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Distance measures, weighted averages, OWA operators and Bonferroni means

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ABSTRACT

The ordered weighted average (OWA) is an aggregation operator that provides a parameterized family of operators between the minimum and the maximum. This paper presents the OWA weighted average distance operator. The main advantage of this new approach is that it unifies the weighted Hamming distance and the OWA distance in the same formulation and considering the degree of importance that each concept has in the analysis. This operator includes a wide range of particular cases from the minimum to the maximum distance. Some further generalizations are also developed with generalized and quasi-arithmetic means. The use of Bonferroni means under this framework is also studied. The paper ends with an application of the new approach in a group decision making problem with Dempster-Shafer belief structure regarding the selection of strategies.

1. Introduction

In the literature, there are a wide range of methods for the aggregation of the information [2,10] and decision making [43]. A very useful technique for doing this is the Hamming distance [11] and more generally all the distance measures [9,13]. The main advantage of using distance measures is that we can compare the alternatives of the problem with some ideal result [13]. Therefore, by doing this comparison, the alternative with closest results to the ideal is the optimal choice.

Usually, when dealing with distance measures in aggregation problems, we normalize it by using the arithmetic mean or the weighted average (WA) obtaining the normalized Hamming distance and the weighted Hamming distance, respectively. However, sometimes it would be interesting to consider other alternatives such as the parameterization of the results from the maximum distance to the minimum distance. Thus, it would be useful to use the ordered weighted averaging (OWA) operator [34]. The OWA operator is a very useful technique for aggregating the information providing a parameterized family of aggregation operators that includes the maximum, the minimum and the average. It has been studied by many authors [42,43].

The use of the OWA operator in different types of distance measures have been studied by many authors [14]. Xu and Chen [32] and Merigó and Gil-Lafuente [20] suggested the OWA distance (OWAD) operator with the objective of introducing a parameterized family of distance operators between the minimum and the maximum distance. Merigó and Casanovas [16,17] developed further extensions by using induced aggregation operators. They also considered the use of heavy aggregation operators [18]. Zeng and collaborators extended these approaches for uncertain environments that can be assessed with interval numbers [45], fuzzy information [47] and linguistic variables [46]. Xu also studied the use of fuzzy information [31]. Zhou et al. [51] presented a new approach that could consider continuous aggregations and distance measures in the same formulation. Merigó and Yager [24] studied the use of moving distance measures. Some other authors studied the use of Choquet integrals [5]. Moreover, note that it is also possible to use other related techniques [4] under this framework such as norms [40] and probabilities [23].

An important issue when dealing with the OWA operator is to consider the importance weights that may also appear in the aggregation process. In order to integrate this issue with the OWA operator, Torra [26] suggested the weighted OWA (WOWA) operator. Yager [37] developed another approach that he

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called importance OWA weights. Xu and Da [33] introduced the hybrid average. Merigó [15] presented the OWA weighted average (OWAWA). Its main advantage is that it can integrate the OWA operator and the weighted average considering the degree of importance that each concept has in the aggregation. He applied this approach in Dempster-Shafer belief structure [19].

The aim of this paper is to develop a new extension of the OWAWA operator by using distance measures in the analysis. We introduce the OWAWA distance (OWAWAD) operator. This operator normalizes the Hamming distance and other distances with the OWAWA operator. Therefore, we are able to include the WA and the OWA operator at the same time in the Hamming distance and taking into account the degree of importance that each concept has in the formulation. Thus, we get a more complete formulation of the Hamming distance because we can consider a parameterized family of operators between the maximum and the minimum and the importance that each distance has in the analysis. We study some of its main properties and particular cases that also represent new distance measures such as the arithmetic weighted Hamming distance, the arithmetic OWAD operator, the weighted Hamming distance, the OWAD operator, the maximum weighted Hamming distance, the minimum weighted Hamming distance, and many others.

Further generalizations of the OWAWAD operator are also developed by using generalized and quasi-arithmetic means obtaining the generalized OWAWAD (GOWAWAD) and the quasi-arithmetic OWAWAD (Quasi-OWAWAD) operator. Their main advantage is that they include a wide range of particular cases including the OWAWAD and the Euclidean OWAWAD. Similar extensions are studied with the WOWAD operator, forming the WOWA distance (WOWAD), the generalized WOWAD (GWOWAD) and the quasi-arithmetic WOWAD (Quasi-WOWAD) operator. Some other aggregation operators are developed by using Bonferroni means. Here, we suggest the use of distance measures building the Bonferroni distances. Several generalizations are considered including the Bonferroni OWAD and the Bonferroni OWAWAD operator.

The applicability of this new approach is studied in a group decision making problem with Dempster-Shafer belief structure. By doing so, it is possible to consider complex probabilistic structures in the environment together with subjective information, the attitudinal character and distance measures that may establish comparisons between different sets of elements. A business problem regarding the selection of optimal strategies is considered where the board of directors of an enterprise has to decide on a key general strategy for the company. The main advantage of the OWAWAD operator is its flexibility because it can consider different sources of information and focus on those that are of highest priority. Moreover, it can be reduced to the classical approach if the information of the problem is very simple.

This paper is organized as follows. Section 2 describes some basic concepts regarding distance measures, OWA operators and the OWAD operator. Section 3 presents the OWAWAD operator. Section 4 introduces some generalizations by using generalized and quasi-arithmetic means and Section 5 studies the use of Bonferroni means with distance measures. Section 6 analyses the applicability of the new approach in a group decision making problem with Dempster-Shafer theory. Section 7 summarizes the main findings and conclusions of the paper.

2. Preliminaries

This Section briefly reviews the Hamming and the Minkowski distance, the OWA and the OWAWA operator and the OWAD.

2.1. The Hamming and the Minkowski distance

The Hamming distance [11] is a very useful technique for calculating the differences between two elements or two sets of variables. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets and intuitionistic fuzzy sets. For two sets A and B, it can be defined as follows.

Definition 1. A normalized Hamming distance of dimension n is a mapping d_H: $[0,1]^n \times [0,1]^n \to [0,1]$ such that:

$$d_H(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} |a_i - b_i| \right) \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

Sometimes, when normalizing the Hamming distance we prefer to give different weights to each individual distance. Then, the distance is known as the weighted Hamming distance. It can be defined as follows.

Definition 2. A weighted Hamming distance of dimension n is a mapping d_{WH}: $[0,1]^n \times [0,1]^n \to [0,1]$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then:

$$d_{WH}(A, B) = \left(\sum_{i=1}^{n} w_i |a_i - b_i| \right) \tag{2}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

More generally, it is possible to use generalized means forming the weighted Minkowski distance as follows:

$$d_{WM}(A, B) \left(\sum_{i=1}^{n} w_i |a_i - b_i|^{\lambda} \right)^{1/\lambda} \tag{3}$$

where λ is a parameter such that $\lambda \in \{-\infty, \infty\} - \{0\}$.

