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Bianchi identities for Yang-Mills fields

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We propose an indirect way to study some solutions of Yang-Mills equations, which is based on the analysis of the following closely related equations: $\partial F + A \times F = 0$ and $\partial \bar{F} + A \times \bar{F} = 0$. The former is one of the standard Yang-Mills equations. The latter is the well- known "Bianchi identity" for Yang-Mills fields; but here we do not assume the usual relation between field strenghts and potentials of the Yang-Mills equations. We give a method that allows the investigation of certain families of solutions of these equations. In particular, we find a family of solutions that does not contain any solution of the Yang-Mills equations. The inverse, of course, cannot happen.

I. INTRODUCTION

In this paper we are concerned with the equations

$$F_{a\mu\nu}{}^{;\mu} + \epsilon_{abc} A_b^\mu F_{c\mu\nu} = 0\,, \tag{1.1}$$

$$\bar{F}_{a\mu\nu}{}^{;\mu} + \epsilon_{abc} A_b^\mu \bar{F}_{c\mu\nu} = 0\,, \tag{1.2}$$

where a bar over an antisymmetric tensor denotes its dual, that is,

$$\bar{F}_{a\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_a^{\rho\sigma}\,. \tag{1.3}$$

Equations (1.1) and (1.2) are written in terms of covariant derivatives since we will be using non-Cartesian coordinates in four-dimensional Minkowski space-time.

Our interest in the study of Eqs. (1.1) and (1.2) is due to their close relationship to the Yang-Mills equations

$$F_{a\mu\nu}{}^{;\mu} + \epsilon_{abc} A_b^\mu F_{c\mu\nu} = 0\,, \tag{1.4}$$

$$F_{a\mu\nu} = A_{a\nu,\mu} - A_{a\mu,\nu} + \epsilon_{abc} A_{b\mu} A_{c\nu}\,. \tag{1.5}$$

In fact Eqs. (1.4) and (1.1) are the same, and if the fields strengths $F_{a\mu\nu}$ are given in terms of the potentials $A_{a\mu}$ by (1.5), then Eqs. (1.2) become the well-known Bianchi identities. Thus, every solution of the Yang-Mills equations (1.4) and (1.5) is also a solution of Eqs. (1.1) and (1.2). However, as we will explicitly show, there are "field strengths" $F_{a\mu\nu}$ and "potentials" $A_{a\mu}$, solutions of Eqs. (1.1) and (1.2), that cannot be written as in (1.5). Nevertheless, if we were able to find families of solutions of the system (1.1) and (1.2), then it would be reasonable to ask which members of those families are also solutions of the Yang-Mills equations.

In this paper we consider the case of O(3) as the gauge group and present an algorithm that allows the investigation of some families of solutions of the system (1.1) and (1.2). Our approach is based on the inversion that expresses the potentials $A_{a\mu}$ in terms of the field strengths $F_{a\mu\nu}$. The algebraic problem of the inversion $A = A(F)$ is worked out starting from the equations

$$F^+_{a\mu\nu}{}^{;\mu} + \epsilon_{abc} A_b^\mu F^+_{c\mu\nu} = 0\,, \tag{1.6}$$

$$F^-_{a\mu\nu}{}^{;\mu} + \epsilon_{abc} A_b^\mu F^-_{c\mu\nu} = 0\,, \tag{1.7}$$

where

$$F^\pm_{a\mu\nu} = F_{a\mu\nu} \pm i\bar{F}_{a\mu\nu}\,. \tag{1.8}$$

Any field configuration $(A_{a\mu}, F_{a\mu\nu})$ that is a solution of (1.1) and (1.2) gives, by (1.8), a configuration $(A_{a\mu}, F^\pm_{a\mu\nu})$ that is a solution of (1.6) and (1.7). And conversely, if $(A_{a\mu}, F^\pm_{a\mu\nu})$ is a solution of (1.6) and (1.7), then $(A_{a\mu}, F_{a\mu\nu} = (F^+_{a\mu\nu} + F^-_{a\mu\nu})/2)$ is a solution of (1.1) and (1.2). We carry out the inversion process $A = A(F)$ in Eqs. (1.6) and (1.7) rather than in Eqs. (1.1) and (1.2), since this is easier. We remark that there are some field configurations $(A_{a\mu}, F_{a\mu\nu})$ for which it is possible to obtain the inversion $A = A(F)$ starting from Eqs. (1.1)–(1.2), while this relation cannot be obtained from Eqs. (1.6)–(1.7) because the corresponding determinant vanishes. The opposite case can also happen.

It turns out that the Newmann-Penrose null-tetrad formalism fits very nicely to our purpose. Using this formalism we find, in the case of real fields, a general and explicit formula that expresses the potentials in terms of the field strengths. This formula allows us to investigate in a rather systematic way some field configurations solutions of (1.1) and (1.2). In particular, we study here in detail the simplest configutation of $F_{a\mu\nu}$ for which the inversion $A = A(F)$ is possible. Although in this case the family of solutions of Eqs. (1.1) and (1.2) that we display explicitly is rather large, it is not large enough to contain a solution of the Yang-Mills equations (1.4) and (1.5).

