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On the q-Weibull distribution for reliability applications: An adaptive hybrid artificial bee colony algorithm for parameter estimation

Meng Xu[a], Enrique López Droguett[a,b], Isis Didier Lins[c,*], Márcio das Chagas Moura[c]

[a] Center for Risk and Reliability, Mechanical Engineering Department, University of Maryland, College Park, MD, USA
[b] Mechanical Engineering Department, University of Chile, Santiago, Chile
[c] Center for Risk Analysis and Environmental Modeling, Department of Production Engineering, Universidade Federal de Pernambuco, Recife, PE, Brazil

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ABSTRACT

The q-Weibull model is based on the Tsallis non-extensive entropy [22] and is able to model various behaviors of the hazard rate function, including bathtub curves, by using a single set of parameters. Despite its flexibility, the q-Weibull has not been widely used in reliability applications partly because of the complicated parameters estimation. In this work, the parameters of the q-Weibull are estimated by the maximum likelihood (ML) method. Due to the intricate system of nonlinear equations, derivative-based optimization methods may fail to converge. Thus, the heuristic optimization method of artificial bee colony (ABC) is used instead. To deal with the slow convergence of ABC, it is proposed an adaptive hybrid ABC (AHABC) algorithm that dynamically combines Nelder-Mead simplex search method with ABC for the ML estimation of the q-Weibull parameters. Interval estimates for the q-Weibull parameters, including confidence intervals based on the ML asymptotic theory and on bootstrap methods, are also developed. The AHABC is validated via numerical experiments involving the q-Weibull ML for reliability applications and results show that it produces faster and more accurate convergence when compared to ABC and similar approaches. The estimation procedure is applied to real reliability failure data characterized by a bathtub-shaped hazard rate.

1. Introduction

The Weibull distribution, frequently used in reliability engineering, has been generalized to a q-Weibull distribution by Picoli et al. [1] in the context of non-extensive statistical mechanics. The q-Weibull distribution can be used to describe complex systems with long-range interactions and long-term memory [2]. Compared to the Weibull distribution, which can only describe monotonic hazard rate functions, the q-Weibull is able to model various behaviors of the hazard rate, including the unimodal, bathtub-shaped, monotonic (monotonically decreasing, monotonically increasing) and constant. Indeed, Assis et al. [2] provided the ranges of the shape parameters q and β related to each type of curve.

Thus, the q-Weibull probabilistic model unifies monotonic and non-monotonic hazard rate functions by using one general formula, which is flexible and elegant for failure data fitting. For example, the well-known bathtub-shaped hazard rate function is reproduced by the q-Weibull distribution using a single set of three parameters for its three characteristic regions. Such a flexibility is important to accurately perform reliability analyses when failure data are characterized by non-monotonic hazard rates. Additionally, q-Weibull model is able to reproduce both short and long tailed distributions.

The Weibull distribution has been modified or generalized in different ways to allow for non-monotonic hazard rate functions. For instance, Murthy et al. [4] provide a taxonomy to integrate the different Weibull models. There are some recent Weibull distribution extensions in the reliability engineering literature. Pham and Lai [5], and Almalki and Nadarajah [6] reviewed the generalizations and modifications of the Weibull distribution. These models are capable of modeling a bathtub-shaped hazard rate functions and can be classified into two categories: (i) methods that add parameters to an existing distribution to obtain classes of more flexible distributions as introduced by Olkin [7], and (ii) methods that combine two or more distributions with one or more being Weibull. Examples include the IDB model [8], the exponentiated Weibull (EW) distribution [9], the generalized Weibull (GW) [10], the additive Weibull (AW) distribution [11], the extended Weibull distribution [12], the modified Weibull (MW) distribution [13], the modified Weibull extension (MWE) [14], the beta Weibull (BW) distribution [15], the flexible Weibull extension (FWE) [16], the generalized modified Weibull (GMW) distribution [17], the ENH distribution [18], the additive modified Weibull (AMW) distribution [19], and the generalized modified Weibull power series (GMWPS)

* Corresponding author.
E-mail addresses: elopezdroguett@ing.uchile.cl (E.L. Droguett), isis.lins@ceerma.org (I.D. Lins).

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distribution [20]. There are also models involving two or more Weibull distributions, for example, sectional method, competing risk approach and multiplicative model introduced by Jiang and Murthy [21].

In this paper, the q-Weibull distribution arises in the context of reliability. It has the advantage of containing only three parameters with flexibility to model various shapes of the hazard rate function, thus positioning itself as an alternative to the existing life distributions in modeling reliability data. Indeed, the q-Weibull distribution has been successfully applied to model lifetime data in the context of reliability engineering. For example, Costa et al. [23] used q-Weibull distribution to properly describe time-to-breakdown data of electronic devices; Sartori et al. [24] considered a q-Weibull distribution to describe the failure rate of a compression unit in a typical natural gas recovery plant based on time-to-failure data.

In these papers, the q-Weibull distribution parameters have been estimated via the least squares estimation (LSE) procedure (see Picoli et al. [1]) or through square correlation coefficient R^2 maximization (in Sartori et al. [24] and Assis et al. [2]). Jose and Naik [25] provided the likelihood function, but claimed that it is very difficult to obtain the maximum likelihood (ML) estimates of the parameters due to the nonlinear set of equations. Alternatively, Jose and Naik [25] employed the method of moments stating, however, that the moment estimates are not easy to evaluate when all the parameters are unknown.

Extensive simulation studies have shown the ML method is better than the LSE in reliability applications when data sets are typically small or moderate in size [26]. Since the distribution of ML parameter estimates are more accurate with smaller variance, we here adopt the ML method. However, the application of ML on q-Weibull distribution presents some challenges: the first derivative equations of the related log-likelihood function are highly nonlinear, and the equations do not have analytical solutions for the parameters' estimators. Such a difficulty can explain the limited number of applications based on the q-Weibull model given that parameter estimation and data fitting are crucial steps for reliability analyses.

In this context, a numerical approach can be adopted to solve the q-Weibull ML problem. In this work, we employ an artificial bee colony (ABC) algorithm [27], which is a nature-based heuristic method that does not require derivative information to solve the q-Weibull distribution ML problem.

However, the convergence performance of ABC for local search is slow due to its solution search method, which is good at exploration but poor at exploitation [32]. In order to improve its performance, some modified versions of ABC have been proposed in the literature. For instance, inspired by Particle Swarm Optimization (PSO), Zhu and Kwong [32] developed an improved ABC algorithm named gbest-guided ABC (GABC) by incorporating the information of global best solution into the solution search equation to improve exploitation. Kang et al. [33] proposed a Hooke-Jeeves ABC (HABC) algorithm that combines Hooke-Jeeves pattern search with ABC algorithm. In the HABC, the exploration phase is performed by ABC and the exploitation stage is completed by pattern search. Karaboga and Gorkemli [34] adopted the quick ABC (qABC), which models the behavior of onlooker bees more accurately and improves the performance of standard ABC in terms of local search ability. In order to achieve an optimization performance with higher convergence speed and an improved exploitation capacity, Shan et al. [35] used a self-adaptive hybrid artificial bee colony (SAHABC) algorithm inspired by self-adaptive mechanism, differential evolution (DE), and PSO algorithm. In the SAHABC, the search equation for employed bees is modified based on the self-adaptive mechanism, which is used to balance the exploration ability and the convergence speed of ABC, and DE mutation strategy, which uses the best solution to improve convergence performance. The search equation for onlooker bees is modified based on PSO to improve the exploitation ability. Kang et al. [36] proposed a hybrid simplex ABC algorithm (HSABCA) that combines Nelder-Mead simplex method with artificial bee colony algorithm for inverse analysis problems. The HSABCA was applied to parameter identification of concrete dam-foundation systems. The Nelder-Mead simplex algorithm proposed by Nelder and Mead [37] is an efficient local search method. It was also combined with other heuristic to improve the convergence accuracy and speed. For example, Fan and Zahara [38] proposed the hybrid NM-PSO algorithm based on the Nelder-Mead simplex search method and PSO for unconstrained optimization.

A method that does not depend on derivative, but also presents fast convergence is necessary in the q-Weibull distribution ML optimization problem. In this direction, this paper proposes an Adaptive Hybrid ABC (AHABC) algorithm, which combines a local Nelder-Mead simplex search method with ABC to enhance the local search capability of ABC. Differently from HSABCA proposed by Kang et al. [36], AHABC dynamically controls the exploration and exploitation, given that the parameter for Nelder-Mead local search is adaptively tuned according to the search status. AHABC is also different from SAHABC [35] in terms of the hybrid strategy and adaptive mechanism.

The proposed AHABC is an efficient manner to tackle the difficult ML problem related to the q-Weibull distribution comprising different behaviors of the hazard rate function. The efficiency of the new algorithm is proved by comparison with the standard ABC and with SAHABC.