Note that it is possible to generalize this definition to all the real numbers by using $R^n \times R^n \to R$. Moreover, observe that distance measures should accomplish the following properties in order to be considered under a standard representation:

- Non-negativity: $D(A_1, A_2) \geq 0$.
- Commutativity: $D(A_1, A_2) = D(A_2, A_1)$.
- Reflexivity: $D(A_1, A_1) = 0$.
- Triangle inequality: $D(A_1, A_2) + D(A_2, A_3) \geq D(A_1, A_3)$.

2.2. OWA and OWAWA operators

The OWA operator was introduced by Yager [34] and it provides a parameterized family of aggregation operators between the maximum and the minimum. It is widely used in decision making under uncertainty problems [12,43] and in many other related disciplines [3,43]. It can be defined as follows.

Definition 3. An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$OWA\ (a_1, a_2 \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

where b_j is the jth largest of the a_i.

The OWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. Different families of OWA operators are available when dealing with the weighting vector in the aggregation [22,43].

The ordered weighted averaging – weighted averaging (OWAWA) operator is an aggregation operator that unifies the WA and the OWA operator in the same formulation considering the degree that each concept has in the analysis [15]. Note that we do not use the name WOWA operator because this name is already used by Torra [26] in his approach. It can be defined as follows.

Definition 4. An OWAWA operator of dimension n is a mapping OWAWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA\,(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{5}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

Note that it is possible to distinguish between descending (DOWAWA) and ascending (AOWAWA) orders. By choosing a different expression in the weighting vector, we are able to obtain a wide range of particular types of OWAWA operators [15]. Especially, when $\beta = 0$, we get the WA, and if $\beta = 1$, the OWA operator. Other interesting cases are found when $w_j = 1/n$, for all a_i, forming the arithmetic mean weighted average (AMWA). And if $v_i = 1/n$, for all a_i, the arithmetic mean OWA (AMOWA) operator [15]. Note that inside the arithmetic OWA we find the arithmetic maximum and the arithmetic minimum, and so on. As we can see, the use of the WA in the OWA creates new semi boundary conditions based on the combination between the maximum and the minimum with the WA.

Note also that some previous models already considered the possibility of using the WA and the OWA in the same formulation including the hybrid average [32] and the WOWA operator [26]. The WOWA operator can be defined as follows:

Definition 5. Let P and W be two weighting vectors of dimension n [$P = (p_1, p_2, \ldots, p_n)$], [$W = (w_1, w_2, \ldots, w_n)$], such that $p_i \in [0,1]$ and $\sum_{i=1}^{n} p_i = 1$, and $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, a mapping *WOWA*: $R^n \times R^n \to R$ is a WOWA operator of dimension n if:

$$WOWA\,(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \omega_i a_{\sigma(i)} \tag{6}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, \ldots, n$, and the weight ω_i is defined as:

$$\omega_i = w * \left(\sum_{j \leq i} p_{\sigma(j)} \right) - w * \left(\sum_{j < i} p_{\sigma(j)} \right) \tag{7}$$

with w^* a monotone increasing function that interpolates the points $(i/n, \sum_{j \leq i} w_j)$ together with the point (0, 0). w^* is required to be a straight line when the points can be interpolated in this way.

The main advantage of the OWAWA operator against the WOWA operator is that it can integrate the OWA and the weighted average in the same formulation taking into account the degree of importance that each concept has in the specific problem considered. The WOWA operator integrates both concepts but does not allow such degree of flexibility. Some other methods considered the use of the OWA operator with the probability including the immediate probability [7,41] and the POWA operator [23].

2.3. The OWA distance

The OWAD (or Hamming OWAD) operator [20,32] is an extension of the traditional normalized Hamming distance by using OWA operators. The main difference is that the reordering of the arguments of the individual distances is developed according to their values. Therefore, it is possible to calculate the distance modifying the results according to the interests or attitude of the decision maker under or overestimating the information. It can be defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 6. An OWAD operator of dimension n is a mapping OWAD: $[0,1]^n \times [0,1]^n \to [0,1]$ that has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ such that:

$$OWAD\,(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \ldots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j D_j, \tag{8}$$

where D_j is the jth largest individual distance of the $|x_i - y_i|$.

Note that this definition can be generalized for all the real numbers R by using OWAD: $R^n \times R^n \to R$. Observe also that it is possible to distinguish between ascending and descending orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the descending OWAD (DOWAD) operator and w^*_{n-j+1} the jth weight of the ascending OWAD (AOWAD) operator. The OWAD operator can be extended with generalized means forming the Minkowski OWAD (MOWAD) as follows:

$$MOWAD\,(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \ldots, \langle x_n, y_n \rangle) = \left(\sum_{j=1}^{n} w_j D_j^{\lambda} \right)^{1/\lambda}, \tag{9}$$

where D_j is the jth largest individual distance of the $|x_i - y_i|$ and λ is a parameter such that $\lambda \in \{-\infty, \infty\} - \{0\}$.

3. The OWA weighted average distance

3.1. Main concepts

The ordered weighted averaging weighted averaging distance (OWAWAD) operator is a distance measure that uses the WA and the OWA operator in the normalization process of the Hamming distance by using the OWAWA operator. Thus, the reordering of the individual distances is developed according to the values of the individual distances formed by comparing two sets. Note that we could also call it the weighted OWA distance (WOWAD). In order to continue with the same names as in previous research we use OWAWAD but it is also possible to use WOWAD. It can be defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 7. An OWAWAD operator of dimension n is a mapping OWAWAD: $R^n \times R^n \to R$ that has an associated weighting vector W such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWAD\,(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \ldots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{10}$$

where b_j is the jth largest individual distance of the $|x_i - y_i|$, each argument $|x_i - y_i|$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$ and v_j is the weight v_i ordered according to b_j, that is, according to the jth largest of the $|x_i - y_i|$.

Note that it is also possible to formulate the OWAWAD operator separating the part that strictly affects the OWAD operator and the part that affects the WAD. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

Definition 8. An OWAWAD operator is a mapping *OWAWAD*: $R^n \times R^n \rightarrow R$ of dimension *n*, if it has an associated weighting vector *W*, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ and a weighting vector *V* that affects the WAD, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, such that:

$$OWAWAD\ (\langle x_1,y_1\rangle, \langle x_2,y_2\rangle, \ldots, \langle x_n,y_n\rangle) = \beta\sum_{j=1}^{n} w_j b_j + (1-\beta)\sum_{i=1}^{n} v_i|x_i - y_i|, \quad (11)$$

where b_j is the *j*th largest of the arguments $|x_i - y_i|$ and $\beta \in [0,1]$.

Both equations are equivalent and provide the same results. Theorem 1 proves this equivalency as follows.

Theorem 1. By using, $\hat{v}_j = \beta w_j + (1-\beta)v_j$ Eq. (8) and (9) are equivalent.

