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Interdependence in Replacement Decisions

Virtually every household in a modern economy owns a refrigerator, a personal computer and an automobile. Given the inter-temporal nature of replacement decisions, the existing literature has resorted to the technique of dynamic programming, and most recently to the theory of stochastic processes. This article focuses on micro replacement decisions. We study replacement of home appliances in the United States, and construct a test statistic that leads to conclude that replacement decisions might be correlated across appliances. Finally, we enrich our analysis by developing a theoretical model in which replacement decisions are interdependent.

Subject classification: Economics. Keywords: replacement, stochastic process.

1 Introduction

Technological innovations have contributed over the years to an increasing stock of durable goods—products that are not immediately consumed but provide a stream of services over a long period of time. Indeed, virtually every household in the United States, and to a great extent in the rest of the world, owns or has access to a microwave oven, a cloth washer, a computer, among many other durable goods. Despite the rich theoretical body of knowledge existing in different fields (e.g., economics and operations research) to analyze durable goods purchases, only in the past few years have applied researchers succeeded in identifying the forces behind replacement of durable goods.

Given that time plays an essential role in replacement decisions, dynamic programming has naturally arisen as an adequate mathematical tool to tackle the replacement problem. The most recent literature has also resorted to the theory of stochastic processes, and characterized the physical decay of a durable good as a Markov process in either discrete time or continuous time. Even though enormous progress has been made on the theoretical ground, most empirical studies of acquisition and replacement of durable goods do not arise from a consumer or firm's optimization process. Instead, they present ad-hoc statistical models developed from the techniques of discrete choice and duration analysis. Exceptions, among others, are the work of Dubin and McFadden (1984), Rust (1987), Lai, Leung, Tao, Wang (2000), and Martin (2001).

2 The Model

Our statistical model is based on work by Ye (1990). Ye's replacement model assumes that the instantaneous maintenance and operation cost increases stochastically with physical deterioration. One appealing feature of his set-up is that it gives rise to a parsimonious structural model that can be fitted to real data. Ye assumes that in every instant of time the consumer or firm must decide either to continue paying a rising maintenance and operation cost for the deteriorating piece of equipment; or, to sell it in the secondary market, and pay a fixed cost to purchase a new piece of equipment with a guaranteed low initial maintenance and operation cost. The objective function in this model is the expected total discounted cost of maintenance and operation as well as of purchasing.

The instantaneous maintenance and operating cost is represented by x_t. This may also be indicative of the state of the equipment. A higher x_t indicates a more physically deteriorated piece of equipment. The evolution of x_t is described by an arithmetic Brownian motion with constant drift, b, and instantaneous volatility, σ, where $b>0$ and $\sigma \geq 0$:

$$dx_t = b\, dt + \sigma\, dW_t \qquad (1)$$

where dW_t represents an increment of a standard Wiener process.

The expected discounted total cost of obtaining the required service from this piece of equipment is given by:

$$K(x)=E[\int_0^{\infty} e^{-rs}x_s ds \mid x_0 = x] \qquad (2)$$

where x_s evolves according to (1), r is the discount rate, and x_0 represents the state of the piece of equipment at time zero, which does not necessarily equal that of a new one, x^*.

The installation cost of new equipment is a fixed amount, $\tilde{C}$, and the scrap value of the previous equipment is zero. When x reaches $\bar{x}$, an upper barrier, replacement takes place and the following condition is satisfied:

$$K(\bar{x}) = \tilde{C} + K(x^*) \qquad (3)$$

That is, the total cost right before replacement, $K(\bar{x})$, equals the total cost after replacement, $K(x^*)$, plus the cost of installing a new piece of equipment, $\tilde{C}$.

The function K(x) is assumed bounded to avoid the problem of explosive behavior:

$$\lim_{x\to\infty} |K(x)| < \infty \qquad (4)$$

Ye shows that the solution of K(x) is:

$$K(x) = \frac{e^{\lambda x}}{e^{\lambda \bar{x}} - e^{\lambda x^*}}[\tilde{C} - \frac{\bar{x} - x^*}{r}] + \frac{x}{r} + \frac{b}{r^2} \qquad (5)$$

where λ is the positive root of the characteristic equation $(1/2)\sigma^2 p^2 + bp - r = 0$. The optimal upper barrier is unique, and can be found from the condition $K'(\bar{x})=0$:

$$1 + \lambda(r\tilde{C} + x^* - \bar{x}) = \exp(\lambda(x^* - \bar{x})) \qquad (6)$$

3 An Empirical Application of Replacement of Home Appliances

This section is divided into four parts. In section 3.1, we estimate replacement models for refrigerators and water heaters using a sample of U.S. households from the Residential Energy Consumption Survey (RECS). Based upon our estimation results of Section 3.1, in Section 3.2 we test the dependence of replacement decisions across appliances. In section 3.3 we develop a more general model where households consider the replacement of more than one appliance at a time. Finally, in Section 3.4, we present a simulation exercise in which replacement decisions are correlated.