The rest of this paper is organized as follows. In Section 2, the ML constrained problem related to the q-Weibull distribution is developed. In Section 3, AHABC algorithm is proposed. Section 4 presents numerical experiments for the validation of the proposed AHABC algorithm to obtain ML estimates for the q-Weibull parameters. Section 5 presents an application example involving ML estimates of the q-Weibull parameters for reliability-related lifetime data via the proposed AHABC. Concluding remarks are given in Section 6. The Appendix contains the development of the asymptotic intervals based on the ML theory [39], as well as parametric and non-parametric bootstrap methods [40] are developed and combined with the proposed AHABC to provide bootstrap confidence intervals for the q-Weibull parameters estimated via ML.

2. q-Weibull maximum likelihood constrained problem

2.1. Characterization of q-Weibull distribution

The probability density function (PDF) of the q-Weibull distribution is as follows:

$$f_q(t)=(2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\exp_q\left[-\left(\frac{t}{\eta}\right)^{\beta}\right],\ t\geq 0, \tag{1}$$

where $\beta > 0$ and $q < 2$ are shape parameters and $\eta > 0$ is a scale parameter. The q-Exponential function $\exp_q(x)$ is defined as:

$$\exp_q(x)=\begin{cases}(1+(1-q)x)^{\frac{1}{1-q}}, \text{if } 1+(1-q)x>0,\\ 0, \text{ otherwise.}\end{cases} \tag{2}$$

Therefore, the q-Weibull PDF can be rewritten as:

$$f_q(t)=(2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}},\ t\geq 0, \tag{3}$$

where

$$t\in\begin{cases}[0,\ \infty), \text{for } 1<q<2,\\ [0,\ t_{max}],\ \text{for } q<1,\end{cases} \text{with } t_{max}=\frac{\eta}{(1-q)^{1/\beta}}. \tag{4}$$

In the limit $q \to 1$, $f_q(t)$ reduces to the Weibull PDF, for $\beta = 1$ it corresponds to the q-Exponential PDF, and when $q \to 1$ and $\beta = 1$ it becomes the Exponential distribution [3].

The q-Weibull cumulative distribution (CDF) and reliability functions are as follows:

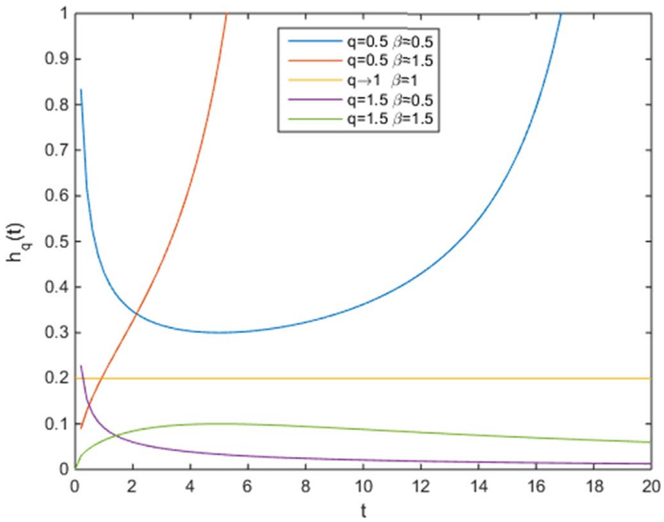

Fig. 1. Behaviors of the q-Weibull hazard rate function for $\eta = 5$ and different values of the shape parameters q and β.

$$F_q(t)=1-\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}}, \tag{5}$$

$$R_q(t)=\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}}. \tag{6}$$

Then, the hazard rate function is defined as:

$$h_q(t)=\frac{f_q(t)}{R_q(t)}=\frac{(2-q)\frac{\beta}{\eta^{\beta}}t^{\beta-1}}{1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}}. \tag{7}$$

Equation (7) is able to represent different types of hazard rate functions according to the values of the shape parameters [2]. Indeed, $h_q(t)$ is monotonically decreasing for $1 < q < 2$ and $0 < \beta < 1$, monotonically increasing for $q < 1$ and $\beta > 1$, unimodal for $1 < q < 2$ and $\beta > 1$, and bathtub-shaped for $q < 1$ and $0 < \beta < 1$.

Fig. 1 illustrates the different behaviors of $h_q(t)$ for $\eta = 5$ and specific values of the shape parameters q and β. Note that for $q = 0.5(q < 1)$, $h_q(t)$ – as well as $f_q(t)$, $F_q(t)$ and $R_q(t)$ – has a limited support. For the cases $\beta = 0.5$ with $q = 0.5$ and $\beta = 1.5$ with $q = 0.5$ depicted in Fig. 1, t_{max} is 20 and 7.937, respectively.

Moreover, random samples may be generated according to the q-Weibull distribution by inverting $F_q(t)$. Indeed, the q-Weibull random number generator is obtained as:

$$t = \eta\left[\frac{1-U^{\frac{1-q}{2-q}}}{1-q}\right]^{\frac{1}{\beta}}, \tag{8}$$

where U is a uniform random number in [0,1].

Suppose that an item has survived to time t_0, then the q-Weibull conditional reliability function is given by:

$$R_q(t\ t_0)=\frac{R_q(t_0+t)}{R_q(t_0)}=\left[\frac{1-(1-q)\left(\frac{t_0+t}{\eta}\right)^{\beta}}{1-(1-q)\left(\frac{t_0}{\eta}\right)^{\beta}}\right]^{\frac{2-q}{1-q}}. \tag{9}$$

2.2. Maximum likelihood constrained problem

In this section, the parameters of the q-Weibull distribution are estimated via the ML method. Let $\underline{t} = (t_1, t_2, \ldots, t_n)$ be an n-dimensional vector of observed failure times t_i, $i = 1, \ldots, n$, independently drawn from a q-Weibull distribution. The likelihood function is given by:

$$L(\underline{t}|\eta,\beta,q)=\prod_{i=1}^{n}f_q(t_i)=\prod_{i=1}^{n}(2-q)\frac{\beta}{\eta}\left(\frac{t_i}{\eta}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}} \tag{10}$$

The log-likelihood function is as follows:

$$\mathcal{L}(\underline{t}|\eta,\beta,q)=n ln(2-q)+n ln(\beta)-n\beta\ln(\eta)+(\beta-1)\sum_{i=1}^{n}\ln(t_i) + \frac{1}{1-q}\sum_{i=1}^{n}\ln\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right] \tag{11}$$

Considering the constraints of parameters and the support, the constrained optimization problem is:

$$\max n\ln(2-q)+n\ln(\beta)-n\beta\ln(\eta)+(\beta-1)\sum\nolimits_{i=1}^{n}\ln(t_i)+\frac{1}{1-q}\sum\nolimits_{i=1}^{n}\ln\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right], \tag{12}$$

$$\text{s. t.}\ 2-q>0, \tag{13}$$

$$1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}>0, i=1,\ldots,n, \tag{14}$$

$$\eta>0, \tag{15}$$

$$\beta>0. \tag{16}$$

The first derivatives of the log-likelihood function w.r.t. parameters are nonlinear, and analytical solutions are very difficult to be obtained. A heuristic based constrained optimization method can be applied to tackle this problem. In this paper, the ML estimates $\hat{\eta}$, $\hat{\beta}$ and $\hat{q}$ are obtained by means of an adaptive hybrid artificial bee colony (AHABC) algorithm developed in the next section.

3. Proposed adaptive hybrid artificial bee colony algorithm

In the ABC algorithm, while onlookers and employed bees carry out the exploitation process in the search space, the scouts control the exploration process [27]. However, the original ABC is good at exploration but bad at exploitation for numerical benchmark functions optimization [32]. From our simulation experiments for ML estimation of the q-Weibull parameters by ABC (see Section 4.2), we also observe similar results: although ABC could find the global optimum, the estimates' variability is large due to the slow convergence speed of ABC for local search.

Thus, in order to make full use of ABC's exploration, and avoid its drawbacks, an adaptive hybrid ABC is proposed that incorporates a local search stage. The main idea of AHABC is that through adaptively tuning the parameters of hybrid ABC according to the search process, the hybrid ABC will gradually change from global ABC search pattern to local search pattern. The general AHABC framework is shown in Fig. 2. The details of the proposed AHABC algorithm are presented in the following subsections.

3.1. Hybrid strategy

"Hybrid Strategy" is the method to combine ABC with a local search algorithm. There are two common types of hybrid strategies: (i) selectively applying either ABC or local search, which means that for a certain population, the next generation is given by ABC or local search method; (ii) merging the local search into ABC, which means that the local search is incorporated into ABC as an operation or a phase.