Proof. Let

$$\beta\sum_{j=1}^{n} w_j b_j + (1-\beta)\sum_{i=1}^{n} v_i|x_i - y_i| = \sum_{j=1}^{n} (\beta w_j + (1-\beta)v_j) \times b_j, \quad (12)$$

being b_j the *j*th largest of the $|x_i - y_i|$ and v_j is the reordering of the *i*th weights. Since $\beta w_j + (1-\beta)v_j = \hat{v}_j$, then

$$\sum_{j=1}^{n} \hat{v}_j b_j = \beta\sum_{j=1}^{n} w_j b_j + (1-\beta)\sum_{i=1}^{n} v_i|x_i - y_i|$$

If *D* is a vector corresponding to the ordered arguments b_j that represent the individual distances, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then the OWAWAD operator can be represented as: $W^T D$.

From a generalized perspective of the reordering step it is possible to distinguish between descending (DOWAWAD) and ascending (AOWAWAD) orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the *j*th weight of the DOWAWAD and w^*_{n-j+1} the *j*th weight of the AOWAWAD operator. In order to understand the OWAWAD numerically, let us present a simple example.

Example 1. Let *X* = (0.4, 0.8, 0.7, 0.5) and *Y* = (0.2, 0.6, 0.4 0.9) be two sets of arguments. *W* = (0.2, 0.2, 0.3, 0.3) and *V* = (0.1, 0.2, 0.3, 0.4) are the weighting vectors and β = 0.3. The OWAWAD is calculated as follows by using Eq. (9):

OWAWAD = 0.3 × [|0.5–0.9| × 0.2 + |0.7–0.4| × 0.2 + |0.4–0.2| × 0.3 + |0.8–0.6| × 0.3] + (1–0.3) × [|0.4–0.2| × 0.1 + |0.8–0.6| × 0.2 + |0.7–0.4| × 0.3 + |0.5–0.9| × 0.4] = 0.289.

As we can see, the reordering process is only produced in the OWAD aggregation while in the WAD the initial ordering remains stable. Therefore, the OWAWAD operator under or over estimates the information partially because it also takes into account the classical subjective information given by the weighted average. Obviously, if β is close to 1, the under or over estimation becomes much stronger and close to the OWAD operator [20]. Note that if the weighting vector is not normalized, i.e. $\hat{V} = \sum_{j=1}^{n} \hat{v}_j \neq 1$, the OWAWAD operator can be expressed as:

$$OWAWAD\ (\langle x_1,y_1\rangle, \langle x_2,y_2\rangle, \ldots, \langle x_n,y_n\rangle) = \frac{1}{\hat{V}}\sum_{j=1}^{n} \hat{v}_j b_j. \quad (13)$$

Observe that the OWAWAD operator accomplishes reflexivity, non-negativity, monotonicity, boundary condition, idempotency and the commutativity of a distance measure.

Theorem 2. *(Reflexivity). Assume f is the OWAWAD operator, then*

$$f(\langle x_1,x_1\rangle, \ldots, \langle x_n,x_n\rangle) = 0. \quad (14)$$

Proof. It is straightforward and thus omitted.

Theorem 3. *(Nonnegativity).* Assume f is the OWAWAD operator, then

$$f(\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) \geq 0. \quad (15)$$

Proof. It is straightforward and thus omitted.

Theorem 4. *(Monotonicity).* Assume f is the OWAWAD operator; if $|x_i - y_i| \geq |c_i - d_i|$, *for all* i_i, *then*

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) \geq f(\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle). \quad (16)$$

Proof. It is straightforward and thus omitted.

Theorem 5. *(Boundary condition). Assume f is the OWAWAD operator, then*

$$\min\{|x_i - y_i|\} \leq f(\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) \leq \max\{|x_i - y_i|\}. \quad (17)$$

Proof. It is straightforward and thus omitted.

Theorem 6. *(Idempotency).* Assume f is the OWAWAD operator; if $|x_i - y_i| = a$, *for all i, then*

$$f(\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = a. \quad (18)$$

Proof. It is straightforward and thus omitted.

Theorem 7. *(Commutativity – distance measure). Assume f is the OWAWAD operator, then*

$$f(\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = f(\langle y_1,x_1\rangle, \ldots, \langle y_n,x_n\rangle). \quad (19)$$

Proof. It is straightforward and thus omitted.

Note that the OWAWAD operator does not accomplish the commutativity of the OWA operator because a part of its formulation uses the weighted average which is not commutative. Therefore, any initial ordering of the data may not always provide the same result because the initial ordering conditions the aggregation developed in the weighted average.

Another interesting issue to consider is the attitudinal character of the OWAWAD operator. Using a similar methodology as it was used for the OWA operator we can define the following measure:

$$\alpha(\hat{V}) = \beta\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}\right) + (1-\beta)\sum_{j=1}^{n} v_j\left(\frac{n-j}{n-1}\right). \quad (20)$$

Under the attitudinal perspective, the weighted average could be seen as neutral, thus forming the following expression:

$$\alpha(\hat{V}) = \beta\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}\right) + (1-\beta) \times 0.5. \quad (21)$$

Note also that other measures [15] could be discussed such as the entropy of dispersion [34], the divergence of *W* or the balance operator. The entropy of dispersion, which includes the Shannon entropy [25], is defined as follows:

$$H(\hat{V}) = -\left(\beta\sum_{j=1}^{n} w_j \ln(w_j) + (1-\beta)\sum_{i=1}^{n} v_i \ln(v_i)\right). \quad (22)$$

For the divergence of *W*, note that $\alpha(\hat{V})$ is decomposed in $\alpha(W)$ for the OWA part and $\alpha(V)$ for the weighted average. Thus, we get the following:

$$Div(\hat{V}) = \beta\left(\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}-\alpha(W)\right)^2\right) + (1-\beta)\left(\sum_{j=1}^{n} v_j\left(\frac{n-j}{n-1}-\alpha(V)\right)^2\right). \tag{23}$$

And for the balance operator:

$$Bal(\hat{V}) = \beta\left(\sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right)w_j\right) + (1-\beta)\left(\sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right)v_j\right). \tag{24}$$

Following the literature on OWA operators, it is also possible to study distance measures with other approaches that integrate the OWA operator with the weighted average. In this context, Xu [30] developed the hybrid averaging distance and Merigó and Gil-Lafuente [21] did the immediate weighted OWA distance (IW-OWAD). Another extension that could be developed is by using the WOWA operator [26] with distance measures forming the weighted OWA distance (WOWAD). It is defined as follows.

