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# Incentive compatible and stable trade mechanisms on networks ☆

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**Abstract**

We study a network of buyers and sellers where each seller owns an indivisible object and has no incentive to keep it, while each buyer has a downward sloping demand curve which is private information. Only the connected buyer-seller pairs can engage in trade. We search for *trade mechanisms* that are *efficient*, *strategy-proof*, *bilateral trade stable* and *individually rational*. In general, there does not exist a trade mechanism simultaneously satisfying these properties. The tension between strategy-proofness and bilateral trade stability is generated by the intersection between sets of competitors of a buyer at different sellers. Such intersections often allow the buyer to manipulate (via *demand reductions*) the prices paid in the network. The observed tension can be resolved if and only if the underlying network is *cycle-free*. In such a case, there is a unique trade mechanism which satisfies our four properties, a *generalized Vickrey auction*.



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## 1. Introduction

We study a network of buyers and sellers in which buyers have downward sloping demand curves and sellers each have one unit of a homogeneous good to sell (henceforth, an object). While buyers' demand curves are private information, there are known gains from trade as each seller attaches zero value to the unit he owns. In our setting, decentralization may lead to inefficiencies due to coordination failures on mutually beneficial trades (Abreu and Manea, 2012; Elliott and Nava, 2015). Following Kranton and Minehart (2001), we take a mechanism design approach and introduce the notion of a *trade mechanism*. Given a network configuration, a trade mechanism simultaneously determines prices and allocations as a function of the buyers' reported valuations. We are interested in a set of properties that a trade mechanism should satisfy. Naturally, we want trades to be both *efficient* and *individually rational*. Since demand curves are private information, we also need to give the correct incentives for buyers to reveal their valuations. We impose a robust notion of incentive compatibility: a trade mechanism must be *strategy-proof*, i.e. truthfully reporting one's set of valuations is a (weakly) dominant strategy.

Sellers' incentives also need to be addressed in our context. Given a price-allocation pair, nothing prevents a seller from canceling his current trade and selling to another buyer who would then either buy an extra unit or cancel one of his trades as well. Such situations imply a failure of the prevailing trade mechanism and would cast doubt on its practical usefulness. Our property, *bilateral trade stability*, implies that no buyer-seller pair can profitably break away from the mechanism at the proposed price-allocation pair.[1] In the special case of our model where demands are single units, Kranton and Minehart (2001) show that there exists a unique mechanism which simultaneously satisfy those four properties, a *generalized Vickrey auction*. Because of the network constraints, several prices need to co-exist at equilibrium for the market to clear.

We first characterize (Proposition 1) the prices paid by buyers in any trade mechanism satisfying our four properties simultaneously. A buyer $i$ pays each seller who sells him an object the highest of the (unsatisfied) valuations of his competitors for this object. This is in the spirit of a Vickrey auction. Yet in a network of buyers and sellers, for each object, the set of competitors faced by buyer $i$ is not fixed but determined both by the allocation and the network structure.

Next, Theorem 1 fully characterizes the network structures for which such a trade mechanism exists or fails to exist. Our paper is the first to relate the possibility of strategy-proof exchange with the structure of the network on which exchange takes place.[2] When the network is *cycle-free*, at any possible allocation $x$, buyer $i$ faces different competitors at each seller to whom he is connected. We say that the competition faced by buyer $i$ for the different objects he gets at $x$ is *independent*, i.e. the sets of competitors that buyer $i$ faces for different objects never intersect. A trade mechanism satisfying our four properties resembles multiple isolated second price

1 Bilateral trade stability is already introduced in Kranton and Minehart (2001) as *Pairwise-Stability*.

2 In a recent paper, Pycia (2016) also studies strategy-proof exchange on networks but in a model without monetary transfers.

auctions, simultaneously computed for each object. This mechanism is unique because there is a unique generalized Vickrey auction that is compatible with bilateral trade stability.[3]

When the network exhibits *cycles* the combination of downward sloping demands with competition amongst buyers in the network creates a tension between strategy-proofness and bilateral trade stability. Contrary to the cycle-free case, at a given allocation $x$, the competition faced by buyer $i$ for the different objects he buys is typically *not independent*, i.e. some of buyer $i$'s competitors are simultaneously bidding for several objects that $i$ receives. To see how this is problematic, suppose there is a buyer $j$ who (a) is a competitor of buyer $i$ at two objects $\alpha$ and $\beta$ that $i$ buys, and (b) has the highest willingness to pay among all of buyer $i$'s competitors for $\alpha$ and $\beta$. It is as if $j$ is the second highest bidder for both $\alpha$ and $\beta$ and, as we show in Proposition 1, buyer $j$'s valuation determines the prices buyer $i$ pays for $\alpha$ and $\beta$. If buyer $j$ were to obtain one object, say $\beta$, then by the downward sloping demand assumption, $j$'s willingness to pay for an extra object would decrease. This softens the competition faced by buyer $i$ for object $\alpha$. Buyer $i$ then faces a trade-off between the number of objects that he gets and the prices he pays for them. Buyer $i$ might gain by giving up one object. We call this type of manipulation a *demand reduction*.[4]