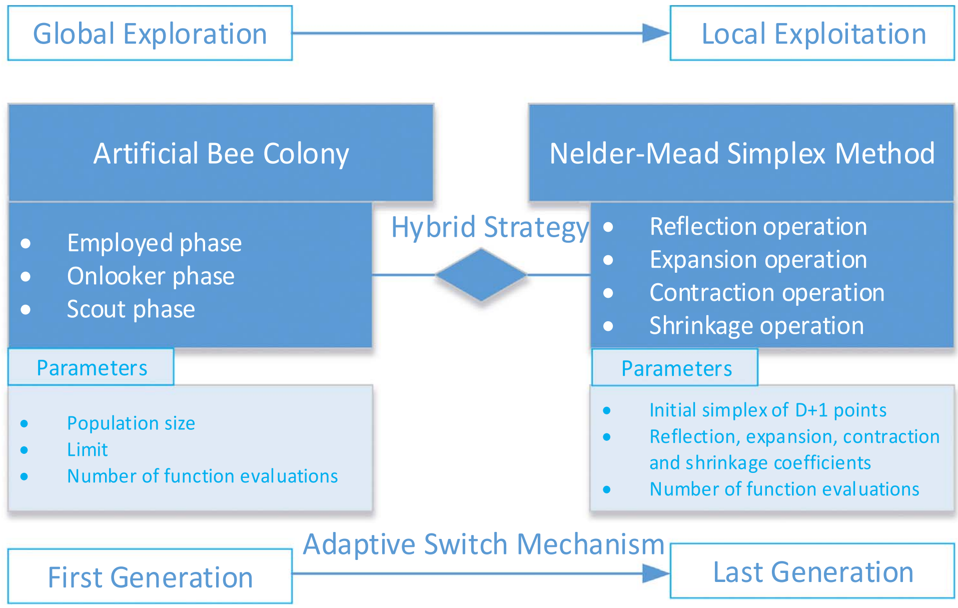

Fig. 2. Framework of adaptive hybrid ABC.

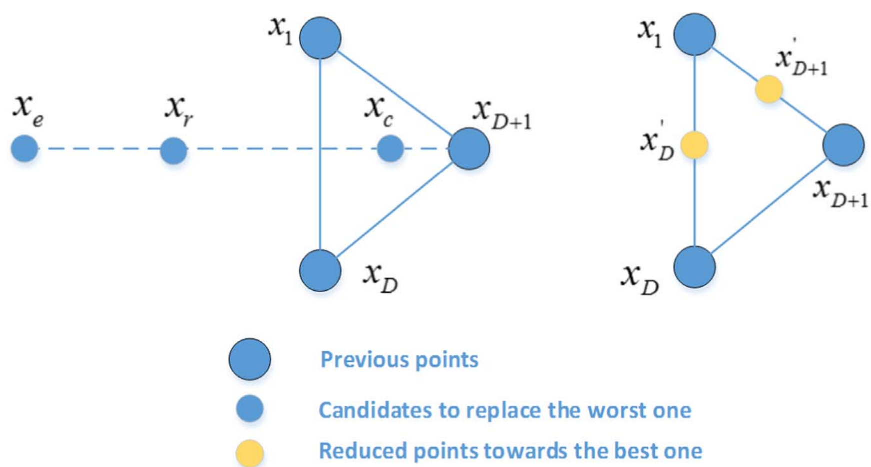

Fig. 3. Scheme of simplex search.

In the proposed algorithm, we adopt the second hybrid strategy. Nelder-Mead simplex search is chosen as the local search method and is added to ABC as an additional step after the original three phases and within every iteration. This method rescales the simplex by four procedures: reflection, expansion, contraction and shrinkage. The input of local search phase is the best $D + 1$ solutions in the population, where D is the dimension of the optimization problem, as shown in Fig. 3. Then, three candidate solutions are generated and evaluated. If the best of these new solutions can outperform the worst solution in current simplex, this new solution replaces the worst one (see Fig. 3). Otherwise, the current simplex shrinks towards the best solution in the current simplex (see Fig. 3). These solutions will be exploited by the Nelder-Mead simplex local search for a number of function evaluations NS.

3.2. Adaptive switch mechanism

"Adaptive switch mechanism" determines how the hybrid algorithm changes from global exploration to local exploitation. Basically, the principle of "adaptive switch mechanism" is to gradually increase the use of local search by tuning algorithm parameters according to the search process. These tunable parameters are search space-related, i.e., changing their values will modify the search property (more global or more local).

In this paper, we adaptively increase the number of simplex search, and the searching process becomes more local. The remaining challenge is how to determine the number of simplex searches NS. We propose the following formula:

$$NS = C * \text{limit} * \text{total number of scout bees}. \tag{17}$$

Firstly, since the total number of scout bees increases over ABC iterations, this definition of NS will guarantee that NS is non-decreasing, which means the search process will become more and more local. Secondly, the total number of scout bees is a symbol of search status. A large number of scout bees indicates that a significant portion of the solution space has been explored, that the exploration is becoming inefficient and a local exploitation is becoming urgent. Also, the "limit" is also an important ABC parameter, which controls the scout bee generation frequency. C is a coefficient that controls the amount of local search. For the q-Weibull distribution ML optimization problem, $C = 1$ provided an acceptable convergence speed (shown in Section 4.1). Thus, we use the product of limit and the total number of scout bees as the number of function evaluations within the local search phase of the AHABC. In summary, NS dynamically increases along the search process and it gradually changes from global to local.

Initialize a randomly distributed population of SN candidate solutions $x_{i,j}, i = 1 \dots SN, j = 1 \dots D$
Set number of function evaluations $k = 0$, number of scout bees $s = 0$
Evaluate the fitness of the initial population, $k = k + 1$ for each evaluation
Repeat
(The employed bees phase)
Each employed bee produces a candidate solution v_i
Evaluate the fitness of v_i, and apply the greedy selection to compare x_i and v_i $k = k + 1$ for each evaluation
Calculate the probability values p_i for solutions x_i
(The onlooker bees phase)
Each onlooker produces a candidate solutions v_i for x_i depending on p_i
Evaluate the fitness of v_i, and apply the greedy selection to compare x_i and v_i $k = k + 1$ for each evaluation
(The scout bees phase)
When a solution cannot be improved further through a predetermined number of cycles, called "limit", then that solution is abandoned and replaced with a new solution v_i randomly generated by a scout bee. Evaluate the fitness of v_i, $s = s + 1, k = k + 1$.
(Adaptively tune parameters)
Calculate the number of function evaluations for Nelder-Mead local search according to the formula $NS = C * limit * s$
Rank the solutions in the population according to fitness values
(Nelder-Mead local search phase)
Select the best $D + 1$ solutions to generate the initial simplex
Set number of function evaluations for local search $k' = 0$
Repeat
apply Nelder-Mead operators: reflection, expansion, contraction and shrinkage, $k' = k' + 1$, $k = k + 1$ for each function evaluation.
Until $k' = NS$ or termination criteria
Memorize the best solution achieved so far
Until $k = MFE$ or termination criteria

Fig. 4. Pseudo-code of AHABC.

3.3. Constraints

For the constraints (13–16) related to the q-Weibull ML problem, we adopt the "throw away" approach, which means that if the generated solution is not feasible, we throw it away and keep the current solution. This is a simplified Deb's rule [46] that involves domination rules between solutions. In our proposed algorithm, we do not allow infeasible solutions in the population, and once an infeasible one is generated, we consider it as inferior to its previous solution and throw it away.

3.4. Proposed algorithm

The pseudo-code of the proposed AHABC algorithm is given in Fig. 4.

There are three commonly used control parameters in the standard ABC: the number of food sources, which is equal to the number of employed or onlooker bees (*SN*); the value of *limit*, which can be obtained from the formula $limit = SN*D$ [27], where *D* is dimension of the optimization problem; and the maximum cycle number (*MCN*).

In the AHABC algorithm, one iteration cycle incorporates iterations of the Nelder-Mead local search. Instead of separately setting the iteration numbers for ABC and Nelder-Mead local search, we use only one parameter of maximum number of function evaluations (*MFE*), totaling all the ABC and Nelder-Mead local search function evaluations. The number of function evaluations for Nelder-Mead local search is set by Eq. (17), which is adaptively tuned according to the search process.

There are three stop criteria employed in the AHABC algorithm:

1) Maximum number of function evaluations (*MFE*).
2) The global best solution is the same for *maxBestTrial* times. In this case, the iteration number in which the best solution has been found is used.
3) The global best objective function value in two consecutive iterations are different, but such a difference is less than a predefined tolerance ε.

4. Numerical experiments

The proposed AHABC was coded in MATLAB environment and simulation experiments were conducted to evaluate its performance. The experimental settings (ES) cover different behaviors of the q-Weibull hazard rate for reliability applications, as they involve different value combinations of the shape parameters q and β. Note that for all ES, $\eta = 5$. Table 1 shows the ES, the q and β values as well as the corresponding hazard rate function behavior.

Sample sizes of 100, 500 and 1000 are taken into consideration. Samples for ES-A, B, D and E were generated by Eq. (8), whereas ES-C samples were directly drawn from the inverse transform of the Exponential cumulative distribution [47]. The parameters' values used in the AHABC simulation experiments are shown in Table 2.

The initial intervals for q, β and η are set to [−10, 1.9], [0.1, 10], [0.1,t_{mean}], respectively, where t_{mean} is the mean of the sample. The initial population of *SN* solutions is randomly generated between these

Table 1
Experimental settings.

ES	q	β	Behavior of hazard rate function
A	0.5	0.5	Bathtub-shaped
B	1.5	0.5	Decreasing
C	1	1	Constant
D	0.5	1.5	Increasing
E	1.5	1.5	Unimodal

Table 2
AHABC parameters.