The contents of this paper is as follows. In

Sec. II we carry out in an explicit way the inversion $A = A(F)$. In Sec. III we particularize this formalism to the simplest case and the corresponding first-order differential equations are completely integrated. The family of solutions of (1.1) and (1.2) thus obtained is examined in the static case in Sec. IV, and it is shown that it does not contain any solution of the Yang-Mills equations. In Sec. IV we also exhibit a field configuration for which it is not possible to carry out the inversion $A = A(F)$ starting from (1.1) and (1.2), although it can be obtained from (1.6) and (1.7). In this paper we do not impose any gauge restriction.

II. THE INVERSION

The technique of the null-tetrad formalism has been applied previously to the Yang-Mills equations by Carmeli,[1] and by Newman.[2] Here we will use the same coordinate system and null tetrad of Newman and Penrose.[3] Then, instead of the usual Cartesian coordinates (t,x,y,z) of Minkowski space, we use

$$x^0=u\ ,\quad x^1=r\ ,\quad x^2=\theta\ ,\quad x^3=\phi\ , \tag{2.1}$$

which are related to (t,x,y,z) by

$$\begin{aligned} &t=2^{-1/2}(2u+r)\ ,\quad x=2^{-1/2}r\sin\theta\cos\phi\ ,\\ &y=2^{-1/2}r\sin\theta\sin\phi\ ,\quad z=2^{-1/2}r\cos\theta\ . \end{aligned} \tag{2.2}$$

The corresponding metric is given by

$$g_{\mu\nu}=\begin{bmatrix} 2 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -r^2/2 & 0\\ 0 & 0 & 0 & -r^2\sin^2\theta/2 \end{bmatrix}\ . \tag{2.3}$$

In particular, we have

$$\epsilon_{\mu\nu\rho\sigma}=\frac{r^2\sin\theta}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}\ , \tag{2.4}$$

where $\hat{\epsilon}_{\mu\nu\rho\sigma}$ is the four-dimensional completely antisymmetric constant tensor with $\hat{\epsilon}_{0123}=+1$.

The null tetrad of Ref. 3 is defined in the coordinate system (2.1) as

$$\begin{aligned} l_\mu &= \delta^0_\mu\ ,\\ n_\mu &= \delta^0_\mu+\delta^1_\mu\ ,\\ m_\mu &= -\frac{r}{2}(\delta^2_\mu+i\sin\theta\,\delta^3_\mu)\ ,\\ m^*_\mu &= -\frac{r}{2}(\delta^2_\mu-i\sin\theta\,\delta^3_\mu)\ . \end{aligned} \tag{2.5}$$

Here and in what follows an asterisk denotes the complex-conjugate quantity. We also use an alternative notation for the null vectors of the tetrad, namely,

$$l_{1\mu}=l_\mu\ ,\quad l_{2\mu}=n_\mu\ ,\quad l_{3\mu}=m_\mu\ ,\quad l_{4\mu}=m^*_\mu\ . \tag{2.6}$$

Let us introduce the bivectors

$$\begin{aligned} L_{\mu\nu}&=l_\mu n_\nu-l_\nu n_\mu\ ,\\ M_{\mu\nu}&=l_\mu m_\nu-l_\nu m_\mu\ ,\\ N_{\mu\nu}&=n_\mu m_\nu-n_\nu m_\mu\ ,\\ P_{\mu\nu}&=m_\mu m^*_\nu-m_\nu m^*_\mu\ . \end{aligned} \tag{2.7}$$

The most general real antisymmetric tensor $F_{a\mu\nu}$ can be written in terms of these quantities in the form

$$\begin{aligned} F_{a\mu\nu}=&\tfrac{1}{2}(\Phi_a+\Phi^*_a)L_{\mu\nu}+\tfrac{1}{2}\chi_a M^*_{\mu\nu}+\tfrac{1}{2}\chi^*_a M_{\mu\nu}\\ &+\tfrac{1}{2}\Psi_a N_{\mu\nu}+\tfrac{1}{2}\Psi^*_a N^*_{\mu\nu}+\tfrac{1}{2}(\Phi_a-\Phi^*_a)P_{\mu\nu}\ , \end{aligned} \tag{2.8}$$

where the scalars Φ_a, χ_a, Ψ_a are in general complex quantities. It is easy to see that the duals of the bivectors (2.7) are given by

$$\begin{aligned} \bar{L}_{\mu\nu}&=-iP_{\mu\nu}\ ,\\ \bar{M}_{\mu\nu}&=iM_{\mu\nu}\ ,\\ \bar{N}_{\mu\nu}&=-iN_{\mu\nu}\ ,\\ \bar{P}_{\mu\nu}&=-iL_{\mu\nu}\ . \end{aligned} \tag{2.9}$$

Therefore, the tensors $F^{\pm}_{a\mu\nu}$ defined in (1.8) can be written as

$$F^+_{a\mu\nu}=\Phi_a(L_{\mu\nu}+P_{\mu\nu})+\chi_a M^*_{\mu\nu}+\Psi_a N_{\mu\nu}\ , \tag{2.10}$$

$$F^-_{a\mu\nu}=\Phi^*_a(L_{\mu\nu}-P_{\mu\nu})+\chi^*_a M_{\mu\nu}+\Psi^*_a N^*_{\mu\nu}\ . \tag{2.11}$$

$F^-_{a\mu\nu}$ is simply the complex conjugate of $F^+_{a\mu\nu}$ because we are considering only real fields $F_{a\mu\nu}$.