At a given allocation $x$, the intersection between the sets of competitors faced by buyer $i$ at different sellers is key. Without cycles, buyer $i$'s sets of competitors never intersect and a demand reduction can thus never change the price of an object that $i$ receives. Demand reductions are also the only possibly profitable manipulations. Since a trade mechanism satisfying our four properties resembles a Vickrey auction, buyer $i$ cannot change the prices he pays unless he changes the prevailing allocation. For this reason, in the unit demand case studied by Kranton and Minehart (2001), the presence of cycles is irrelevant since demand reduction cannot be profitable (by individual rationality).

The paper proceeds as follows. In Section 2 we introduce the model. In Section 3 we present our results.

## 2. The model

There are $N$ sellers and $M$ buyers. We denote representative buyers with the Latin letters $\{i, j, k\}$ and representative sellers with the Greek letters $\{\alpha, \beta, \gamma\}$. We use subscript $t$ or $z$ when we refer to a buyer $i_t$ or a seller $\alpha_z$ in a sequence of buyers and sellers. Let $S$ be the set of sellers and $B$ be the set of buyers.

Sellers each sell a unit of a *homogeneous* and *indivisible* good—henceforth, *object*. Each buyer $i$ *receives privately a signal* $v_i \in \mathbb{R}^N_{++}$ interpreted as his vector of *valuations* for the $N$ objects available. A typical valuation for buyer $i$ is $v_i = (v_{1i}, ..., v_{Ni})$ where $v_{ri}$ is the value that buyer $i$ attaches to the $r$th unit of the object he consumes. A typical valuation profile is $v = (v_1, ..., v_M) \in \mathbb{R}^{NM}_{++}$. In addition, we assume throughout that buyers have *downward sloping demands*: $v_{ri} \geq v_{si}$ for each buyer $i$ and each $r < s$, $1 \leq r, s \leq N$. It is understood that we consider the set of all possible valuation profiles that satisfy the previous restrictions.

[3] Recall that in a multi-unit setting such as ours, the set of generalized Vickrey auctions is infinite. Indeed, a Vickrey–Clarke–Groves mechanism only determines the sum of total payments made. There is an infinite number of ways to divide these payments across sellers—since sellers do not value objects.

[4] The downward sloping demand assumption is essential here. If buyers' marginal utility is constant ("horizontal" demand curves), a demand reduction can never be profitable. In such a case, the presence of cycles is irrelevant.

A non-directed bipartite graph $G =< B \cup S, L >$ consists of a set of buyers $B$, a set of sellers $S$, and a set of links $L$, each link connecting a buyer and a seller. A buyer can obtain an object from a seller if and only if the two are linked. Let $g_{i\alpha} = 1$ indicate that buyer $i$ and seller $\alpha$ are linked, i.e. that $i\alpha \in L$, and $g_{i\alpha} = 0$ indicate that $i\alpha \notin L$. For each buyer $i$, let $l_i(G)$ be the set of sellers linked to buyer $i$ in $G$. Given a subset $\tilde{B} \subseteq B$ of buyers, let $l_{\tilde{B}}(G)$ be the set of sellers linked to buyers in $\tilde{B}$. Likewise, we define $l_\alpha(G)$ for a seller $\alpha$ and $l_{\tilde{S}}(G)$ for a set of sellers $\tilde{S} \subseteq S$. Whenever this causes no confusion we simply write $l_i$, $l_{\tilde{B}}$, $l_\alpha$ and $l_{\tilde{S}}$.

A *buyer-path* $\pi \subseteq L$ in $G$ connecting two buyers $i$ and $j$ is a subset of links formed by $t$ distinct sellers $\alpha_1$ to $\alpha_t$, and $t+1$ (some possibly repeated) buyers $i_1$ to $i_{t+1}$ such that

$$(i_1, \alpha_1), (\alpha_1, i_2), ..., (\alpha_t, i_{t+1}) \in L, \ i_1 = i, \ i_{t+1} = j$$

A graph $G$ is connected if for any two buyers $i$ and $j$ there exists a path connecting them. We only consider connected graphs.[5] A buyer-path $\pi$ is a *cycle* if $i_{t+1} = i_1$. We say that $G$ is *cycle-free* if it does not contain a buyer-path $\pi$ that is a cycle.

An *allocation* $x \in \{0, 1, ..., M\}^N$ describes the destination of trade, i.e. to whom sellers are selling their units. A typical allocation is $x = (x_1, ..., x_N)$, where $x_\alpha = i$ indicates that buyer $i$ receives an object from seller $\alpha$, and $x_\alpha = 0$ indicates that seller $\alpha$ keeps the object.