Part of AHABC approach	Parameter	Value
ABC	SN	50
	$limit$	150
	MFE	200,000
	$maxBestTrial$	1000
	ε	1e−16
Nelder-Mead simplex method	α	1
	γ	2
	ρ	−0.5
	δ	0.5
Adaptive hybrid coefficient	C	1

intervals. In the initialization, we also adopt the "throw away" method to ensure that all the initial solutions are feasible.

4.1. Effect of parameter C on AHABC

The effect of parameter C on AHABC is tested on ES-A with sample size n=100. Parameter C is set to 0, 0.5, 1, 1.5, 2, 25, and 125. To assess the convergence performance of AHABC, we take the difference between the objective function value $\mathcal{L}(\underline{t}|\hat{\eta}, \hat{\beta}, \hat{q})$ and the true optimum value as the convergence performance. Since the true parameters of the sample are unknown, we take the best objective function value $\max\{\mathcal{L}(\underline{t}|\hat{\eta}, \hat{\beta}, \hat{q})\}$ found among 30 replication runs as the true optimum value. The mean and the standard deviations of this difference for 30 replication runs are shown in Fig. 5 and Fig. 6, respectively.

The results reveal that a proper value of C can improve the performance of AHABC by providing faster convergence and more accurate solutions. It is observed that both for $C = 1$ and $C = 125$, satisfactory convergence can be obtained. For the sake of simplicity, $C = 1$ is adopted in the subsequent experiments. Thus, Eq. (17) for the number of function evaluations for local search can be simplified to $NS = limit*\text{total number of } scout\ \text{bees}$.

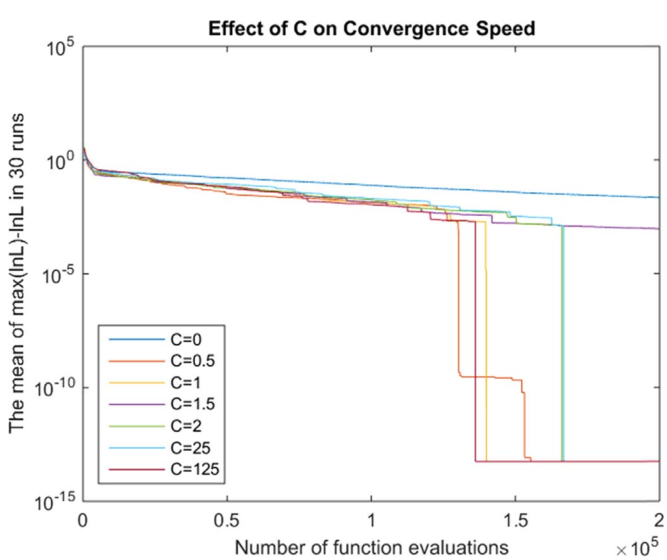

Fig. 5. Effect of C on convergence speed.

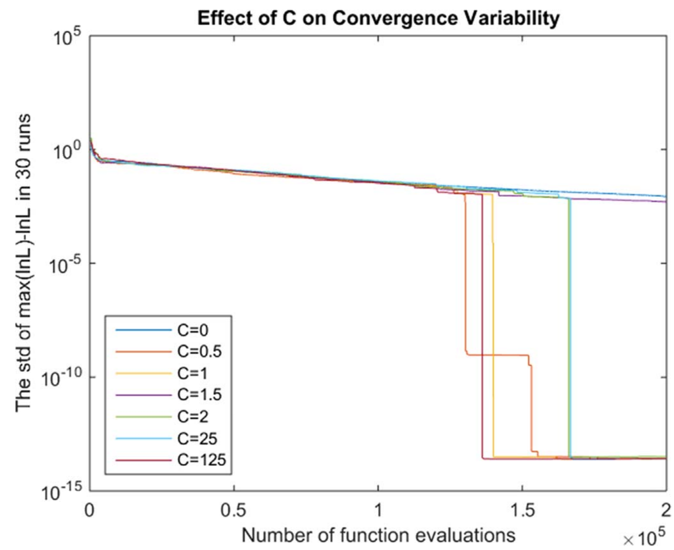

Fig. 6. Effect of C on convergence variability.

4.2. Comparison with ABC and SAHABC

The proposed AHABC algorithm is compared with the standard ABC and with SAHABC for the q-Weibull ML problem in terms of variability and convergence speed. AHABC uses the same parameters given in Table 2, ABC is fed with the parameters' values of the ABC part also shown in Table 2 and SAHABC uses the parameters provided in Table 3. The algorithms are replicated 30 times for each sample (with $n = 100,500,1000$) and ES (A, B, C, D, E), which yields 15 different scenarios. The mean and standard deviations of ML estimates for parameters q, β, η, as well as log-likelihood function $\mathcal{L}$ over 30 runs are shown in Table 4.

For a given sample size and an ES, AHABC can provide accurate estimates for the parameters and for the log-likelihood. Indeed, as we can see in Table 4, all the standard deviations for parameters estimates are in the order of 10^{-6} or less, and for the log-likelihood in the order of 10^{-12} or less. The mean values of the parameter estimates are close to the true values of the q-Weibull distribution shown in Table 1.

By comparing the results from AHABC, ABC and SAHABC in Table 4, the best result for each scenario is highlighted in bold, and it is clear that most of the standard deviations for parameters and for the log-likelihood by AHABC are smaller than those provided by ABC and SAHABC algorithms. These results indicate that AHABC can give more accurate estimates than both ABC and SAHABC. We also compare the convergence speed, for ES-A and E (see Fig. 7 and Fig. 8). AHABC converges faster than ABC and SAHABC in both cases. Therefore, one can expect the proposed AHABC to be more efficient and to provide better solutions than ABC and SAHABC for the q-Weibull ML optimization problem.

Table 3
SAHABC parameters.

Parameters	Values
SN	50
$limit$	150
MCN	2000
$maxBestTrial$	1000
ε	1e−16

Table 4
ML estimates for 30 replications of AHABC, ABC and SAHABC.

