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Stability Properties of the Solutions of the Nonlinear Functional Differential Systems

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1. INTRODUCTION

We study the asymptotic behavior of the functional differential equation

$$y' = f(t, y) + g(t, y, Ty) \tag{1}$$

knowing some asymptotic properties about the solutions of the ordinary differential equation

$$x' = f(t, x). \tag{2}$$

Let $t \in I = [0, \infty)$, $x \in \mathbf{R}^n$, $f \in C(I \times \mathbf{R}^n, \mathbf{R}^n)$, $f(t, 0) \equiv 0$, and the derivative $f_x \in C(I \times \mathbf{R}^n, \mathbf{R}^n)$. The functional perturbation $g = g(t, y, z)\colon I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and T is a continuous operator mapping $C(I, \mathbf{R}^n)$ into $C(I, \mathbf{R}^n)$. In this way, Eq. (1) may represent several interesting cases, namely, integrodifferential equations [14–17] as

$$y' = f(t, y) + g\left(t, y, \int_{t_0}^{t} k(t, s, y(s))\, ds\right),$$

functional (delay) differential equations as

$$y'(t) = f(t, y(t)) + g(t, y(t), y(t - \tau)),$$

etc., taking

$$Ty(t) = \int_{t_0}^{t} k(t, s, y(s))\, ds$$

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and

$$Ty(t) = y(t-\tau),$$

respectively.

In Section 2, we will study the asymptotic behavior of the solutions of some functional differential equations which include these classes of equations. Moreover, we determine the range of validity of the results. Thus, for example, we make precise the initial conditions (the radius of attraction) for which the solutions tend to zero as $t \to \infty$. Further, we obtain nice estimates for the solutions of (1) depending on the integral-norm (L_1-norm) of the variable coefficients of g. All that yields a more natural approach to the nonlinear situation than the approach of Pachpatte [6, 7]. Finally, in Section 3 we give several examples illustrating the results.

2. Main Results

In this section we will prove theorems which relate the asymptotic behavior and boundedness of the solutions of Eqs. (1) and (2). Particularily, we will get several asymptotic properties of the solutions of (2). Before stating and proving any result it is necessary to recall some basic notions. Let $x(t, t_0, x_0)$ be the solution of (2) such that $x(t_0, t_0, x_0) = x_0$ and $\Phi = \Phi(t, t_0, x_0)$ the fundamental matrix of the variational equation

$$z'(t) = f_x(t, x(t, t_0, x_0))\, z(t) \tag{3}$$

such that $\Phi(t_0, t_0, x_0)$ is the identity matrix (see [2]).

Definition 1. The null solution $x \equiv 0$ of (2) is exponentially asymptotically stable in variation if there exist positive constants δ, α, and M such that

$$\|\Phi(t, t_0, x_0)\| \leqslant M e^{-\alpha(t-t_0)} \qquad \forall t \geqslant t_0 \geqslant 0 \tag{4}$$

for $\|x_0\| \leqslant \delta$.

Remark 1. The last definition implies (see [9]) that for $\|x_0\| \leqslant \delta$

$$\|x(t, t_0, x_0)\| \leqslant M\, \|x_0\|\, e^{-\alpha(t-t_0)} \qquad (t \geqslant t_0 \geqslant 0). \tag{5}$$

Now, we will need a "solution" of the functional inequalities

$$u(t) \leqslant c + \sum_{i=1}^{p} \int_a^t \lambda_i(s)\, w_i(u(s))\, ds, \qquad t \in [a, b] \tag{6}$$

and

$$u(t) \leqslant c + \int_a^t \lambda_1(s)\, w_1(u(s))\, ds + \int_a^t \lambda_2(s)\, w_2\left[\int_a^s \lambda_3(\tau)\, w_3(u(\tau))\, d\tau\right] ds, \tag{7}$$

where the functions w_i $(1 \leqslant i \leqslant p)$ satisfy the conditions:

(H) The functions w_i $(1 \leqslant i \leqslant p)$ are continuous and nondecreasing on $[0, \infty)$ and positive on $(0, \infty)$ such that w_{i+1}/w_i $(1 \leqslant i \leqslant p-1)$ are nondecreasing on $(0, \infty)$.