Definition 9. Let $P=(p_1, p_2, \ldots, p_n)$ and $W=(w_1, w_2, \ldots, w_n)$ be two weighting vectors of dimension n such that $p_i \in [0,1]$ and $\sum_{i=1}^{n} p_i = 1$, and $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. A set $D=\{D_1, D_2, \ldots, D_n\}$ is defined as the individual distances $D_i=|x_i - y_i|$ of the sets $X=\{x_1, x_2, \ldots, x_n\}$ and $Y=\{y_1, y_2, \ldots, y_n\}$. In this case, a mapping $WOWAD$: $R^n \times R^n \to R$ is a WOWAD operator of dimension n if:

$$WOWAD\ (D_1, D_2, \ldots, D_n) = \sum_{i=1}^{n} \omega_i D_{\sigma(i)}, \tag{25}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ such that $D_{\sigma(i-1)} \geq D_{\sigma(i)}$ for all $i=2, \ldots, n$, and the weight ω_i is defined as:

$$\omega_i = w*\left(\sum_{j\leq i} p_{\sigma(j)}\right) - w*\left(\sum_{j<i} p_{\sigma(j)}\right), \tag{26}$$

with w^* a monotone increasing function that interpolates the points $(i/n, \sum_{j\leq i} w_j)$ together with the point (0, 0) and w^* is required to be a straight line when the points can be interpolated in this way.

3.2. Families of OWAWAD operators

In the OWAWAD operator, we find different families of aggregation operators. First, we are going to consider the two main cases of the OWAWAD operator that are found by analysing the coefficient β.

Remark 1. If $\beta=0$, we get the weighted Hamming distance and if $\beta=1$, the OWAD operator. Note also that if $v_i=1/n$, for all i, and $w_j=1/n$, for all j, then, we get the normalized Hamming distance.

Remark 2. If $v_i=1/n$, for all i, then, we get the unification between the normalized Hamming distance and the OWAD operator. This operator is named the arithmetic OWAD operator and it is formulated as follows:

$$OWAWAD\ (\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = \beta\sum_{j=1}^{n} w_j b_j + (1-\beta)\frac{1}{n}\sum_{i=1}^{n}|x_i - y_i|. \tag{27}$$

Remark 3. If $w_j=1/n$, for all j, it is formed the unification between the normalized Hamming distance and the weighted Hamming distance that we call the arithmetic weighted Hamming distance. It is expressed in the following way:

$$OWAWAD\ (\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = \beta\frac{1}{n}\sum_{j=1}^{n} b_j + (1-\beta)\sum_{i=1}^{n} v_i|x_i - y_i|. \tag{28}$$

Remark 4. If one of the sets is empty, we get the OWAWA operator.

Remark 5. By choosing a different manifestation of the weighting vector in the OWAWAD operator, we are able to obtain different types of aggregation operators. For example, we can obtain the maximum weighted Hamming distance and the minimum weighted Hamming distance. The maximum weighted Hamming distance is found when $w_1=1$ and $w_j=0$ for all $j \neq 1$. It is formulated as:

$$OWAWAD\ (\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = \beta \times Max\{|x_i - y_i|\} + (1-\beta) \times \sum_{i=1}^{n} v_i|x_i - y_i|. \tag{29}$$

And the minimum weighted Hamming distance is formed when $w_n=1$ and $w_j=0$ for all $j \neq n$. Thus:

$$OWAWAD\ (\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) = \beta \times Min\{|x_i - y_i|\} + (1-\beta) \times \sum_{i=1}^{n} v_i|x_i - y_i|. \tag{30}$$

Remark 6. Many other particular cases could be developed following the OWA literature [22,36] including:

- The step-OWAWAD is formed when $w_k=1$ and $w_j=0$ for all $j \neq k$. Note that if $k=1$, the step-OWAWAD is transformed into the maximum weighted Hamming distance, and if $k=n$, the step-OWAWAD becomes the minimum weighted Hamming distance.
- The median-OWAWAD: If n is odd we assign $w_{(n+1)/2}=1$ and $w_j=0$ for all others. If n is even we assign for example, $w_{n/2}=w_{(n/2)+1}=0.5$ and $w_j=0$ for all others.
- The weighted median-OWAWAD: We select the argument b_k that has the kth largest argument such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k-1$ is less than 0.5.
- The olympic-OWAWAD is generated when $w_1=w_n=0$, and for all others $w_j=1/(n-2)$.
- Note that it is possible to develop a general form of the olympic-OWAWAD by considering that $w_j=0$ for $j=1, 2, \ldots, k, n, n-1, \ldots, n-k+1$, and for all others $w_{j*}=1/(n-2k)$, where $k<n/2$. Note that if $k=1$, then this general form becomes the usual olympic-OWAWAD operator.
- Another family of aggregation operator that could be used is the centered-OWAWAD operator. We can define it as a centered aggregation operator if it is symmetric, strongly decaying and inclusive. That is, symmetric if $w_j=w_{n-j+1}$ for all j, strongly decaying when $i<j\leq(n+1)/2$ then $w_i<w_j$ and when $i>j\geq(n+1)/2$ then $w_i<w_j$, and inclusive if $w_j>0$, for all j.
- The S-OWAWAD operator. It can be subdivided into three classes: the "or-like," the "and-like" and the generalized S-OWAWAD operators. The generalized S-OWAWAD operator is obtained if $w_1=(1/n)(1-(\alpha+\beta))+\alpha$, $w_n=(1/n)(1-(\alpha+\beta))+\beta$, and $w_j=(1/n)(1-(\alpha+\beta))$ for $j=2$ to $n-1$, where $\alpha, \beta \in [0,1]$ and $\alpha+\beta \leq 1$. Note that if $\alpha=0$, it becomes the "and-like" S-OWAWAD operator and if $\beta=0$, the "or-like" S-OWAWAD operator.
- A further interesting type is the non-monotonic-OWAWAD operator. It is obtained when at least one of the weights w_j is lower than 0 and $\sum_{j=1}^{n} w_j = 1$. Note that a key aspect of this operator is that it does not always achieve monotonicity.

4. Generalized aggregation operators in the OWAWAD operator

The OWAWAD operator can also be further extended by using generalized and quasi-arithmetic means [8,38]. Thus, it becomes a more general formulation that can represent a wide range of particular cases. By using generalized means we get the generalized (or Minkowski) OWAWAD (GOWAWAD) operator. It is defined as follows.

Definition 10. A GOWAWAD operator is a mapping *GOWAWAD*: $R^n \times R^n \to R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ and a weighting vector V, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, such that:

$$GOWAWAD\left(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle\right) = \beta\left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} + (1-\beta)\left(\sum_{i=1}^{n} v_i |x_i - y_i|^{\delta}\right)^{1/\delta}, \tag{31}$$

where b_j is the jth largest of the arguments $|x_i - y_i|$, $\beta \in [0,1]$ and is a parameter such that λ and $\delta \in \{-\infty, \infty\} - 0$.

As we can see, if $\beta = 1$, the GOWAWAD operator becomes the Minkowski OWA distance (MOWAD) and if $\beta = 0$, it becomes the weighted Minkowski distance.

Next, let us look into the quasi-arithmetic OWAWAD (Quasi-OWAWAD) operator. As the name indicates it generalized the OWAWAD operator by using quasi-arithmetic means. It can be defined as follows.

