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Matching colored points with rectangles

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Abstract Let S be a point set in the plane such that each of its elements is colored either red or blue. A matching of S with rectangles is any set of pairwise-disjoint axis-aligned closed rectangles such that each rectangle contains exactly two points of S. Such a matching is monochromatic if every rectangle contains points of the same color, and is bichromatic if every rectangle contains points of different colors. We study the following two problems: (1) Find a maximum monochromatic matching of S with rectangles. (2) Find a maximum bichromatic matching of S with rectangles. For each problem we provide a polynomial-time approximation algorithm that constructs a matching with at least $1/4$ of the number of rectangles of an optimal matching. We show that the first problem is NP-hard even if either the matching rectangles are restricted to axis-aligned segments or S is in general position, that is, no two points of S share the same x or y coordinate. We further show that the second problem is also NP-hard, even if S is in general position. These NP-hardness results follow by showing that deciding the existence of a matching that covers all points is NP-complete in each case. Additionally, we prove that it is NP-complete to decide the existence of

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a matching with rectangles that cover all points in the case where all the points have the same color, solving an open problem of Bereg et al. (Comput Geom 42(2):93–108, 2009).

Keywords Computational geometry · Matching colored points · Maximum independent set · Rectangles · Approximations

1 Introduction

Matching points in the plane with geometric objects consists in, given an input point set S and a class $\mathcal{C}$ of geometric objects, to find a collection $M \subseteq \mathcal{C}$ such that each element of M contains exactly two points of S and every point of S lies in at most one element of M. A geometric matching is called *strong* if the geometric objects are disjoint, and *perfect* if every point of S belongs to some element of M. This class of geometric matching problems was considered by Ábrego et al. (2009). They studied the existence and properties of matchings for point sets in the plane when $\mathcal{C}$ is the class of axis-aligned squares, or the class of disks.

A generalization is when $S = R \cup B$ is a set of n colored points in the plane, each one being either red or blue, where R and B are the sets of red and blue points, respectively. In this setting, a matching of S is called *monochromatic* if all matching objects cover points of the same color, and *bichromatic* if all matching objects cover points of different colors. We study both monochromatic and bichromatic strong matchings of S with axis-aligned closed rectangles. Every rectangle in this paper will be considered axis-aligned and closed.

For the monochromatic case, one can build examples in which no matching rectangle exists, and examples in which a perfect strong matching with rectangles exists. For the bichromatic case, there always exists at least one matching rectangle (e.g. match the red point and the blue point such that their minimum enclosing rectangle has minimum area among all combinations of a red point and a blue point). Similar to the monochromatic case, one can build examples in which exactly one matching rectangle exists, and examples in which a perfect strong matching exists. Therefore, we focus on the following two optimization problems:

Maximum Monochromatic Rectangle Matching (MonoMRM) *problem*: *Given $S = R \cup B$, find a monochromatic strong matching of S with the maximum number of rectangles.*

Maximum Bichromatic Rectangle Matching (BicMRM) problem: *Given $S = R \cup B$, find a bichromatic strong matching of S with the maximum number of rectangles.*

Unless otherwise specified, we will consider that the elements of S are not necessarily in general position. We say that S is in *general position* if no two elements of S share the same x or y coordinate.

This work is also motivated by an open question posed by Bereg et al. (2009). They studied the case of the MonoMRM problem where all elements of S have the same color, and showed that every point set of n points in the plane admits a strong matching that matches at least $2\lfloor n/3 \rfloor$ of the points. They proved that, if the point set is not in general position and the matching rectangles are restricted to squares, then it is NP-hard to decide whether a perfect strong matching exists. They left open the computational complexity of finding a maximum strong matching with general rectangles. We show that it is NP-hard to find such a maximum strong matching. Problems similar to the MonoMRM and BicMRM problems have been studied before. Dumitrescu and Steiger (2000) considered strong monochromatic matchings of two-colored point sets in the plane with straight segments. The best results known are due to Dumitrescu and Kaye (2001): Every two-colored point set $S = R \cup B$ of n points admits a strong straight segment matching that matches at least $\frac{6}{7}n - O(1)$ of the points, which can be found in $O(n^2)$ time; furthermore, there exist n-point sets such that every strong matching with straight segments matches at most $\frac{94}{95}n + O(1)$ points. The computational complexity of deciding whether a given two-colored point set admits a perfect monochromatic strong matching with straight segments is still an open problem (Dumitrescu and Steiger 2000). Additionally, it is well known that every point set $S = R \cup B$ in the plane such that $|R| = |B|$, and where no three points are collinear, admits a perfect bichromatic strong matching with straight segments (Larson 1990).

Soto and Telha (2011) considered a special case of the BicMRM problem: the matching rectangles are restricted to have a red point as bottom-left corner and a blue point as top-right corner. They solved it in polynomial time (see Sect. 2 for more details).

Ahn et al. (2011) studied the (p, k)-Rectangle Covering problem: given n points in the plane, find p pairwise-disjoint rectangles that, together, cover at least $n-k$ of the points, while minimizing the area of the largest rectangle. Among other results, they showed that given n points and an input p, deciding whether the points can be covered with p pairwise-disjoint rectangles, each of area at most one, is NP-complete. The arguments used in their proof rely on the fact that rectangles can cover either two or three points, being not straightforward to use them to prove the results we describe here. We show that the MonoMRM problem is NP-hard when the rectangles are restricted to axis-aligned segments, proving that their decision problem is NP-complete.

The MonoMRM and BicMRM problems are special cases of the Maximum Independent Set of Rectangles (MISR) problem, a classical NP-hard problem in computational geometry and combinatorics (Adamaszek and Wiese 2013; Agarwal and Mustafa 2006; Chalermsook 2011, 2009; Fowler et al. 1981; Imai and Asano 1983; Rim and Nakajima 1995). The MISR problem is to find a maximum pairwise-disjoint subset of rectangles in a given set of rectangles. Any α-approximation algorithm for the MISR problem implies an α-approximation algorithm for each of the MonoMRM and BicMRM problems. The general MISR problem admits a polynomial-time approximation algorithm, which with high probability produces an independent set of rectangles with at least $\Omega(\frac{1}{\log\log m})$ times the number of rectangles in an optimal solution, m being the number of rectangles in the input (Chalermsook 2011, 2009). There also exist deterministic polynomial-time $\Omega(\frac{1}{\log m})$-approximation algorithms for the MISR problem (Agarwal et al. 1998; Khanna et al. 1998). However, finding a constant-approximation algorithm, or a PTAS, is still an open question.

Results: For each of the MonoMRM and BicMRM problems, we provide a polynomial-time 1/4-approximation algorithm. We complement the approximation results by showing that the MonoMRM problem is NP-hard, even if either the matching rectangles are restricted to axis-aligned segments or the points are in general position. We further show that the BicMRM problem is also NP-hard, even if the points are in general position. Additionally, we are able to prove that if all elements of S have the same color, then the MonoMRM problem keeps being NP-hard, solving an open question of Bereg et al. (2009). These NP-hardness results follow by showing that deciding the existence of a perfect matching is NP-complete in each case.

Outline: We introduce some notations and definitions in Sect. 2, and analyze in detail the relation between the MonoMRM and BicMRM problems and the MISR problem. We also discuss some previous work on the MISR problem relevant to the results introduced here. In Sect. 3 we present the approximations algorithms mentioned, and complement the approximation results with hardness results in Sect. 4. Finally, we summarize the results in Sect. 5.