To state these results, we define:

1. The functions

$$W_k(u) = \int_{u_k}^{u} \frac{ds}{w_k(s)}, \qquad u > 0,\ u_k > 0\ (1 \leqslant k \leqslant p) \tag{8}$$

and W_k^{-1} their inverse function.

2. For $b \geqslant b_1 \geqslant a$ and $\lambda_i\colon [a, b] \to [0, \infty)$ $(1 \leqslant i \leqslant p)$ integrable functions, we define the functions $\varphi_0(u) = u$ and

$$\begin{aligned} \varphi_k &= \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1, \\ \psi_k(u) &= W_k^{-1}[W_k(u) + \alpha_k(a, b_1)], \end{aligned} \tag{9}$$

where

$$\alpha_k(a, b_1) = \int_a^{b_1} \lambda_k(s)\, ds.$$

The function φ_k (and ψ_k) does not depend on the choice of u_k in (8). Any φ_k $(1 \leqslant k \leqslant p)$ is a continuous, positive, and nondecreasing function on its domain (see Remark 4 in [8]).

Thus, we have the following theorems:

THEOREM A [8]. *Assume that the functions w_i $(1 \leqslant i \leqslant p)$ satisfy* (H), *the functions u and λ_i $(1 \leqslant i \leqslant p)$ are continuous and nonnegative on the interval $[a, b]$, and the constant c is positive. If* (6) *holds, then for $t \in [a, b_1)$*

$$u(t) \leqslant W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + \int_a^t \lambda_p(s)\, ds\right], \tag{10}$$

where $b_1 \in [a, b]$ *is a number such that*

$$\alpha_k(a, b_1) := \int_a^{b_1} \lambda_k(s)\, ds \leqslant \int_{\varphi_{k-1}(c)}^{\infty} \frac{ds}{w_k(s)} \qquad (1 \leqslant k \leqslant p). \tag{11}$$

THEOREM B. *Under the conditions of Theorem* A, *if* (7) *holds then* (10) *is true for* $p = 3$, *where* b_1 *satisfies* (11) *for* $p = 3$.

Several applications of this theorem can be founded in [3–5; 7, 8].

We remark that b_1 in (11) can be taken as large as possible if

$$\int_1^{\infty} \frac{ds}{w_i(s)} = \infty \qquad (1 \leqslant i \leqslant p), \tag{12}$$

which implies that any φ_k (and ψ_k) is defined for all u and $b_1 \geqslant a$. Then (10) is valid for all $t \geqslant a$. The dual condition to (12), namely,

$$\int_{0^+}^{1} \frac{ds}{w_i(s)} = \infty \qquad (1 \leqslant i \leqslant p) \tag{13}$$

implies that any φ_k (and ψ_k) is defined for all u small enough and any $b_1 \geqslant a$. Then (10) is valid for any $t \geqslant a$ if c is small enough. Moreover, (13) implies

$$\varphi_k(0^+) = 0 \qquad (1 \leqslant k \leqslant p), \tag{14}$$

which is actually the stability condition.

Further, the inequalities (11) allow us to compute b_1. See [8, 10, 11, 13].

In the following, we consider the functions φ_i $(1 \leqslant i \leqslant p)$ given by (9) with $b_1 = \infty$.

THEOREM 1. *Let* ω_i $(1 \leqslant i \leqslant p)$ *be as in Theorem* A *and let us assume the following hypotheses.*

(i) λ_i $(1 \leqslant i \leqslant p)$ *are continuous nonnegative functions on* I *and* $\lambda_i \in L_1(I)$;

(ii) *for* $t \geqslant s \geqslant 0$ *and* $z, y \in \mathbf{R}^n$ *we have* $\|\Phi(t, s, y)\, g(s, y, z)\| \leqslant \sum_{i=1}^{p} \lambda_i(s)\, \omega_i(\|y\|)$; *and*

(iii) *there is a positive constant* c *such that*

$$\int_0^{\infty} \lambda_p(s)\, ds < \int_{\varphi_{p-1}(c)}^{\infty} \frac{ds}{\omega_p(s)}.$$

