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ASYMPTOTICS FOR SECOND ORDER DELAYED DIFFERENTIAL EQUATIONS

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Abstract

In this work we present a way to find asymptotic formulas for some solutions of second order linear differential equations with a retarded functional perturbation.

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1. Introduction

The initial motivation of this work is the conjecture of Haddock-Sacker [16] (1980). They consider a differential system

$$y' = \Lambda y + R(t)y(t-r), \tag{1.1}$$

where Λ and R are $N \times N$ matrix valued functions of $t \geq 0$ such that $\Lambda = \text{diag}(\lambda_1, ..., \lambda_N)$, with $\Re e\lambda_i \neq \Re e\lambda_j$ for $i \neq j$ and the operator norm $\|R(t)\|$ is in L^2. Then, they conjecture that the fundamental matrix of (1.1) $Y = Y(t)$ such that $Y(0) = I$ satisfies

$$Y(t) = (I + o(1)) \exp\left(t\Lambda + e^{-\Lambda r}\int_0^t \text{diag}[R(s)]ds\right),$$

as $t \to +\infty$. This conjecture is a version of the asymptotic theorem of Hartman-Wintner [18] (1955) for an autonomous diagonal differential system with a L^p $(p = 2)$ linear perturbation with delayed argument. This and similar problems were considered by Arino-Győri [2] (1989), Ai [1] (1992) and Cassel-Hou [6] (1993) (here $p \geq 2$ is considered) for a system where the non-perturbed system is diagonal and satisfies the hypotheses of the Hartman-Wintner's asymptotic theorem.

A result which extend the Conjecture of Haddock and Sacker [16] and the result of Cassel and Hou [6] is the following:

Proposition 1. *(See Castillo [7, 2003] Consider the linear differential system*

$$y'(t) = B(t)y(t) + R(t, y_t), \tag{1.2}$$

where $y_t(s) = y(t+s)$ *the matrix* B *is in Jordan form, that is*

$$B(t) = \left[\oplus_{i=1}^{k-1} J_{n_i}(\lambda_i(t))\right] \oplus \lambda_k(t) \oplus \left[\oplus_{i=k+1}^{m} J_{n_i}(\lambda_i(t))\right],$$

where $J_{n_i}(\lambda)$ *are the* $n_i \times n_i$ *Jordan matrices*

$$J_{n_i}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ ... & ... & ... & ... & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

and the λ_i*'s are functions from* $\mathbf{R}$ *into* $\mathbf{C}$ *satisfying*

$$\Re e(\lambda_i(t)-\lambda_k(t)) < -\eta < 0, \qquad i=1,\cdots,k-1$$

$$\Re e(\lambda_i(t)-\lambda_k(t)) > \eta > 0, \qquad i=k+1,\cdots,m,$$

where η is a constant, $y_t : [-\tau,0] \to \mathbf{C}^N$ defined as $y_t(s) = y(t+s)$ for all $t \geq 0$, $\{R(t,\cdot)\}_{t\geq 0}$ is a family of bounded linear functionals from the set of the essentially bounded functions $[-\tau,0] \to \mathbf{C}^N$ into $\mathbf{C}^N$, $N = \sum_{j=1}^m n_j$ and
$\|R(t, \exp\left(\int_t^{t+\cdot} \lambda_k(\tau)d\tau\right) I)\| \in L^p, 1 \leq p \leq 2$. *Then (2) has a solution $y = y_0(t)$ such that*

$$\begin{aligned} y_0(t) &= \exp\left(\int_0^t [\lambda_k(s) + e_{\tilde{k}}^* \cdot R(s, \exp\left(\int_s^{s+\cdot} \lambda_k(\tau)d\tau\right) e_{\tilde{k}})]\right) \\ &\times \quad (e_{\tilde{k}} + o(1)) \end{aligned} \tag{1.3}$$

as $t \to \infty$, where $\tilde{k} = n_1 + \cdots + n_{k-1} + 1$.