2 Preliminaries

For every point p of S, let $x(p)$, $y(p)$, and $c(p)$ denote the x-coordinate, the y-coordinate, and the color of p, respectively. Given two points a and b of the plane with $x(a) < x(b)$, or $x(a) = x(b)$ and $y(a) < y(b)$, let $D(a, b)$ denote the rectangle which has the segment connecting a and b as diagonal, which is in fact the minimum enclosing axis-aligned rectangle of a and b. If a and b are horizontally or vertically aligned, we say that $D(a, b)$ is a *segment*, otherwise we say that $D(a, b)$ is a *box*. We say that $D(a, b)$ is *red* if both a and b are colored red. If both a and b are colored blue, we say that $D(a, b)$ is *blue*. Given S, consider the following two sets of axis-aligned rectangles:

$$\mathcal{R}(S) = \big\{D(p,q) \mid p,q \in S; c(p) = c(q); \text{ and } D(p,q) \cap S = \{p,q\}\big\}$$
$$\overline{\mathcal{R}}(S) = \big\{D(p,q) \mid p,q \in S; c(p) \neq c(q); \text{ and } D(p,q) \cap S = \{p,q\}\big\}$$

Observe that the MonoMRM problem is equivalent to finding a maximum subset of $\mathcal{R}(S)$ of independent rectangles. Two rectangles are *independent* if and only if they are disjoint. Similarly, the BicMRM problem is equivalent to finding a maximum subset of $\overline{\mathcal{R}}(S)$ of independent rectangles. Thus, the MonoMRM and BicMRM problems are special cases of the MISR problem.

There exist polynomial-time exact algorithms, constant-approximation algorithms, and PTAS's for special cases of the MISR problem, based on the intersection graph of the rectangles. The *intersection graph* is the undirected graph with the rectangles of the input as vertices, and two rectangles are adjacent if and only if they are not independent. For any set $\mathcal{H}$ of rectangles, let $G(\mathcal{H})$ denote the intersection graph of $\mathcal{H}$. Given two rectangles R_1 and R_2, we say that R_1 *pierces* R_2 if onto the x-axis the orthogonal projection of R_1 contains the orthogonal projection of R_2, and onto the y-axis the orthogonal projection of R_2 contains the orthogonal projection of R_1. We say that two intersecting rectangles *pierce* if one of them pierces the other one (see Fig. 1a) (Agarwal and Mustafa 2006; Lewin-Eytan et al. 2004; Soto and Telha 2011). Independently, Agarwal and Mustafa (2006) and Lewin-Eytan et al. (2004) showed

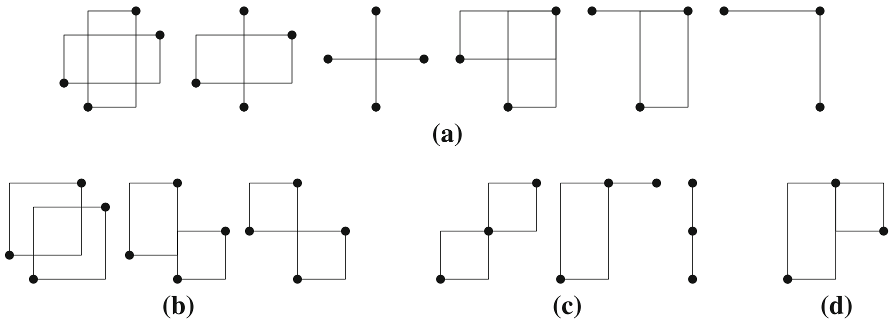

Fig. 1 Up to symmetry, the four types of intersection: **a** piercing; **b** corner; **c** point; and **d** side

that if every pair of intersecting rectangles pierce, then the MISR problem can be solved in polynomial time since, in this case, the intersection graph of the rectangles is perfect. A graph is *perfect* if the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. Using a classical result of Grötschel et al. (1984), a maximum independent set of a perfect graph can be computed in polynomial time. Agarwal and Mustafa (2006) generalized this fact, claiming that the spanning subgraph (i.e. factor) of the intersection graph, with edge set the edges corresponding to the piercing intersections, is also perfect. A graph $G' = \langle V', E' \rangle$ is a *spanning* subgraph of $G = \langle V, E \rangle$ if $V' = V$ and $E' \subseteq E$. We will use these results on piercing rectangles in the approximation algorithms described in Sect. 3. If q is the clique number of the intersection graph, there exists a $(1/4q)$-approximation (Agarwal and Mustafa 2006; Lewin-Eytan et al. 2004). For both problems we study here, we can build examples in which the size of the optimal solution is either arbitrarily big or small, and independently of that, the clique number q is either arbitrarily big or small. Then, applying this result does not always guarantee a good approximation.

In the sets $\mathcal{R}(S)$ and $\overline{\mathcal{R}}(S)$, two intersecting rectangles realize only one of the following four types of intersection: (1) a *piercing* intersection in which the two rectangles pierce (see Fig. 1a); (2) a *corner* intersection in which each rectangle contains exactly one of the corners of the other one, and these corners are not elements of S (see Fig. 1b); (3) a *point* intersection where the intersection of the rectangles is precisely an element of S and it is not a piercing intersection (see Fig. 1c); and (4) a *side* intersection which is the complement of the above three intersection types (see Fig. 1d). This is due to the fact that every rectangle in these sets is defined by two points of S being opposed vertices, is closed, and does not contain any other point of S.

Let $G = G(\mathcal{R}(S))$. Observe that if we consider the spanning subgraph G' of G with edge set the edges corresponding to the piercing intersections, and compute in polynomial time the maximum independent set for G' (Agarwal and Mustafa 2006; Lewin-Eytan et al. 2004), then we will obtain a set $H \subseteq \mathcal{R}(S)$ of pairwise non-piercing rectangles. In that case the set H (after a slight perturbation that maintains the same intersection graph) is a set of pseudo-disks and the PTAS of Chan and Har-Peled (2012), for approximating the maximum independent set in a set of pseudo-disks, can be applied in H to obtain an independent set $H' \subseteq H \subseteq \mathcal{R}(S)$. Unfortunately, it is not straightforward to compare $|H'|$ with the optimal value of the MISR problem for

$\mathcal{R}(S)$. The same arguments apply for $\overline{\mathcal{R}}(S)$. On the other hand, there exist PTAS's for the MISR problem when the rectangles have unit height (Chan 2004), or bounded aspect ratio (Chan 2003; Erlebach et al. 2005).

Soto and Telha (2011) studied the following problem to model cross-free matchings in two-directional orthogonal ray graphs (2-dorgs): Given finite point sets X and Y in the plane, find a maximum set of independent rectangles of the set $\mathcal{R}(X, Y)$ of the rectangles having an element of X as bottom-left corner and an element of Y as top-right corner. For $X = R$ and $Y = B$, where $S = R \cup B$, this problem is equivalent to the MISR problem over the rectangles $\overline{\mathcal{R}}(S)$ that have a red point as bottom-left corner and a blue point as top-right corner. The authors solved this problem in polynomial time with the next observations: the rectangles of $\mathcal{R}(X, Y)$ have only two types of intersections, piercing and corner, and $\mathcal{R}(X, Y)$ can be reduced to a small one $\mathcal{R}_0 \subseteq \mathcal{R}(X, Y)$ whose intersection graph is perfect since the elements of $\mathcal{R}_0$ are pairwise piercing, and a maximum independent set in $\mathcal{R}_0$ is a maximum independent set of $\mathcal{R}$. We use these observations to obtain the approximation algorithms that we present in the next section.