Then, for each bounded solution $x(t, t_0, x_0)$ of (2) *such that* $\|x(t, t_0, x_0)\| \leq c$ *for* $t \geq t_0 \geq 0$, *the solution* $y(t, t_0, x_0)$ *of* (1) *is defined and bounded on* $[t_0, \infty)$ *and*

$$\|y(t, t_0, x_0)\| \leq \varphi_p(\|x\|_\infty). \tag{15}$$

Proof. Given any $t_0 \in I$, the nonlinear variation of constants formula of Alekseev [1] allows us to relate the solution of (1) and (2) with the same initial condition as follows

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s), Ty(s))\, ds. \tag{16}$$

Therefore

$$\|y(t)\| \leq \|x\|_\infty + \sum_{i=1}^{p} \int_{t_0}^{t} \lambda_i(s)\, \omega_i(\|y(s)\|)\, ds,$$

where $\|x\|_\infty = \sup\{|x(t)|/t \in I\}$. Then for all $t \geq t_0 \geq 0$ from Theorem A we have

$$\|y(t)\| \leq W_p^{-1}\left[W_p(\varphi_{p-1}(c)) + \int_{t_0}^{t} \lambda_p(s)\, ds\right] \leq \varphi_p(\|x\|_\infty). \tag{17}$$

Condition (iii) implies $\varphi_p(c) < \infty$. Then $\varphi_p(\|x\|_\infty) \leq \varphi_p(c) < \infty$ and we get the boundedness for the solution $y(t)$ in its interval of definition $[t_0, t_1)$. Then for any t fixed $\Phi(t, s, y(s))\, g(s, y(s), Ty(s)) \in L_1([t_0, t_1))$ as a function of s and $\lim_{t \to t_1^-} y(t)$ exists. Then we can continue the solution $y(t)$ beyond t_1.

Finally, by (17), $y(t)$ is bounded on $[t_0, +\infty)$ and (15) follows. So, the proof is complete.

The method used in Theorem 1 can be applied to delay-differential equations [1, 12] and, in general, to those equations satisfying

$$\|\Phi(t, s, y(s))\, g(s, y(s), Ty(s))\| \leq \sum_{i=1}^{p} \lambda_i(s)\, \omega_i(\|y\|_s) \qquad (t \geq s \geq 0),$$

where for some $t^* = t^*(t) \leq t$ (as $t^* = t - r$ in difference equations):

$$\|y\|_t = \sup_{\tau \in I_t} |y(\tau)|, \qquad I_t = [t^*, t] \subset [0, t].$$

In fact, in this case from (16) we deduce

$$\|y\|_t \leq \|x\|_\infty + \sum_{i=1}^{p} \int_{t_0}^{t} \lambda_i(s)\, \omega_i(\|y\|_s)\, ds$$

and we apply Theorem A to $u(t) = \|y\|_t$. See [1, 12].

Remark 2. (1) If (12) holds then condition (iii) of Theorem 1 is fulfilled for all $c>0$. (2) If (13) holds then (see [8]) there exists always c small enough satisfying condition (iii). (3) Finally, in the case that $1/\omega_i \in L_1((0, \infty))$ $(1 \leqslant i \leqslant p)$, the inequality

$$\int_0^\infty \lambda_i(s)\, ds \geqslant \int_0^\infty \frac{ds}{\omega_i(s)}$$

for some i, implies that there is no $c>0$ satisfying condition (iii) of Theorem 1. Otherwise, there always exists c small enough satisfying condition (iii). In every case, the biggest c satisfying condition (iii) is

$$c = \varphi_p^{-1}(\infty) \tag{18}$$

(see Sect. 3). So, we get

COROLLARY 1. *(1) If (12) holds then the result of Theorem 1 is true for all solutions. (2) If (13) holds then the result of the Theorem 1 is true only for x such that $\|x\|_\infty$ is small enough, exactly $\|x\|_\infty < \varphi_p^{-1}(\infty)$.*

Remark 3. The equation $y' = y^2/t^2$, $y(1) = y_0$, $t \geqslant 1$, and the solution $y = t$ shows that the result of Theorem 1 is not true for arbitrary solutions. In fact, here $x' = 0$, $x(t, t_0, x_0) = x_0$, and $\omega_1(u) = u^2$, $\lambda_1(s) = s^{-2} \in L_1([1, \infty))$. The condition

$$\int_1^\infty \lambda_1(s)\, ds < \int_c^\infty \frac{du}{u^2}$$

is only true for $c < 1$.