Sample size	ES	Statistic	AHABC		ABC		SAHABC	
			Mean	Std.	Mean	Std.	Mean	Std.
n=100	A	$\hat{q}$	**0.4700**	4.65E−08	0.5616	0.0471	0.4681	0.0112
		$\hat{\beta}$	**0.5497**	7.96E−09	0.5642	0.0069	0.5494	0.0017
		$\hat{\eta}$	**5.4926**	5.16E−07	4.5407	0.5390	5.5153	0.1304
		$\mathcal{L}$	**−158.2313**	2.42E−14	−158.2537	0.0066	−158.2317	0.0004
	B	$\hat{q}$	**1.4236**	1.36E−08	**1.4236**	4.58E−08	**1.4236**	2.8624e−08
		$\hat{\beta}$	**0.4556**	6.83E−09	**0.4556**	2.39E−08	**0.4556**	1.8592e−08
		$\hat{\eta}$	**5.9228**	5.95E−07	**5.9229**	1.62E−06	**5.9228**	1.2351e−06
		$\mathcal{L}$	**−531.0407**	3.00E−13	**−531.0407**	3.17E−13	**−531.0407**	1.3352e−13
	C	$\hat{q}$	**0.9926**	2.27E−08	**0.9928**	4.47E−05	0.9942	0.0041
		$\hat{\beta}$	**0.9735**	1.46E−08	**0.9736**	2.86E−05	0.9743	0.0026
		$\hat{\eta}$	**4.9522**	2.01E−07	**4.9508**	0.0004	4.9376	0.0393
		$\mathcal{L}$	**−259.5912**	4.95E−14	**−259.5912**	1.98E−07	−259.5916	0.0005
	D	$\hat{q}$	**0.5583**	2.65E−08	0.5711	0.0060	0.5549	0.0186
		$\hat{\beta}$	**1.4949**	1.96E−08	1.5013	0.0035	1.4932	0.0101
		$\hat{\eta}$	**4.8668**	1.03E−07	4.8141	0.0253	4.8817	0.0782
		$\mathcal{L}$	**−186.6367**	1.11E−13	−186.6379	0.0004	−186.6394	0.0050
	E	$\hat{q}$	**1.5853**	9.78E−09	**1.5853**	4.38E−08	**1.5853**	2.4355e−06
		$\hat{\beta}$	**1.6157**	3.21E−08	**1.6157**	1.31E−07	**1.6157**	6.4441e−06
		$\hat{\eta}$	**4.4819**	1.16E−07	**4.4819**	5.02E−07	**4.4819**	2.6417e−05
		$\mathcal{L}$	**−387.1655**	6.06E−14	**−387.1655**	2.05E−13	**−387.1655**	1.9181e−09
n=500	A	$\hat{q}$	**0.5338**	3.35E−08	0.5474	0.0021	0.5328	0.0148
		$\hat{\beta}$	**0.5115**	7.43E−09	0.5139	0.0004	0.5114	0.0028
		$\hat{\eta}$	**4.2319**	3.58E−07	4.0921	0.0193	4.2467	0.1491
		$\mathcal{L}$	**−677.2932**	1.48E−13	−677.2990	0.0014	−677.3011	0.0083
	B	$\hat{q}$	**1.4665**	2.08E−08	**1.4665**	4.90E−08	**1.4665**	4.0271e−08
		$\hat{\beta}$	**0.4790**	1.79E−08	**0.4790**	3.71E−08	**0.4790**	3.1588e−08
		$\hat{\eta}$	**5.9459**	8.30E−07	**5.9459**	2.11E−06	**5.9459**	1.2849e−06
		$\mathcal{L}$	**−2807.6055**	6.03E−13	**−2807.6055**	2.57E−12	**−2807.6055**	1.3695e−12
	C	$\hat{q}$	**1.0429**	1.06E−07	**1.0429**	5.65E−07	1.0427	0.0011
		$\hat{\beta}$	**1.0509**	6.27E−08	**1.0509**	4.61E−07	1.0508	0.0009
		$\hat{\eta}$	**4.6741**	9.26E−07	**4.6741**	4.93E−06	4.6767	0.0089
		$\mathcal{L}$	**−1302.9980**	7.24E−12	**−1302.9980**	2.40E−10	−1302.9982	0.0004
	D	$\hat{q}$	**0.4999**	3.44E−08	0.5072	0.0009	0.4972	0.0222
		$\hat{\beta}$	**1.5091**	2.66E−08	1.5131	0.0005	1.5081	0.0120
		$\hat{\eta}$	**5.0255**	1.34E−07	4.9949	0.0038	5.0391	0.0929
		$\mathcal{L}$	**−924.4560**	5.39E−13	−924.4581	0.0005	−924.4776	0.0236
	E	$\hat{q}$	**1.5044**	1.38E−08	**1.5044**	1.34E−07	**1.5043**	9.2122e−05
		$\hat{\beta}$	**1.5687**	3.98E−08	**1.5687**	3.01E−07	**1.5687**	0.0002
		$\hat{\eta}$	**5.2296**	1.80E−07	**5.2296**	1.51E−06	**5.2298**	0.0008
		$\mathcal{L}$	**−1831.8860**	9.76E−13	**−1831.8860**	3.19E−12	**−1831.8860**	1.5732e−05
n=1000	A	$\hat{q}$	**0.5519**	3.31E−08	0.5582	0.0020	0.5509	0.0109
		$\hat{\beta}$	**0.5048**	7.87E−09	0.5059	0.0004	0.5045	0.0024
		$\hat{\eta}$	**4.5260**	4.09E−07	4.4546	0.0224	4.5408	0.1242
		$\mathcal{L}$	**−1438.8339**	7.67E−13	−1438.8371	0.0009	−1438.8471	0.0208
	B	$\hat{q}$	**1.5040**	2.31E−08	**1.5040**	4.07E−08	**1.5040**	4.7864e−08
		$\hat{\beta}$	**0.5035**	2.28E−08	**0.5035**	3.28E−08	**0.5035**	4.7886e−08
		$\hat{\eta}$	**4.5311**	7.36E−07	**4.5311**	1.22E−06	**4.5311**	1.4466e−06
		$\mathcal{L}$	**−5616.3389**	1.51E−12	**−5616.3389**	6.27E−12	**−5616.3389**	5.8845e−12
	C	$\hat{q}$	**0.9919**	8.42E−07	**0.9919**	2.85E−06	0.9915	0.0027
		$\hat{\beta}$	**0.9442**	5.25E−07	**0.9442**	1.72E−06	0.9441	0.0014
		$\hat{\eta}$	**4.5989**	7.78E−06	**4.5989**	2.52E−05	4.6028	0.0234
		$\mathcal{L}$	**−2532.4055**	8.50E−10	**−2532.4055**	8.82E−09	−2532.4074	0.0032
	D	$\hat{q}$	**0.5062**	3.18E−08	0.5132	0.0008	0.5100	0.0193
		$\hat{\beta}$	**1.5061**	2.57E−08	1.5099	0.0004	1.5083	0.0099
		$\hat{\eta}$	**4.9962**	1.39E−07	4.9669	0.0031	4.9793	0.0814
		$\mathcal{L}$	**−1848.9886**	1.17E−12	−1848.9924	0.0008	−1849.0244	0.0532
	E	$\hat{q}$	**1.5134**	1.64E−08	**1.5134**	1.09E−07	**1.5134**	3.0101e−05
		$\hat{\beta}$	**1.5076**	4.18E−08	**1.5076**	2.66E−07	**1.5077**	9.6409e−05
		$\hat{\eta}$	**4.7017**	2.32E−07	**4.7017**	1.08E−06	**4.7016**	0.0002
		$\mathcal{L}$	**−3657.4987**	9.67E−13	**−3657.4987**	3.71E−12	**−3657.4987**	4.5104e−06

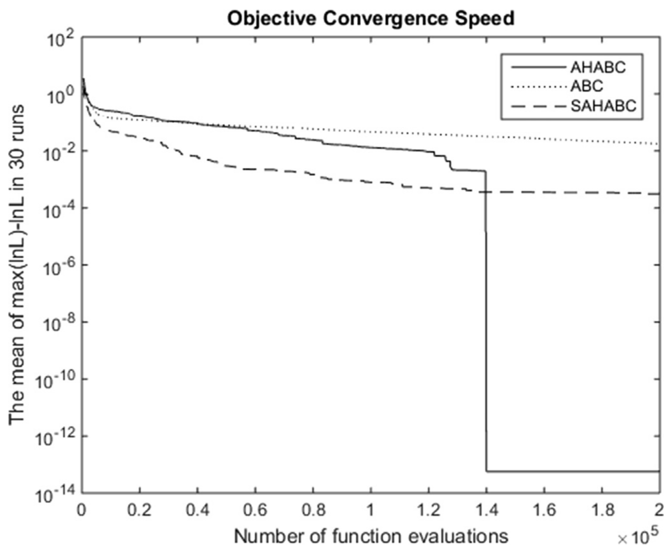

Fig. 7. Convergence comparison of AHABC, ABC and SAHABC for ES-A, n=100.

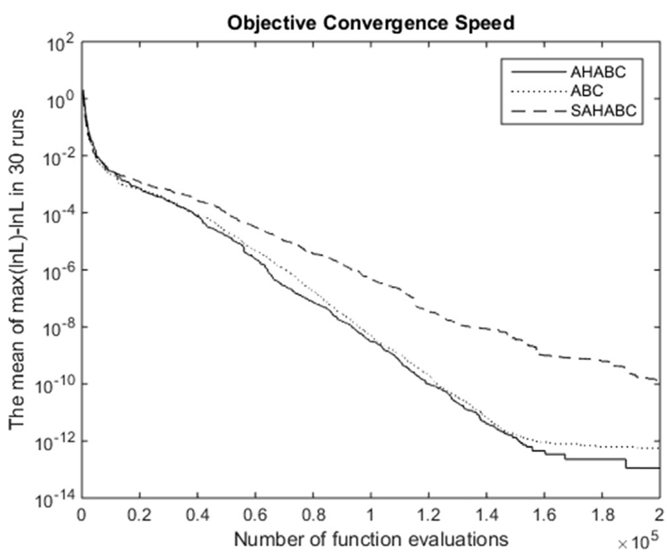

Fig. 8. Convergence comparison of AHABC, ABC and SAHABC for ES-E, n=100.

4.3. Bias and mean squared error

We also used the bias and MSE as additional criteria to evaluate the quality of the ML estimators via AHABC. For this purpose, we generate 1000 samples for each ES-A, B, C, D and E for each sample size $n = 100,500,1000$. Then, AHABC algorithm was executed once for each sample. For each scenario, we have 1000 estimates for parameters q, β, η. Bias and MSE are computed according to the following equations:

$$bias(\hat{\theta},\ \theta)=\frac{1}{m}(\sum_{i=1}^{m}\hat{\theta}_i)-\theta,$$

$$MSE(\hat{\theta})=var(\hat{\theta})+[bias(\hat{\theta},\ \theta)]^2,$$

with $m = 1000$, $\hat{\theta}=\hat{q},\ \hat{\beta}$ or $\hat{\eta}$, and $var(\hat{\theta})$ as the variance of the 1000 estimates.

Results of bias and MSE are shown in Table 5 and Table 6 respectively. From these results, for larger sample sizes n=500, and 1000, bias and MSE are very small for the q-Weibull parameters' estimates. Thus, AHABC is able to provide accurate and precise estimates for the q-Weibull parameters.

Table 5
Bias of ML estimates for q-Weibull parameters.

ES	Statistic	n=100	n=500	n=1000
A	$\hat{q}$	−0.2802	−0.0431	−0.0227
	$\hat{\beta}$	−0.0059	−0.0018	−0.0010
	$\hat{\eta}$	11.6693	0.7646	0.3583
B	$\hat{q}$	−0.0087	−0.0035	−0.0036
	$\hat{\beta}$	0.0173	0.0021	0.0002
	$\hat{\eta}$	1.9045	0.3734	0.2226
C	$\hat{q}$	−0.0812	−0.0157	−0.0086
	$\hat{\beta}$	0.0044	−0.0018	−0.0023
	$\hat{\eta}$	1.0719	0.1756	0.0909
D	$\hat{q}$	−0.2689	−0.0496	−0.0278
	$\hat{\beta}$	−0.0171	−0.0080	−0.0046
	$\hat{\eta}$	1.1526	0.2032	0.1161
E	$\hat{q}$	−0.0078	0.0003	−0.0026
	$\hat{\beta}$	0.0518	0.0159	0.0016
	$\hat{\eta}$	0.2154	0.0420	0.0452

Table 6
MSE of ML estimates for q-Weibull parameters.