3 Approximation algorithms

Given a point set P in the plane, we say that $\mathcal{H}$ is a *set of rectangles on* P if every element of $\mathcal{H}$ is of the form $D(a, b)$, where $a, b \in P$ and $D(a, b)$ contains exactly the points a and b of P. We say that the set $\mathcal{H}$ is *complete* if for every pair of elements $D(a, b)$ and $D(a', b')$ of $\mathcal{H}$ that have a corner intersection, the two rectangles $D(a, b')$ and $D(a', b)$ realizing a piercing intersection also belong to $\mathcal{H}$. Let $G_{p,c}(\mathcal{H})$ denote the spanning subgraph of $G(\mathcal{H})$ with edge set the edges that correspond to the piercing and the corner intersections.

Lemma 1 *Let P be a point set and $\mathcal{H}$ be any complete set of rectangles on P. A maximum independent set of $G_{p,c}(\mathcal{H})$ can be found in polynomial time.*

Proof Given finite point sets X and Y in the plane, Soto and Telha (2011) showed how to find in polynomial time a maximum independent set and a minimum hitting set of a complete set $\mathcal{R}(X, Y)$ of rectangles on $X \cup Y$, where each rectangle has an element of X as bottom-left corner, and an element of Y as top-right corner. A *hitting set* is a finite point set H such that each rectangle of $\mathcal{R}(X, Y)$ contains at least one of the points of H. They noted that the rectangles of $\mathcal{R}(X, Y)$ have only two types of intersections, piercing and corner. Their overall algorithm and arguments are the following ones (refer to Sect. 4 of Soto and Telha 2011):

1. Sort the rectangles $\mathcal{R}(X, Y)$ in *right-top* order: the rectangle $D(a, b)$ is before the rectangle $D(a', b')$ if and only if $x(a) < x(a')$, or $x(a) = x(a')$ and $y(b) < y(b')$.
2. Construct a subset $\mathcal{K} \subseteq \mathcal{R}(X, Y)$ by processing the rectangles of $\mathcal{R}(X, Y)$ in right-top order and adding only those that keep $\mathcal{K}$ free of corner intersections.
3. Using that only piercing intersections are possible in $\mathcal{K}$, compute in polynomial time a maximum independent set $\mathcal{R}_0$ and a minimum hitting set $H^\star$ for $\mathcal{K}$, which always satisfy $|\mathcal{R}_0| = |H^\star|$ since $G(\mathcal{K})$ is perfect.

4. Prove that $H^\star$ is also a (minimum) hitting set of $\mathcal{R}(X, Y)$ (Soto and Telha 2011, see Lemma 1), which implies that $\mathcal{R}_0$ is a maximum independent set of $\mathcal{R}(X, Y)$.

We extend steps (1–3) to find a maximum independent set of $G_{p,c}(\mathcal{H})$. It is as follows: We partition $\mathcal{H}$ into the following three sets: $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$. The set $\mathcal{H}_0$ contains all the segments; $\mathcal{H}_1$ contains the boxes $D(a, b)$ such that $y(a) < y(b)$; and $\mathcal{H}_2$ contains the boxes $D(a, b)$ such that $y(a) > y(b)$. By definition, for every box $D(a, b)$ we have $x(a) < x(b)$. Sort the boxes of $\mathcal{H}_1$ by using the right-top order of Step 1. Then, one can construct the subset $\mathcal{K}_1 \subseteq \mathcal{H}_1$, having no corner intersection, by processing the boxes of $\mathcal{H}_1$ in such an order, and adding to $\mathcal{K}_1$ only those that keep $\mathcal{K}_1$ free of corner intersections. Similarly and using symmetry, one can construct the subset $\mathcal{K}_2 \subseteq \mathcal{H}_2$, having no corner intersection, by processing the boxes of $\mathcal{H}_2$ in the following order: the box $D(a, b)$ is before the box $D(a', b')$ if and only if $x(a) < x(a')$, or $x(a) = x(a')$ and $y(b) > y(b')$. By construction, and the fact that a box of $\mathcal{H}_1$ and a box of $\mathcal{H}_2$ cannot realize a corner intersection, and a segment of $\mathcal{H}_0$ cannot realize corner intersections with any other rectangle of $\mathcal{H}$, the set $\mathcal{K} = \mathcal{H}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \subseteq \mathcal{H}$ is free of corner intersections. A maximum independent set of $G_{p,c}(\mathcal{K})$ can be found in polynomial time as done in Step 3 since $\mathcal{K}$ is free of corner intersections.

To show that a maximum independent set of $G_{p,c}(\mathcal{K})$ is indeed a maximum independent set of $G_{p,c}(\mathcal{H})$, the arguments given in Step 4 cannot directly be applied. A reason for this is the following: Pairs of rectangles in $\mathcal{H}$ having a point or a side intersection can be hit by the same point, whereas they both can appear in $\mathcal{K}$ without being adjacent in $G_{p,c}(\mathcal{K})$ (see Fig. 2 for an example). Then, it cannot be guaranteed that the cardinality of a maximum independent set of $G_{p,c}(\mathcal{K})$ equals the cardinality of a minimum hitting set of $\mathcal{K}$. However, we can adapt the arguments of Step 4 to work directly on independent sets of rectangles, not using hitting sets.

We will prove that any box $B \in I$ that is not in $\mathcal{K}_1 \cup \mathcal{K}_2$ can be *replaced* by some suitable rectangle R, obtaining other independent set of $G_{p,c}(\mathcal{K})$ with cardinality $|I|$. Suppose that B belongs to $\mathcal{H}_1$ (the case where B belongs to $\mathcal{H}_2$ is analogous). Box B is replaced by a rectangle R that satisfies either of the following two conditions:

(i) R is in $\mathcal{H}_0 \cup \mathcal{K}_1$.
(ii) R is not in $\mathcal{H}_0 \cup \mathcal{K}_1$, but appears after B in the right-top order defined for $\mathcal{H}_1$.

Hence, if we iteratively apply this replacement procedure, each time for any box of I that is not in $\mathcal{K}_1 \cup \mathcal{K}_2$ obtaining a new set I, we will obtain at the end an independent set of size $|I|$ consisting only of elements from $\mathcal{H}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. The replacement can be done as follows. Assume that a box $B = D(a', b') \in \mathcal{H}_1 \backslash \mathcal{K}_1$ belongs to I. Then,

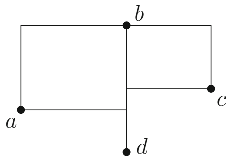

Fig. 2 The set $\mathcal{K} = \{D(a, b), D(b, c), D(d, b)\}$ is a complete set of rectangles on $\{a, b, c, d\}$, and free of corner intersections. The maximum independent set of $G_{p,c}(\mathcal{K})$ is the set $\{D(a, b), D(b, c)\}$, whereas the minimum hitting set has cardinality one