3.1 A Replacement Model

Consider the Wiener process, $\{x(t)\}$ of (1) when the upper barrier, $\bar{x}$, is determined by (6). From the theory of stochastic processes (see Cox and Miller, 1965, pp. 219-221), the transition probability density function (p.d.f.) of $\{x(t)\}$, $p(x, t)$, must be the solution to the differential equation:

$$\frac{1}{2}\sigma^2 \frac{\partial p}{\partial x^2} - b\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t} \quad (x<\bar{x}), \tag{7}$$

subject to the boundary conditions:

$$p(x, 0)=\delta(x-x^*), \tag{8}$$

$$p(\bar{x}, t)=0 \quad (t>0), \tag{9}$$

where $\bar{x}$ is defined by the implicit function $H(\bar{x}, b, \sigma^2)\equiv 1+\lambda(rC+x^*-\bar{x})-\exp[\lambda(x^*-\bar{x})]=0$.

Equation (7)—the Kolmogorov forward equation—describes the evolution of the p.d.f. $p(x, t)$ over time. Condition (8) states that, at time $t=0$, $p(x, t)$ is located entirely at the point $x=x^*$, where $\delta(.)$ represents the Dirac delta function. Condition (9) states that $p(x, t)$ must vanish at $x=\bar{x}$ for all t. That is, the process is terminated if $\bar{x}$ is ever reached.

The p.d.f. of the first passage time for the Wiener process {x(t)}, T, can be obtained once we find the solution for the density p(x, t) from (7), (8) and (9) (see Cox and Miller, pp. 219-225, or Lancaster, 1990, pp. 118-121):

$$g_T(t|b, \sigma, \bar{x}, x^*) = -\frac{d}{dt}\int_{-\infty}^{\bar{x}} p(x, t)dx = \frac{\bar{x}-x^*}{\sigma\sqrt{2\pi t^3}}\exp\left(\frac{-(\bar{x}-x^*-bt)^2}{2\sigma^2 t}\right), \; t\geq 0. \qquad (10)$$

Its survivor function, $G_T(t)$, is given by:

$$G_T(t|b,\sigma,\bar{x}, x^*) = \Phi\left(\frac{\bar{x}-x^*-bt}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2(\bar{x}-x^*)b}{\sigma^2}\right)\Phi\left(\frac{-(\bar{x}-x^*)-bt}{\sigma\sqrt{t}}\right), \qquad (11)$$

where $\Phi(.)$ represents the cumulative distribution function of a standard normal, and $\bar{x}$ is given by the implicit function $H(\bar{x}, b, \sigma^2)=0$.

In order to calibrate our model, we take a sample from the Residential Energy Consumption Survey (RECS). The RECS is a statistical survey of the U.S. Department of Energy that collects energy-related data for occupied primary housing units in the 50 states and District of Columbia. Our sample was taken from the RECS 1990, which contains approximately 5,100 households, out of which 3,398 are homeowners.

One shortcoming of the RECS is that it does not provide information on replacement times. It only records current equipment ages in intervals: category 01=equipment is less than 2 years old, category 02=equipment is between 2 and 4 years old, category 03= equipment is between 5 and 9 years old, category 04= equipment is between 10 and 19 years old, and category 05= equipment is 20 years old or older. Therefore, the model parameters cannot be estimated directly from the p.d.f. of replacement times in (10). Instead, the p.d.f. of equipment age, U, must be used. It can be shown that an asymptotic approximation for the p.d.f. of U can be obtained from the renewal theorem (see Fernandez, 2000, for the technical details):

$$f_U(u|b,\sigma,\bar{x},x^*)=\frac{b}{\bar{x}-x^*}\left[\Phi\left(\frac{\bar{x}-x^*-bu}{\sigma\sqrt{u}}\right)-\exp\left(\frac{2(\bar{x}-x^*)b}{\sigma^2}\right)\Phi\left(\frac{-(\bar{x}-x^*)-bu}{\sigma\sqrt{u}}\right)\right] \qquad u\geq 0 \qquad (12)$$

In turn, the cumulative distribution function of equipment age or elapsed duration is given by

$$F_U(u|b,\sigma,\bar{x},x^*)=1+\frac{-(\bar{x}-x^*)+bu}{\bar{x}-x^*}\Phi\left(\frac{\bar{x}-x^*-bu}{\sigma\sqrt{u}}\right) - \frac{\bar{x}-x^*+bu}{\bar{x}-x^*}\exp\left(\frac{2(\bar{x}-x^*)b}{\sigma^2}\right)\Phi\left(\frac{-(\bar{x}-x^*)-bu}{\sigma\sqrt{u}}\right). \qquad (13)$$