THEOREM 2. *Assume that ω_i $(1 \leqslant i \leqslant 3)$ and λ_i $(1 \leqslant i \leqslant 3)$ satisfy Theorem 1 and that*

(ii) *For $0 \leqslant s \leqslant t < +\infty$ and $y \in C(I, \mathbf{R}^n)$ we have*

$$\|\Phi(t, s, y(s))\, g(s, y(s), Ty(s))\| \leqslant \lambda_1(s)\, \omega_1(\|y(s)\|) + \lambda_2(s)\, \omega_2\left(\int_{t_0}^s \lambda_3(\tau)\, \omega_3(\|y(\tau)\|)\, d\tau\right)$$

and

(iii) *there is a positive constant c such that*

$$\int_0^{+\infty} \lambda_3(s)\, ds < \int_{\varphi_2(c)}^{+\infty} \frac{ds}{\omega_3(s)}.$$

Then for each bounded solution $x(t, t_0, x_0)$ of (2) such that $\|x(t, t_0, x_0)\| \leqslant c$ for $t \geqslant t_0 \geqslant 0$, the corresponding solution $y(t, t_0, x_0)$ of (1) is defined and bounded on $[t_0, \infty)$ and

$$\|y(t, t_0, x_0)\| \leqslant \varphi_3(\|x\|_\infty). \tag{19}$$

Proof. As in Theorem 1, from Alekseev's formula (16) we get that

$$\|y(t)\| \leqslant \|x\|_\infty + \int_{t_0}^{t} \left[\lambda_1(s)\, \omega_1(\|y(s)\|) + \lambda_2(s)\, \omega_2\left(\int_{t_0}^{s} \lambda_3(\tau)\, \omega_3(\|y(\tau)\|)\, d\tau \right) \right] ds.$$

Using Theorem B and a technique analogous to that used in Theorem 1 we can prove that $y(t, t_0, x_0)$ is defined and bounded on $[t_0, +\infty)$.

THEOREM 3. *Assume that the null solution of (2) is exponentially asymptotically stable in variation. Suppose also that the hypotheses of Theorem 2 are fulfilled, where* (ii) *is replaced by*

(ii) *For $s \geqslant 0$ and $y \in C(I, \mathbf{R}^n)$ we have*

$$\|g(s, y(s), Ty(s))\| \leqslant \lambda_1(s)\, \omega_1(\|y(s)\|) + \lambda_2(s)\, \omega_2\left(\int_{t_0}^{s} \lambda_3(\tau)\, \omega_3(\|y(\tau)\|)\, d\tau \right).$$

Then every solution y of (1) *which satisfies $\|y(t_0)\| \leqslant cM^{-1}$ is defined on $[t_0, \infty)$, tends zo zero as $t \to \infty$, and satisfies*

$$\|y(t, t_0, y_0)\| \leqslant \varphi_3(M\, \|y_0\|). \tag{20}$$

Moreover, the zero solution of the functional equation (1) *is asymptotically stable if* (12) *holds.*

Proof. Again, from Alekseev's formula (16) and (5) we obtain that

$$\|y(t)\| \leqslant M\, \|x_0\| + \int_{t_0}^{t} \|\Phi(t, s, y(s))\, g(s, y(s), Ty(s))\|\, ds.$$

By (5), the matrix Φ is bounded on $[t_0, +\infty)$. Therefore by (ii)′ the hypotheses of Theorem 2 are satisfied and then y is defined and bounded on $[t_0, \infty)$ and it verifies (20). Then $\tilde{g}(s) := g(s, y(s), Ty(s))$ is an integrable function on $[t_0, \infty)$. To end the proof of the theorem we need to show

$$\lim_{t \to +\infty} \int_{t_0}^{t} \|\Phi(t, s, y(s))\| \, \|g(s, y(s), Ty(s))\| \, ds = 0.$$