A scalar version with the more general perturbation is given by the following result:

Proposition 2. *(Castillo-Pinto [11, 2004]) Consider the linear functional differential equation*

$$y'(t) = \lambda_0(t)y(t) + b(t, y_t), t \geq 0, \tag{1.4}$$

where $\lambda_0 : [0,+\infty[\to \mathbf{C}$ is a locally integrable function and

$$b : [0,+\infty[\times L^\infty([-\tau,0],\mathbf{C}) \to \mathbf{C}$$

is a continuous function and linear in the second variable function such that

$$\sup_{t\geq T} \int_{t-\tau}^t |b(s, \exp\left(\int_s^{s+\cdot} \lambda_0(\xi)d\xi\right)|ds < \frac{1}{e},$$

for T large enough. Then, every solution of (1.4) has the following asymptotic formula

$$\begin{aligned} y(t) &= \exp\left(\int_\tau^t \left[\lambda_0(s) + b(s, e^{\int_s^{s+\cdot} \lambda_0(\xi)d\xi}) + \sum_{n=1}^{+\infty} \Delta_n(s)\right] ds\right) \\ &\times \quad (c + o(1)), \end{aligned} \tag{1.5}$$

as $t \to +\infty$, where $\Delta_n(t) = b(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi}[e^{\int_t^{t+\cdot} \mu_n(\xi)d\xi} - e^{\int_t^{t+\cdot} \mu_{n-1}(\xi)d\xi}])$, $\mu_n(t) = b(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} e^{\int_t^{t+\cdot} \mu_{n-1}(\xi)d\xi})$, for all $t \geq n\tau$, $\mu_n(t) = 0$ for all $t \in [0, n\tau[$, $\mu_0 = 0$ and $n \in \mathbf{N}$. Conversely, given $c \in \mathbf{C}$ there is a solution $y = y(t)$ of (1.4) satisfying (1.5).

2. Preliminaries

It is assumed that:

Hypothesis 1. *The second order ordinary differential equation*

$$x''(t) = a(t)x'(t) + b(t)x(t),\ t \geq 0, \tag{2.1}$$

has a complex valued solution $x = x_0(t)$ *such that* $x_0(t) \neq 0$ *for all* $t \geq 0$, *where* $a, b : [0, +\infty[\to \mathbf{R}$ *are locally integrable functions.*

Here, it is considered as a perturbation of the equation (2.1), the second order retarded functional differential equation

$$y''(t) = [a(t) + \hat{a}(t)]y'(t) + b(t)y(t) + c(t, y_t),\ t \geq 0, \tag{2.2}$$

where $a, b : [0, +\infty[\to \mathbf{R}$ are locally integrable functions, $\{c(t, \cdot)\}_{t \geq 0}$ is a family of linear functional from the set of the locally integrable functions $[-\tau, 0] \to \mathbf{C}$, with $\tau > 0$, into $\mathbf{C}$ and for every function $y : [-\tau, +\infty[\to \mathbf{C}$ it is denoted $y_t(s) = y(t + s)$ for all $(t, s) \in [0, +\infty[\times [-\tau, 0]$.

Remark 1. *Notice that the equation (2.2) is the perturbation of equation (2.1). The terms* $\hat{a}(t)y'(t)$ *and* $c(t, y_t)$ *are the perturbations which will satisfy smallness conditions as it is mentioned below.*

Hypothesis 2. *Every member of the family* $\{c(t, \cdot)\}_{t \geq 0}$ *leads the set of the locally integrable functions* $[-\tau, 0] \to \mathbf{R}$, *with* $\tau > 0$, *into* $\mathbf{R}$.

It is wanted to get an asymptotic formula for a solution of (2.2) which is a small perturbation of x_0. For that, one aditional definition and a hypothesis are given.

Let

$$\Phi(t, s) = \exp\left(\int_s^t [a(\xi) - 2\lambda_0(\xi)]d\xi\right),$$

for all $t, s \geq 0$, where $\lambda_0 = \frac{x_0'}{x_0}$.

Assume that:

Hypothesis 3. *There are positive constants* α *and* K *such that*

$$\int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi \leq \ln K - \alpha|t - s|,$$

for all $t, s \geq 0$, *where* $\lambda_0 = \frac{x_0'}{x_0}$ *and* x_0 *is a solution of a non perturbed second order linear equation (2.1) which satifies the Hypothesis 1.*

Then, under the Hypothesis 3, we have the following inequality:

$$|G(t,s)| \leq Ke^{-\alpha|t-s|}, \tag{2.3}$$

for all $t, s \geq 0$, where

$$G(t,s) = \begin{cases} \Phi(t,s), & \text{if} \quad \Re e \int_s^t [a(\xi) - 2\lambda_0(\xi)]d\xi < 0 \\ 0, & \text{if} \quad \text{else} \end{cases}$$