ES	Statistic	n=100	n=500	n=1000
A	$\hat{q}$	0.4205	0.0214	0.0086
	$\hat{\beta}$	0.0046	0.0008	0.0004
	$\hat{\eta}$	1771.1883	5.1727	1.6128
B	$\hat{q}$	0.0099	0.0017	0.0010
	$\hat{\beta}$	0.0079	0.0012	0.0007
	$\hat{\eta}$	42.7344	2.6959	1.4148
C	$\hat{q}$	0.0689	0.0064	0.0029
	$\hat{\beta}$	0.0208	0.0032	0.0016
	$\hat{\eta}$	10.4633	0.6140	0.2621
D	$\hat{q}$	0.4851	0.0226	0.0085
	$\hat{\beta}$	0.0383	0.0071	0.0031
	$\hat{\eta}$	9.8914	0.3953	0.1515
E	$\hat{q}$	0.0105	0.0018	0.0009
	$\hat{\beta}$	0.0770	0.0119	0.0058
	$\hat{\eta}$	1.7203	0.2723	0.1409

4.4. Confidence intervals

In order to construct confidence intervals for the parameters of the q-Weibull distribution, asymptotic confidence intervals (ACI), parametric bootstrap confidence intervals (BCI-P) and non-parametric bootstrap confidence intervals (BCI-NP) are developed (see Appendix). Once again, $n = 100,500,1000$ and ES-A, B, C, D and E. For all bootstrap experiments, $B = 999$. For BCI-P sampling, the estimates obtained from the first sample are used as q-Weibull parameters to generate B bootstrap samples by Eq. (8). For BCI-NP sampling, in turn, the first sample is used to generate B bootstrap samples by resampling with replacement. Then, AHABC is applied to each sample to compute ML estimates. The 5th and 95th percentiles are obtained to construct the corresponding 90% confidence interval. The resulting confidence intervals for parameters η, β, q are presented in Table 7–9 for sample sizes $n = 100,500,1000$ respectively. The values in parentheses are the corresponding interval lengths.

From the results, it can be observed that all intervals contain the true values of parameters η, β, q. For larger sample sizes, asymptotic and bootstrap approaches tend to provide similar and more accurate interval estimates for the q-Weibull parameters. Note also that for the experimental settings A and B with n =100 (Table 7), ACI provided negative lower bounds related to parameter eta. In spite of being infeasible values for this parameter, the asymptotic approach does not guarantee valid bounds and their results become more accurate and precise as the sample size increases.

Table 7
Interval estimates for the parameters, n=100.

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$	−4.240	2.413	1.200	0.406	0.431	0.396	−0.378	−1.028	−4.782
	$\beta = 0.5$	15.226	55.818	837.669	0.693	0.676	0.734	1.318	0.819	1.066
	$q = 0.5$	(19.466)	(53.405)	(836.469)	(0.287)	(0.245)	(0.338)	(1.696)	(1.846)	(5.849)
B	$\eta = 5$	−0.607	2.148	2.207	0.353	0.358	0.377	1.267	1.189	1.241
	$\beta = 0.5$	12.453	23.876	19.314	0.558	0.607	0.565	1.580	1.575	1.550
	$q = 1.5$	(13.060)	(21.728)	(17.107)	(0.205)	(0.249)	(0.188)	(0.313)	(0.386)	(0.309)
C	$\eta = 5$	1.967	3.092	2.863	0.749	0.777	0.741	0.669	0.461	0.525
	$\beta = 1$	7.938	11.048	10.877	1.198	1.222	1.342	1.316	1.218	1.278
	$q = 1$	(5.971)	(7.956)	(8.014)	(0.450)	(0.445)	(0.601)	(0.646)	(0.757)	(0.752)
D	$\eta = 5$	2.871	3.555	3.392	1.184	1.202	1.209	0.077	−0.523	−0.871
	$\beta = 1.5$	6.863	9.793	11.176	1.805	1.827	1.813	1.040	0.893	0.907
	$q = 0.5$	(3.992)	(6.238)	(7.784)	(0.621)	(0.625)	(0.604)	(0.962)	(1.415)	(1.777)
E	$\eta = 5$	2.910	3.158	3.194	1.191	1.268	1.277	1.457	1.428	1.417
	$\beta = 1.5$	6.054	6.665	6.811	2.041	2.284	2.170	1.714	1.718	1.699
	$q = 1.5$	(3.144)	(3.507)	(3.617)	(0.850)	(1.017)	(0.893)	(0.257)	(0.290)	(0.283)

Based on the validation results presented in this section, the AHABC can provide accurate estimates for the q-Weibull parameters for all the ES-A, B, C, D and E covering different behaviors of the q-Weibull hazard rate. Therefore, with the proposed AHABC, the q-Weibull distribution is used to tackle a real reliability problem in the next section.

5. Application example

In this section, the proposed procedure to obtain the ML estimates of the q-Weibull parameters is illustrated by means of one application example involving lifetime data of engineering devices in reliability studies. The example deals with a data set of the time to first failure of 500 MW generators [48] that results in a bathtub-shaped hazard rate. For the data with non-monotonic hazard rate, commonly used distributions like Weibull are barely suitable to fit the failure data. Thus, the use of the q-Weibull illustrates the ability of this distribution in dealing with non-monotonic hazard rate function, which encompasses a set of problems with relevant applications in the reliability context [49,50].

Table 10 shows the time to first failure for a group of 36 generators of 500 MW [48]. The AHABC is replicated 30 times and the estimated ML parameters and the associated standard deviations are shown in Table 11.

To check the goodness-of-fit, we use the Kolmogorov-Smirnov (KS) test, which compares the empirical and the cumulative distribution function (CDF). However, the traditional KS test is not applicable to our situation, where the parameters of the theoretical distribution have been estimated from the same data used to apply this goodness-of-fit test [51]. Therefore, a bootstrapped version of the KS test [52] has been developed and applied in this paper. The KS test statistic is computed as follows:

$$D^0 = max\{|F_n(t_i) - F(t_i|\hat{q},\hat{\beta},\hat{\eta})|, |F_n(t_{i-1}) - F(t_i|\hat{q},\hat{\beta},\hat{\eta})|\},$$

where $F_n(t_i)=i/n$ is the empirical CDF and $F(t_0)=0$, $F(t_i|\hat{q},\ \hat{\beta},\ \hat{\eta})$ is the theoretical CDF with estimated parameters. B bootstrap samples $\underline{t}^j = \{t_1^j,\ t_2^j,\ \dots,\ t_n^j\},\ j = 1, 2, \dots, B$ are generated using Eq. (8) with $\hat{q},\ \hat{\beta},\ \hat{\eta}$. The ML estimates $\hat{q}^j,\ \hat{\beta}^j,\ \hat{\eta}^j$ for the j^{th} sample are obtained by the proposed AHABC. The test statistic D^j is computed with $F(t_i^j|\hat{q}^j,\ \hat{\beta}^j,\ \hat{\eta}^j)$ in place of $F(t_i|\hat{q},\ \hat{\beta},\ \hat{\eta})$. Then, we get B+1 observations of the KS test statistic D. The p-value is computed as the number of observations where D^j exceeds D^0divided by B+1.

For comparison purpose, we consider the standard Weibull and some alternative bathtub-shaped hazard rate models: the modified Weibull extension [14] and the ENH [18] models, as shown in Table 12. We then apply the proposed AHABC procedure to obtain the ML estimates of the parameters not only for the q-Weibull but also for the modified Weibull extension and the ENH models. The fitted parameters and log-likelihoods are given in Table 13, which also gives the KS test statistic D^0 and p-value. Fig. 9 presents the empirical and

Table 8
Interval estimates for the parameters, n=500.