Characteristics of household 'i' are incorporated into the model through $(\bar{x}-x^*)_i/\sigma_i$:

$$\frac{(\bar{x}-x^*)_i}{\sigma_i}=\exp(\boldsymbol{\beta}'\mathbf{z}_i) \qquad (14)$$

where $\boldsymbol{\beta}$ is a vector of parameters, and $\mathbf{z}_i$ represents a vector of household characteristics. This functional form ensures the non-negativity of $(\bar{x}-x^*)_i/\sigma_i$. For simplicity, the ratio b_i/σ_i is assumed to be constant across households, and equal to b/σ. Under these extra assumptions, an asymptotic approximation to the likelihood function of n independent observations is given by:

$$f_{U_1,U_2,\ldots,U_n}(u_1,u_2,\ldots,u_n|\tilde{b},\boldsymbol{\beta},\mathbf{z}_1,\ldots,\mathbf{z}_n)=$$

$$\prod_{i=1}^{n}\tilde{b}\exp(-\boldsymbol{\beta}'\mathbf{z}_i)\left[\Phi\left(\frac{\exp(\boldsymbol{\beta}'\mathbf{z}_i)-\tilde{b}u_i}{\sqrt{u_i}}\right)-\exp\left(2\tilde{b}\exp(\boldsymbol{\beta}'\mathbf{z}_i)\right)\Phi\left(\frac{-\exp(\boldsymbol{\beta}'\mathbf{z}_i)-\tilde{b}u_i}{\sqrt{u_i}}\right)\right] \qquad (15)$$

where $\tilde{b}\equiv b/\sigma$.

Estimates of $(\bar{x}-x^*)_i$, b_i, σ_i for household 'i' can be obtained by numerical solution from equation (16), once $\tilde{b}$ and $\boldsymbol{\beta}$ have been estimated.

$$1+\lambda_i(r\tilde{C}_i+(x^*-\bar{x})_i)=e^{\lambda_i(x^*-\bar{x})_i} \qquad i=1, 2, ..., n \qquad (16)$$

where λ_i represents the positive root of the characteristic equation $(1/2)\sigma_i^2p^2+b_ip-r=0$.

However, the likelihood function in (15) cannot be fitted to the RECS data because we do not observe the equipment ages. Instead, we are given only the intervals into which the age of each household's appliance falls. Therefore, the relevant likelihood function takes the form:

$$L=\prod_{i=1}^{n}[F_U(u_1\mid\beta,\tilde{b},\mathbf{z}_i)]^{d_1}\prod_{i=1}^{n}\prod_{j=2}^{4}[F_U(u_j\mid\beta,\tilde{b},\mathbf{z}_i)-F_U(u_{j-1}\mid\beta,\tilde{b},\mathbf{z}_i)]^{d_j}$$

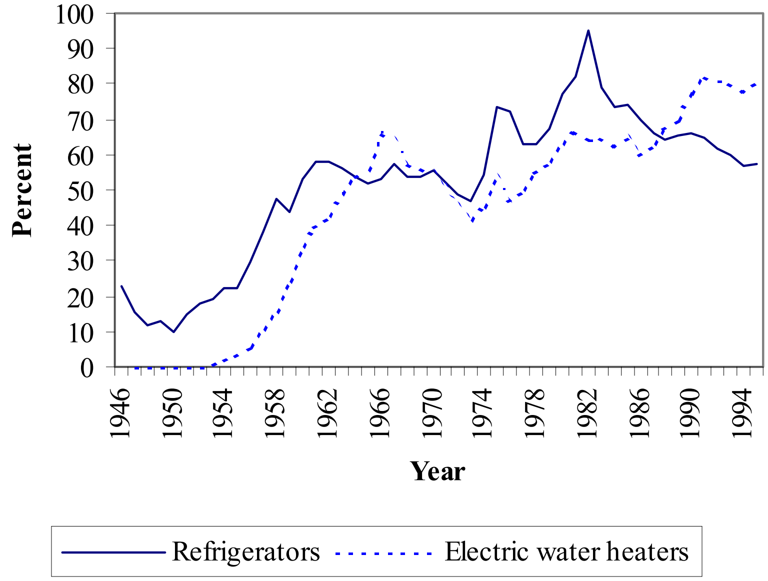
$$\cdot\prod_{i=1}^{n}[1-F_U(u_4\mid\beta,\tilde{b},\mathbf{z}_i)]^{1-d_1-d_2-d_3-d_4}\text{, } (17)$$

where $F_U(.)$ is given by (13), d_j=1 if age category = j, with j=1, 2, 3, 4 (for instance, d_1=1 if age is less than two years old, and 0 otherwise), and u_1=2, u_2=5, u_3=10, and u_4=20.

As before, estimates of $(\bar{x}-x^*)_i$, b_i, σ_i can be obtained from equation (16) for household 'i' once we have obtained estimates for $\tilde{b}$ and β.