When (2.1) is autonomous, i.e., $a(t) = a_0$ and $b(t) = b_0$ are real constants we have

$$\lambda_0 = \frac{1}{2}(a_0 \pm \sqrt{a_0^2 + 4b_0}),$$

which are roots of the characteristic equation of (2.1), i.e,

$$\lambda^2 - a_0\lambda - b_0 = 0 \tag{2.4}$$

and the term

$$a(\xi) - 2\lambda_0(\xi) = a_0 - 2\lambda_0 = -\frac{d}{d\lambda}[\lambda^2 - a_0\lambda - b_0]\Big|_{\lambda=\lambda_0} = \mp\sqrt{a_0^2 + 4b_0},$$

is the difference between the two roots of the characteristic equation (2.4).

2.1. Procedure

We make the change of variables, $y(t) = x_0(t)z(t)$ for $t \geq 0$ and we arrive to the differential equation:

$$\begin{aligned} z''(t) &= [a(t) - 2\lambda_0(t)]z'(t) + \hat{a}(t)[\lambda_0 z + z'] \\ &\quad + c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} z_t), \; t \geq \tau. \end{aligned} \tag{2.5}$$

For simplifying notations it is made $\tilde{c}(t,\varphi) = c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi}\varphi)$ for every locally integrable function $\varphi : [-\tau, 0] \to \mathbf{C}$. Let

$$f_1(t) = \tilde{c}(t,1) + \hat{a}(t)\lambda_0(t),$$

and $f(t,\mu) = \hat{a}(t)\mu(t) + \tilde{c}(t, [e^{\int_t^{t+\cdot} \mu(\xi)d\xi} - 1]) - \mu(t)^2$. Then, for every solution of the equation

$$\mu'(t) = [a(t) - 2\lambda_0(t)]\mu(t) + f_1(t) + f(t,\mu), \quad t \geq 2\tau + t_0, \tag{2.6}$$

we have that $z(t) = e^{\int_{2\tau}^t \mu(s)ds}$ is a solution of the equation (10), for $t \geq 2\tau$.

Notice that $f(t,0) = 0$.

The following steps are

a) Consider

$$\mathcal{T}: X \to X,$$

where $X = (X, \|\cdot\|)$ is a suitable Banach space, defined by

$$(\mathcal{T}\mu)(t) = \begin{cases} \int_{t_0+2\tau}^{+\infty} G(t,s)[f_1(s) + f(s,\mu)]ds, & \text{if} \quad t \geq 2\tau + t_0 \\ 0, & \text{if} \quad t \in [t_0 - \tau, t_0 + 2\tau[. \end{cases} \tag{2.7}$$

b) Find an attractor fixed point for $\mathcal{T}$: $\mu_\infty \in X$ ($\mu_\infty = \mathcal{T}\mu_\infty$), so equation (2.2) has a solution y such that

$$\begin{aligned} y(t) &= x_0(t)e^{\int_{t_0}^{t} \mu_\infty(\xi)d\xi}, \\ y'(t) &= [\lambda_0(t) + \mu_\infty(t)]y(t). \end{aligned} \tag{2.8}$$

c) Since μ_∞ is an attractor we have that μ_∞ can be written as

$$\mu_\infty = \mathcal{T}(0) + \sum_{n=1}^{+\infty}[\mathcal{T}^{n+1}(0) - \mathcal{T}^n(0)], \tag{2.9}$$

in the chosen norm.

3. Cases

3.1. The perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi}-)$ are bounded

Proposition 3. *Let M, k_1 and k_2 be such that $M \in]0, \frac{\alpha}{2K}[$, $k_2 \in]0, 1 - \frac{2MK}{\alpha}[$ and $k_1 \in]0, \frac{M^2K}{2\alpha}[$. Suppose that in equation (2.1), hypotheses 1, 2 and 3 are satisfied. Assume that in equation (2.2), $\hat{a}(t)$ and $|c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi}-)|$ are so small that satisfy*

$$\left| \int_{2\tau}^{+\infty} G(t,s)f_1(s)ds \right| \leq k_1 \tag{3.1}$$

and

$$\int_{2\tau}^{+\infty} e^{-\alpha|t-s|}[|\hat{a}(s)| + e^{\frac{\alpha\tau}{2K}}\tau|c(s, e^{\int_s^{s+\cdot} \lambda_0(\xi)d\xi}-)|]ds \leq k_2. \tag{3.2}$$