Let us denote by $\mathcal{A}_{ai}$ the components of the potentials $A_{a\mu}$ over the null tetrad, that is,

$$A_{a\mu}=\mathcal{A}_{ai}l_{i\mu}\ , \tag{2.12}$$

where the internal index a takes the values 1, 2, 3; and the index $i=1,2,3,4$ labels the vectors of the tetrad according to (2.6). When we project Eq. (1.6) on the tetrad, we get

$$F^+_{a\mu\nu}{}^{;\mu}l^\nu_i+(\epsilon_{abc}F^+_{c\mu\nu}l^\nu_i l^\mu_j)\mathcal{A}_{bj}=0\ . \tag{2.13}$$

We denote by $\mathfrak{F}^+$ the 12×12 matrix that appears in this equation, that is,

$$\mathfrak{F}^+_{abij}=\epsilon_{abc}F^+_{c\mu\nu}l^\nu_i l^\mu_j\ . \tag{2.14}$$

If we write this matrix in terms of 4×4 submatrices (a), $a=1,2,3$, it is easy to see that it has the structure

$$\mathfrak{F}^+=\begin{bmatrix} (0) & (3) & -(2)\\ -(3) & (0) & (1)\\ (2) & -(1) & (0) \end{bmatrix}, \tag{2.15}$$

where each submatrix (a) can be written in terms of the scalars Φ_a, χ_a, and Ψ_a, which define the fields in (2.8) in the form

$$(a)=\begin{bmatrix} 0 & \Phi_a & 0 & \Psi_a \\ -\Phi_a & 0 & \chi_a & 0 \\ 0 & -\chi_a & 0 & \Phi_a \\ -\Psi_a & 0 & -\Phi_a & 0 \end{bmatrix}. \tag{2.16}$$

In a similar way, from Eq. (1.7) we obtain the matrix $\mathfrak{F}^-$ defined by

$$\mathfrak{F}^-_{abij}=\epsilon_{abc}F^-_{c\mu\nu}l^\nu_i l^\mu_j\,, \tag{2.17}$$

which has the same structure as (2.15), but with $(a)\to(\hat{a})$, where

$$(\hat{a})=\begin{bmatrix} 0 & \Phi^*_a & \Psi^*_a & 0 \\ -\Phi^*_a & 0 & 0 & \chi^*_a \\ -\Psi^*_a & 0 & 0 & -\Phi^*_a \\ 0 & -\chi^*_a & \Phi^*_a & 0 \end{bmatrix}. \tag{2.18}$$

The problem of finding the inverse of the matrices $\mathfrak{F}^+$ and $\mathfrak{F}^-$ is simplified considerably due to the simple structure of the submatrices (a) and $(\hat{a})$. It is straightforward to show that the inverse of $\mathfrak{F}^+$ is given by

$$(\mathfrak{F}^+)^{-1}=|\mathcal{E}_{ab}|\,,\quad a,b=1,2,3 \tag{2.19}$$

where the $\mathcal{E}_{ab}$ are the 4×4 matrices

$$\mathcal{E}_{ab}=\frac{1}{\epsilon_{lmn}\Phi_l\Psi_m\chi_n}\begin{bmatrix} 0 & \Phi_a\Phi_b+\Psi_a\chi_b & 0 & \chi_a\Phi_b-\Phi_a\chi_b \\ \Phi_a\Phi_b+\chi_a\Psi_b & 0 & \Phi_a\Psi_b-\Psi_a\Phi_b & 0 \\ 0 & \Psi_a\Phi_b-\Phi_a\Psi_b & 0 & -\Phi_a\Phi_b-\chi_a\Psi_b \\ \Phi_a\chi_b-\chi_a\Phi_b & 0 & -\Phi_a\Phi_b-\Psi_a\chi_b & 0 \end{bmatrix}. \tag{2.20}$$

For the inverse of $\mathfrak{F}^-$ we find

$$(\mathfrak{F}^-)^{-1}=|\hat{\mathcal{E}}_{ab}|\,,\quad a,b=1,2,3 \tag{2.21}$$

where

$$\hat{\mathcal{E}}_{ab}=\frac{1}{\epsilon_{lmn}\Phi^*_l\Psi^*_m\chi^*_n}\begin{bmatrix} 0 & \Phi^*_a\Phi^*_b+\Psi^*_a\chi^*_b & \chi^*_a\Phi^*_b-\Phi^*_a\chi^*_b & 0 \\ \Phi^*_a\Phi^*_b+\chi^*_a\Psi^*_b & 0 & 0 & \Phi^*_a\Psi^*_b-\Psi^*_a\Phi^*_b \\ \Phi^*_a\chi^*_b-\chi^*_a\Phi^*_b & 0 & 0 & \Phi^*_a\Phi^*_b-\Psi^*_a\chi^*_b \\ 0 & \Psi^*_a\Phi^*_b-\Phi^*_a\Psi^*_b & -\Phi^*_a\Phi^*_b-\chi^*_a\Psi^*_b & 0 \end{bmatrix}. \tag{2.22}$$