An allocation $x$ is *feasible* if for each $i, j \in B$ and $\alpha \in S$ such that $x_\alpha = i$, $\alpha \in l_i(G)$ and $x_\alpha \neq j$ for $j \neq i$.[6] Given graph $G$, let $X(G)$ be the set of feasible allocations. When this causes no confusion, we simply write $X$.

Given $x \in X$ and $i \in B$, let $d_i(x)$ be the number of objects that buyer $i$ receives at allocation $x$. Given $i \in B$ and $v_i \in \mathbb{R}^N_{++}$, let $v_i(x)$ to be the total value obtained by buyer $i$ at $x$, i.e.

$$v_i(x) = \begin{cases} \sum_{r=1}^{d_i(x)} v_{ri} & \text{if } d_i(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

We refer to $v_{d_i(x)i}$ as the *lowest satisfied valuation* and $v_{(d_i(x)+1)i}$ as the *highest unsatisfied valuation* of buyer $i$ at allocation $x$.

Let $p \in \mathbb{R}^N_+$ be a vector of *prices*. A typical *price vector* is $p = (p_1, ..., p_N)$, where $p_\alpha$ is the price of the object held by seller $\alpha$. Given $(p, x) \in \mathbb{R}^N_+ \times X$, and $v_i \in \mathbb{R}^N_{++}$, the *utility* that buyer $i$ gets is

$$u_i(p, x; v_i) = v_i(x) - \sum_{\alpha: x_\alpha = i} p_\alpha$$

Notice that we implicitly assume that (i) a buyer pays only if he receives objects, and (ii) a buyer pays only for the objects he gets.

The utility that seller $\alpha$ gets is simply

$$u_\alpha(p, x) = \begin{cases} p_\alpha & \text{if } x_\alpha \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given a graph $G$, a *trade mechanism* is summarized as $(M, h)(G)$, where (i) $M = \times_{i \in B} M_i$, and with $M_i = \mathbb{R}^N_{++}$ is buyer $i$'s set of messages while $M$ is the set of message profiles, and (ii) $h : \mathbb{R}^{NM}_{++} \to \mathbb{R}^N_+ \times X$ is a function which determines a price vector and a feasible allocation for

[5] Each disconnected component of a graph can be treated separately.

[6] Notice that the definition of feasibility implies that a seller has only one unit to sell.

any profile of valuations $v$. In the sequel, we simply refer to a trade mechanism as $h$. This should cause no confusion. Given a trade mechanism $h$ and $v \in \mathbb{R}^{NM}_{++}$, the utility that buyer $i$ and seller $\alpha$ get at $h(v)$ are, respectively, $u_i(h(v); v_i)$ and $u_\alpha(h(v))$—where utilities are defined as before, given $h(v) = (p, x)$.

Given a graph $G$, we are interested in trade mechanisms which satisfy the following four properties.

**Efficiency:** A trade mechanism is *efficient* if for each $v \in \mathbb{R}^{NM}_{++}$, given $h(v) = (p, x)$, there does not exist $\tilde{x} \in X$ such that $\sum_{i \in B} v_i(\tilde{x}) > \sum_{i \in B} v_i(x)$.

**Strategy-proofness:** A trade mechanism is *strategy-proof* if for each $i \in B$, each $v_i \in \mathbb{R}^N_{++}$, and each $v_{-i} \in \mathbb{R}^{N(M-1)}_{++}$,

$$u_i(h(v); v_i) \geq u_i(h(v'_i, v_{-i}); v_i) \text{ for any } v'_i \in \mathbb{R}^N_{++}$$

Strategy-proofness states that truthtelling is a (weakly) dominant strategy for each buyer.

**Bilateral trade stability:** A trade mechanism is *bilateral trade stable* if for each pair $(i, \alpha) \in M \times N$ with $x_\alpha \neq i$ and $\alpha \in l_i(G)$, each $v \in \mathbb{R}^{NM}_{++}$, and given $h(v) = (p, x)$,

$$p_\alpha \geq \max\{\bar{p}_i(p, x), v_{(d_i(x)+1)i}\}$$

where $\bar{p}_i(p, x) = \max_{\alpha : x_\alpha = i} p_\alpha$.

Bilateral trade stability states that a (non-trading) buyer-seller $(i, \alpha)$ pair cannot improve by making a deal on the side, where either (i) $i$ cancels one of his trades and buys from $\alpha$, who would also be canceling his trade, or (ii) $i$ buys one additional unit from $\alpha$ who would be canceling his trade. In the former case with a price less than $\bar{p}_i(p, x)$, in the latter with a price lower than $v_{(d_i(x)+1)i}$. If bilateral trade stability does not hold a mutually beneficial exchange between $i$ and $\alpha$ can occur at a price greater than $p_\alpha$.