Then, equation (2.2) has a solution with the following formula

$$y(t) = x_0(t)\exp\left(\int_{2\tau}^{t}\left[\mu_1(s)+\sum_{n=1}^{+\infty}\Delta_n(s)\right]ds\right),$$
$$y'(t) = \left[\lambda_0(t)+\mu_1(t)+\sum_{n=1}^{+\infty}\Delta_n(t)\right]y(t), \tag{3.3}$$

where

$$\Delta_n(t) = \int_{2\tau}^{+\infty} G(t,s)[f(s,\mu_n) - f(s,\mu_{n-1})]ds, \tag{3.4}$$

and the functions $\{\mu_n\}_{n=0}^{+\infty}$ *are given by* $\mu_0 = 0$
and

$$\mu_n(t) = \begin{cases} \int_{2\tau}^{+\infty} G(t,s)[f_1(s)+f(s,\mu_{n-1})]ds, & \text{for} \quad t \geq (n+1)\tau \\ 0, & \text{for} \quad t \in [0,(n+1)\tau[\end{cases} \tag{3.5}$$

for $n \in \mathbf{N}$.

Proof of Proposition 3 :
It can be proved that M, k_1 and k_2 satisfy

$$\begin{aligned} k_1 + Mk_2 + \tfrac{M^2K}{\alpha} &\leq \tfrac{M^2K}{2\alpha} + M\left(1-\tfrac{2MK}{\alpha}\right) + \tfrac{M^2K}{\alpha} \\ &= M\left(1-\tfrac{MK}{2\alpha}\right) \\ &< M \end{aligned}$$

and $k_2 + \frac{2MK}{\alpha} < 1$.

Then, if we consider $B[0,M]$ as the closed ball centered in 0 with radius M in $(\mathcal{B}([0,+\infty[,\mathbf{R}), \|\cdot\|_\infty)$, then by the relations (3.1) and (3.2), $\mathcal{T}(B[0,M]) \subseteq B[0,M]$ and the restriction

$$\mathcal{T} : B[0,M] \to B[0,M]$$

is a contraction operator. So, there exists a unique

$$\mu_\infty \in B[0,M] : \; \mathcal{T}\mu_\infty = \mu_\infty.$$

Then,

$$\mu_\infty(t) = \int_{2\tau}^{+\infty} G(t,s)[f_1(s) + f(s,\mu_\infty)]ds, \tag{3.6}$$

for $t \geq 2\tau$. From (2.8), the formula (3.3) is obtained.

To make the formula (3.3) easier to be understood, the following remarks are given.

Remark 2. *Let* $\lambda_n(t) = \lambda_0(t)$, *for* $t \in [0, \tau[$ *and*

$$\lambda_n(t) = \lambda_0(t) + \mu_1(t) + \sum_{j=1}^{n} \Delta_j(t), \text{ for } t \geq \tau$$

and $n \in \mathbf{N}$. *Then, given* $n_0 \in \mathbf{N}$, $\lambda_n(t) = \lambda_{n_0}(t)$, *for* $t \in [0, (n_0+1)\tau[$ *and* $n \geq n_0$. *So, (3.3) can be written as:*

$$y(t) = \exp\left(\int_0^t \left[\lim_{n\to+\infty} \lambda_n(s)\right] ds\right)(c + o(1)),$$

as $t \to +\infty$.

Remark 3. *If* $\Delta_k \in L^1$ *for some* $k \in \mathbf{N}$, *then* $\sum_{m=k}^{+\infty} \Delta_m \in L^1$. *In this case (3.3) can be written as*

$$y(t) = \exp\left(\int_{2\tau}^t \left[\lambda_0(s) + \mu_1(s) + \sum_{j=1}^{n-1} \Delta_j(s)\right] ds\right)(\hat{c} + o(1)),$$

as $n \to +\infty$, *for some* $\hat{c} \in \mathbf{C}$. *Here* $\sum_{j=1}^{n-1} \Delta_j(s) = 0$ *if* $n = 1$.

3.2. The perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)$ are in L^p

Similar relations to (3.1) and (3.2) can be given for the cases where $f_1(t)$ and $f(t,\mu)$ and μ are in L^p. So, we have the following result.