Definition 11. A Quasi-OWAWAD operator is a mapping *Quasi-OWAWAD*: $R^n \times R^n \to R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ and a weighting vector V that affects the WAD, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, such that:

$$QOWAWAD\left(\langle x_1, y_1\rangle,\ldots,\langle x_n, y_n\rangle\right) = \beta g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right) + (1-\beta) h^{-1}\left(\sum_{i=1}^{n} v_i h\left(|x_i - y_i|\right)\right), \tag{32}$$

where b_j is the jth largest of the arguments $|x_i - y_i|$, $\beta \in [0,1]$ and g and h are strictly continuous monotonic functions.

Note that if $\beta = 1$, it becomes the Quasi-OWAD operator [17] and if $\beta = 0$, it becomes the weighted quasi-arithmetic Minkowski distance. Moreover, it includes the GOWAWAD operator as a particular case when $g = b^{\lambda}$ and $h = |x_i - y_i|^{\lambda}$. Observe that λ and δ, and g and h may be equal or different depending on the specific needs of the aggregation process considered. Both the GOWAWAD and the Quasi-OWAWAD operator include a wide range of particular cases worth mentioning. Table 1 presents some of these particular cases.

Note that a similar analysis could be developed with the hybrid average, immediate weights and WOWAD operator. The use of generalized aggregation operators with immediate weights and distances measures was studied by Merigó and Gil-Lafuente [21] and with hybrid averages by Xu [30]. The quasi-arithmetic WOWAD (Quasi-WOWAD) operator is very similar to Definition 9 with the difference that it uses quasi-arithmetic means as follows:

$$Quasi\text{-}WOWAD\left(D_1,D_2,\ldots,D_n\right) = g^{-1}\left(\sum_{i=1}^{n} \omega_i g(D_{\sigma(i)})\right), \tag{33}$$

Table 1
Particular cases of GOWAWAD and Quasi-OWAWAD operators.

Particular case	GOWAWAD	Quasi-OWAWAD
OWAWAD	$\lambda = 1, \delta = 1$	$g = b, h = \lvert x-y \rvert$
Harmonic OWAWAD	$\lambda = -1, \delta = -1$	$g = b^{-1}, h = \lvert x-y \rvert^{-1}$
Quadratic (Euclidean) OWAWAD	$\lambda = 2, \delta = 2$	$g = b^2, h = \lvert x-y \rvert^2$
Geometric OWAWAD	$\lambda \to 0, \delta \to 0$	$g \to b^0, h \to \lvert x-y \rvert^0$
Cubic OWAWAD	$\lambda = 3, \delta = 3$	$g = b^3, h = \lvert x-y \rvert^3$
Maximum distance	$\lambda = \infty, \delta = \infty$	$g = b^{\infty}, h = \lvert x-y \rvert^{\infty}$
Minimum distance	$\lambda = -\infty, \delta = -\infty$	$g = b^{-\infty}, h = \lvert x-y \rvert^{-\infty}$
OWA weighted Euclidean distance	$\lambda = 1, \delta = 2$	$g = b, h = \lvert x-y \rvert^2$
Euclidean OWA weighted distance	$\lambda = 2, \delta = 1$	$g = b^2, h = \lvert x-y \rvert$
OWA weighted cubic distance	$\lambda = 1, \delta = 3$	$g = b, h = \lvert x-y \rvert^3$
Cubic OWA weighted distance	$\lambda = 3, \delta = 1$	$g = b^3, h = \lvert x-y \rvert$
Cubic OWA weighted quadratic distance	$\lambda = 3, \delta = 2$	$g = b^3, h = \lvert x-y \rvert^2$
GOWAWAD	–	$g = b^{\lambda}, h = \lvert x-y \rvert^{\lambda}$

where g is a strictly continuous monotonic function. Note that a similar expression could be developed by using generalized means forming the generalized WOWAD (GWOWAD) operator as follows:

$$GWOWAD\left(D_1,D_2,\ldots,D_n\right) = \left(\sum_{i=1}^{n} \omega_i D_{\sigma(i)}^{\lambda}\right)^{1/\lambda}, \tag{34}$$

where λ is a parameter such that $\lambda \in \{-\infty, \infty\} - \{0\}$.

5. Bonferroni means and OWAWAD operators

The Bonferroni mean [6] is another type of mean that can be used in the aggregation process in order represent the information. Recently, several authors have used it when dealing with the OWA operator [1,39] and also with other types of information including linguistic variables [27] and intuitionistic [28,29] and hesitant fuzzy representations [52,53]. First, let us briefly recall the Bonferroni mean and the Bonferroni OWA (BON-OWA).

$$B\left(a_1,a_2,\ldots,a_n\right) = \left(\frac{1}{n}\frac{1}{n-1}\sum_{\substack{ij=1\\ i\neq j}}^{n} a_i^r a_j^q\right)^{\frac{1}{r+q}}, \tag{35}$$

where r and q are parameters such that $r, q \geq 0$ and the arguments $a_i, a_j \geq 0$. By rearranging the terms [1], it can be formulated in the following way:

$$B\left(a_1,a_2,\ldots,a_n\right) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)\right)^{\frac{1}{r+q}}. \tag{36}$$

In this paper, let us introduce the OWA operator in the Bonferroni mean in a different way as it was done by Yager [39] by using the following expression:

$$BON\text{-}OWA\ (a_1,a_2,\ldots,a_n) = \left(\sum_{k=1}^{n} w_k a_k^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq k}}^{n} a_j^q\right)\right)^{\frac{1}{r+q}}, \tag{37}$$

where a_k is the kth smallest of the a_i.

When dealing with distance measures, it is also possible to use the Bonferroni mean. The difference is that the arguments a_i and a_j now are two set of variables instead of one but the methodology is the same. Thus, we get the Bonferroni distance as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$:

$$BD\ (\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle) = \left(\frac{1}{n}\frac{1}{n-1}\sum_{\substack{ij=1\\ i\neq j}}^{n} d_i^r d_j^q\right)^{\frac{1}{r+q}}, \tag{38}$$

where d_i and d_j are the individual distances such that $d_i = |x_i - y_i|$ and $d_j = |x_j - y_j|$.

Similarly, it is also possible to extend the BON-OWA with distance measures forming the BON-OWA distance (BON-OWAD) as follows:

$$BON\text{-}OWAD\ (\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle) = \left(\sum_{k=1}^{n} w_k D_k^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq k}}^{n} |x_j-y_j|^q\right)\right)^{\frac{1}{r+q}}, \tag{39}$$

where D_k is the kth smallest of the individual distances $|x_i - y_i|$.