Finally we introduce one last requirement which fits the spirit of a trade mechanism, i.e. participation must be voluntary.

**Individual rationality:** Given $v \in \mathbb{R}^{NM}_{++}$, a price-allocation pair $(p, x) \in \mathbb{R}^N_+ \times X$ is *individually rational* if $u_i(p, x; v_i) \geq 0$ for each $i \in B$. A trade mechanism is individually rational if for each $v \in \mathbb{R}^{NM}_{++}$, $u_i(h(v); v_i) \geq 0$ for each $i \in B$.

Notice that our definition of individual rationality is one-sided as it does not include sellers. It is clear that in any trade mechanism, sellers are guaranteed at least their reservation value. A more stringent notion that plays an important role in our proofs (see Corollary 1) is *object-by-object individual rationality*.

**Object-by-object individual rationality:** Given $v \in \mathbb{R}^{NM}_{++}$, a price-allocation pair $(p, x) \in \mathbb{R}^N_+ \times X$ is *object-by-object individually rational* if for each $i \in B$, $v_{d_i(x)i} \geq p_\alpha$ for each $\alpha \in S$ such that $x_\alpha = i$. A trade mechanism is object-by-object individually rational if for each $v \in \mathbb{R}^{NM}_{++}$, $h(v)$ is object-by-object individually rational.

Object-by-object individual rationality is a strengthening of individual rationality. Given $(p, x)$, buyer $i$ will never pay more than his lowest satisfied valuation $v_{d_i(x)i}$ for any of the objects he buys. We will show (Corollary 1 of Proposition 1) that an efficient, strategy-proof, bilateral trade stable and individually rational trade mechanism must be object-by-object individually rational.

We now define the class of *generalized Vickrey auctions*, $\Psi$ on a network $G$. Given $j \in M$, let $G \setminus j$ be the graph obtained from $G$ when $j$ is removed along with all his links. Given $v \in \mathbb{R}^{NM}_{++}$, let $v_{\max}(G) \equiv \max_{x \in X(G)} \{\sum_{i \in B} v_i(x)\}$. Likewise, given $G \setminus j$, let $v_{\max}(G \setminus j) \equiv \max_{x \in X(G \setminus j)} \{\sum_{i \in B \setminus \{j\}} v_i(x)\}$. Given a graph $G$, a *Vickrey auction* $\psi \in \Psi$ is a trade mechanism such that for each $v \in \mathbb{R}^{NM}_{++}$, $\psi(v) \equiv (p^*, x^*)$, $x^*$ is an allocation that maximizes the total sum of valuations, and $p^*$ is such that for each $i$,

$$\sum_{\alpha : x_\alpha = i} p^*_\alpha = (v_i(x^*) - (v_{\max}(G) - v_{\max}(G \setminus i)))$$

For each $v \in \mathbb{R}^{NM}_{++}$ and $i \in B$, let $t_i(v) \equiv \sum_{\alpha : x_\alpha = i} p^*_\alpha$ be the *social opportunity cost of giving $d_i(x)$ objects to buyer $i$*, or simply the total *transfer* that buyer $i$ should make to get $d_i(x)$ objects in any Vickrey auction $\psi$. Let $t(v)$ be the vector of transfers.[7]

## 3. Results

We are interested in trade mechanisms which simultaneously satisfy efficiency, strategy-proofness, bilateral trade stability and individual rationality. Our first result provides a characterization of prices in such trade mechanisms.

Before we proceed, let us introduce a definition. For a buyer $i$ and an allocation $x \in X$ such that $x_\alpha = i$ we define the following set,

$$A(i, \alpha, x) \equiv \{j \in B \setminus \{i\} : \exists x' \in X \text{ such that } x'_\alpha \neq i, d_k(x') = d_k(x) \text{ for all } k \neq i, j,$$
$$d_i(x') = d_i(x) - 1 \text{ and } d_j(x') = d_j(x) + 1\}$$