Proposition 4. *Suppose that in equation (2.1), hypotheses 1, 2 and 3 are satisfied. Assume that in equation (2.2),* $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -) \in L^p$ *for some* $p \in [2^{n-1}, 2^n[$ *and* $n \in \mathbf{N}$. *Then, (2.2) has a solution with the following asymptotic formula*

$$y(t) = x_0(t) \exp\left(\int_{2\tau}^t \left[\mu_1(s) + \sum_{j=1}^{n-1} \Delta_j(s)\right] ds\right)(c + o(1)), \tag{3.7}$$

as $t \to +\infty$, *where the functions* $\{\Delta_j\}_{j=1}^{+\infty}$, $\{\mu_j\}_{j=0}^{+\infty}$ *are given by the relation (3.4) and (3.5).*

Example 1: Consider the second order delayed differential equation

$$(3.8)\quad y''(t) = (a_0 + \hat{a}(t))y'(t) + (b_0 + \hat{b}(t))y(t) + \hat{c}(t)y(t-\tau),\ t \geq 0,$$

where $a_0, b_0 \in \mathbf{R}$ such that $a_0^2 + 4b_0 > 0$ and $\hat{a}, \hat{b}, \hat{c} \in L^2$. If we consider

$$x''(t) = a_0x'(t) + b_0x(t),\ t \geq 0,$$

as the equation (2.1) which has the following characteristic roots $\lambda_{0\pm} = \frac{1}{2}(a_0 \pm \sqrt{a_0^2 + 4b_0})$, then

$$a_0 - 2\lambda_{0\pm} = \mp\sqrt{a_0^2 + 4b_0} = \lambda_{0\mp} - \lambda_{0\pm}.$$

Since $a_0^2 + 4b_0 > 0$, (2.3) is satisfied. In (2.2) we take,

$$c(t, \varphi) = \hat{b}(t)\varphi(0) + \hat{c}(t)\varphi(-\tau).$$

Then, when we consider $\lambda_0 = \lambda_{0-}$. So,

$$\mu_\infty(t) = \int_t^{+\infty} e^{(\lambda_{0+}-\lambda_{0-})(t-s)}[\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0-}\tau}\hat{c}(s)]ds + \Theta(t),$$

for $t \geq 2\tau$, where

$$\begin{array}{rcl}\Theta(t) & = & \int_t^{+\infty} e^{(\lambda_{0+}-\lambda_{0-})(t-s)} \\ & \times & [\hat{a}(s)\mu_\infty(s) \\ & + & \hat{c}(s)e^{-\lambda_{0-}\tau}\left[e^{\int_s^{s+\cdot}\mu_\infty(\xi)d\xi} - 1\right] \\ & - & \mu_\infty(s)^2]ds.\end{array}$$

Clearly, $\Theta \in L^1$.

Then, by integration by parts

$$\int_0^t \mu_\infty(s)ds = \frac{1}{\lambda_{0-} - \lambda_{0+}}\int_0^t [\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0+}\tau}\hat{c}(s)]ds + \gamma(t),$$

where $\lim_{t\to+\infty}\gamma(t) = \text{const}$. So, equation (3.8) has a solution $y(t)$ such that

$$y(t) = x_0(t)\exp\left(\frac{1}{\lambda_{0+} - \lambda_{0-}}\int_0^t [\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0-}\tau}\hat{c}(s)]ds\right)(c+o(1)),$$

as $t \to +\infty$.

Now, when we consider $\frac{x_0'}{x_0} = \lambda_0 = \lambda_{0+}$ Then, (2.2) has a solution y such that

$$y(t) = x_0(t)\exp\left(\frac{1}{\lambda_{0-} - \lambda_{0+}}\int_0^t [\hat{a}(s)\lambda_{0+} + \hat{b}(s) + e^{-\lambda_{0+}\tau}\hat{c}(s)]ds\right)(c+o(1)),$$

as $t \to +\infty$.

3.3. Equation (2.1) satisfies Hyp. 1 and a weaker hypothesis than Hyp. 3 but perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)$ are in L^1

Consider the following hypothesis

Hypothesis 4. *$\int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi$ is bounded for all $t, s : s \geq t \geq 0$ or*

$$\lim_{t \to +\infty} \int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi = -\infty.$$

Under Hypothesis 4, the relations (3.1) and (3.2) can be extended for the cases where $f_1(t)$ and $f(t, \mu)$ and μ are in L^1 for a parameter t large enough. So, we have the following result.

