)\,\mathrm{d}s\right)$$

and if, in addition, (H4) holds then

$$y_\pm = x_\pm + h_\pm \cdot o(1) \qquad \text{as } t \to \infty.$$

Proof. First, we prove that

$$\lim_{t\to\infty} h_\pm(t)^{-1}\Phi(t)P_-v_\pm(t) = 0.$$

From the estimate

$$|\Phi(t)P_-v_\pm(t)| \le K\int_t^\infty h_-(t)^{-1}h_-(s)^{-1}h_\pm(s)m_\pm(s)|h_\pm(s)^{-1}y_\pm(s)|\,\mathrm{d}s$$

we have

$$|h_\pm(t)^{-1}\Phi(t)P_-v_\pm(t)| \le \rho CK\int_t^\infty m_\pm(s)\,\mathrm{d}s \to 0 \qquad \text{as } t\to\infty.$$

We have only to prove that $\Phi(t)P_+u_\pm(t) = h_\pm(t)o(1)$ if (H4) holds. Let $(t_n)_{n=1}^\infty$ be any sequence of real numbers such that $\lim_{n\to\infty} t_n = \infty$. We define the sequence of functions

$$f_n(s) := h_\pm(t_n)^{-1}\Phi(t_n)P_+\Phi^{-1}(s)G(s, y_\pm(s)), \qquad \text{if } t_0 \le s \le t_n,$$

and $f_n(s) := 0$, if $s > t_n$. We have that $|f_n(s)| \le \rho CKm_\pm(s)$, and by virtue of (H4) $\lim_{n\to\infty} f_n(s) = 0$ for any fixed s. By the dominated convergence theorem we have that

$$\lim_{n\to\infty}\int_{t_0}^\infty f_n(s)\,\mathrm{d}s = 0.$$

Since

$$\int_{t_0}^\infty f_n(s)\,\mathrm{d}s = \int_{t_0}^{t_n} h_\pm(t_n)^{-1}\Phi(t_n)P_+\Phi^{-1}(s)G(s, y_\pm(s))\,\mathrm{d}s,$$

then

$$\lim_{t\to\infty} h_\pm(t)^{-1}\Phi(t)P_+\int_{t_0}^{t}\Phi^{-1}(s)G(s, y_\pm(s))\,\mathrm{d}s = 0.$$

We establish now our main result.

THEOREM 1. Let (σ, t_0) satisfy (3.1). If system (1.1) satisfies (H1)–(H3) then for any solution $x_\pm \in C_\pm(t_0, \sigma)$ of (1.2), there exists a solution $y_\pm \in C_\pm(t_0, \rho)$ of (1.1) such that

$$\|y_\pm - x_\pm\|_\pm \leq \rho - \sigma. \tag{3.4}$$

If, in addition, (H4) holds, then

$$y_\pm(t) = x_\pm(t) + h_\pm(t)o(1), \qquad t \leq t_0 \leq 0. \tag{3.5}$$

Conversely, for any solution $y_\pm \in C_\pm(t_0, \rho)$ of (1.1), if (t_0, ρ) satisfies (3.1) then there exist solutions $x_\pm \in C_\pm(t_0, \rho)$, of (1.2) satisfying (3.4). Moreover, if (H4) is satisfied, then (3.5) holds.

Proof. Let $t_0 = t_0(\rho)$ be the number obtained in lemma 1. By lemma 3, for $x_\pm \in C_\pm(t_0, \rho)$, we know that the operator $T_\pm$: $C_\pm(t_0, \rho) \to C_\pm(t_0, \rho)$ has a fixed point $y_\pm(t)$ satisfying the integral equation

$$y_\pm(t) = x_\pm(t) + \int_{t_0}^{\infty} \Gamma(t, s)G(s, y_\pm(s))\,\mathrm{d}s, \qquad t \geq t_0.$$

It is easy to see that these functions are solutions of equation (1.1). As in lemma 1, we estimate $h_\pm(t)^{-1}(y_\pm(t) - x_\pm(t)) = h_\pm(t)^{-1} \int_{t_0}^{\infty} \Gamma(t, s)G(s, y(s))\,\mathrm{d}s$ and, taking into account the definition (3.1) of t_0, we obtain $|h_\pm(t)^{-1}(y_\pm(t) - x_\pm(t))| \leq \rho$. Formula (3.5) follows from lemma 4.