The set $A(i, \alpha, x)$ is the *set of (direct and indirect) competitors* of buyer $i$ for seller $\alpha$'s unit at allocation $x$. It includes all buyers $j$ such that $i$'s object coming from seller $\alpha$ could be transferred in such a way that $j$ receives one more object without changing the bundles of buyers other than $i$ and $j$, i.e. à la Pigou–Dalton. Note that it is possible that $A(i, \alpha, x) = \emptyset$, in which case $l_\alpha = \{i\}$. Note also that $j \in A(i, \alpha, x)$ does not necessarily imply that $j \in l_\alpha(G)$. To see this, pick an allocation $x$, a seller $\alpha_1$, $i \in l_{\alpha_1}(G)$, $j \in A(i, \alpha_1, x)$ with $j \notin l_{\alpha_1}(G)$. By definition of $A(i, \alpha_1, x)$, there exists $x'$ such that $x'_{\alpha_1} \neq i$, $d_j(x') = d_j(x) + 1$ and $d_k(x') = d_k(x)$ for each $k \neq i, j$. Then there exists $k_1 \neq i, j$, $k_1 \in l_{\alpha_1}(G)$ such that $x'_{\alpha_1} = k_1$. Note that if $d_{k_1}(x) = 0$, then $k_1 = j$, a contradiction. Hence $d_{k_1}(x) > 0$. By definition $d_{k_1}(x) = d_{k_1}(x')$. There exists $\alpha_2 \in l_{k_1}(G)$, $k_2 \neq i, k_1$, $k_2 \in l_{\alpha_2}(G)$ such that $x_{\alpha_2} = k_1$, $x'_{\alpha_2} = k_2$. Note that if $d_{k_2} = 0$ then $k_2 = j$ and we are done: there is a path $(i, \alpha_1), (\alpha_1, k_1), (k_1, \alpha_2), (\alpha_2, k_2)$. If $d_{k_2} > 0$ but $k_2 \neq j$, we continue this iterating process. Since the network is finite, there exists a number $m > 2$, $k_m \neq i, k_1, ..., k_{m-1}$,

[7] Note that in our model for any graph $G$ there exists a continuum of generalized Vickrey auctions. They all have the same vector of transfer $t(v)$ for each $v \in \mathbb{R}^{NM}_{++}$ but differ only in the prices paid to the sellers—i.e. in the way the transfer of buyer $i$ is divided across those who sell him an object.

$\alpha_m \neq \alpha_1, ..., \alpha_{m-1}$ such that $x_{\alpha_m} = k_{m-1}$ and $x'_{\alpha_m} = k_m$. Since this is the end of the iteration, $d_{k_m}(x') = d_{k_m}(x) + 1$. By definition of $A(i, \alpha_1, x)$ and $x'$, $k_m = j$ and we are done: there is a path $(i, \alpha_1), (\alpha_1, k_1), ..., (\alpha_m, k_m)$, a path that goes from $i$ to $j$ and goes through $\alpha_1$.[8]

If there are no cycles in the graph then between any two nodes there is a unique path, meaning that for sellers $\alpha$ and $\beta$ such that $x_\alpha = i = x_\beta$, we always have $A(i, \alpha, x) \cap A(i, \beta, x) = \emptyset$. To see this, consider a cycle-free network $G$ and $j \in A(i, \alpha, x) \cap A(i, \beta, x)$. Then there exists a path from $i$ to $j$ passing through $\alpha$ and another path from $i$ to $j$ passing through $\beta$. The union of these paths forms a cycle, a contradiction with the fact that $G$ is cycle-free. Without cycles, the sets of competitors of a buyer for two different objects never intersect, while they typically do in a network with cycles.

In Proposition 1, we provide a characterization of prices in trade mechanisms that satisfy our four properties. Let us first prove an instructive Lemma which will be used repeatedly for our results. It shows that the number of objects a buyer $i$ gets at an efficient allocation does not decrease as long as his lowest satisfied valuation is higher than the highest unsatisfied valuation of his competitors.

**Lemma 1.** *Fix a network $G$. Pick $v \in \mathbb{R}^{NM}_{++}$, $x \in X$ that is efficient at $v$ and $i \in B$ such that $v_{d_i(x)i} > \max_{x_\alpha = i} [\max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}]$. Then for any $v'_i \in \mathbb{R}^N_{++}$ such that $\max_{x_\alpha = i} [\max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}] < v'_{d_i(x)i} < v_{d_i(x)i}$, there does not exist an allocation $\tilde{x}$ that is efficient at $(v'_i, v_{-i})$ and such that $d_i(\tilde{x}) < d_i(x)$.*

**Proof.** Before proving the claim, we start with an observation. Pick $v \in \mathbb{R}^{NM}_{++}$ and $x \in X$. If $x$ is efficient at $v$, then for each $i \in B$ such that $d_i(x) \neq 0$, $v_{d_i(x)i} \geq \max_{x_\alpha = i} [\max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}]$. Suppose the inequality is not true. That is, there exists $i \in B$ such that $v_{d_i(x)i} < \max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ for some $\alpha \in S$ with $x_\alpha = i$. Let $\{j\} = \underset{j \in A(i,\alpha,x)}{argmax} \{v_{(d_j(x)+1)j}\}$. Since $j \in A(i, \alpha, x)$ there exists an allocation $\hat{x} \neq x$ such that $\hat{x}_\alpha \neq i$, $d_i(\hat{x}) = d_i(x) - 1$, $d_j(\hat{x}) = d_j(x) + 1$ while $d_k(\hat{x}) = d_k(x)$ for each $k \neq i, j$. Observe that $\sum_{k \in B} v_k(\hat{x}) - \sum_{k \in B} v_k(x) = v_{(d_j(x)+1)j} - v_{(d_i(x))i} > 0$, a contradiction with our assumption that $x$ is an efficient allocation at $v$.