Thus, the matrices $\mathfrak{F}^+$ and $\mathfrak{F}^-$ are invertible if and only if

$$\epsilon_{lmn}\Phi_l\Psi_m\chi_n\neq 0\,. \tag{2.23}$$

Denoting by F^+ the column vector $F^+_{a\mu\nu}{}^{;\mu}l^\nu_i$ and by $\mathfrak{A}$ the column vector $\mathfrak{A}_{bj}$, we can write Eq. (2.13) as

$$F^++\mathfrak{F}^+\mathfrak{A}=0\,, \tag{2.24}$$

and the corresponding equation associated with (1.7) is

$$F^-+\mathfrak{F}^-\mathfrak{A}=0\,. \tag{2.25}$$

Now, under the condition (2.23), Eqs. (2.24) and (2.25) are equivalent to

$$-\mathfrak{A}=(\mathfrak{F}^+)^{-1}F^+=(\mathfrak{F}^-)^{-1}F^-\,, \tag{2.26}$$

which are 12 first-order differential equations for the scalars Φ_a, χ_a, and Ψ_a that appear in the definition of the field strengths in (2.8). In particular, these equations imply that the potentials $A_{a\mu}$ in (2.12) are real.

The projections $F^+_{a\mu\nu}{}^{;\mu}l^\nu_i$ are given by

$$\begin{aligned} F^+_{a\mu\nu}{}^{;\mu}l^\nu_1&=\frac{\partial\Phi_a}{\partial r}+2r^{-1}\Phi_a-r^{-1}(\Psi_a\cot\theta+\eth\Psi_a)\,,\\ F^+_{a\mu\nu}{}^{;\mu}l^\nu_2&=\frac{\partial\Phi_a}{\partial r}-\frac{\partial\Phi_a}{\partial u}+2r^{-1}\Phi_a\\ &\quad-r^{-1}(\chi_a\cot\theta+\eth^*\chi_a)\,,\\ F^+_{a\mu\nu}{}^{;\mu}l^\nu_3&=-\frac{\partial\chi_a}{\partial r}-r^{-1}\chi_a-r^{-1}\eth\Phi_a\,,\\ F^+_{a\mu\nu}{}^{;\mu}l^\nu_4&=\frac{\partial\Psi_a}{\partial r}-\frac{\partial\Psi_a}{\partial u}+r^{-1}\Psi_a+r^{-1}\eth^*\Phi_a\,, \end{aligned} \tag{2.27}$$

where $\eth$ is the angular differential operator defined by

$$\eth=\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{2.28}$$

The projections $F^-_{a\mu\nu}{}^{;\mu}l^\nu_i$ can be inferred easily from (2.27) since $F^-_{a\mu\nu}$ is the complex conjugate of $F^+_{a\mu\nu}$.

III. A SPECIAL CASE

A simple field configuration consistent with (2.23) is given by the ansatz

$$\begin{aligned} &\Phi_1=\Phi_2=0\,, \quad \Phi_3\equiv\Phi\,,\\ &\Psi_2=\Psi_3=0\,, \quad \Psi_1\equiv\Psi\,,\\ &\chi_1=\chi_3=0\,, \quad \chi_2\equiv\chi\,. \end{aligned} \tag{3.1}$$

In addition, we assume that the functions Φ, χ, and Ψ do not have zeros. In this case (2.23) reduces to

$$\epsilon_{lmn}\Phi_l\Psi_m\chi_n=\Phi\chi\Psi\,. \tag{3.2}$$

Then, according to (2.8), the corresponding fields $F_{a\mu\nu}$ are given by

$$\begin{aligned} &F_{1\mu\nu}=\tfrac{1}{2}(\Psi N_{\mu\nu}+\Psi^* N^*_{\mu\nu})\,,\\ &F_{2\mu\nu}=\tfrac{1}{2}(\chi M^*_{\mu\nu}+\chi^* M_{\mu\nu})\,,\\ &F_{3\mu\nu}=\tfrac{1}{2}(\Phi+\Phi^*)L_{\mu\nu}+\tfrac{1}{2}(\Phi-\Phi^*)P_{\mu\nu}\,. \end{aligned} \tag{3.3}$$

If we insert the quantities (3.1) in the matrix given in (2.19) and (2.20), we get

$$\begin{pmatrix}
0&0&0&0&0&\frac{1}{\Phi}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{\chi}&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{\chi}&0&0\\
0&0&0&0&0&0&-\frac{1}{\Phi}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{\Psi}\\
\frac{1}{\Phi}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-\frac{1}{\Phi}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-\frac{1}{\Psi}&0&0&0\\
0&0&0&0&0&0&0&-\frac{1}{\Psi}&0&\frac{\Phi}{\chi\Psi}&0&0\\
0&0&\frac{1}{\chi}&0&0&0&0&0&\frac{\Phi}{\chi\Psi}&0&0&0\\
0&-\frac{1}{\chi}&0&0&0&0&0&0&0&0&0&-\frac{\Phi}{\chi\Psi}\\
0&0&0&0&\frac{1}{\Psi}&0&0&0&0&0&-\frac{\Phi}{\chi\Psi}&0
\end{pmatrix}. \tag{3.4}$$