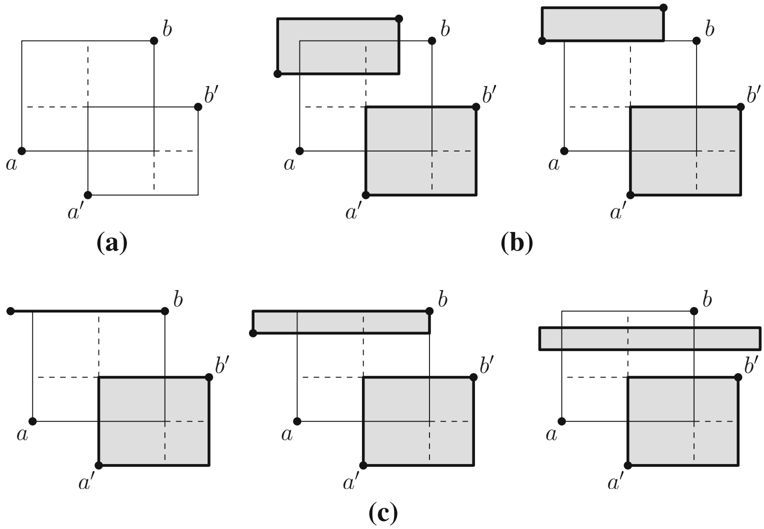

Fig. 3 (**a**, **b**, **c**) Cases in the proof of Lemma 1

by construction of $\mathcal{K}_1$, there exists a box $D(a, b) \in \mathcal{K}_1$ with $x(a) < x(a')$ that has a corner intersection with $D(a', b')$, and let $D(a, b)$ be the first box in the order of $\mathcal{H}_1$ that satisfies this condition (see Fig. 3a). Since $\mathcal{H}$ is complete, and the choice of $D(a, b)$, the rectangle $D(a, b')$ belongs to $\mathcal{H}_0 \cup \mathcal{K}_1$. The rectangle $D(a', b)$ either belongs to $\mathcal{H}_0 \cup \mathcal{K}_1$, or does not belong to $\mathcal{H}_0 \cup \mathcal{K}_1$ (i.e. belongs to $\mathcal{H}_1 \setminus \mathcal{K}_1$) and appears after $D(a', b')$ in the order of $\mathcal{H}_1$. Now, we only need to prove that $D(a', b')$ can be replaced in I by either $D(a, b')$ or $D(a', b)$. In any case, one of conditions (i) and (ii) is ensured. That is, we need to prove that

$$(I \setminus \{D(a', b')\}) \cup \{D(a', b)\} \text{ or } (I \setminus \{D(a', b')\}) \cup \{D(a, b')\}$$

is also an independent set of $G_{p,c}(\mathcal{H})$. Indeed, if $(I \setminus \{D(a', b')\}) \cup \{D(a', b)\}$ is an independent set of $G_{p,c}(\mathcal{H})$, then we are done. Otherwise, at least one of the next two cases is satisfied: (1) there is a rectangle of $I \setminus \{D(a', b')\}$ that has a corner intersection with both $D(a, b)$ and $D(a', b)$ (see Fig. 3b); or (2) there is a rectangle (either a box or a segment) of $I \setminus \{D(a', b')\}$ that has a piercing intersection with both $D(a, b)$ and $D(a', b)$ (see Fig. 3c). In both cases $D(a, b')$ is independent in $G_{p,c}(\mathcal{H})$ from any rectangle in $I \setminus \{D(a', b')\}$. Hence, $(I \setminus \{D(a', b')\}) \cup \{D(a, b')\}$ is an independent set of $G_{p,c}(\mathcal{H})$. This completes the proof. □

Let $S = R \cup B$ be a two-colored point set in the plane. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the next two subsets of rectangles of $\mathcal{R}(S)$ (see Fig. 4):

- $\mathcal{R}_1$ contains the blue rectangles $D(a, b) \in \mathcal{R}(S)$ such that $y(a) \le y(b)$ (i.e. the rectangles with bottom-left corner a blue point), and the red rectangles $D(a', b') \in \mathcal{R}(S)$ such that $y(a') \ge y(b')$ (i.e. the rectangles with bottom-right corner a red point).

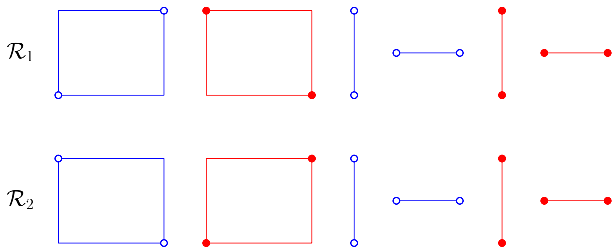

Fig. 4 The types of rectangles in the subsets $\mathcal{R}_1$ and $\mathcal{R}_2$

- $\mathcal{R}_2$ contains the blue rectangles $D(a, b) \in \mathcal{R}(S)$ such that $y(a) \geq y(b)$ (i.e. the rectangles with bottom-right corner a blue point), and the red rectangles $D(a', b') \in \mathcal{R}(S)$ such that $y(a') \leq y(b')$ (i.e. the rectangles with bottom-left corner a red point).

The following lemma will be used in the proof of Lemma 3.

Lemma 2 *Let H_1 and H_2 be independent sets of $G_{p,c}(\mathcal{R}_1)$ and $G_{p,c}(\mathcal{R}_2)$, respectively. The graphs $G(H_1)$ and $G(H_2)$ are acyclic.*

Proof We prove the lemma for $G(H_1)$. The proof for $G(H_2)$ is analogous. Note that in H_1 every blue rectangle is independent from every red rectangle, and rectangles of the same color can have point intersections only. For every $k \geq 1$, every single path of length k in $G(H_1)$ is a sequence $\langle D(a_0, a_1), D(a_1, a_2), D(a_2, a_3), \ldots, D(a_k, a_{k+1})\rangle$ of rectangles of H_1, with point intersections between consecutive rectangles, such that: $x(a_0) \leq x(a_1) \leq \cdots \leq x(a_{k+1})$, and $y(a_0) \leq y(a_1) \leq \cdots \leq y(a_{k+1})$ if the rectangles $D(a_0, a_1), \ldots, D(a_k, a_{k+1})$ are all blue or $y(a_0) \geq y(a_1) \geq \cdots \geq y(a_{k+1})$ if they are red. Under these monotone properties, the graph $G(H_1)$ cannot have any cycle. □

Lemma 3 *For $\mathcal{R} \in \{\mathcal{R}_1, \mathcal{R}_2\}$, there exists a polynomial-time (1/2)-approximation algorithm for the maximum independent set of $G(\mathcal{R})$.*

Proof Consider the set $\mathcal{R}_1$, the arguments for the set $\mathcal{R}_2$ are analogous. Let OPT_1 denote the size of a maximum independent set in $\mathcal{R}_1$. Observe that a blue and a red rectangle in $\mathcal{R}_1$ can have only a piercing intersection, that two rectangles of the same color cannot have a side intersection, and that $\mathcal{R}_1$ is a complete set of rectangles on S. Let H_1 denote a maximum independent set of $G_{p,c}(\mathcal{R}_1)$, which can be found in polynomial time by Lemma 1. The graph $G(H_1)$ is acyclic (Lemma 2) and thus 2-colorable. Such a 2-coloring of $G(H_1)$ can be found in polynomial time and gives an independent set I_1 of H_1 with at least $|H_1|/2$ rectangles, which is an independent set in $\mathcal{R}_1$ as well. The set I_1 is the approximation and satisfies $\mathsf{OPT}_1 \leq |H_1| \leq 2|I_1|$. The result thus follows. □

Theorem 4 *There exists a polynomial-time (1/4)-approximation algorithm for the* MonoMRM *problem.*

Proof Let OPT denote the size of a maximum independent set in $\mathcal{R}(S)$, and OPT_1 and OPT_2 denote the sizes of the maximum independent sets in $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Let I_1 be a (1/2)-approximation for the maximum independent set in $\mathcal{R}_1$ and I_2 be a (1/2)-approximation for the maximum independent set in $\mathcal{R}_2$ (Lemma 3). The approximation for the MonoMRM problem is to return the set with maximum cardinality between I_1 and I_2. Since $\mathsf{OPT} \leq \mathsf{OPT}_1 + \mathsf{OPT}_2 \leq 2|I_1| + 2|I_2| \leq 4\max\{|I_1|, |I_2|\}$, the result follows. □

Consider now the BicMRM problem and the set $\overline{\mathcal{R}}(S)$. Let $\overline{\mathcal{R}}_1$, $\overline{\mathcal{R}}_2$, $\overline{\mathcal{R}}_3$, and $\overline{\mathcal{R}}_4$ be the next four subsets of rectangles of $\overline{\mathcal{R}}(S)$:

- $\overline{\mathcal{R}}_1$ contains the rectangles with a blue point in the bottom-left corner.
- $\overline{\mathcal{R}}_2$ contains the rectangles with a red point in the bottom-left corner.
- $\overline{\mathcal{R}}_3$ contains the rectangles with a blue point in the bottom-right corner.
- $\overline{\mathcal{R}}_4$ contains the rectangles with a red point in the bottom-right corner.