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$	2.002	2.853	2.376	0.464	0.467	0.462	0.320	0.269	0.250
	$\beta = 0.5$	6.462	7.701	8.090	0.559	0.556	0.570	0.748	0.683	`0.746
	$q = 0.5$	(4.460)	(4.848	(5.715)	(0.095)	(0.090)	(0.108)	(0.429)	(0.414)	(0.496)
B	$\eta = 5$	3.074	3.712	3.642	0.428	0.429	0.437	1.401	1.381	1.393
	$\beta = 0.5$	8.817	10.403	9.736	0.530	0.535	0.535	1.534	1.529	1.529
	$q = 1.5$	(5.744)	(6.691	(6.094)	(0.102)	(0.107)	(0.098)	(0.135)	(0.148)	(0.136)
C	$\eta = 5$	3.727	3.889	3.862	0.955	0.953	0.952	0.933	0.906	0.892
	$\beta = 1$	5.622	6.003	6.075	1.147	1.152	1.155	1.152	1.136	1.143
	$q = 1$	(1.895)	(2.115)	(2.213)	(0.192)	(0.199)	(0.203)	(0.219)	(0.230)	(0.252)
D	$\eta = 5$	4.243	4.347	4.371	1.377	1.365	1.373	0.314	0.196	0.249
	$\beta = 1.5$	5.809	6.293	6.084	1.641	1.655	1.642	0.686	0.664	0.656
	$q = 0.5$	(1.566)	(1.946)	(1.712)	(0.264)	(0.289)	(0.269)	(0.373)	(0.467)	(0.406)
E	$\eta = 5$	4.385	4.515	4.450	1.387	1.409	1.404	1.435	1.433	1.430
	$\beta = 1.5$	6.074	6.186	6.255	1.751	1.777	1.785	1.574	1.569	1.573
	$q = 1.5$	(1.689)	(1.671)	(1.805)	(0.364)	(0.367)	(0.381)	(0.139)	(0.136)	(0.143)

Table 9
Interval estimates for the parameters, n=1000.

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$	2.951	3.456	3.167	0.473	0.472	0.467	0.416	0.390	0.389
	$\beta = 0.5$	6.101	6.750	6.869	0.537	0.537	0.546	0.688	0.654	0.682
	$q = 0.5$	(3.150)	(3.294)	(3.702)	(0.064)	(0.065)	(0.079)	(0.272)	(0.265)	(0.293)
B	$\eta = 5$	2.957	3.215	3.341	0.463	0.466	0.464	1.456	1.452	1.455
	$\beta = 0.5$	6.105	6.650	6.423	0.544	0.547	0.549	1.552	1.550	1.549
	$q = 1.5$	(3.148)	(3.436)	(3.082)	(0.082)	(0.081)	(0.085)	(0.096)	(0.098)	(0.093)
C	$\eta = 5$	3.855	3.932	3.933	0.884	0.883	0.882	0.909	0.891	0.877
	$\beta = 1$	5.343	5.549	5.707	1.005	1.008	1.010	1.074	1.070	1.070
	$q = 1$	(1.488)	(1.617)	(1.774)	(0.121)	(0.126)	(0.129)	(0.165)	(0.179)	(0.193)
D	$\eta = 5$	4.435	4.530	4.556	1.412	1.406	1.413	0.372	0.328	0.379
	$\beta = 1.5$	5.557	5.729	5.536	1.600	1.604	1.595	0.640	0.619	0.614
	$q = 0.5$	(1.121)	(1.199)	(0.981)	(0.188)	(0.199)	(0.182)	(0.268)	(0.292)	(0.235)
E	$\eta = 5$	4.154	4.229	4.159	1.386	1.397	1.391	1.466	1.467	1.462
	$\beta = 1.5$	5.250	5.235	5.349	1.629	1.637	1.639	1.560	1.555	1.562
	$q = 1.5$	(1.096)	(1.005)	(1.190)	(0.243)	(0.240)	(0.248)	(0.094)	(0.088)	(0.0998)

Table 10
Time to first failure (1000′s of hours) of 500 MW generators.

0.058	0.070	0.090	0.105	0.113	0.121	0.153	0.159
0.224	0.421	0.570	0.596	0.618	0.834	1.019	1.104
1.497	2.027	2.234	2.372	2.433	2.505	2.690	2.877
2.879	3.166	3.455	3.551	4.378	4.872	5.085	5.272
5.341	8.952	9.188	11.399				

Table 11
ML estimates for 30 replications of AHABC.

	Mean	Std.
$\hat{q}$	0.4318	2.5555e−08
$\hat{\beta}$	0.6697	4.8570e−09
$\hat{\eta}$	6.6087	2.9609e−07
$\mathcal{L}$	−68.0595	1.4211e−14

Table 12
Some bathtub-shaped hazard rate models.

Model	$h(t)$	Parameters
Modified Weibull Extension	$\lambda\beta(t/\alpha)^{(\beta-1)}\exp\,[(t/\alpha)^{\beta}]$	$\alpha,\,\beta,\,\lambda>0$
ENH	$\alpha\beta\lambda\frac{(1+\lambda t)^{\alpha-1}\exp[1-(1+\lambda t)^{\alpha}]\{1-\exp[1-(1+\lambda t)^{\alpha}]\}^{\beta-1}}{1-\{1-\exp\,[1-(1+\lambda t)^{\alpha}]\}^{\beta}}$	$\alpha,\,\beta,\,\lambda>0$

Table 13
Results for the example.

Model	ML estimates	log$L$	D^0	p
q-Weibull	$\hat{q}$=0. 4318, $\hat{\beta}$=0. 6697, $\hat{\eta}$=6. 6087	−68.0595	0.0983	0.5080
Weibull	$\hat{\beta}$=0.8156, $\hat{\eta}$=2. 3118	−68.6906	0.1219	0.1880
Modified Weibull Extension	$\hat{\alpha}$=10. 0923, $\hat{\beta}$=0. 6920, $\hat{\lambda}$=0. 2130	−68.2628	0.1046	0.2900
ENH	$\hat{\alpha}$=1. 6347, $\hat{\beta}$=0. 6415, $\hat{\lambda}$=0. 1430	−68.3560	0.1021	0.3330

fitted CDFs for the example and Fig. 10 shows the hazard rate functions. Note that except for the standard Weibull that models the data as decreasing hazard rate, all the other models result in a bathtub-shaped hazard rate, which has been also observed by Bebbington et al. [49].

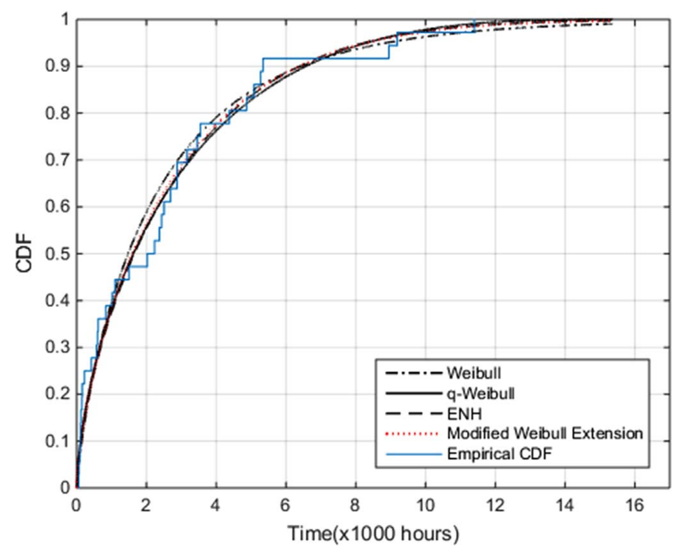

Fig. 9. Empirical and fitted CDFs.

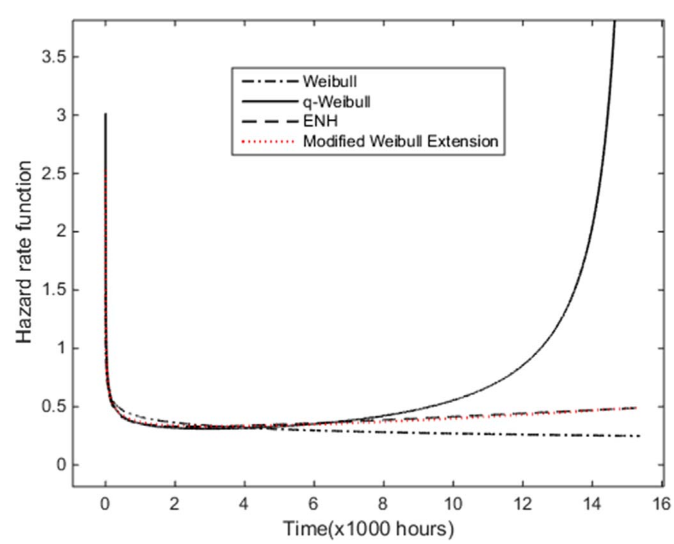

Fig. 10. Hazard rate functions.

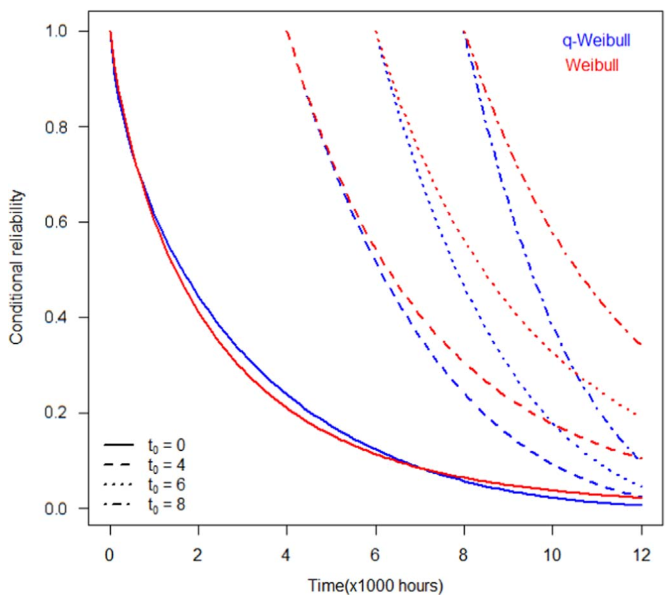

Fig. 11. Conditional reliability of q-Weibull and Weibull.