On the other hand, for the matrix in (2.21) and (2.22) we find

$$\begin{pmatrix}
0&0&0&0&0&\frac{1}{\Phi^*}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{\chi^*}\\
0&0&0&0&0&0&0&-\frac{1}{\Phi^*}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{\chi^*}&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{\Psi^*}&0\\
\frac{1}{\Phi^*}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-\frac{1}{\Psi^*}&0&0&0\\
0&0&-\frac{1}{\Phi^*}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{\Psi^*}&0&0&\frac{\Phi^*}{\chi^*\Psi^*}&0&0\\
0&0&0&\frac{1}{\chi^*}&0&0&0&0&\frac{\Phi^*}{\chi^*\Psi^*}&0&0&0\\
0&0&0&0&\frac{1}{\Psi^*}&0&0&0&0&0&0&-\frac{\Phi^*}{\chi^*\Psi^*}\\
0&-\frac{1}{\chi^*}&0&0&0&0&0&0&0&0&-\frac{\Phi^*}{\chi^*\Psi^*}&0
\end{pmatrix} \tag{3.5}$$

From these matrices and Eq. (2.27), particularized to the case (3.1), we find that Eq. (2.26) can be written as

$$\Phi^*(\chi\cot\theta+\eth^*\chi)=\Phi(\chi^*\cot\theta+\eth\chi^*)\,, \tag{3.6}$$

$$\chi^*\eth\Phi=\chi\eth^*\Phi^*\,, \tag{3.7}$$

$$\chi\left(\frac{\partial\chi^*}{\partial r}+r^{-1}\chi^*\right)=\Phi^*\left(\frac{\partial\Phi}{\partial r}-\frac{\partial\Phi}{\partial u}+2r^{-1}\Phi\right)\,, \tag{3.8}$$

$$\Psi^*\eth^*\Phi=\Psi\eth\Phi^*\,, \tag{3.9}$$

$$\Phi^*(\Psi\cot\theta+\eth\Psi)=\Phi(\Psi^*\cot\theta+\eth^*\Psi^*)\,, \tag{3.10}$$

$$\Psi^*\left(\frac{\partial\Psi}{\partial r}-\frac{\partial\Psi}{\partial u}+r^{-1}\Psi\right)=\Phi\left(\frac{\partial\Phi^*}{\partial r}+2r^{-1}\Phi^*\right), \tag{3.11}$$

$$\Phi\Psi^*\chi^*\left(\frac{\partial\Phi}{\partial r}-\frac{\partial\Phi}{\partial u}+2r^{-1}\Phi\right)=\Phi^*\Psi\chi\left(\frac{\partial\Phi^*}{\partial r}-\frac{\partial\Phi^*}{\partial u}+2r^{-1}\Phi^*\right), \tag{3.12}$$

$$\Phi\Psi^*\chi^*\left(\frac{\partial\Phi}{\partial r}+2r^{-1}\Phi\right)=\Phi^*\Psi\chi\left(\frac{\partial\Phi^*}{\partial r}+2r^{-1}\Phi^*\right), \tag{3.13}$$

$$\Phi\Psi^*\chi^*\eth^*\Phi=-\Phi^*\Psi\chi\eth^*\Phi^*\,. \tag{3.14}$$

The three remaining equations are the complex conjugates of (3.8), (3.11), and (3.14).

From Eqs. (3.6)–(3.14) we see that the choice (3.1) has the remarkable property of giving rise to equations where the variables (r, u) are separated from the angular variables (θ, ϕ). Although Eqs. (3.6)–(3.14) seem to be tractable in the general case, in what follows we will consider only real valued scalars Φ, χ, and Ψ, that is,

$$\Phi = \Phi^*, \quad \chi = \chi^*, \quad \Psi = \Psi^*. \qquad (3.15)$$

Under this hypothesis Eqs. (3.12) and (3.13) do not impose any restrictions on the functions Φ, χ, and Ψ; and Eqs. (3.7), (3.9), and (3.14) become

$$\begin{aligned} \eth\Phi &= \eth^*\Phi, \\ \eth^*\Phi &= -\eth^*\Phi, \end{aligned} \qquad (3.16)$$

which imply $\partial\Phi/\partial\theta = 0$ and $\partial\Phi/\partial\phi = 0$. That is, Φ is independent of the angular variables (θ, ϕ). On the other hand, from Eqs. (3.6) and (3.10) it follows that χ and Ψ are independent of the angular variable ϕ. Thus Eqs. (3.6)–(3.14) reduce to

$$\chi\left(\frac{\partial\chi}{\partial r} + 2r^{-1}\chi\right) = \Phi\left(\frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial u} + 2r^{-1}\Phi\right), \qquad (3.17)$$

$$\Psi\left(\frac{\partial\Psi}{\partial r} - \frac{\partial\Psi}{\partial u} + r^{-1}\Psi\right) = \Phi\left(\frac{\partial\Phi}{\partial r} + 2r^{-1}\Phi\right). \qquad (3.18)$$