Our application deals with replacement of refrigerators and water heaters. We first estimate separate replacement models for each appliance, and then test whether replacement decisions are correlated. Figure 1 illustrates how replacement sales have become a sizeable share of total annual shipments of refrigerators and electric water heaters. Indeed, this holds for all consumer durable goods with high market penetrations.

Figure 1 Estimated Annual Replacement Units as a Percentage of Total Annual Shipments



Source: Own elaboration based upon the distribution of lifetime equipment calibrated with the RECS data, and data on annual shipments of appliances from the *Statistical Abstract of the United States*, various issues.

Based on the annual average operation costs of new refrigerators from the RECS, and on the "Consumer Reports" (December 1992), we estimated the average price of a new refrigerator in 1990 to be $1,355. Our estimate of the annual operating and maintenance costs of a new refrigerator is $105, which corresponds with the annual operation cost of equipment aged two years or less reported in the RECS. Our estimate of the average price of a new electric water heater in 1990 is $662, and it is based on information provided in the RECS 1990 and the "National Construction Estimator" (1990). From the RECS 1990 its annual operating and maintenance cost is estimated to be $241.

The regressors in the replacement model of refrigerators are a constant, the age of the head of the household (per 10 years), monthly income (per $10,000), a dummy variable that takes on the value of 1 if the household lives in an urban area and 0 otherwise, the size of the refrigerator (cubic feet), family size (number of members), and a dummy variable that takes on the value of 1 if the household has a poor credit rating and 0 otherwise. Households are classified as having a poor credit rating in case they have received aid in terms of food stamps, unemployment benefits or income from AFDC (Aid to Families with Dependent Children) during the 12 months prior to the conduction of the survey.

Table 1 presents the estimation results for the refrigerator data. The exogenous variables that are statistically significant at the 5 percent level are the age of the head of the household and the size of the refrigerator. In particular, the older the head of the household, the less likely he/she will replace his/her piece of equipment. It is possible that older people have higher discount rates or, alternatively, that their preferences may change more slowly. By contrast, a greater refrigerator size accelerates replacement. This may be due to the fact that size is highly correlated with operating costs, after controlling for income, family size, and electricity rate, among others factors, as Table 2 shows. Although income and family size are not statistically significant at the conventional levels, they have the expected sign. That is, as income and family size increase, the gap between $\bar{x}$ and x^* shrinks making replacement more likely.

Table 1. Replacement Model for Refrigerators

Regressor	Parameter estimate	Standard error	Asymptotic t-statistic
Constant	2.529	0.145	17.444*
Age head of household (per 10 years)	0.099	0.013	7.679*
Monthly income (per \$10,000)	-0.006	0.010	–0.538
Urban area dummy (=1 if yes)	0.046	0.040	1.149
Family size (number of members)	–0.010	0.015	–0.703
Refrigerator size (cubic^3)	–0.040	0.006	7.218*
Poor credit rating dummy (=1 if yes)	–0.067	0.082	–0.819
Standardized drift, b/σ	0.624	0.032	19.375*

Log of likelihood function at convergence =–3,612; n= 2,440

*: Statistically significant at 5% level for H_0: β=0 against H_1: $\beta\neq$0.

Table 2. Refrigerator Monthly Operating Cost Modeled as a Linear Function of Exogenous Regressors

Regressor	Parameter estimate	Standard error	t-statistic
Constant	-48.145	7.567	–6.362*
Monthly income (per $10,000)	1.847	0.679	2.721*
Urban area dummy (=1 if yes)	7.831	2.707	2.893*
Family size (number of members)	–0.634	0.831	–0.764
Refrigerator size (feet^3)	5.053	0.348	14.541*
Average electricity rate ($/kwh)	0.947	0.056	16.906*

R^2=0.194, Adjusted R^2= 0.193; n=2,440
*: Statistically significant at 5% level for H_0: β=0 against H_1: β≠0.

The estimation results for water heaters are presented in Table 3. The regressors are in this case a constant, age of the head of the household, monthly income, a dummy variable for those households that live in an urban area, a dummy variable for those households for which natural gas is available in their neighborhood, the tank size of the water heater (gallons), family size, and a dummy variable for those households with a poor credit rating. As we see, the regressors statistically significant at the 5 and 10 percent levels are the age of the head of the household, natural gas availability, tank size, and the poor credit rating dummy.

Table 3 Replacement Model for Electric Water Heaters

Regressor	Parameter estimate	Standard error	Asymptotic t-statistic
Constant	1.249	0.179	6.939*
Age head of household (per 10 years)	0.137	0.018	7.449*
Monthly income (per $10,000)	-0.069	0.140	–0.491
Urban area dummy (=1 if yes)	-0.064	0.059	–1.086
Natural gas availability (=1 if yes)	0.219	0.057	3.830*
Tank size (gallons)	–0.004	0.002	–1.757**
Family size (number of members)	0.146	0.090	1.617
Poor credit rating dummy (=1 if yes)	0.039	0.021	1.866**
Standardized drift, b/σ	0.516	0.025	20.615*

Log of likelihood function at convergence =–2,612.9; n= 1,057
*: Statistically significant at 5% level for H_0: β=0 against H_1: β≠0.
** Statistically significant at 10% level for H_0: β=0 against H_1: β≠0.