We have

$$\begin{aligned}
&\int_{t_0}^{t} \|\Phi(t, s, y(s))\| \, \|g(s, y(s), Ty(s))\| \, ds \\
&\quad \leqslant Me^{-\alpha t} \int_{t_0}^{t} e^{\alpha s} \|\tilde{g}(s)\| \, ds \\
&\quad \leqslant Me^{-\alpha t} \int_{t_0}^{(t+t_0)/2} e^{\alpha s} \|\tilde{g}(s)\| \, ds + M \int_{(t+t_0)/2}^{t} \|\tilde{g}(s)\| \, ds \\
&\quad \leqslant Me^{-\alpha t} e^{((t+t_0)/2)} \int_{t_0}^{t} \|\tilde{g}(s)\| \, ds + \int_{(t+t_0)/2}^{t} \|\tilde{g}(s)\| \, ds \\
&\quad \leqslant Me^{-\alpha((t-t_0)/2)} \int_{t_0}^{t} \|\tilde{g}(s)\| \, ds + M \int_{(t+t_0)/2}^{t} \|\tilde{g}(s)\| \, ds \to 0
\end{aligned}$$

as $t \to +\infty$. Finally, the stability follows at once from (20) using (14).

3. Examples

In this section we will illustrate the above results, showing explicitly the radius of attraction and the estimates for the solutions.

Example 1. Consider the functional differential equation

$$y' = -e^t y^3 + g(t, y, Ty), \tag{21}$$

where $g(t, y, z) = \lambda_1(t)\, y + \lambda_2(t)\, z$ and

$$Ty(t) = \int_{t_0}^{t} \lambda_3(s)\, y^k(s)\, ds, \qquad t \geqslant t_0,$$

$k \geqslant 1$; and λ_i, $i = 1, 2, 3$, are integrable continuous function on I.

Equation (21) is a Volterra integrodifferential equation [14–17]. Solving equation

$$x' = -e^t x^3 \tag{22}$$

we have

$$x(t, t_0, x_0) = \frac{|x_0|}{(1 + 2x_0^2(e^t - e^{t_0}))^{1/2}}.$$

It is easy to see that the null solution of Eq. (22) is exponentially asymptotically stable in variation with $\alpha = \frac{1}{2}$.

Now, we have

(I) $\omega_1(u) = \omega_2(u) = u$, $\omega_3(u) = u^k$. Then conditions (i) and (ii) of Theorem 3 are automatically verified.

(II) To verify conditions (iii) of Theorem 2,

$$\int_0^{+\infty} \lambda_3(s)\, ds < \int_{\varphi_2(c)}^{+\infty} \frac{ds}{\omega_3(s)},$$

let us consider two cases:

(a) $k = 1$. In this case condition (II) is satisfied for any $c > 0$ since $\lambda_3 \in L_1(I)$ and the second integral has value $+\infty$.

(b) $k > 1$. In this case it is necessary to choose the correct constant c. We have that

$$\int_{\varphi_2(c)}^{+\infty} \frac{ds}{s^k} = \frac{-\varphi_2(c)^{1-k}}{1-k},$$

and if $\alpha_i = \int_0^{+\infty} \lambda_i(s)\, ds$ $(i = 1, 2, 3)$ then condition (II) is equivalent to the inequality

$$\alpha_3 < \frac{\varphi_2(c)^{1-k}}{k-1}. \tag{23}$$

Since φ_2 is a monotone function and by (14) $\lim_{u \to 0^+} \varphi_2(u) = 0$, choosing c small enough we will get that (23) is satisfied. Solving the equation

$$\alpha_3 = \frac{\varphi_2(c^*)^{1-k}}{k-1}$$

we can see that (actually $c^* = \varphi_3^{-1}(\infty)$)

$$c^* = \varphi_2^{-1}\left(\sqrt[1-k]{\alpha_3(k-1)}\right).$$

For determining c^* we will calculate φ_2^{-1} explicitly. By definition, we have that

$$\varphi_1(u) = W_1^{-1}[W_1(u) + \alpha_1] = ue^{\alpha_1}$$

and

$$\varphi_2(u) = W_2^{-1}[W_2(\varphi_1(u)) + \alpha_2]$$
$$= \varphi_1(u)\, e^{\alpha_2} = ue^{(\alpha_1+\alpha_2)}.$$

Then

$$\varphi_2^{-1}(u) = ue^{-(\alpha_1+\alpha_2)}$$

and therefore

$$c^* = e^{-(\alpha_1+\alpha_2)}\, \sqrt[1-k]{\alpha_3(k-1)}.$$

Then taking $c \leqslant c^*$ condition (23) is satisfied.