Next, let us study the use of the OWAWA operator in the Bonferroni mean. Thus, we get the BON-OWAWA operator that includes the weighted Bonferroni mean and the BON—OWA operator in the same formulation considering the degree of importance that both concepts have in the aggregation. It is expressed as:

$$BON\text{-}OWAWA\ (a_1, a_2, \ldots, a_n) = \beta \times \left(\sum_{k=1}^{n} w_k a_k^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq k}}^{n} a_j^q\right)\right)^{\frac{1}{r+q}} + (1-\beta) \times \left(\sum_{i=1}^{n} w_i a_i^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)\right)^{\frac{1}{r+q}}, \tag{40}$$

where a_k is the kth smallest of the a_i and $\beta \in [0,1]$. Note that here we assume that r and q have the same values for the OWA and the weighted average. But we may find situations where they have different values.

As we can see, if $\beta = 1$, the BON-OWAWA operator becomes the BON-OWA operator and if $\beta = 0$, the weighted Bonferroni mean that can be formulated as:

$$BON\text{-}WA\ (a_1,a_2,\ldots,a_n) = \left(\sum_{i=1}^{n} w_i a_i^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)\right)^{\frac{1}{r+q}}. \tag{41}$$

By using the OWAWAD operator, it is also possible to develop a new type of Bonferroni mean. In this case, we get the Bonferroni OWAWAD (BON-OWAWAD) operator that can be formulated in the following way:

$$BON\text{-}OWAWAD\ (\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \beta \times \left(\sum_{k=1}^{n} w_k D_k^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq k}}^{n} |x_j-y_j|^q\right)\right)^{\frac{1}{r+q}} + (1-\beta) \times \left(\sum_{i=1}^{n} w_i |x_i-y_i|^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} |x_j-y_j|^q\right)\right)^{\frac{1}{r+q}}, \tag{42}$$

where D_k is the kth smallest of the individual distances $|x_i - y_i|$ and $\beta \in [0,1]$. Observe that $\beta = 1$ forms the BON-OWAD operator and $\beta = 0$ does the weighted Bonferroni distance (BON-WAD) that is expressed as:

$$BON\text{-}WAD\ (\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle) = \left(\sum_{i=1}^{n} w_i |x_i-y_i|^r \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} |x_j-y_j|^q\right)\right)^{\frac{1}{r+q}}. \tag{43}$$

Finally, note that many particular cases of the previous extensions regarding the Bonferroni mean could be studied following the methodology explained in Sections 3 and 4 including the use of immediate weights, hybrid averages and weighted OWAs.

6. Decision making with Dempster-Shafer belief structure and OWAWAD operators

In this Section, we present an application of the OWAWAD operator in order to see its applicability. An illustrative example in group decision making with Dempster-Shafer belief structure is studied.

6.1. Decision making approach

The OWAWAD operator can be used in decision making problems with D-S theory [19,35]. Its main advantage is the possibility of combining weighted averages, OWA operators and distance measures in a general framework that also deals with probabilistic information. Therefore, this approach provides a more general representation of the information because it can consider different sources and at the same time be reduced to the classical approach [35,44]. The decision process with the OWAWAD operator can be summarized as follows.

Assume a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. d_{ih} is the payoff if the decision maker selects alternative A_i and the state of nature is S_h. It is an individual distance that considers the differences between two sets $X = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$ and $Y = \{y_{i1}, y_{i2}, \ldots, y_{in}\}$ such that $d_{ih} = |x_{ih} - y_{ih}|$. These sets represent the collective information provided by a group of experts $E = \{e_1, \ldots, e_p, \ldots, e_z\}$ from the subsets $X_{ep} = \{x_{p1}, x_{p2}, \ldots, x_{pn}\}$ and $Y_{ep} = \{y_{p1}, y_{p2}, \ldots, y_{pn}\}$. The knowledge of the state of nature is captured in terms of a belief structure m with focal elements $B_1, B_2, \ldots, B_r$. A weight $m(B_k)$ is associated with each of these focal elements. The aim of the problem is to select the alternative which provides the best solution to the decision maker. In doing so, the following steps should be followed:

Step 1: Analyse the results of the payoff matrix and the individual distances.

Step 2: Calculate the collective results by using the weighting vector $U = (u_1, u_2, \ldots, u_z)$ with $x_i = \sum_{p=1}^{z} u_{pi} x_{pi}$ and $y_i = \sum_{p=1}^{z} u_{pi} y_{pi}$.

Step 3: Calculate the belief function m regarding the states of nature.

Step 4: Find the attitudinal character (or degree of optimism) of the decision maker $\alpha(W)$ [34] and the weights $W = (w_1, \ldots, w_n)$ and $V = (v_1, \ldots, v_n)$ to be used in the OWAWAD operator for each cardinality of focal elements. Also calculate the degree of importance of each concept by determining the value of β.

Step 5: Calculate the individual distances between X and Y.

Step 6: Determine the results of the collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{d_{ih} \mid S_h \in B_k\}$.

Step 7: Calculate the aggregated results, $V_{ik} = OWAWAD(M_{ik})$, using Eq. (8) or (9), for all the values of i and k.

Step 8: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k). \tag{44}$$

Step 9: Select the alternative with the lowest C_i as the optimal because usually we look for the alternative with the lowest distance. Usually, this happens because when calculating distances in a selection process [20] we compare the alternatives with ideal situations.

Studying the aggregation in *Steps* 6 and 7 of the previous decision making process, it is possible to formulate in one equation the whole aggregation system. We call this process the belief structure – OWAWAD (BS-OWAWAD) aggregation. It can be defined as follows.

Definition 12. A BS-OWAWAD operator is defined by

$$C_i = \sum_{k=1}^{r} \sum_{j=1}^{q} m(B_k) \hat{v}_{jk} b_{jk}, \tag{45}$$

where $\hat{v}_{jk}$ is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} \hat{v}_{jk} = 1$ and $\hat{v}_{jk} \in [0,1]$, b_{jk} is the jth largest of the ith d_{ik}, each argument $d_{ik} = |x_{ih} - y_{ih}|$ with $M_{ik} = \{d_{ih} \mid S_h \in B_k\}$, has an associated weight v_{ik} with $\sum_{j=1}^{n} \hat{v}_{ik} = 1$ and $v_{ik} \in [0,1]$, and a weight w_{jk} with $\sum_{j=1}^{n} w_{ik} = 1$ and $w_{jk} \in [0,1]$, $\hat{v}_{jk} = \beta w_{jk} + (1-\beta) v_{jk}$ with $\beta \in [0,1]$, $m(B_k)$ is the basic probability assignment, $x_{ih} = \sum_{p=1}^{z} u_{pih} x_{pih}$ and $y_{ih} = \sum_{p=1}^{z} u_{pih} y_{pih}$ being u_{pih} the weights of the experts opinions such that $u_{pih} \in [0,1]$ and $\sum_{p=1}^{z} u_{pih} = 1$.

Note that by choosing a different manifestation in the weighting vector of the OWAWA operator, we are able to develop different families of BS-OWAWAD operators in a similar way as it has been explained in the previous sections.