The converse part of the theorem is immediate since for any solution $y_\pm \in C_\pm(t_0, \rho)$, the function $x_\pm$ defined by

$$x_\pm(t) = y_\pm(t) - \int_{t_0}^{\infty} \Gamma(t, s)G(s, y_\pm(s))\,\mathrm{d}s$$

is a solution of (1.2) and satisfies (3.4) and (3.5).

We remark that theorem 1 can be adapted in an obvious way for a local function G: $[0, \infty) \times \Omega \to \mathbf{R}^n$, where Ω is an open subset of $\mathbf{R}^n$.

4. DICHOTOMIC CHARACTER OF SOLUTIONS $y_\pm$

For a system of differential equations

$$u' = f(t, u), \qquad t \geq 0,\ u \in R^n \tag{4.1}$$

we say that the set $\Sigma = \{(t, u);\ t \geq 0,\ u \in R^n\}$ is an integral manifold of (4.1) if $(t_0, u_0) \in \Sigma$ implies that the solution $u(t)$ of (4.1), with initial condition $u(t_0) = 3u_0$ satisfies $(t, u(t)) \in \Sigma$ for all $t \geq 0$. System (1.2) with the properties (1.3) and (1.4) has the integral manifold $\Sigma_\pm = \{(t, \Phi(t)P_\pm\Phi^{-1}(t)u);\ t \geq 0,\ u \in R^n\}$. For a function $x(t)$: $[0, \infty) \to \mathbf{R}^n$ we define the distance from $x(t)$ to $\Sigma_\pm$ as

$$d_\pm(x(t)) = \inf\{h_\pm(t)^{-1}|x(t) - u|;\ (t, u) \in \Sigma_\pm\}.$$

With these definitions we have the following theorems.

THEOREM 2. If hypotheses (H1)–(H4) hold then the solutions $y_\pm(t)$ of system (1.1) given by theorem 1 satisfy

$$d_\pm(y_\pm(t)) = o(1).$$

The proof follows immediately from the definition of $d_{\pm}(x(t))$ and from the asymptotic formulae (3.5).

This last theorem says that the conditions that we have imposed on system (1.1) imply that the integral manifolds $\Sigma_{\pm}$ are asymptotic integral manifolds for system (1.1). The following question arises: can we infer the existence of integral manifolds for the nonlinear equation (1.1)? Investigation into this question is the subject of the author's current research (see [10]).

5. APPLICATIONS

5.1. *Equations with convergent solutions*

Let us consider the ordinary differential equation

$$y' = G(t, y), \qquad t \geq 0,\ y \in R^n,\ |y| < \rho_0. \tag{5.1}$$

We study the following question: under what conditions can we ensure the existence of a neighborhood $B(0, \nu)$ of $y = 0$, such that for any initial condition y_0 in this neighborhood the solution $y(t, t_0, y_0)$ is defined and

$$\lim_{t\to\infty} y(t, t_0, y_0) = \xi \tag{5.2}$$

exists? Conversely, given $\xi \in B(0, \nu)$ does there exist a solution y of (5.1) satisfying (5.2)?

Theorem 1 gives conditions under which this result is valid when $\nu = \rho < \rho_0$. Consider the system (5.1) as a perturbation of the system

$$x' = 0. \tag{5.3}$$

We can suppose that the system (5.3) satisfies (1.3)–(1.4) for $P_+ = 0$, $h(t) = 1$, $k(t) = 1$, $K = 1$. We assume that the function $G(t, y)$ can be majorized by a scalar function $r(t, y)|y|$

$$|G(t, y)| \leq r(t, y)|y|,$$

and that

$$m_\rho(t) = \sup_{|u|\leq\rho} |r(t, u)| \in L^1[0, \infty).$$

According to theorem 1, if

$$\sigma + \rho \int_{t_0}^{\infty} m_\rho(s)\,\mathrm{d}s \leq \rho, \qquad 0 \leq \sigma \leq \rho < \rho_0 \tag{5.4}$$

then for any x_0, $|x_0| \leq \sigma$, a solution $y = y(t)$ of (5.1) can be defined on $[t_0, \infty)$, such that $|y(t) - x_0| \leq \rho - \sigma$ and

$$y(t) = x_0 + 0\left(\int_t^{\infty} m_\rho(s)\,\mathrm{d}s\right). \tag{5.5}$$

The converse is also true: if $y(t)$ is a solution of (5.1) with $|y(t)| \leq \rho$ and if (t_0, σ) satisfies (5.4), then there exists x_0, $|x_0| \leq \sigma$, such that $|y(t) - x_0| \leq \rho - \sigma$, and $y(t)$ satisfies (5.5).