Compared to the standard Weibull, q-Weibull is more flexible to perform reliability analyses when failure data are characterized by non-monotonic hazard rates. Moreover, with the low KS test statistic and high p-value (see Table 13), the q-Weibull distribution is a good alternative to the other bathtub-shaped hazard rate models, namely the modified Weibull extension and the ENH.

For the sake of comparison, the estimates of q-Weibull and Weibull parameters shown in Table 13 are used for obtaining the conditional reliability (Eq. (9)) as shown in Fig. 11. Note that as t_0 increases, the Weibull provides higher conditional reliability, which is in accordance with the decreasing behavior of the hazard rate resulting from the application of the Weibull to this data set. On the other hand, the q-Weibull conditional reliability decreases more rapidly as t_0 increases. Given the nature of the reliability data set in Table 10, one can argue that the Weibull model results in an optimistic performance of the generators when compared to the q-Weibull distribution.

Note that these results are representative of the failure data set in Table 10, and different outcomes might be obtained for different sets of reliability data. However, based on the experimental results discussed in the previous section and the ones from this application example, the q-Weibull distribution is a flexible and capable model that might be considered as one more alternative distribution when engineers are faced with modeling of reliability data sets.

6. Conclusion

The q-Weibull distribution is able to describe various behaviors of the hazard rate - monotonically decreasing, monotonically increasing, constant, unimodal and bathtub-shaped - with a single set of parameters. This flexibility provided by the q-Weibull probabilistic model is important to describe accurately failure data characterized by both monotonic and non-monotonic hazard rate functions. Although there are other 3-parameter distributions with that flexibility (e.g. modified Weibull extension [14], ENH [18]), the q-Weibull distribution constitutes another alternative to the arsenal of options available for the reliability analyst.

However, it is impractical to analytically obtain the ML estimates for the q-Weibull parameters, and also the classic numerical optimization approach fails to efficiently find the global solution for the associated ML problem. Thus, the q-Weibull distribution is a flexible and useful distribution in the context of reliability engineering as it allows for the modeling and analysis of a variety of failure data behaviors, in particular data with non-monotonic hazard rate functions. However, its intricate likelihood function imposes significant numerical difficulties in estimating its parameters, which has limited the number of applications of this distribution so far.

In this paper, an adaptive hybrid artificial bee colony (AHABC) algorithm has been proposed to tackle this problem, which combines the global exploration of ABC and the local exploitation of Nelder-Mead simplex search. The exploitation ability of Nelder-Mead improves the local search performance of ABC.

Numerical results show that the proposed AHABC algorithm efficiently finds the optimal solution for the q-Weibull ML problem, comprising different behaviors of the hazard rate and sample sizes. The ML estimates of the q-Weibull parameters obtained via AHABC are accurate and precise with small bias and MSE. Using the proposed AHABC algorithm, intervals estimates for the q-Weibull parameters are provided, including asymptotic intervals based on the ML theory, parametric and non-parametric bootstrapped confidence intervals. Based on the results presented in Section 4, the proposed AHABC outperformed both ABC and similar algorithm in terms of accuracy and convergence speed in the context of the maximum likelihood problem for the q-Weibull distribution. The proposed method for the ML constrained q-Weibull problem was also applied to an example involving failure data characterized by bathtub-shaped hazard rate function.

To conclude, the proposed AHABC for parameter estimation showed that the q-Weibull is a promising alternative distribution for reliability modeling and constitutes in another distribution model in the reliability engineers' toolbox. With AHABC in hand, a wider range of reliability problems can incorporate the q-Weibull model such as stress-strength analysis [41], optimal preventive maintenance policies [42,43], optimal system design [44], competitive risks [45].

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Appendix

Confidence intervals for the q-Weibull distribution parameters, including asymptotic confidence intervals (ACI), parametric bootstrap confidence intervals (BCI-P) and non-parametric bootstrap confidence intervals (BCI-NP) are developed as follows.

Asymptotic confidence intervals

The related covariance matrix associated to the ML estimators for the q-Weibull distribution parameters can be estimated by the inverse of the observed information matrix $I(\underline{t}|\hat{\eta}, \hat{\beta}, \hat{q})$

$$\widehat{\mathrm{var}}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q}) = \mathrm{I}^{-1}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q}) = -\frac{1}{\nabla^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}$$

$$=-\begin{bmatrix} \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta^2} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta\partial\beta} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta\partial q} \\ \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta\partial\eta} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta^2} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta\partial q} \\ \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial q\partial\eta} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial q\partial\beta} & \frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial q^2} \end{bmatrix}^{-1},$$

whose entries are

$$\frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta^2} = \frac{\hat{\beta}}{\hat{\eta}^2}\left\{n - \sum_{i=1}^{n}\frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}}\right\} - \hat{\beta}^2\hat{\eta}^{\hat{\beta}-2}\sum_{i=1}^{n}\frac{1}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2}$$

$$\frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta\partial\beta} = \frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta\partial\eta} = -\frac{1}{\hat{\eta}}\left\{n - \sum_{i=1}^{n}\frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}}\right\} - \hat{\beta}\hat{\eta}^{\hat{\beta}-1}\sum_{i=1}^{n}\frac{\ln\left(\frac{\hat{\eta}}{t_i}\right)}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2},$$

$$\frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\eta\partial q} = \frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial q\partial\eta} = -\frac{\hat{\beta}}{\hat{\eta}}\sum_{i=1}^{n}\frac{1}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2},$$

$$\frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta^2} = -\frac{n}{\hat{\beta}^2} + \hat{\eta}^{\hat{\beta}}\sum_{i=1}^{n}\frac{\ln\left(\frac{\hat{\eta}}{t_i}\right)\ln\left(\frac{t_i}{\hat{\eta}}\right)}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2},$$

$$\frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial\beta\partial q} = \frac{\partial^2\mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial q\partial\beta} = \sum_{i=1}^{n}\frac{\ln\left(\frac{t_i}{\hat{\eta}}\right)}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2},$$

$$\frac{\partial^2\mathcal{L}(\underline{t}\mid\hat{\eta},\hat{\beta},\hat{q})}{\partial q^2} = -\frac{\mathrm{n}}{(2-\hat{q})^2} + \frac{2}{(1-\hat{q})^3}\sum_{i=1}^{n} ln\left[1-(1-\hat{q})\left(\frac{t_i}{\hat{\eta}}\right)^{\hat{\beta}}\right] + \frac{2}{(1-\hat{q})^2}\sum_{i=1}^{n}\frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}} - \frac{1}{1-\hat{q}}\sum_{i=1}^{n}\frac{1}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2}.$$

Once we have this covariance matrix, the asymptotic confidence intervals could be constructed for the q-Weibull distribution parameters. The asymptotic confidence intervals with $(1-\alpha)100\%$ confidence for η, β and q are given below:

$$\mathrm{CI}[\eta:(1-\alpha)100\%]=\left[\hat{\eta}+z_{\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{11}},\ \hat{\eta}+z_{1-\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{11}}\right],$$

$$\mathrm{CI}[\beta:(1-\alpha)100\%]=\left[\hat{\beta}+z_{\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{22}},\ \hat{\beta}+z_{1-\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{22}}\right],$$

$$\mathrm{CI}[q:(1-\alpha)100\%]=\left[\hat{q}+z_{\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{33}},\ \hat{q}+z_{1-\frac{\alpha}{2}}\sqrt{\hat{\mathrm{var}}_{33}}\right],$$

in which $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of the standard normal distribution, and $\hat{\mathrm{var}}_{11}$, $\hat{\mathrm{var}}_{22}$ and $\hat{\mathrm{var}}_{33}$ are the diagonal elements of the covariance matrix.

Parametric and non-parametric bootstrapped confidence intervals

The bootstrap is a computer-based method for assigning measures of accuracy to sample estimates [40]. This technique allows us to generate confidence intervals for the parameters of the q-Weibull distribution by using simple sampling methods so as to infer the precision of the ML estimators.

The bootstrap approaches are classified as parametric and non-parametric depending on how the samples are generated [54]. Given the original data set and the estimates of the parameters obtained from it, parametric and non-parametric bootstrap samples can be generated. For parametric bootstrap, the q-Weibull distribution uses the estimates to generate other B new samples by Eq. (8). For the non-parametric bootstrap, B samples are generated by resampling with replacement from the original data set. Along with the original sample, a total of $B+1$ samples are obtained and we apply the ML method via AHABC to these samples. By sorting the $B+1$ resulting estimates, the $\frac{\alpha}{2}100\%$ and $(1-\frac{\alpha}{2})100\%$ percentiles are set as the lower and upper bounds to construct the confidence intervals with α level of significance.

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