Table 2
Payoff matrix – Group of experts 1.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
A_1	0.5	0.7	0.7	0.9	0.6	0.3	0.6
A_2	0.6	0.8	0.5	0.3	0.7	0.8	0.5
A_3	0.6	0.7	0.9	0.8	0.5	0.4	0.3
A_4	0.5	0.2	0.7	0.8	0.4	0.5	0.3
A_5	0.8	0.4	0.3	0.6	0.6	0.7	0.6

Table 3
Payoff matrix – Group of experts 2.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
A_1	0.4	0.7	0.4	0.3	0.5	0.2	0.7
A_2	0.5	0.8	0.4	0.3	0.6	0.2	0.5
A_3	0.6	0.8	0.4	0.1	0.3	0.5	0.8
A_4	0.6	0.3	0.2	0.7	0.9	0.4	0.6
A_5	0.5	0.3	0.4	0.8	0.7	0.7	0.6

6.2. Illustrative example

In this Section, we are going to develop a numerical example in order to understand the methodology to be used with the OWAWAD operator in group decision making problems with Dempster-Shafer belief structure.

Step 1: Let us assume an enterprise that is planning the general strategy for the next year. A key issue the company is considering is the possibility of expanding to new markets. Currently, they operate in Europe and North America. After careful review of the information, they consider five general strategies.

- A_1 = Expand to the South American market.
- A_2 = Expand to the Asian market.
- A_3 = Expand to the African market.
- A_4 = Expand to the three continents.
- A_5 = Do not make any expansion.

The evaluation of these strategies is developed by three groups of experts. They analyse each strategy considering seven general characteristics that explain the potential benefits that each alternative may produce. These characteristics are classified as follows:

- S_1 = Expected benefits in the short term.
- S_2 = Expected benefits in the mid-term.
- S_3 = Expected benefits in the long term.
- S_4 = Risk of the strategy.
- S_5 = Available information for the strategy.
- S_6 = Additional growth in the long term.
- S_7 = Other variables.

Step 2: The three groups of experts provide their own opinion regarding each strategy and characteristic considered. The results are shown in Tables 2–4.

Table 4
Payoff matrix – Group of experts 3.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
A_1	0.6	0.7	0.4	0.5	0.3	0.6	0.5
A_2	0.6	0.8	0.4	0.6	0.5	0.8	0.3
A_3	0.5	0.6	0.3	0.8	0.6	0.7	0.4
A_4	0.7	0.5	0.7	0.7	0.5	0.4	0.7
A_5	0.8	0.6	0.5	0.7	0.5	0.7	0.6

Table 5
Payoff matrix – Collective results.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
A_1	0.49	0.7	0.55	0.64	0.51	0.33	0.61
A_2	0.57	0.8	0.45	0.36	0.63	0.62	0.46
A_3	0.58	0.71	0.63	0.59	0.46	0.49	0.47
A_4	0.57	0.29	0.55	0.75	0.57	0.45	0.47
A_5	0.71	0.41	0.37	0.68	0.61	0.7	0.6

In this example, we assume the following weighting vector for the three groups of experts: $U = (0.5, 0.3, 0.2)$. Thus, we can aggregate their opinions obtaining a single collective payoff matrix. Note that these weights indicate that the first group has a degree of importance of 50% (Table 2), the second group 30% (Table 3) and the third group 20% (Table 4). The results are shown in Table 5.

Observe that the collective results are calculated with the weighting vector U and Tables 2–4. For example, for A_1S_1, we proceed as follows: $0.5 \times 0.5 + 0.4 \times 0.3 + 0.6 \times 0.2 = 0.49$.

Next, they develop a similar analysis when defining the ideal strategy which represents the optimal results that a strategy should produce. In the real world usually there is not such an ideal strategy but the objective is to find the strategy closest to this ideal one. Each group provides its own opinions and are also aggregated with the weighted vector U as shown in Table 6.

Step 3: The complexities of the environment can be assessed with probabilistic information that is represented in terms of belief structures.

Focal element

$m(B_1) = m(\{S_1, S_2, S_3\}) = 0.4$

$m(B_2) = m(\{S_3, S_4, S_5\}) = 0.3$

Table 6
Ideal strategies.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
Ideal – Group 1	0.8	0.9	1	1	0.9	0.8	0.8
Ideal – Group 2	0.7	0.8	1	0.9	0.9	0.8	0.7
Ideal – Group 3	0.7	0.9	1	1	0.8	0.9	0.8
Ideal – Collective	0.75	0.87	1	0.97	0.88	0.82	0.77

Table 7
Individual distances.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
A_1	0.26	0.17	0.45	0.33	0.37	0.49	0.16
A_2	0.18	0.07	0.55	0.61	0.25	0.2	0.31
A_3	0.17	0.16	0.37	0.38	0.42	0.33	0.3
A_4	0.18	0.58	0.45	0.22	0.31	0.37	0.3
A_5	0.04	0.46	0.63	0.29	0.27	0.12	0.17

$m(B_3) = m(\{S_5, S_6, S_7\}) = 0.3$

Step 4: The attitudinal character of the enterprise is defined according to the attitudes of the different groups of experts that analyse the problems since it is assumed that the directors of the company are included in these groups. The experts establish the following weighting vectors for the weighted average and the OWA operator: $W = (0.3, 0.3, 0.4)$ and $V = (0.4, 0.4, 0.2)$. They assume that the OWA has a degree of importance of 40% and the weighted average a degree of 60%.

Step 5: Calculate the distances between the collective results available in Table 5 and the ideal collective strategy shown in Table 6. The results are shown in Table 7.

Note that the results of Table 7 are formed as follows. For example, for A_1S_1: | Table 5 (A_1S_1) – Table 6 (Ideal Collective S_1)| = |0.49–0.75| = 0.26. And so on.

Step 6 and 7: With the information studied in the previous steps, the experts can obtain the aggregated results by using the distance measures studied in this paper and the focal information. Here, we consider the minimum (MinD) and maximum distance (MaxD), the minimum (Min-HD) and maximum Hamming distance (Max-HD), the minimum (Min-WAD) and maximum WAD (Max-WAD), the normalized (HD) and weighted Hamming distance

Table 8
Aggregated results.