The inequality (5.4) establishes a dependence between t_0, σ and ρ. In general it will satisfy for large values of t_0. This inequality is necessary for our result; in the concrete example

$$y' = y^2/t^2,$$

we observe that $\tilde{y}(t) = t$ is a solution of this equation whose initial condition $y(t_0) = t_0 \leq \rho$ does not satisfy inequality (5.4) for any pair of positive numbers t_0 and σ.

5.2. *Asymptotic equivalence*

We want to compare our results with those of [2, 3]. Although a more general nonlinear term could be considered, we shall work with the system

$$y' = A(t)y + \lambda(t)f(y)y, \qquad t \geq 0,\ y \in \mathbf{R}^n \tag{5.6}$$

where $\lambda(t)$ is a matrix with L^1 coefficients and the function $f(y)$ is a bounded function on $\mathbf{R}^n$. With this condition hypothesis (H2) is fulfilled with

$$m_{\pm}(t) = |\lambda(t)| \operatorname{Sup}\{|f(x)|;\ x \in \mathbf{R}^n\}.$$

We will apply theorem 1 to three homogeneous systems.

(I) Let us consider system (1.2) with the matrix

$$A(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h(t) = \mathrm{e}^{-\alpha t}, \qquad 0 < \alpha < 1, \qquad k(t) = \mathrm{e}^{-t}.$$

The homogeneous system (1.2) has the h-bounded solution

$$x(t) = \mathrm{e}^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the k^{-1}-bounded solution

$$x(t) = \mathrm{e}^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is easy to verify that the hypotheses of theorem 1 are satisfied. Therefore, system (5.6) has solutions of the form

$$y_+(t) = \xi\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-t} + \mathrm{e}^{-\alpha t}o(1), \qquad 0 < \alpha < 1$$

$$y_-(t) = \xi\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{t} + \mathrm{e}^{t}o(1).$$

This is a much more precise result than the one given by theorem 5.11 in [3], because the latter does not treat unbounded solutions. Neither can this result be obtained from [2]. We further note that we have not imposed a monotonicity condition on $f(y)$.

(II) Let us consider now system (1.2) with the matrix

$$A(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{5.7}$$

We define $h(t) = e^{-\alpha t}$, $0 < \alpha < 1$, $k(t) = e^{-t}$ and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

In this case, we have the following asymptotic correspondence between systems (1.2) and (5.6)

$$y_+(t) = \xi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + e^{-\alpha t} o(1) \tag{5.9}$$

and

$$y_-(t) = \xi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} + e^{t} o(1). \tag{5.10}$$

In this case we are not able to relate the solutions

$$x(t) = \xi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} \tag{5.11}$$

with solutions of system (5.6), because $x(t)$ is not k^{-1}-bounded. Nevertheless, we can consider the system (1.2) with the same matrix (5.7) where instead of the projection matrix (5.8) we use

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

with $h(t) = e^{\beta t}$, $\beta > 1$, and $k(t) = e^{-2t}$. In this case, the following correspondences are obtained

$$y_+(t) = \xi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + e^{\beta t} o(1) \tag{5.13}$$

$$y_+(t) = \xi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} + e^{\beta t} o(1) \tag{5.14}$$

and

$$y_-(t) = \xi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + e^{2t} o(1). \tag{5.15}$$

Correspondences (5.13) and (5.14) are not of great interest because (5.9) and (5.10) are more precise asymptotic correspondences. But correspondence (5.15) says that the unbounded solutions (5.11) correspond to unbounded solution of system (5.6). Grouping (5.13)–(5.15), we obtain a complete asymptotic correspondence between solutions of (1.2) and (5.6).