The integration of these equations is trivial because they are linear in χ^2 and Ψ^2. Let us introduce the notation

$$X(u, r) \equiv 2\Phi\left(\frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial u} + 2r^{-1}\Phi\right), \qquad (3.19)$$

$$Y(u, r) \equiv 2\Phi\left(\frac{\partial\Phi}{\partial r} + 2r^{-1}\Phi\right), \qquad (3.20)$$

and

$$\omega \equiv \chi^2, \quad \tau \equiv \Psi^2. \qquad (3.21)$$

Then we can write (3.17) and (3.18) as

$$\frac{\partial\omega}{\partial r} + 2r^{-1}\omega = X, \qquad (3.22)$$

$$\frac{\partial\tau}{\partial r} - \frac{\partial\tau}{\partial u} + 2r^{-1}\tau = Y. \qquad (3.23)$$

The general solution of (3.22) is given by

$$\omega(u, r, \theta) = r^{-2}\int_0^r t^2 X(u, t)\,dt + r^{-2}F(u, \theta), \qquad (3.24)$$

where $F(u, \theta)$ is a rather arbitrary function of two variables. The solution $\tau = \tau(u, r, \theta)$ of Eq. (3.23) is implicitly defined through an almost arbitrary function Q of three variables as

$$Q\left[u + r;\, \theta;\, r^2\tau - \int_0^r t^2 Y(u + r - t, t)\,dt\right] = 0. \qquad (3.25)$$

The arbitrariness of the functions F and Q, which appear in (3.24) and (3.25), respectively, is limited by the condition of producing functions χ and Ψ which are real and without zeros. In spite of this, the set of solutions of (3.22) and (3.23) is very large. As an illustration of this point let us choose Φ as a positive function that depends only on the variable r. Then it is easy to see that

$$\omega = \Phi^2(r) + 2r^{-2}\int_0^r t\Phi^2(t)\,dt + r^{-2}F_1(u, \theta) \qquad (3.26)$$

and

$$\tau = \Phi^2(r) + 2r^{-2}\int_0^r t\Phi^2(t)\,dt + r^{-2}F_2(u + r, \theta) \qquad (3.27)$$

are solutions of (3.22) and (3.23), respectively, for arbitrary F_1 and F_2. Now, if we choose F_1 and F_2 positive but otherwise arbitrary, the functions $\chi = \omega^{1/2}$ and $\Psi = \tau^{1/2}$ are real-valued solutions without zeros of Eqs. (3.17) and (3.18), respectively.

For the choice given by (3.1) and (3.15), we get from (2.26), (3.4), and (3.5) the following expressions for the components of the potentials $\mathfrak{A}_{ai}$:

$$\begin{aligned}
\mathfrak{A}_{11} &= r^{-1}\Phi^{-1}\left(\chi\cot\theta + \frac{\partial\chi}{\partial\theta}\right), \\
\mathfrak{A}_{12} &= 0, \\
\mathfrak{A}_{13} &= \mathfrak{A}_{14} = -\chi^{-1}\left(2r^{-1}\Phi + \frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial u}\right), \\
\mathfrak{A}_{21} &= 0, \\
\mathfrak{A}_{22} &= r^{-1}\Phi^{-1}\left(\Psi\cot\theta + \frac{\partial\Psi}{\partial\theta}\right), \\
\mathfrak{A}_{23} &= \mathfrak{A}_{24} = \Psi^{-1}\left(2r^{-1}\Phi + \frac{\partial\Phi}{\partial r}\right), \\
\mathfrak{A}_{31} &= -\Psi^{-1}\chi^{-1}\Phi\left(2r^{-1}\Phi + \frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial u}\right), \\
\mathfrak{A}_{32} &= -\chi^{-1}\Psi^{-1}\Phi\left(2r^{-1}\Phi + \frac{\partial\Phi}{\partial r}\right), \\
\mathfrak{A}_{33} &= \mathfrak{A}_{34} = 0,
\end{aligned} \qquad (3.28)$$

where $\Phi(u, r)$ is an arbitrary real function without zeros and χ, Ψ are determined by means of (3.24) and (3.25), respectively. The potentials $A_{a\mu}$ obtained by replacing (3.28) in (2.12), together with the field strengths $F_{a\mu\nu}$ given by

$$\begin{aligned}
F_{1\mu\nu} &= \tfrac{1}{2}\Psi(N_{\mu\nu} + N^*_{\mu\nu}), \\
F_{2\mu\nu} &= \tfrac{1}{2}\chi(M_{\mu\nu} + M^*_{\mu\nu}), \\
F_{3\mu\nu} &= \Phi L_{\mu\nu},
\end{aligned} \qquad (3.29)$$

are solutions of Eqs. (1.1) and (1.2).