	MinD	MaxD	Min-HD	Max-HD	Min-WAD	Max-WAD	HD	WAD	OWAD	OWAWAD
V_{11}	0.170	0.450	0.244	0.356	0.225	0.337	0.293	0.262	0.281	0.270
V_{12}	0.330	0.450	0.362	0.410	0.364	0.412	0.383	0.386	0.378	0.383
V_{13}	0.160	0.490	0.268	0.400	0.290	0.422	0.340	0.376	0.322	0.354
V_{21}	0.070	0.550	0.188	0.380	0.154	0.346	0.266	0.210	0.247	0.225
V_{22}	0.250	0.610	0.382	0.526	0.408	0.552	0.470	0.514	0.448	0.488
V_{23}	0.200	0.310	0.232	0.276	0.225	0.269	0.253	0.242	0.248	0.244
V_{31}	0.160	0.370	0.204	0.288	0.188	0.272	0.233	0.206	0.226	0.214
V_{32}	0.370	0.420	0.382	0.402	0.378	0.398	0.390	0.384	0.388	0.386
V_{33}	0.300	0.420	0.330	0.378	0.336	0.384	0.350	0.360	0.345	0.354
V_{41}	0.180	0.580	0.314	0.474	0.308	0.468	0.403	0.394	0.381	0.389
V_{42}	0.220	0.450	0.284	0.376	0.286	0.378	0.326	0.330	0.316	0.324
V_{43}	0.300	0.370	0.316	0.344	0.319	0.347	0.326	0.332	0.324	0.329
V_{51}	0.040	0.630	0.242	0.478	0.212	0.448	0.376	0.326	0.343	0.333
V_{52}	0.270	0.630	0.346	0.490	0.361	0.505	0.396	0.422	0.384	0.407
V_{53}	0.120	0.270	0.160	0.220	0.162	0.222	0.186	0.190	0.180	0.186

Table 9
Generalized expected value.

	MinD	MaxD	Min-HD	Max-HD	Min-WAD	Max-WAD	HD	WAD	OWAD	OWAWAD
A_1	0.215	0.462	0.287	0.385	0.286	0.385	0.334	0.333	0.322	0.329
A_2	0.163	0.496	0.259	0.393	0.252	0.385	0.323	0.311	0.308	0.310
A_3	0.265	0.400	0.295	0.349	0.289	0.343	0.315	0.306	0.310	0.308
A_4	0.228	0.478	0.306	0.406	0.305	0.405	0.357	0.356	0.344	0.352
A_5	0.133	0.522	0.249	0.404	0.242	0.397	0.325	0.314	0.306	0.311

Table 10
Ranking of the strategies.

	Ranking		Ranking
MinD	$A_5\}A_2\}A_1\}A_4\}A_3$	Max-WAD	$A_3\}A_1=A_2\}A_5\}A_4$
MaxD	$A_3\}A_1\}A_4\}A_2\}A_5$	HD	$A_3\}A_2\}A_5\}A_1\}A_4$
Min-HD	$A_5\}A_2\}A_1\}A_3\}A_4$	WAD	$A_3\}A_2\}A_5\}A_1\}A_4$
Max-HD	$A_3\}A_1\}A_2\}A_5\}A_4$	OWAD	$A_5\}A_2\}A_3\}A_1\}A_4$
Min-WAD	$A_5\}A_2\}A_1\}A_3\}A_4$	OWAWAD	$A_3\}A_2\}A_5\}A_1\}A_4$

(WAD), the OWAD and the OWAWAD operator. The results are shown in Table 8.

Observe that the results of Table 8 are calculated by using the weights β, W and V and the results of Table 7. For example, V_{11} in the OWAWAD operator is calculated as follows: $0.4 \times (0.45 \times 0.3 + 0.26 \times 0.3 + 0.17 \times 0.4)$ + $0.6 \times (0.26 \times 0.4 + 0.17 \times 0.4 + 0.45 \times 0.2) = 0.270$.

Step 8: Once we have found the aggregated results, we calculate the generalized expected value. The results are shown in Table 9.

The generalized expected value is calculated by using the results of Table 8 and the probabilistic weights of the focal elements explained in Step 3. For example, the MinD in A_1 is calculated as: $0.170 \times 0.4 + 0.330 \times 0.3 + 0.160 \times 0.3 = 0.215$.

As we can see, depending on the distance aggregation operator used, the results and decisions may be different. In this particular example, it seems that A_4 is the optimal choice because it has the lowest distance. This implies that it provides results closest to the ideal ones.

Finally, a further interesting issue is to establish a ranking of the strategies because sometimes it is good to consider more than one alternative. This issue arises when for some specific reason not only the first choice may be used in the decision. The results are shown in Table 10.

As we can see, depending on the aggregation operator used, the ranking of the strategies may vary. In this example, A_3 seems to provide the best results although sometimes A_5 is also optimal.

Observe that the minimum distance looks for the smallest distance in the aggregation and the maximum distance for the highest one. From a decision making perspective, this is connected to the concept of optimism and pessimism. The Min-HD, Max-HD, Min-WAD and Max-WAD have similar interpretations but now without looking into the extreme case. Instead, the decision maker adopts a conservative position but taking into account the minimum and maximum distances. The HD is the classical situation where the decision maker assumes that all the variables are equally important and the WAD gives different degrees of importance to the variables. The OWAD assumes that the importance of the variables is unknown so instead, it uses the attitudinal character of the decision maker. And the OWAWAD operator integrates the WAD and the OWAD in the same formulation. That is, it considers the attitudinal character and the importance of the variables at the same time.

7. Conclusions

This paper has introduced the OWAWAD operator and some of its main generalizations. It is an aggregation operator that unifies the weighted Hamming distance and the OWAD in the same formulation considering the degree of importance that each concept has in the analysis. Its fundamental advantage is the possibility of combining distance measures with subjective information and the attitudinal character of the decision maker. Therefore, it is able to under or overestimate partially the information taking into account the weighted average. This implies a partial ordering of the individual distances when dealing with the OWA operator. However, no change in the initial ordering occurs in the weighted average because this operator aggregates the information without using any additional reordering process. We have studied some of its key properties and particular cases including the arithmetic weighted Hamming distance, the arithmetic OWAD, the OWAD, the WAD, the maximum WAD and the minimum WAD.

The use of generalized and quasi-arithmetic means under this framework has also been studied obtaining the GOWAWAD and the Quasi-OWAWAD operator. These aggregation operators include a wide range of particular cases including the OWAWAD, the quadratic (or Euclidean) OWAWAD and the harmonic OWAWAD operator. This approach has also been considered by using the WOWAD operator forming the GWOWAD operator. Further extensions by using the Bonferroni mean have also been developed. This paper has suggested the use of distance measures with Bonferroni means and OWA operators obtaining the BON-OWAD and the BON-OWAWAD operators.

These new distance measures can be applied in a wide range of problems. In principle, they can be applied in all the studies that use any kind of Hamming distance because it is included as particular cases of these general formulations. Specially, they can be applied in a lot of statistical problems because they often use some kind of distance measure. Moreover, it can also be implemented in a lot of fields that use some type of average including decision theory, economics, physics or engineering. This paper has focused on a group decision making problem with Dempster-Shafer belief structure. The selection of strategies has been studied under this approach. We have seen that the OWAWAD operator is very flexible because it can consider many particular cases adapting to the needs of the specific problem considered.

In future research, we expect to develop further developments by using more complex formulations including the use of order-inducing variables, other types of probabilities, norms and moving averages. Other applications in decision making [48,50] or in other areas of research [49] will be considered including statistics, economics and engineering.

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