(III) Let us now consider system (1.2) with the matrix

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The importance of this example is that no solution of this equation is bounded. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $h(t) = \mathrm{e}^{\alpha t}$, $\alpha > 1$, $k(t) = \mathrm{e}^{-2t}$. In this case the following correspondences are obtained

$$y_+(t) = \xi\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{t} + \mathrm{e}^{\alpha t}o(1) \qquad \alpha < 1$$

$$y_-(t) = \xi\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{2t} + \mathrm{e}^{2t}o(1).$$

6. AN EXAMPLE

We consider the nonautonomous system

$$\begin{aligned} u' &= -t^{-1}u + \lambda_1(t)v + \lambda_2(t)uv, \qquad t \geq 1 \\ v' &= t^{-1}v + \mu_1(t)u + \mu_2(t)uv, \end{aligned} \tag{6.1}$$

with measurable coefficients λ_i, μ_i, $i = 1, 2$, as a perturbation of the linear equation

$$u' = -t^{-1}u$$

$$v' = t^{-1}v$$

whose fundamental matrix is given by

$$\Phi(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \qquad t \geq 1.$$

We consider

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = I - P_+.$$

Thus, we have (1.3) and (1.4) with $K = 1$ and $h(t) = t^{-\alpha}$, $\alpha < 1$, $k(t) = t^{-1}$.

In this example the perturbation term in (6.1) can be written in the form

$$G(t, u, v) = R(t, u, v)\begin{pmatrix} u \\ v \end{pmatrix}, \qquad R(t, u, v) = \begin{pmatrix} \lambda_2 v & \lambda_2 \\ \mu_1 & \mu_2 u \end{pmatrix}.$$

Since $P_\pm\Phi(t) = \Phi(t)P_\pm$, hypothesis (H2) can be expressed in the form (2.4) and (2.5)

$$m_+(t) := \sup_{|x| \leq \rho} |P_+R(t, h(t)x)| = |\lambda_1(t)| + |t^{-\alpha}\lambda_2(t)|\rho$$

$$m_-(t) := \sup_{|x| \leq \rho} |P_-R(t, k(t)t^{-1}x)| = |\mu_1(t)| + |t\mu_2(t)|\rho$$

for some $\rho > 0$. That $m_+(t) \in L^1$ follows from λ_1, (t), $t^{-\alpha}\lambda_2(t) \in L^1$ and that $m_-(t) \in L^1$ follows from μ_1, $t\mu_2 \in L^1$. If hypothesis (H2) is used in the form (2.3) then $m_+(t) \in L^1$ if λ_1, μ_1, $t^{-\alpha}\lambda_2$, $t^{-\alpha}\mu_2 \in L^1$ and $m_-(t) \in L^1$ if λ_1, μ_1, $t\lambda_2$, $t\mu_2 \in L^1$. For this last set of conditions we observe that there is an additional restriction that $t\lambda_2 \in L^1$. This example shows that hypothesis (H2) in the form (2.5) is less restrictive on the coefficients than condition (2.3).

Hypothesis (H3) is satisfied with $C = 1$.

According to theorem 1 we can conclude: given $\lambda_1, \mu_1, t^{-\alpha}\lambda_2, t\mu_2 \in L^1[0, \infty)$, then for $\rho > 0$ and t_0 satisfying

$$\sigma + \rho \max\left[\int_{t_0}^{\infty} (|\lambda_1(s)| + |s^{-\alpha}\lambda_2(s)|\rho)\, ds, \int_{t_0}^{\infty} (|\mu_1(s)| + |s\mu_2(s)|\rho)\, ds\right] \le \rho$$

and any $|\xi| \le \sigma$, $\xi \in \mathbf{R}$, system (6.1) has the solutions $y_+(t)$ and $y_-(t)$ such that $|t^{\alpha}y_+(t)| \le \rho$, $|t^{-1}y_-(t)| \le \rho$ for $t \ge t_0$ and

$$y_-(t) = t^{-1}(\xi e_1 + t^{1-\alpha}o(1)), \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\xi| \le \sigma, \qquad t \ge t_0,$$

$$y_-(t) = t(\xi e_2 + o(1)), \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |\xi| \le \sigma, \qquad t \ge t_0.$$

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