Thus, finally, a direct application of Theorem 3 establishes that every solution $y(t, t_0, x_0)$ tends to zero as $t \to +\infty$ if $\|x_0\| < c^* M^{-1}$, i.e., if

$$\|x_0\| < M^{-1} e^{(\alpha_1+\alpha_2)}/\sqrt[k-1]{\alpha_3(k-1)}, \tag{24}$$

which gives an explicit radius of attraction. Moreover, the zero solution of the functional equation (21) is asymptotically stable and the following estimate is true,

$$\|y(t, t_0, x_0)\| \leqslant \frac{Me^{(\alpha_1+\alpha_2)}\|x_0\|}{\sqrt[k-1]{1-(M\|x_0\|)^{k-1}\, e^{(k-1)(\alpha_1+\alpha_2)}\alpha_3(k-1)}} \tag{25}$$

which is valid, in effect, only if (24) holds. The radius of attraction (24) and the estimate (25) depend directly on the integrals α_i of the coefficients $\lambda_i(t)$ $(i = 1, 2, 3)$.

EXAMPLE 2. Consider the delay-differential equation

$$y' = Ty, \qquad t \geqslant \max(\tau_1, \tau_2, \tau_3), \tag{26}$$

where

$$Ty(t) = \lambda_1(t)\, y(t-\tau_1)^{n_1} + \lambda_2(t)\, y(t-\tau_2)^{n_2} + \lambda_3(t)\, y(t-\tau_3)^{n_3}$$

and τ_i, n_i $(1 \leqslant i \leqslant 3)$ are real numbers such that $1 < n_1 \leqslant n_2 \leqslant n_3$, $\tau_i > 0$. In this case, we have

(I) If we take $w_i(u) = u^{n_i}$ $(1 \leqslant i \leqslant 3)$ then conditions (i) and (ii) are verified.

(II) To see that condition (iii) in Theorem 1 is satisfied we observe that

$$\varphi_0(c) = c,$$
$$\varphi_1(c) = [c^{1-n_1} + \alpha_1(1-n_1)]^{1/(1-n_1)}$$

for $0 < c < c_1$, where

$$c_1 = (\alpha_1(n_1 - 1))^{1/(1-n_1)}$$

$$\varphi_2(c) = [[c^{1-n_1} + \alpha_1(1-n_1)]^{(n_2-1)/(n_1-1)} + \alpha_2(1-n_2)]^{1/(1-n_2)}$$

for $0 < c < c_2$, where

$$c_2 = [\alpha_1(n_1 - 1) + [\alpha_2(n_2 - 1)]^{(n_1-1)/(n_2-1)}]^{1/(1-n_1)}.$$

We have

$$\int_{\varphi_{i-1}(c)}^{\infty} \frac{ds}{w_i(s)} = \frac{1}{n_i - 1} (\varphi_{i-1}(c))^{1-n_i}$$

for $c \in \mathrm{Dom}\, \varphi_{i-1}$. Then condition (iii) in Theorem 1 is equivalent to

$$\alpha_3 < \frac{1}{n_3 - 1} (\varphi_2(c))^{1-n_3}$$

or

$$\varphi_2(c) < \left(\frac{1}{\alpha_3(n_3 - 1)}\right)^{1/(n_3-1)}. \tag{27}$$

Since φ_i $(i = 1, 2, 3)$ are monotone functions and by (14) $\lim_{c \to 0^+} \varphi_i(c) = 0^0$, choosing c small enough we get that (27) is satisfied. Solving the equation

$$\varphi_2(c^*) = \left(\frac{1}{\alpha_3(n_3 - 1)}\right)^{1/(n_3-1)}$$

we obtain

$$c^* = \{[\alpha_3(n_3 - 1)^{(n_2-1)/(n_3-1)} + \alpha_2(n_2 - 1)]^{(n_1-1)/(n_2-1)} + \alpha_1(n_1 - 1)\}^{1/(1-n_1)}$$

(actually $c^* = \varphi_3^{-1}(\infty)$). Then taking $c \leqslant c^*$, condition (iii) in Theorem 1 is satisfied.