As before, replacement is less likely as the head of the household becomes older. Natural gas availability and a poor credit rating have the same effect. In particular, natural gas availability might delay replacement because of differentials in operation costs between natural gas and electric powered equipment. Indeed, those households that would like to reduce operation costs by switching from electric to natural gas equipment cannot do it when natural gas is not available in their neighborhoods. Consequently, replacement of electric equipment becomes less likely. A poor credit rating delaying replacement is self-evident. Like in the case of refrigerators, a larger equipment size makes replacement more likely because of its high and positive correlation with operation costs.

Table 4 presents estimates for the difference between the threshold operation cost $\bar{x}$ and the operation cost of new equipment x^*, equipment lifetime, the drift and the standard deviation of the arithmetic Brownian process, b and σ, respectively, and for the total expected discounted costs for both appliances. Our estimate of the expected lifetime of a refrigerator is approximately 16.5 years. If we start up with new equipment, the expected total discounted cost amounts to $4,271.87.

Table 4. Estimates of $\bar{x} - x^*$, b, σ, Expected Equipment Lifetime and Total Discounted Cost

Estimates	Mean		Standard deviation	
	Refrigerator	Water heater	Refrigerator	Water heater
$\bar{x} - x^*$ ($)	243.2	139.9	42.5	26.1
b ($)	16.3	11.3	6.5	4.7
σ ($)	26.1	21.8	10.4	9.2
Expected lifetime (years)	16.5	13.7	4.3	3.4
Total discounted cost ($)	4,271.6	5,539.3	596.8	309.6

We should point out that our lifetime estimate is fairly close to that of the industry: an average lifetime of 16 years with a range of 10 (low)-20 (high) years (source: "A Portrait of the U.S. Appliance Industry 1992", Appliance, September 1992, Dana Chase Publications). For water heaters, our estimates of the expected lifetime and expected total cost are 13.7 years and $5,526.6, respectively. Like for refrigerators, our estimate of equipment lifetime is quite close to that given by the industry in 1992: 14 years with a range of 10 (low)-18 (high) years.

The overall fit for both models is quite good, as Table 5 shows. The percent error for all age categories of refrigerators is below 10 percent, being the greatest for equipment that are less than 2 years old, and between 5 and 9 years old. For water heaters in turn, the greatest percent error is below 5 percent in absolute value. Finally, Table 6 shows the impact of marginal changes in the value of the regressors on the probability of replacement over time. For example, a cubic-foot increase in refrigerator size leads to an increase of 9.93 per cent in the probability of replacement within 20 years. The overall probability of replacement is very small within the first 9 years, as we might have expected.

Table 5. Fitted and Actual Frequency for Each Age Category

Age Category	Fitted		Actual		Percent error	
	Refrigerator	Water heater	Refrigerator	Water heater	Refrigerator	Water heater
Less than 2 years old	0.124	0.156	0.136	0.153	8.8	–1.9
2-4 years old	0.185	0.228	0.187	0.227	1.1	–0.4
5-9 years old	0.289	0.289	0.270	0.299	–7.0	3.3
10-19 years old	0.311	0.242	0.318	0.234	2.2	–4.7
Over 20 years old	0.090	0.084	0.088	0.086	–2.3	2.3

Table 6 Impact on the Probability of Replacement due to Marginal Changes in the Regressors

Regressor	Time Period (years) 1-3		7-9		1-20	
	Refrigerator	Water heater	Refrigerator	Water heater	Refrigerator	Water heater
Age of head of household (per 10 years)	−5.61e-4	−0.065	−0.033	−0.062	−0.256	−0.366
Monthly income (per $10,000)	1.07e-6	0.003	0.002	0.031	0.016	0.285
Equipment size (*)	2.17e-4	1.90e-4	0.013	0.002	0.099	0.011
Probability of replacement	4.4e-4	0.011	0.088	0.131	0.730	0.796

Notes: Marginal impacts are evaluated at sample means. (*): Equipment size is measured in feet for refrigerators and in gallons for water heaters.

3.2 Are Replacement Decisions Independent?

In the previous section we modeled household decisions to replace a given set of appliances independently. However, it may be the case that such decisions are indeed correlated. Theoretically, the demand for durable goods is derived from a utility function that depends on the services these goods provide over time. Therefore, it would not be surprising to observe some degree of either substitution or complementarity in replacement decisions of different durable goods.

In order to test the hypothesis of independent replacement decisions for an individual household, we constructed a set of generalized residuals (Gourieroux and Monfort, 1987) based on the difference between observed and expected elapsed duration.