IV. YANG-MILLS EQUATIONS

In this section we discuss the relation of the family of solutions of Eqs. (1.1) and (1.2) found in

Sec. III with the Yang-Mills equations (1.4) and (1.5). Equation (1.4) is obviously satisfied because it is identical to (1.1); it remains to study Eq. (1.5). For simplicity we will consider the static case, that is, when Φ, χ, and Ψ do not depend on u. For a given $\Phi(r)$, real and without zeros, let $G(r)$ be defined by

$$G(r)=r^2\Phi^2+2\int_0^r t\Phi^2(t)dt. \tag{4.1}$$

From Eqs. (3.19), (3.20), (3.21), (3.22), and (3.23) it follows that

$$\chi=\pm r^{-1}[G(r)+F(\theta)]^{1/2}\equiv r^{-1}R, \tag{4.2}$$

$$\Psi=\pm r^{-1}[G(r)+H(\theta)]^{1/2}\equiv r^{-1}S. \tag{4.3}$$

The functions $F(\theta)$ and $H(\theta)$ must be such that they give rise to functions χ and Ψ, which are real and without zeros, but otherwise they are arbitrary. The explicit form of the components $\mathfrak{A}_{ai}$ of the potentials $A_{a\mu}$ are given in this case by

$$\begin{aligned}
&\mathfrak{A}_{12}=\mathfrak{A}_{21}=\mathfrak{A}_{33}=\mathfrak{A}_{34}=0\,,\\
&\mathfrak{A}_{11}=r^{-2}\Phi^{-1}\left(R\cot\theta+\frac{\partial R}{\partial\theta}\right),\\
&\mathfrak{A}_{13}=\mathfrak{A}_{14}=-r^{-1}R^{-1}\frac{d}{dr}(r^2\Phi)\,,\\
&\mathfrak{A}_{22}=r^{-2}\Phi^{-1}\left(S\cot\theta+\frac{\partial S}{\partial\theta}\right),\\
&\mathfrak{A}_{23}=\mathfrak{A}_{24}=r^{-1}S^{-1}\frac{d}{dr}(r^2\Phi)\,,\\
&\mathfrak{A}_{31}=\mathfrak{A}_{32}=-\Phi R^{-1}S^{-1}\frac{d}{dr}(r^2\Phi)\,.
\end{aligned} \tag{4.4}$$

Let us denote by $G_{a\mu\nu}$ the following 18 quantities:

$$G_{a\mu\nu}=A_{a\nu,\mu}-A_{a\mu,\nu}+\epsilon_{abc}A_{b\mu}A_{c\nu}\,. \tag{4.5}$$

Projecting this equation over the tetrad we get

$$G_{a\mu\nu}l^\mu n^\nu=\mathfrak{A}_{a1,r}-\mathfrak{A}_{a2,u}+\mathfrak{A}_{a2,r}+\epsilon_{abc}\mathfrak{A}_{b2}\mathfrak{A}_{c1}\,, \tag{4.6a}$$

$$G_{a\mu\nu}l^\mu m^\nu=-\mathfrak{A}_{a4,r}-r^{-1}\eth\mathfrak{A}_{a2}-r^{-1}\mathfrak{A}_{a4}-\epsilon_{abc}\mathfrak{A}_{b2}\mathfrak{A}_{c4}\,, \tag{4.6b}$$

$$G_{a\mu\nu}n^\mu m^\nu=-r^{-1}\eth\mathfrak{A}_{a1}-\mathfrak{A}_{a4,u}+\mathfrak{A}_{a4,r}+r^{-1}\mathfrak{A}_{a4}-\epsilon_{abc}\mathfrak{A}_{b1}\mathfrak{A}_{c4}\,, \tag{4.6c}$$

$$G_{a\mu\nu}m^\mu m^{*\nu}=r^{-1}\eth^*\mathfrak{A}_{a4}-r^{-1}\eth\mathfrak{A}_{A3}+(\mathfrak{A}_{a4}-\mathfrak{A}_{a3})r^{-1}\cot\theta+\epsilon_{abc}\mathfrak{A}_{b4}\mathfrak{A}_{c3}\,. \tag{4.6d}$$

The projections $G_{a\mu\nu}l^\mu m^{*\nu}$ and $G_{a\mu\nu}n^\mu m^{*\nu}$ are the complex conjugates of (4.6b) and (4.6c), respectively, because we are dealing with reals potentials $A_{a\mu}$. On the other hand, the projections of (2.8) over the tetrad are given by

$$F_{a\mu\nu}l^\mu n^\nu=-\tfrac{1}{2}(\Phi_a+\Phi_a^*)\,, \tag{4.7a}$$

$$F_{a\mu\nu}l^\mu m^\nu=-\tfrac{1}{2}\Psi_a^*\,, \tag{4.7b}$$

$$F_{a\mu\nu}n^\mu m^\nu=-\tfrac{1}{2}\chi_a\,, \tag{4.7c}$$

$$F_{a\mu\nu}m^\mu m^{*\nu}=\tfrac{1}{2}(\Phi_a^*-\Phi_a)\,. \tag{4.7d}$$

When we particularize Eqs. (4.6) to the components (4.4), and (4.7) to the case (3.29), we see that

$$G_{a\mu\nu}m^\mu m^{*\nu}=F_{a\mu\nu}m^\mu m^{*\nu}\,,\quad a=1,2,3\,. \tag{4.8}$$