Since the solutions of differential equation $x' = 0$ are $x(t, t_0, x_0) \equiv x_0$, then by a remark immediately after Theorem 1 the conditions of this theorem are satisfied and we can conclude that every solution $y(t, t_0, x_0)$ of the delay-differential equation (26) such that $\|x_0\| < c^*$ is defined and bounded on $[t_0, \infty)$. Moreover, by (15)

$$\begin{aligned} \|y(t, t_0, x_0)\| &\leqslant \varphi_3(\|x_0\|) \\ &= [[[\|x_0\|^{1-n_1} + \alpha_1(1-n_1)]^{(n_2-1)/(n_1-1)} \\ &\quad + \alpha_2(1-n_2)]^{(n_3-1)/(n_2-1)} + \alpha_3(1-n_3)]^{1/(n_3-1)} \end{aligned}$$

which directly also establishes the stability of the delay-differential equation (26). Again, the radius of attraction and the estimate of the solutions depend directly on the integrals α_i of the coefficients $\lambda_i(t)$ $(i = 1, 2, 3)$.

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REFERENCES

1. V. M. ALEKSEEV, An estimate for the perturbation of the solutions of ordinary differential equations, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2** (1961), 28–36. [Russian]
2. V. LAKSHMIKANTHAM AND S. LEELA, "Differential and Integral Inequalities," Vols. 1 and 2, Academic Press, New York, 1969.
3. J. GUZMAN AND M. PINTO, Global existence and asymptotic behavior of solutions of nonlinear differential equations, *J. Math. Anal. Appl.*, in press.
4. R. MEDINA AND M. PINTO, On the asymptotic behavior of higher-order nonlinear differential equations, *J. Math. Anal. Appl.* **146** (1990), 128–140.
5. R. MEDINA AND M. PINTO, On the asymptotic behavior of certain second order nonlinear differential equations, *J. Math. Anal. Appl.* **135** (1988), 399–405.
6. B. G. PACHPATTE, Stability and asymptotic behavior of perturbed nonlinear systems, *J. Differential Equations* **16** (1974), 14–25.
7. B. G. PACHPATTE, Perturbations of nonlinear stability of differential equations, *J. Math. Anal. Appl.* **51** (1975), 550–556.
8. M. PINTO, Integral inequalities Bihari-type and applications, *Funkcial. Ekvac.* **33**, No. 3 (1990), 387–404.
9. M. PINTO, Asymptotic integration of systems resulting from a perturbation of an h-system, *J. Math. Anal. Appl.* **131** (1988), 144–216.
10. R. MEDINA AND M. PINTO, Asymptotic behavior of solutions of second order nonlinear difference equations, *Nonlinear Analysis T.M.A.* **19** (1992), 187–195.
11. M. PINTO, Stability of nonlinear differential systems, *Appl. Anal.* **43** (1992), 1–20.
12. M. PINTO, Asymptotic integration of the functional differential equation $y'(t) = a(t)\,y(t - r(t, y))$, *J. Math. Anal. Appl.* **175**, No. 1 (1993), 46–52.
13. M. PINTO, Continuation, nonoscillation and asymptotic formulae of solutions of second order nonlinear differential equations, *Nonlinear Anal. T.M.A.* **26**, No. 11 (1991), 981–993.
14. S. ELAYDI AND R. RAMA MOHAN RAO, Lipschitz stability for nonlinear Volterra integro-differential systems, *Appl. Math. Comput.* **27** (1990), 191–199.
15. L. J. LIN, The stability and growth for the perturbed nonlinear systems of differential equations, *Bull. Inst. Math. Acad. Sinica* **8** (1980), 535–551.
16. S. K. CHOI AND H. S. RYU, h-stability in differential system, *Bull. Inst. Math. Acad. Sinica*, in press.
17. P. S. SIMEONOV AND D. D. BAINOV, Asymptotic equivalence of ordinary and operator-differential equations, *Tamkang J. Math.* **17**, No. 3 (1986), 83–90.

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