In our model, the expectation of equipment age (elapsed duration) is given by:

$$E(U) = \frac{\mu_2'}{2\mu} = \frac{(\bar{x} - x^*)\sigma^2 / b^3 + (\bar{x} - x^*)^2 / b^2}{2(\bar{x} - x^*)/b} = \frac{\sigma^2 + (\bar{x} - x^*)b}{2b^2} \qquad (18)$$

In order to test the independence of replacement decisions, we utilized a score test of the form:

$$\xi_n = \frac{\left(\sum_{i=1}^{n} \hat{\omega}_{1i}\hat{\omega}_{2i}\right)^2}{\sum_{i=1}^{n} (\hat{\omega}_{1i}\hat{\omega}_{2i})^2} \xrightarrow{d} \chi^2(1) \qquad (19)$$

where $\hat{\omega}_{1i}$ and $\hat{\omega}_{2i}$ represent the estimates of the generalized residuals of appliance 1 (refrigerator) and 2 (water heater) for household i, respectively. This test is asymptotically distributed as chi-square with 1 degree of freedom (see Gourieroux and Monfort for further details).

Each residual is computed as the difference between the observed equipment age and its expected value, evaluated at the parameter estimates of Section 3.1:

$$\hat{\omega}_{mi} = u_i - \frac{\hat{\sigma}_i^2 + \widehat{(\bar{x} - x^*)_i}\,\hat{b}_i}{2\hat{b}_i^2} \qquad m=1, 2; i=1, 2,.., n \qquad (20)$$

Only those households that own both appliances are considered in our computations. We found positive correlation among residuals of refrigerators and water heaters: 11.6 percent, and $\tilde{\xi}_n$=20.8, being the 95-percent critical value for a χ^2 (1) equal to 3.84. That is, we reject at the 95-percent confidence level the null hypothesis of independence of the residuals of the replacement models of refrigerators and water heaters. In addition, the positive correlation between the residuals indicates that unobservable factors that either accelerate or delay replacement of one appliance will also affect the other in the same direction.

3.3 Estimating Replacement Decisions Simultaneously: Minimum and Maximum Stopping Times

Let t_1, t_2, ..., be a sequence of independent random variables. An integer-valued random variable T is said to be a stopping time for the sequence t_1, t_2, .., if the event {T=n} is independent of t_{n+1}, t_{n+2}, for all n=1, 2, ... This means that we observe the t_n's in sequential order and N denotes the number observed before stopping. If T=n, then we have stopped after observing t_1, t_2, ..., t_n and before observing t_{n+1}, t_{n+2}, ...(see Ross, 1996, page 104).

Let us now consider two independent stopping times, T_1 and T_2, with corresponding cumulative density functions F_{T_1} and F_{T_2}, and density functions f_{T_1} and f_{T_2}. For example, let us think of two home appliances whose times of either technical failure or obsolescence are independent of one another. This is the assumption in Section 3.1. But, how do we reconcile the assumption of independent stopping times with the evidence in Section 3.2? One way to go about it is by thinking that, although stopping times are independent, households replace their appliances jointly.

For instance, a household might wait and replace its obsolete microwave oven until the cutting-edge technology of refrigerators becomes available at the market place.[1] Or, alternatively, the household might decide to replace its microwave oven and refrigerator at once, as soon as the technology of the former falls behind the new trends.

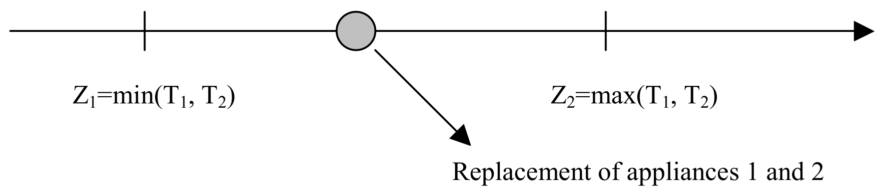
If that is so, then we can find the minimum and maximum bounds for the replacement time of both appliances. Let Z_1=max(T_1, T_2) and Z_2=min (T_1, T_2). The probability density function of Z_1 and Z_2 are given, respectively, by:

$$f_{Z_1}(z) = f_{T_1}(z)F_{T_2}(z) + F_{T_1}(z)f_{T_2}(z) \qquad z \geq 0 \qquad (21)$$

$$f_{Z_2}(z) = f_{T_1}(z) + f_{T_2}(z) - f_{Z_1}(z) \qquad z \geq 0 \qquad (22)$$

One way to model joint replacement decisions is by assuming that replacement will take place somewhere between the minimum and the maximum stopping times . Therefore, we can define a new random variable $W=Z_2-Z_1$ that denotes the time elapsed between the minimum and the maximum stopping times (Figure 2). Intuitively, households might replace both appliances right after any of them either becomes obsolete or breaks down. Or, alternatively, they might as well wait until both appliances render inadequate to their needs. The exact time at which households will replace both appliances is therefore random, and will be located somewhere between Z_1 and Z_2.