Proposition 5. *Suppose that in equation (2.1), hypotheses 1, 2 and 4 are satisfied. Assume that in equation (2.2), $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -) \in L^1$. Then, (2.2) has a solution with the following asymptotic formula*

$$y(t) = x_0(t)(c + o(1)), \tag{3.9}$$

as $t \to +\infty$, where the functions $\{\Delta_n\}_{n=1}^{+\infty}$, $\{\mu_n\}_{n=0}^{+\infty}$ are given by the relation (3.4) and (3.5).

Example 1: Consider the equation

$$y''(t) + a^2 y(t) = \frac{1}{t^2} y(t - \tau), \ t \geq 0, \tag{3.10}$$

has solutions y_0, y_1 with the following asymptotic formulas

$y_0(t) = \cos(at)(1 + o(1))$ and $y_1(t) = \sin(at)(1 + o(1)),$ as $t \to +\infty$.

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References

[1] S. Ai, Asymptotic integration of delay differential systems, *J. Math. Anal. Appl.* **165**, pp. 71-101, (1992).

[2] O. Arino and I. Győri, Asymptotic integration of delay differential systems, *J. Math. Anal. Appl.* **138**, pp. 311-327, (1989).

[3] O. Arino, I. Győri and M. Pituk, Asymptotic diagonal delay differential systems, *J. Math. Anal. Appl* **204**, pp. 701-728, (1996).

[4] O. Arino and M. Pituk, More on linear differential systems with small delays, *J. Differential Equations* **170**, pp. 381-407, (2001).

[5] F. Atkinson and J. Haddock, Criteria for asymptotic constancy of solutions of functional differential equations, *J. Math. Anal. Appl* **91**, pp. 410-423, (1983).

[6] J. Cassell and Z. Hou, Asymptotically diagonal linear differential equations with retardation, *J. London Math. Soc.* (2) 47 (1993) 473-483.

[7] S. Castillo, Asymptotic formula for functional dynamic equations in time scale with functional perturbation, *Functional Differential Equations*, The Research Institute, The College of Judea and Samaria, Ariel Israel Vol **10**. No. 1-2, pp. 107-120, (2003).

[8] S. Castillo and M. Pinto, Asymptotic integration of ordinary differential systems. *J. Math. Anal. Appl.* **218**, pp. 1-12, (1998).

[9] S. Castillo and M. Pinto, Levinson theorem for functional differential systems, Nonlinear Analysis, Vol 47/6, pp. 3963-3975, (2001).

[10] S. Castillo and M. Pinto, An asymptotic theory for nonlinear functional differential equations. *Comput. Math. Appl.* **44**, N. 5-6, pp. 763-775, (2002).

[11] S. Castillo and M. Pinto, Asymptotics of Scalar Functional Differential Equations, *Functional Differential Equations*, The Research Institute, The College of Judea and Samaria, Ariel Israel Vol **11**. N 1-2, pp. 29-36, (2004).

[12] K. Cooke, Functional differential equations close to differential equations, *Bull. Amer. Math. Soc.* **72**, pp. 285-288, (1966).

[13] R. Driver, Linear differential systems with small delays, *J. Differential Equations* **21**, pp. 149-167, (1976).

[14] M. Eastham, The Asymptotic Solution of Linear Differential Systems, Applications of the Levinson Theorem, Clarendon, Oxford, (1989).

[15] I. Győri and M. Pituk, L^2-Perturbation of a linear delay differential equation, *J. Math. Anal. Appl.,* **195**, pp. 415-427, (1995).

[16] J. Haddock and R. Sacker, Stability and asymptotic integration for certain linear systems of functional differential equations, *J. Math. Anal. Appl.*, **76**, pp. 328-338, (1980).

[17] W. Harris and D. Lutz, A unified theory of asymptotic integration, *J. Math. Anal. Appl.* 57, pp. 571-586, (1977).

[18] P. Hartman and A. Wintner, Asymptotic integration of linear differential equations, *Amer. J. Math.* **77**, pp. 48-86 and 932, (1955).

[19] N. Levinson, The asymptotic behavior of system of linear differential equations, *Amer. J. Math.* **68**, pp. 1-6, (1946).

[20] M. Pituk, Asymptotic characterization of solutions of functional differential equations, *Boll. Un. Math. Ital.* **7-B**, pp. 653-689, (1993).

[21] M. Pituk, The Hartman-Wintner theorem for functional-differential equations. *J. Differential Equations* **155**, N. 1, 1-16, (1999).

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