Each of the above four subsets is a complete set of rectangles on S, where every two rectangles have either a corner or a piercing intersection. Then, the maximum independent set in each subset can be found in polynomial time (Lemma 1). These observations imply the next result:

Theorem 5 *There exists a polynomial-time* (1/4)*-approximation algorithm for the* BicMRM *problem.*

In the next section we complement the results of Theorems 4 and 5 by showing that the corresponding problems are NP-hard.

4 Hardness

We prove that the MonoMRM and BicMRM problems are NP-hard, even if further conditions are assumed. To this end, we consider the following decision problems:
PERFECT MONOCHROMATIC RECTANGLE MATCHING (MonoPRM) problem: *Is there a perfect monochromatic strong matching of S with rectangles?*
PERFECT BICHROMATIC RECTANGLE MATCHING (BicPRM) problem: *Is there a perfect bichromatic strong matching of S with rectangles?*
Proving that the MonoPRM and BicPRM problems are NP-complete, even on certain additional conditions, implies that the MonoMRM and the BicMRM problems are NP-hard under the same conditions.

In the proofs that follow, we use reductions from the PLANAR 1-IN-3 SAT problem which is NP-complete (Mulzer and Rote 2008). The input of the PLANAR 1-IN-3 SAT problem is a Boolean formula in 3-CNF whose associated graph[1] is planar, and the formula is satisfiable if and only if there exists an assignment to its variables such that in each clause exactly one literal is satisfied (Mulzer and Rote 2008). Given any planar 3-SAT formula, the main idea is to construct a point set $S = S_1 \cup S_2$, such that: (1) the elements of S_2 force to match certain pairs of points in S_1, and those pairs can only

[1] The associated graph is the bipartite graph with vertices the variables and the clauses, and there exists an edge between a variable and a clause if and only if the variable participates in the clause.

be matched with (axis-aligned) segments; (2) there always exists a perfect matching with segments for S_2 independently of S_1; and (3) there exists a perfect matching with segments for S_1 independently of S_2 if and only the formula is satisfiable.

The above method can be applied in the construction that Kratochvíl and Nešetřil (1990) used to prove that finding a maximum independent set in a set of axis-aligned segments is NP-hard. Indeed, we can put the elements of S_1 at the endpoints of the segments T of their construction, by first modelling the parallel overlapping segments by segments sharing an endpoint. Then, the elements of S_2 are added in such a way that every two elements of S_1 can be matched if and only if they are endpoints of the same segment in T. This approach would give us a proof that the optimization problems we consider are NP-hard, but not that the perfect matching decision problems are NP-complete, which are stronger results. On the other hand, the hardness proofs here give an alternative NP-hardness proof for the problem of finding a maximum independent set in a set of axis-aligned segments (Kratochvíl and Nešetřil 1990).

Theorem 6 *The* MonoPRM *problem is* NP*-complete, even if we restrict the matching rectangles to segments.*

Proof Given a combinatorial matching of S, certifying that such a matching is monochromatic, strong, and perfect can be done in polynomial time. Then, the MonoPRM problem is in NP. We prove now that the MonoPRM problem is NP-hard. Let φ be a planar 3-SAT formula. The (planar) graph associated with φ can be represented in the plane as in Fig. 5, where all variables lie on an horizontal line, and all clauses are represented by *non-intersecting* three-legged combs (Knuth and Raghunathan 1992). Using this embedding, which can be constructed in a grid of polynomial size (Knuth and Raghunathan 1992), we construct a set S of red and blue integer-coordinate points in a polynomial-size grid, such that there exists a perfect monochromatic strong matching with (axis-aligned) segments in S if and only if φ is satisfiable.

For an overview of our construction of S, refer to Fig. 6. We use variable gadgets (the dark-shaded rectangles called variable rectangles) and clause gadgets (the light-shaded orthogonal polygon representing the three-legged comb).

Variable gadgets: For each variable v, its rectangle Q_v has height 4 and width $6 \cdot d(v)$, where $d(v)$ is the number of clauses in which v appears. We assume that each variable appears in every clause at most once. Along the boundary of Q_v, starting from a vertex, we put blue points so that every two successive points are at distance 2 from each other. We number consecutively in clockwise order these $4 + 6 \cdot d(v)$ points, starting from the top-left vertex of Q_v which is numbered 1.

Clause gadgets: Let C be a clause with variables u, v, and w, appearing in this order from left to right in the embedding of φ. Assume w.l.o.g. that the gadget of C is above

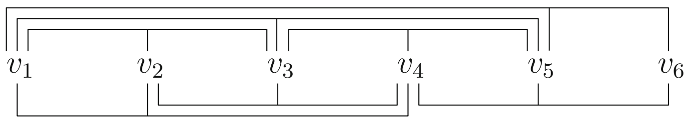

Fig. 5 Planar representation of $\varphi = (v_1 \vee \overline{v_2} \vee v_3) \wedge (v_3 \vee \overline{v_4} \vee \overline{v_5}) \wedge (\overline{v_1} \vee \overline{v_3} \vee v_5) \wedge (v_1 \vee \overline{v_2} \vee v_4) \wedge (\overline{v_2} \vee \overline{v_3} \vee \overline{v_4}) \wedge (\overline{v_4} \vee v_5 \vee \overline{v_6}) \wedge (\overline{v_1} \vee v_5 \vee v_6)$

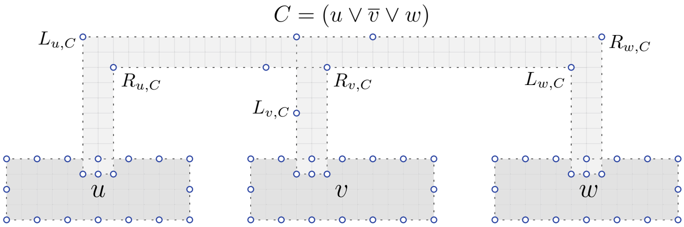

Fig. 6 The variable gadgets and the clause gadgets. In the figure, each variable u, v, w might participate in other clauses

the horizontal line through the variables. Every leg of the gadget of C overlaps the rectangles of its corresponding variable (denoted x) in a rectangle $Q_{x,C}$ of height 1 and width 2, so that the midpoint of the top side of $Q_{x,C}$ is a blue point in the boundary of Q_x. The overlapping satisfies that such a midpoint is numbered with an even number if and only if x appears positive in C. We further put three blue points equally spaced at distance 1 in the bottom side of $Q_{x,C}$, and other nine blue points in the boundary of the gadget, as shown in Fig. 6. Among these nine points, for $x \in \{u, v, w\}$, let $R_{x,C}$ denote the blue point in the right side of the vertical leg corresponding to x in gadget of C, and $L_{x,C}$ the bottommost blue point in the left side (see Fig. 6).