However, the other equations obtained by equating (4.6a) with (4.7a), (4.6b) with (4.7b), and (4.6c) with (4.7c) do not reduce to an identity for the components (4.4) and field strengths (3.29). These equations are

$$\frac{\partial\mathfrak{A}_{11}}{\partial r}+\mathfrak{A}_{22}\mathfrak{A}_{31}=0\,, \tag{4.9a}$$

$$\frac{\partial\mathfrak{A}_{22}}{\partial r}+\mathfrak{A}_{11}\mathfrak{A}_{31}=0\,, \tag{4.9b}$$

$$2\frac{\partial\mathfrak{A}_{31}}{\partial r}-\mathfrak{A}_{22}\mathfrak{A}_{11}=-\Phi\,, \tag{4.9c}$$

$$\left(\frac{\partial}{\partial r}+r^{-1}\right)\mathfrak{A}_{13}-\mathfrak{A}_{31}\mathfrak{A}_{23}=\tfrac{1}{2}\Psi\,, \tag{4.9d}$$

$$\left(\frac{\partial}{\partial r}+r^{-1}\right)\mathfrak{A}_{23}+r^{-1}\frac{\partial\mathfrak{A}_{22}}{\partial\theta}+\mathfrak{A}_{31}\mathfrak{A}_{13}=0\,, \tag{4.9e}$$

$$r^{-1}\frac{\partial\mathfrak{A}_{31}}{\partial\theta}-\mathfrak{A}_{22}\mathfrak{A}_{13}=0\,, \tag{4.9f}$$

$$r^{-1}\frac{\partial\mathfrak{A}_{11}}{\partial\theta}-\left(\frac{\partial}{\partial r}+r^{-1}\right)\mathfrak{A}_{13}-\mathfrak{A}_{31}\mathfrak{A}_{23}=0\,, \tag{4.9g}$$

$$-\left(\frac{\partial}{\partial r}+r^{-1}\right)\mathfrak{A}_{23}+\mathfrak{A}_{31}\mathfrak{A}_{13}=\tfrac{1}{2}\chi\,, \tag{4.9h}$$

$$r^{-1}\frac{\partial\mathfrak{A}_{31}}{\partial\theta}+\mathfrak{A}_{11}\mathfrak{A}_{23}=0\,. \tag{4.9i}$$

Let us analyze the restrictions that these equations impose on the functions $\Phi(r)$, $F(\theta)$, and $H(\theta)$ that appear in (4.2) and (4.3). It is easy to see that the choice $\Phi=cr^{-2}$, with c a constant, gives rise to $\mathfrak{A}_{13}=\mathfrak{A}_{31}=\mathfrak{A}_{23}=0$, which are in contradiction with (4.9d). Now, under the hypothesis

$$\frac{d}{dr}(r^2\Phi)\neq 0\,, \tag{4.10}$$

Eqs. (4.9f) and (4.9i) tell us that

$$H=F+\frac{C}{\sin^2\theta}\,, \tag{4.11}$$

where C is an arbitrary constant. On the other hand, from Eqs. (4.9a) and (4.9b) it follows that

$$\frac{\partial}{\partial r}(\mathfrak{A}_{11}{}^2-\mathfrak{A}_{22}{}^2)=0\,. \tag{4.12}$$

But from (4.4) and (4.11) we obtain

$$\mathfrak{a}_{11}{}^2 - \mathfrak{a}_{22}{}^2 = r^{-4}\Phi^{-2}\left[\frac{C\cot^2\theta}{\sin^2\theta} + \frac{1}{4R^2}\left(\frac{dF}{d\theta}\right)^2 - \frac{1}{4S^2}\left(\frac{dF}{d\theta} - \frac{2C\cot\theta}{\sin^2\theta}\right)^2\right],$$

which does not depend on r only if $C=0$. That is, we must put $R=S$ in (4.4). In this case Eq. (4.9a) takes the form

$$2\cot\theta R^4 + \left(\frac{dF}{d\theta}\right)R^2 + 2r^2\Phi^2\left(\frac{dF}{d\theta}\right) = 0\,, \qquad (4.13)$$

from which we obtain an R^2 in contradiction with its definition given by (4.2). Therefore, in the static case, the family of solutions of (1.1) and (1.2) given in Sec. III does not contain any solution of the Yang-Mills equations (1.4) and (1.5).

Finally, we want to make a remark on the inversion $A=A(F)$. Projecting Eq. (1.1) on the tetrad we obtain

$$F_{a\mu\nu}{}^{;\mu} l_i^\nu + (\epsilon_{abc} F_{c\mu\nu} l_i^\nu l_j^\mu)\mathfrak{a}_{bj} = 0\,.$$

It can be shown that the matrix

$$\mathfrak{F} = \epsilon_{abc} F_{c\mu\nu} l_i^\nu l_j^\mu$$

in the case (3.3) is such that its determinant is given by

$$\det\mathfrak{F} = 2^{-12}(\Phi^2 - \Phi^{*2})^2(\chi\Psi - \chi^*\Psi^*)^4\,,$$

which is zero if Φ, χ, and Ψ satisfy (3.15). Thus, in this case the inversion $A=A(F)$ cannot be accomplished by starting from (1.1), but such inversion can be obtained from (1.6).

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[1]M. Carmeli, Phys. Lett. 68B, 463 (1977); M. Carmeli, Ch. Charach, and M. Kaye, Nuovo Cimento 45B, 310 (1978).

[2]E. T. Newman, Phys. Rev. D 18, 2901 (1978); 22, 3023 (1980).

[3]E. T. Newman and R. Penrose, Proc. R. Soc. London A305, 205 (1968).