Now let us prove the Lemma. Pick $v \in \mathbb{R}^{NM}_{++}$, $x \in X$ that is efficient at $v$, and $i \in B$ such that $v_{d_i(x)i} > \max_{x_\alpha = i} [\max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}]$. Pick $v'_i \in \mathbb{R}^N_{++}$ such that (i) $\max_{x_\alpha = i} [\max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}] < v'_{d_i(x)i} < v_{d_i(x)i}$ and (ii) the rest of the valuation vector is unchanged from $v_i$ to $v'_i$. Suppose contrary to the statement of the Lemma that there exists an allocation $\tilde{x}$ that is efficient at $(v'_i, v_{-i})$ and such that $d_i(\tilde{x}) < d_i(x)$. Hence there exists $\alpha \in S$, $k_1 \in l_\alpha$ such that $x_\alpha = i$ while $x'_\alpha = k_1$. Suppose first that $d_{k_1}(\tilde{x}) > d_{k_1}(x)$. Since $k_1 \in A(i, \alpha, x)$, we have that $v_{d_{k_1}(\tilde{x})k_1} \leq v_{(d_{k_1}(x)+1)k_1} < v'_{d_i(x)i}$. This is a contradiction with the assumption that $\tilde{x}$ is efficient at $(v'_i, v_{-i})$. Hence, suppose next that $d_{k_1}(\tilde{x}) \leq d_{k_1}(x)$. This implies that there exists $k_2 \neq k_1$ such that $x_\beta = k_1$ while $\tilde{x}_\beta = k_2$. Observe that $k_2 \in A(i, \alpha, x)$. If $d_{k_2}(\tilde{x}) > d_{k_2}(x)$, then $v_{d_{k_2}(\tilde{x})k_2} \leq v_{(d_{k_2}(x)+1)k_2} < v'_{d_i(x)i}$. This is a contradiction with the assumption that $\tilde{x}$ is efficient at $(v'_i, v_{-i})$. Hence, for all such buyers

[8] Consider buyers $i, j, k$, sellers $\alpha, \beta$, $x_\alpha = i$, $x_\beta = j$, and $l_\alpha(G) = \{i, j\}$ and $l_\beta(G) = \{j, k\}$. We obtain $A(i, \alpha, x) = \{j, k\}$. Buyer $j$ can get an additional unit while $k$ still gets none, or $k$ can get a unit from $\beta$ instead of $j$, while $j$ takes the unit from $\alpha$. Buyer $j$ is a *direct competitor* to buyer $i$, while buyer $k$ is an *indirect competitor* to buyer $i$ at allocation $x$.

$k_2$, it must be the case that $d_{k_2}(\tilde{x}) \leq d_{k_2}(x)$ and each has at least one seller $\gamma$ such that $x_\gamma = k_2$ and $\tilde{x}_\gamma \neq k_2$. Since the network is finite, a finite iteration of this argument leads to a buyer $k_m$ for whom $d_{k_m}(\tilde{x}) > d_{k_m}(x)$, contradicting the assumption that $\tilde{x}$ is efficient at $(v'_i, v_{-i})$. We conclude that $d_i(\tilde{x}) \geq d_i(x)$.[9] □

**Proposition 1.** *Fix a network $G$. Let $h$ be a trade mechanism satisfying efficiency, strategy-proofness, bilateral trade stability and individual rationality. Pick $v \in \mathbb{R}^{NM}_{++}$, with $h(v) = (p, x)$. For each $i \in B$ and $\alpha \in S$ with $x_\alpha = i$, then either (i) $A(i, \alpha, x) \neq \emptyset$ and $p_\alpha = \max\limits_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ or (ii) $A(i, \alpha, x) = \emptyset$ and $p_\alpha = 0$.*

**Proof.** Pick a trade mechanism $h$ that satisfies our four properties. Pick $v \in \mathbb{R}^{NM}_{++}$ with $h(v) = (p, x)$, and choose $i \in B$ with $d_i(x) \geq 1$. The result can be proven by an induction on the number $\ell$ of objects $d_i(x)$ that buyer $i$ receives at $x$.

**Step 1:** $d_i(x) = 1$
Let $\alpha \in S$ with $x_\alpha = i$ and let $p_\alpha \geq 0$ be the price charged by seller $\alpha$.

If $A(i, \alpha, x) = \emptyset$, then by individual rationality, $p_\alpha \leq v_{1i}$, and by strategy-proofness, $p_\alpha = 0$. To see that the latter is true, consider $v'_i$ such that $v'_{ri} = \varepsilon$ for all $r$, for arbitrarily small $\varepsilon > 0$. Since $l_\alpha = \{i\}$, by efficiency, $x_\alpha = i$. By individual rationality, $p'_\alpha \leq \varepsilon$. Hence the only price at which there is no possible manipulation is $p_\alpha = 0$.