Forcing convenient matchings of the blue points: We add red points (a polynomial number of them) in such a way that for every pair $a \neq b$ of blue points it holds that if $D(a, b)$ is not a segment of a dotted line (see Fig. 6) then $D(a, b)$ contains a colored point in its interior.[2] This implies that two blue points a and b can be matched if and only if $D(a, b)$ is a segment of any dotted line and does not contain any other colored point. This can be done as follows: Since blue points have all integer coordinates, we can scale the blue point set (multiplying by 2) so that every element has even x- and y-coordinates. Then, a quadratic number of red points each of which with at least one odd coordinate can be added so that to ensure the above condition. We finally scale again the points, the blue and the red ones, and make a copy of the scaled red points and move it one unit downwards.

Intuition: Consider the blue point at the top-left vertex of the rectangle Q_v of variable v. This point can be matched only with either the blue point immediately to its right or the blue point immediately below. If we decide to match this point as in the first case (see Fig. 7a), then we consider that $v = 1$. Otherwise, if we match as in the second case, we consider that $v = 0$ (see Fig. 7b). Then, trying to find a maximum matching, this decision *propagates* a matching of the other blue points in the boundary of Q_v, as shown in Fig. 7a and b. Furthermore, if the value of v satisfies some clause C, it induces to match the point $R_{v,C}$ with the point in the bottom-right vertex of $Q_{v,C}$. Otherwise, the point $L_{v,C}$ is induced to be matched with the point in the bottom-left vertex of $Q_{v,C}$. Let C be a clause with variables u, v, and w. It can be verified that all the blue points in the gadget of C can be matched (throughout the matching that starts with the decisions made at the top-left vertices of Q_u, Q_v, and Q_w, which propagate

[2] If $D(a, b)$ is a box, then its interior is the interior of the box. Otherwise, if $D(a, b)$ is a segment, then its interior is the set $D(a, b) \setminus \{a, b\}$.

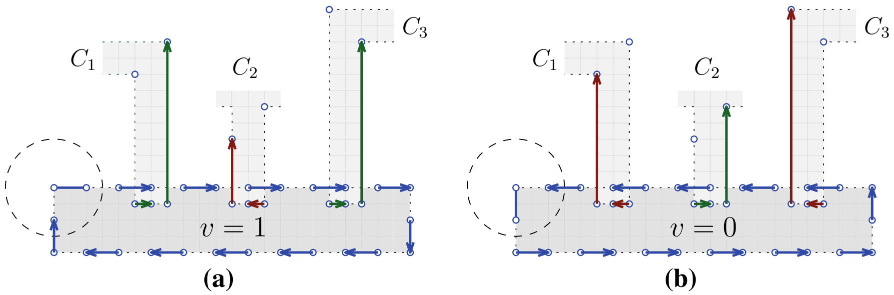

Fig. 7 The variable v appears positive in the clauses C_1 and C_3, and appears negative in the clause C_2. **a** If the *blue point* at the *top-left vertex* of Q_v is matched to the *right* (i.e. $v = 1$), then each of R_{v,C_1}, L_{v,C_2}, and R_{v,C_3} is matched with a point in Q_v, since $v = 1$ satisfies both C_1 and C_3, but not C_2. (b) If the *blue point* at the *top-left vertex* of Q_v is matched *downwards* (i.e. $v = 0$), then each of L_{v,C_1}, R_{v,C_2}, and L_{v,C_3} is matched with a point in Q_v, since $v = 0$ satisfies C_2, but neither C_1 nor C_3. In both **a** and **b**, the *arrows* are matching segments. Each *arrow* represents the fact that the *blue point* at the *source vertex* needs to be matched with the *blue point* at the *target one*, due to the match inside the *dashed circle* which is the first one that was made

to the other blue points) if and only if precisely one of u, v, and w satisfies C. This statement is described in Fig. 8.

Reduction: Based on the intuition, observe that in each variable v, the blue points along the boundary of Q_v can be matched independently of the other points, and that they have two perfect strong matchings: the *1-matching* that matches the ith point with the $(i + 1)$th point for all odd i; and the *0-matching* that matches the ith point with the $(i + 1)$th one for all even i. In each clause C in which v appears, each of these two matchings induces a maximum strong matching on the blue points in the leg of the gadget of C that overlaps Q_v, until reaching the points in the union of the three legs. We consider that variable $v = 1$ if we use the 1-matching, and consider $v = 0$ if the 0-matching is used. Let C be a clause with variables u, v, and w; and draw perfect strong matchings on the blue points of the boundaries of Q_u, Q_v, and Q_w, respectively, giving values to u, v, and w. Notice that if exactly one among u, v, and w makes C positive, then the strong matching induced in the blue points of the gadget of C is perfect (see Fig. 9). Otherwise, if none or at least two among u, v, and w make C positive, then the strong matching induced on the blue points of the gadget of C is not perfect since two blue points are unmatched (see Figs. 10 and 11). Finally, note that the red points admit a perfect strong matching with segments such that no segment contains a blue point. Therefore, we can ensure that the 3-SAT formula φ can be satisfied if and only if the point set S admits a perfect strong matching with segments. □

Suppose now that the two-colored point set S is in general position. In what follows we show that the MonoPRM problem remains NP-complete under this assumption. To this end, we first perturb the two-colored point set of the construction of the proof of Theorem 6 so that no two points share the same x- or y-coordinate, and second show that two points of S can be matched in the perturbed point set if and only if they can be matched in the original one.

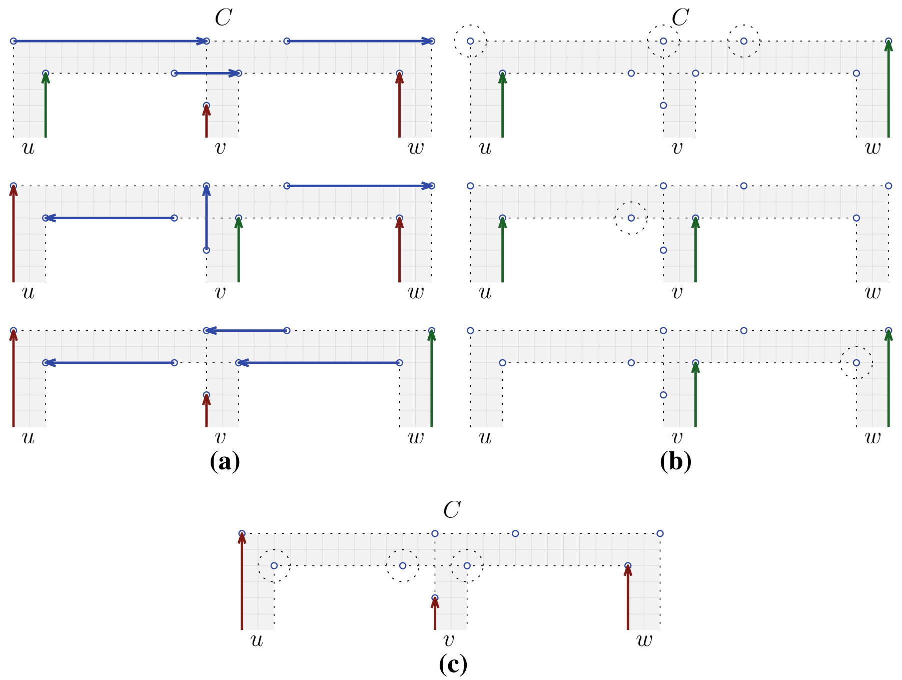

Fig. 8 **a** If exactly one of u, v, and w satisfies C, then all *blue points* in the gadget of C can be matched. Note that C is satisfied only by u (resp. v, w) in the *top* (resp. *middle*, *bottom*) figure. **b** In each case (*top*, *middle*, and *bottom*) at least two variables among u, v, and w satisfy C. Then, at least one of the *blue points* inside the *dotted circles* cannot be matched. **c** If none of u, v, and w satisfies C, then one of the *blue points* inside a *dotted circle* cannot be matched