Figure 2 Joint Replacement Decisions



Now, in order to determine the distribution function of W, we make use of convolutions:

$$\Rightarrow \qquad f_W(w) = \frac{d}{dw} F_W(w) = \int_0^{\infty} \left(\frac{d}{dw} \int_0^{w+z_1} f_{Z_1Z_2}(z_1, z_2)\, dz_2 \right) dz_1$$

$$= \int_0^{\infty} f_{Z_1Z_2}(z_1, w + z_1) dz_1 \qquad \text{by Leibnitz's rule}$$

Now, given that both Z_1 and Z_2 are stopping times, Z_2 is independent of Z_1. Therefore, the distribution function of W boils down to:

$$f_W(w) = \int_0^{\infty} f_{Z_1}(z_1) f_{Z_2}(w + z_1) dz_1 \qquad w \geq 0 \qquad (23)$$

where $f_W(w)$ is the convolution of $f_{Z_1}(z_1)$ and $f_{Z_2}(w)$.

In our model, the distribution function of the maximum is given by:

[1] On the other hand, a household's postponing the purchase of new durable goods might be indicative of borrowing constraints.

$$f_{Z_1}(z) = \left(\frac{\bar{x}_1 - x_1^*}{\sigma_1 \sqrt{2\pi z^2}} \exp\left(\frac{-(\bar{x}_1 - x_1^* - b_1 z)^2}{2\sigma_1^2 z} \right) \right)$$

$$* \left(1 - \Phi\left(\frac{\bar{x}_2 - x_2^* - b_2 z}{\sigma_2 \sqrt{z}} \right) + \exp\left(\frac{2(\bar{x}_2 - x_2^*) b_2}{\sigma_2^2} \right) \Phi\left(\frac{-(\bar{x}_2 - x_2^*) - b_2 z}{\sigma_2 \sqrt{z}} \right) \right)$$

$$+ \left(1 - \Phi\left(\frac{\bar{x}_1 - x_1^* - b_1 z}{\sigma_1 \sqrt{z}} \right) + \exp\left(\frac{2(\bar{x}_1 - x_1^*) b_1}{\sigma_1^2} \right) \Phi\left(\frac{-(\bar{x}_1 - x_1^*) - b_1 z}{\sigma_1 \sqrt{z}} \right) \right)$$

$$* \left(\frac{\bar{x}_2 - x_2^*}{\sigma_2 \sqrt{2\pi z^2}} \exp\left(\frac{-(\bar{x}_2 - x_2^* - b_2 z)^2}{2\sigma_2^2 z} \right) \right) \qquad z \geq 0 \qquad (24)$$

In turn the distribution function of the minimum is given by:

$$f_{Z_2}(z) = \left(\frac{\bar{x}_1 - x_1^*}{\sigma_1 \sqrt{2\pi z^2}} \exp\left(\frac{-(\bar{x}_1 - x_1^* - b_1 z)^2}{2\sigma_1^2 z} \right) \right) + \left(\frac{\bar{x}_2 - x_2^*}{\sigma_2 \sqrt{2\pi z^2}} \exp\left(\frac{-(\bar{x}_2 - x_2^* - b_2 z)^2}{2\sigma_2^2 z} \right) \right)$$

$$- f_{Z_1}(z)$$

$$z \geq 0 \qquad (25)$$

Now suppose we have a cross section of n independent pairs of stopping times for n households. Then the likelihood function of the sample is given by:

$$\prod_{i=1}^{n} f_{W_i}(w_i) = \prod_{i=1}^{n} \left(\int_0^\infty f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1 \right) \qquad w_i \geq 0,\ i = 1, 2, \ldots n$$

$$\Leftrightarrow \qquad \sum_{i=1}^{n} \ln(f_{W_i}(w_i)) = \sum_{i=1}^{n} \ln\left(\int_0^\infty f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1 \right) \qquad (26)$$

where $w_i = Z_{2i} - Z_{1i}$, $Z_{2i} = \max(T_{1i}, T_{2i})$ and $Z_{1i} = \min(T_{1i}, T_{2i})$.

How do we go about approximating the integral $\int_0^\infty f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1$? We know, from our estimation results, that the probability mass for a stopping time greater than some constant M, large enough (say M=60), goes to zero. Therefore the above improper integral can be suitably truncated:

$$\int_0^{\infty} f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1 \approx \int_0^{M} f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_i) dz_1$$

Therefore the log likelihood function of the sample becomes:

$$\sum_{i=1}^{n} \ln(f_{W_i}(w_i)) \approx \sum_{i=1}^{n} \ln\left(\int_0^{M} f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1 \right) \tag{27}$$

In turn the integral $\int_0^{M} f_{Z_{1i}}(z_{1i}) f_{Z_{2i}}(w_i + z_{1i}) dz_{1i}$ can be approximated by some numeric method, such as the trapezoidal rule:[2]

$$\int_0^{M} f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1) dz_1 \approx \frac{\Delta z_1}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{m-2} + y_m\right) \tag{28}$$

where $\Delta z_1 = \frac{M}{m}$, $y = g(z_1) \equiv f_{Z_{1i}}(z_1) f_{Z_{2i}}(w_i + z_1)$, and $g(z_{1k})=g(k\Delta z_1)$, k=0, 1, 2, ..., m.