So suppose $A(i, \alpha, x) \neq \emptyset$ and let $j \equiv \underset{k \in A(i,\alpha,x)}{argmax} \{v_{(d_k(x)+1)k}\}$. Hence, there is a path $\pi$ between buyers $i$ and $j$,

$$(i_1, \beta_1), (\beta_1, i_2), ..., (\beta_t, i_{t+1}) \in G, \; i_1 = i, \; \beta_1 = \alpha, \; i_{t+1} = j$$

such that

$$x_{\beta_1} = i, x_{\beta_2} = i_2, x_{\beta_3} = i_3, ..., x_{\beta_t} = i_t$$

Suppose $p_\alpha < v_{(d_j(x)+1)j}$. When buyer $i$ is charged $p_\alpha$, by bilateral trade stability, all buyers which are connected to seller $\alpha$ must be paying at most $p_\alpha$ for the objects they buy. In particular, this implies that buyer $i_2$ on the path $\pi$ must be paying at most $p_\alpha$ for each object he buys. Iteratively, this means that all buyers on the path $\pi$ must also be paying at most $p_\alpha$ for each object they buy. This is in contradiction with bilateral trade stability because $x_{\beta_t} = i_t$, $p_{\beta_t} \leq p_{\beta_{t-1}} \leq ... \leq p_\alpha < v_{(d_j(x)+1)j}$ and $\beta_t$ is connected to buyer $j$.

Suppose now that $v_{1i} = v_{(d_j(x)+1)j}$. By individual rationality, buyer $i$ cannot be charged an amount $p_\alpha > v_{(d_j(x)+1)j} = v_{1i}$. So suppose $v_{1i} > v_{(d_j(x)+1)j}$. If buyer $i$ with valuation $v_i$ is charged an amount $p_\alpha > v_{(d_j(x)+1)j}$, he then has an incentive to misreport $v'_i$ such that (i) $v_{(d_j(x)+1)j} < v'_{1i} < p_\alpha$, and (ii) $v'_{(d_i(x)+1)i} < v_{d_k(x)k}$ for each $k \in B$ such that $i, k \in l_\beta(G)$ for some $\beta \in S$ with $x_\beta = k$. By Lemma 1, buyer $i$ would still obtain one object (and by (ii), one object only) and by individual rationality, he would pay a price lower than $p_\alpha$. This is a contradiction with strategy-proofness. Hence in such a case $p_\alpha = \max\limits_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}$.

**Step 2:** $d_i(x) = \ell$

[9] If in addition, $v'_i$ is such that $v'_{(d_i(x)+1)i} < v_{d_j(x)j}$ for each $j \in B$ such that $i, j \in l_\beta$ for some $\beta \in S$ with $x_\beta = j$, then $d_i(\tilde{x}) = d_i(x)$.

Suppose now it is true that if buyer $i$ receives $\ell$ objects at $x$, then for any seller $\alpha$ such that $x_\alpha = i$ either (i) $A(i,\alpha,x) \neq \emptyset$ and $p_\alpha = \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$, or (ii) $A(i,\alpha,x) = \emptyset$ and $p_\alpha = 0$. We will show that this continues to hold when buyer $i$ receives $\ell+1$ objects.

**Step 3:** $d_i(x) = \ell + 1$

Let buyer $i$ receive $\ell+1$ objects at allocation $x$. Using the observation on efficiency made in Lemma 1, note that for all sellers $\beta$ such that $x_\beta = i$ and $A(i,\beta,x) \neq \emptyset$ it holds that $v_{(\ell+1)i} \geq \max_{j\in A(i,\beta,x)} v_{(d_j(x)+1)j}$. Note that for each $\alpha$ such that $x_\alpha = i$, either $A(i,\alpha,x) \neq \emptyset$ or $A(i,\alpha,x) = \emptyset$.