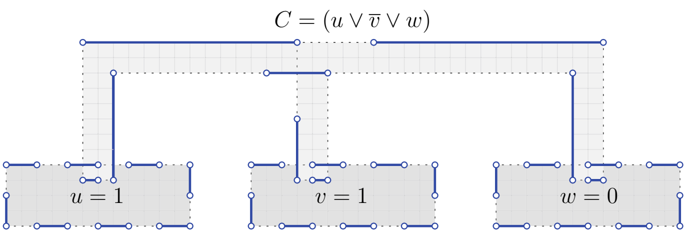

Fig. 9 If $u = 1$, $v = 1$, and $w = 0$, then only u satisfies C and there exists a perfect strong matching on the *blue points*

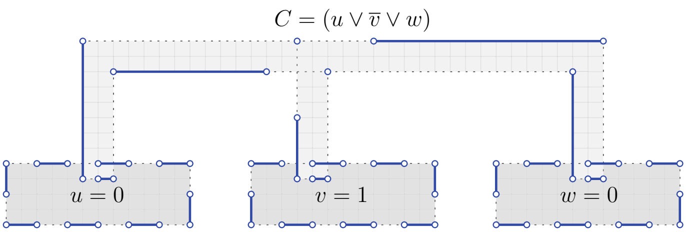

Fig. 10 If $u = 0$, $v = 1$, and $w = 0$, then no variable satisfies C and there does not exist any perfect strong matching on the *blue points*

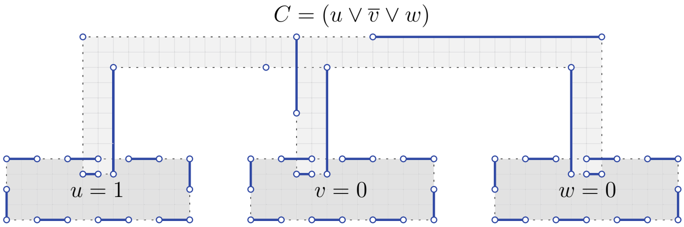

Fig. 11 If $u = 1$, $v = 0$, and $w = 1$, then two variables satisfies C and there does not exist any perfect strong matching on the *blue points*

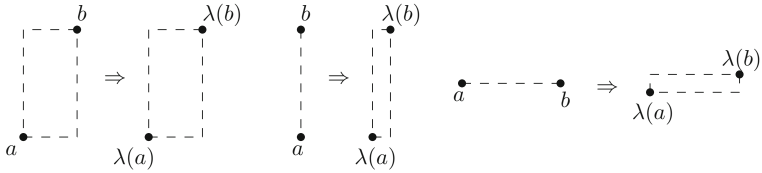

Fig. 12 Perturbation of the point set to put S in general position

Alliez et al. (1997) proposed the transformation that replaces each point $p = (x, y)$ by the point $\lambda(p) = ((1 + \varepsilon)x + \varepsilon^2 y, \varepsilon^3 x + y)$ for some small enough $\varepsilon > 0$, with the aim of removing the degeneracies in a point set for computing the Delaunay triangulation under the L_∞ metric. Although this transformation can be used for our purpose, by using the fact that the points in the proof of Theorem 6 belong to a grid $[0..N]^2$, where N is polynomially-bounded, we use the simpler transformation $\lambda(p) = ((1 + \varepsilon)x + \varepsilon y, \varepsilon x + (1 + \varepsilon)y)$ for $\varepsilon = 1/(2N + 1)$, which is linear in ε. Both transformations change the relative positions of the initial points in the manner shown in Fig. 12. Some useful properties of the transformation we use, stated in the next lemma, were not stated by Alliez et al. (1997).

Lemma 7 *Let N be a natural number and $P \subseteq [0 \ldots N]^2$. The function $\lambda : P \to \mathbb{Q}^2$ such that*

$$\lambda(p) = \left(x(p) + \frac{x(p) + y(p)}{2N + 1}, y(p) + \frac{x(p) + y(p)}{2N + 1} \right)$$

satisfies the next properties:

(a) *λ is injective and the point set $\lambda(P) = \{\lambda(p) : p \in P\}$ is in general position.*
(b) *For every two distinct points $a, b \in P$ such that $x(a) = x(b)$ or $y(a) = y(b)$, we have that $D(a, b) \cap P = \{a, b\}$ if and only if $D(\lambda(a), \lambda(b)) \cap \lambda(P) = \{\lambda(a), \lambda(b)\}$.*
(c) *For every three distinct points $a, b, c \in P$ such that $x(a) \neq x(b)$ and $y(a) \neq y(b)$, we have that c belongs to the interior of $D(a, b)$ if and only if $\lambda(c)$ belongs to the interior of $D(\lambda(a), \lambda(b))$.*

Proof Properties (a–c) are a consequence of $0 \leq \frac{x(p)+y(p)}{2N+1} \leq \frac{2N}{2N+1} < 1$. □

Theorem 8 *The* MonoPRM *problem remains* NP*-complete on point sets in general position.*

Proof Let S be the colored point set generated in the reduction of the proof of Theorem 6. Let N be a polynomially-bounded natural number such that $S \subset [0 \ldots N]^2$, and let $S' = \lambda(S)$, where λ is the function of Lemma 7. Consider the next observations:

(a) If $a, b \in S$ are red points that can be matched in S because $x(a) = x(b)$ and $y(b) = y(a) - 1$, then $\lambda(a)$ and $\lambda(b)$ can also be matched in S' (Property (b) of Lemma 7).
(b) If $a, b \in S$ are blue points that can be matched in S, then we have that either $x(a) = x(b)$ or $y(a) = y(b)$, which implies that $\lambda(a)$ and $\lambda(b)$ can also be matched in S' by Property (b) of Lemma 7.
(c) If $a, b \in S$ are blue points that cannot be matched in S because $D(a, b)$ is a segment containing a point $c \in S$ in its interior, then neither $\lambda(a)$ and $\lambda(b)$ can be matched in S' (Property (b) of Lemma 7).
(d) If $a, b \in S$ are blue points that cannot be matched in S because $D(a, b)$ is a box containing a point $c \in S$ in the interior, then neither $\lambda(a)$ and $\lambda(b)$ can be matched in S' since the box $D(\lambda(a), \lambda(b))$ contains $\lambda(c)$ (Property (c) of Lemma 7).

The above observations imply that there exists a perfect strong rectangle matching in S if and only if it exists in S'. The result thus follows since S' is in general position by Property (a) of Lemma 7.

Combining the construction of Theorem 6 with the perturbation of Lemma 7, we can prove that the MonoPRM problem is also NP-complete when all points have the same color, and that the BicPRM problem is also NP-complete.