As in Section 3.1, the parameters of the distributions of Z_{1i} and Z_{2i}, i=1, 2,..., n, may be modeled as functions of household characteristics and appliances features.

3.4 A Simple Simulation Example on Replacement Interdependence

The model of the above section cannot be fitted to our RECS data set because it requires that we observe replacement times (i.e., complete durations). Therefore, in order to illustrate interdependence in replacement decisions, we present a simplified framework, but which goes along the lines of the two previous sections.

Let $\mathbf{X}=(X_1,\ldots, X_N)$ be a random vector with cdf F. F can be decomposed into its univariate margins and another cdf called a copula. Let us assume that the univariate cdfs $F_{(j)}$

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[2] The area of the first trapezoid is $1/2(y_0+y_1)\Delta z_1$, the area of the second trapezoid is $1/2(y_1+y_2)\Delta z_1$, etc. up to the area of the nth trapezoid, which is $1/2(y_{n-1}+y_n)\Delta z_1$.

are continuous. Then the random variables X_j can be transformed to (0, 1)-uniforms $F_{(j)}(X_j)$. The cdf C of the vector **Y** of transformed random variables $Y_j=F_{(j)}(X_j)$ is called the copula pertaining to F. That is,

$$C(\mathbf{u})=P(Y_1\leq u_1, \ldots, Y_N\leq u_N)=F(F_{(1)}^{-1}(u_1),\ldots,F_{(N)}^{-1}(u_N)) \qquad (29)$$

where $F_{(j)}^{-1}$ is the quantal function of $F_{(j)}$. Conversely, one can go back to the original cdf F from the copula

$$F(\mathbf{x})= C(F_{(1)}(x_1),\ldots, F_{(N)}(x_N)) \qquad (30)$$

(see Reiss and Thomas, 2001).

Let us take two random variables T_1 and T_2, which are exponentially distributed, and that represents complete durations. For simplicity, we take a normal copula

$$C(\mathbf{u},\rho)=\Phi(G_1^{-1}(u_1),G_2^{-1}(u_2)) \qquad (31)$$

where $G_i(.)$ is the cdf of an exponential random variable. The density is

$$c(\mathbf{u},\rho) =|\rho|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\zeta'(\rho^{-1}-1)\zeta\right) \qquad (32)$$

with $\zeta = (G_1^{-1}(u_1),G_2^{-1}(u_2))$.

The advantage of taking the normal copula is that the parameter ρ is the Pearson correlation coefficient of the transformed data.

Now let us define $W=\max(T_1,T_2)-\min(T_1,T_2)$, as in section 3.3. Table 7 presents simulations for W under independent random variables T_1 and T_2, and assuming a degree of correlation between the two random variables using the copulas approach. As ρ increases, the time elapsed between complete stopping times (Figure 2) decreases. That is, it becomes more likely that both appliances are replaced short after the first one either breaks down or becomes obsolete.

Table 7 Time elapsed between the minimum and the maximum stopping times, W

		Normal Copula					
Statistic (years)	**Independent r.v.**	**ρ=0**	**ρ=0.1**	**ρ= 0.5**	**ρ=0.7**	**ρ=0.9**	**ρ=1**
Average	7.60	7.61	7.24	5.49	4.34	2.76	1.64
Median	5.24	5.22	4.92	3.53	2.72	1.64	1.13
Std	7.65	7.77	7.49	6.03	4.94	3.31	1.66
Interquartile range	8.30	8.32	7.95	6.00	4.83	3.09	1.79
Minimum	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Maximum	71.41	86.54	82.28	68.70	66.09	46.94	16.76
Sample size	10,000	10,000	10,000	10,000	10,000	10,000	10,000

Notes: T_1 and T_2 are assumed exponentially distributed with parameters 0.12 and 0.15, respectively.

It is worth noticing that the case of ρ=0 gives a very similar result to that obtained under the assumption of independent exponentially distributed random variables.

5 Summary and conclusions

The core of this paper is the empirical results for replacement of home appliances in the United States, and the theoretical model of multiple replacement decisions. Based upon individual replacement models for electric water heaters and refrigerators, we concluded that demographics and appliance features might either accelerate or delay replacement. In addition, we constructed a test statistic that led us to conclude that replacement decisions might be correlated across appliances. Based upon this evidence, we enriched our model by allowing households to replace a set of appliances simultaneously rather than each one in isolation. Although the estimation process of this extension may be computationally intensive, it is still tractable.

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