Lemma 9 *Let* $M_1 = \{(0, 0), (5, 0), (5, 5), (0, 5)\}$ *and* $M_2 = \{(1, 3), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 2)\}$ *be two point sets. The point set* $M_1 \cup M_2$ *has a perfect strong matching with rectangles, and for every proper subset* $M_1' \subset M_1$ *the point set* $M_1' \cup M_2$ *does not have any perfect strong matching with rectangles.*

Proof The proof is straightforward (see Fig. 13a–c). □

Theorem 10 *The* MonoPRM *problem remains* NP*-complete if all elements of* S *have the same color.*

Proof Let R_0 and B_0 be the sets of the red points and the blue points, respectively, in the proof of Theorem 6. Let Q be a set of (artificial) *green* points to block the forbidden matching rectangles in B_0, that is, for every two points $p, q \in B_0$ we have that $D(p, q)$ contains elements of $B_0 \cup Q$ in its interior if and only if $D(p, q)$ is not a matching rectangle in $R_0 \cup B_0$. In other words, p, q can be matched in $R_0 \cup B_0$ if and only if they can be matched in $B_0 \cup Q$. The point set $S_1 = B_0 \cup Q$ belongs to the grid $[0..N]^2$, where N is polynomially-bounded in $|R_0| + |B_0|$, and is not in general position. Let $S_2 = \lambda(S_1)$, where λ is the function of the Lemma 7. We now replace each green point g of Q by a translated and stretched copy S_g of the set $M_1 \cup M_2$ of Lemma 9, with all elements colored blue (see Fig. 13a). Let $S = B_0 \cup (\bigcup_{g \in Q} S_g)$. Putting the elements of S_g close enough one another for every g, we can guarantee that if we want to obtain a perfect strong matching in S then we must have by Lemma 9 a perfect strong matching in each S_g in particular (see Fig. 13b). Therefore, the set S_g acts as the

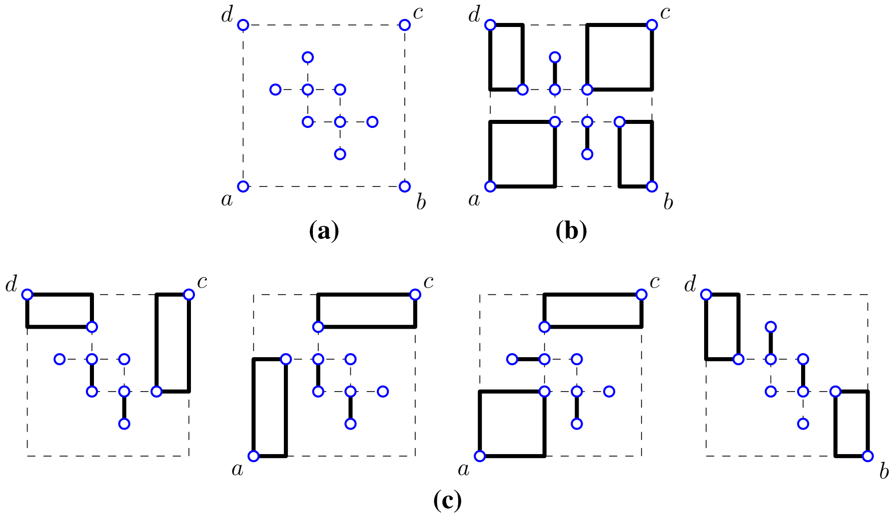

Fig. 13 **a** The point set $M_1 \cup M_2$. **b** A perfect strong matching of $M_1 \cup M_2$. **c** If exactly two points among a, b, c, d are removed, then the remaining points do not have any perfect strong matching

green point g blocking the forbidden matching rectangles in B_0. The construction of S starts from the planar 3-SAT formula φ of the proof of Theorem 6, and using all the above arguments, we can claim that there exists a perfect strong matching in S if and only if the formula φ is satisfiable. Hence, the MonoPRM problem with input points of the same color is NP-complete since there exists a polynomial-time reduction from the Planar 1- in- 3 SAT problem. □

Theorem 11 *The* BicPRM *problem is* NP*-complete, even if the point set S is in general position.*

Proof Let R_0 and B_0 be the sets of the red points and the blue points, respectively, in the proof of Theorem 6. Change to color red elements of B_0, to obtain the colored point set S_0, so that for every segment matching two blue points in $R_0 \cup B_0$ exactly one of the matched points is changed to color red (see Fig. 14a). For every point $p \in B_0$, let p' denote the corresponding point in S_0, and vice versa. Let Q be a set of (artificial) green points to block the forbidden matching rectangles in S_0, that is, for every two distinct points $p', q' \in S_0$ we have that $D(p', q')$ contains elements of $S_0 \cup Q$ in its interior if and only if $D(p, q)$ is not a matching rectangle in $R_0 \cup B_0$. The point set $S_1 = S_0 \cup Q$ belongs to the grid $[0 \dots N]^2$, where N is polynomially-bounded, and is not in general position. Let $S_2 = \lambda(S_1)$, where λ is the function of the Lemma 7. We now replace each green point g of Q by the set S_g of eight red and blue points in general position (see Fig. 14b). Let $S = S_0 \cup (\bigcup_{g \in Q} S_g)$. Putting the elements of S_g close enough one another for every g, we can guarantee that S is also in general position and that for every g the points of S_g appear together in both the left-to-right and the top-down order of S. This last condition ensures that if we want to obtain a perfect strong matching in S then we must have a perfect strong matching for each

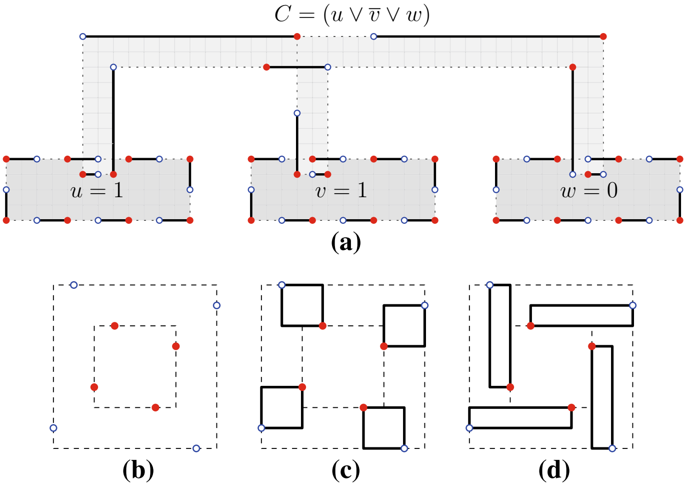

Fig. 14 Proof of Theorem 11. **a** Changing the *colors* of the *blue points* in the gadgets of the proof of Theorem 6. **b** The eight points (close enough one another) that replace each *green point*. **c** One of the only two ways to match the points corresponding to a *green point* in order to obtain a perfect matching. **d** The other way

S_g in particular (see Fig. 14c and d) because for all g every red point of S_g cannot be matched with any blue point not in S_g. Therefore, the set S_g acts as the green point g blocking the forbidden matching rectangles in S_0. The BicPRM problem is thus NP-complete, even on points in general position. □

5 Summary

We have proved that finding a maximum strong matching of a two-colored point set, with either rectangles containing points from the same color or rectangles containing points of different colors, is NP-hard and provide a polynomial-time $(1/4)$-approximation algorithm for each case. These approximation algorithms provide a (1/4)-approximation algorithm for the problem of finding a maximum strong rectangle matching of points of the same color, studied by Bereg et al. (2009). However, the approximation ratio is smaller than $2/3$, the one given by Bereg et al. We leave as an open problem to find a better $O(1)$-approximation algorithm, or a PTAS, for the MonoMRM and BicMRM problems. On the other hand, finding a $O(1)$-approximation algorithm for the general Maximum Independent Set of Rectangles problem is still an intriguing open question.

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