Suppose first that $A(i,\alpha,x) \neq \emptyset$. The proof that $p_\alpha \geq \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ is the same as the case when buyer $i$ receives a single object. Let $T \equiv \max_{x_\alpha = i} [\max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}]$ and fix a seller $\gamma$ such that $p_\gamma \geq T$. Suppose buyer $i$ is charged an amount $p_\alpha > \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ by some seller $\alpha$. We have two different possibilities to consider. First, if $T < p_\alpha$, then buyer $i$ with valuation $v_i$ has an incentive to misreport $v'_i$ such that (i) $v'_{(\ell+1)i} < \min_{j\neq i}\{v_{Nj}\}$ and (ii) $v'_{\ell i} > \max_{j\neq i}\{v_{1j}\}$. By the observation in Lemma 1 on efficient allocations and restrictions (i) and (ii) above, at any $x' \in X$ that is efficient at $(v'_i, v_{-i})$, $d_i(x') = \ell$. By the induction axiom the prices buyer $i$ pays for those $\ell$ objects will not increase, his payment would decrease at least by $p_\alpha$ and for $v_{(\ell+1)i} < p_\alpha$ he would obtain a higher utility. This is a contradiction with strategy-proofness.[10] In turn, this implies that if $d_i(x) = \ell+1$, then buyer $i$ can never be charged a price above $v_{(\ell+1)i}$. Suppose instead that $T \geq p_\alpha > \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$. Hence $\alpha \neq \gamma$ and buyer $i$ is paying at least $T + p_\alpha - \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ for an extra object. Indeed he pays at least $T$ to $\gamma$ and with valuation $v'_i$, he would be paying only $p'_\alpha = \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$, by the induction axiom. Then for $v_{(\ell+1)i} < T + p_\alpha - \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$, buyer $i$ would gain by misreporting $v'_i$ and receiving $\ell$ objects, a contradiction with strategy-proofness. Since the choice of $\alpha$ was arbitrary, we conclude that $p_\alpha = \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ for each $\alpha$ such that $A(i,\alpha,x) \neq \emptyset$.

Suppose next that $A(i,\alpha,x) = \emptyset$. If for all $\beta \neq \alpha$ such that $x_\beta = i$, $A(i,\beta,x) = \emptyset$ then by efficiency, strategy-proofness and individual rationality, buyer $i$ would pay 0 for all the objects he buys. The argument is similar to the one in Step 1. So we assume that there is a seller $\beta \neq \alpha$ such that $x_\beta = i$ and $A(i,\beta,x) \neq \emptyset$. For a contradiction, suppose that $p_\alpha > v_{(\ell+1)i}$, then buyer $i$ with valuation $v_i$ has an incentive to misreport $v'_i$ such that (i) $v'_{(\ell+1)i} < \min_{j\neq i} v_{Nj}$, and (ii) $v'_{\ell i} > \max_{j\neq i} v_{1j}$. By the same argument given above for the case $A(i,\alpha,x) \neq \emptyset$, $i$ receives now $\ell$ objects following the report $v'_i$ and no longer receives an object from some buyer $\beta$ with $x_\beta = i$, $A(i,\beta,x) \neq \emptyset$. The prices do not increase for the objects that $i$ continues to receive and, by the induction axiom, $i$ now pays $p_\alpha = 0$. His utility after the misreport is $u_i(p',x';v_i) \geq C + v_{\ell i} > C + v_{\ell i} - p_\beta + v_{(\ell+1)i} - p_\alpha = u_i(p,x;v_i)$, where $C = \sum_{a=1}^{(\ell-1)} v_{ai} - \sum_{\lambda\neq\alpha,\beta,\ x_\lambda = i} p_\lambda$. This is a profitable deviation since $v_{(\ell+1)i} - p_\beta - p_\alpha < 0$. Therefore we must have $p_\alpha \leq v_{(\ell+1)i}$. Since

[10] The result that $p_\alpha \geq \max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}$ is necessary to reach this conclusion. If buyer $i$ receives $\ell$ objects instead of $\ell+1$, the price of any object $\beta$ he continues to get at $x'$ is exactly equal to $\max_{j\in A(i,\beta,x')} v_{(d_j(x')+1)j}$ and giving up one object will not increase the competition for the objects he continues to receive.

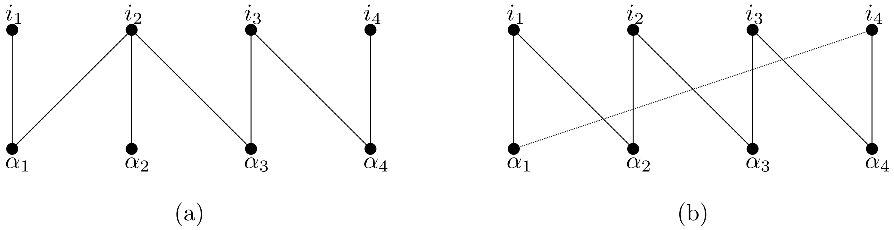


Fig. 1. (a) A cycle-free graph. (b) A graph with one cycle.

buyer $i$ can claim to have $v_{(\ell+1)i}$ arbitrarily small, the only price at which there is no profitable manipulation is $p_\alpha = 0$. □

**Corollary 1.** *Let h be a trade mechanism satisfying efficiency, strategy-proofness, bilateral trade stability and individual rationality. Then h is object-by-object individually rational.*

**Proof.** We have established that, for each object that a buyer gets, he pays the highest unsatisfied valuation of the buyers in the set of competitors for that object. By efficiency, this is less than buyer $i$'s lowest satisfied valuation. Hence, given allocation $x$, for each buyer $i$ and seller $\alpha$ with $x_\alpha = i$, $v_{d_i(x)i} \geq p_\alpha$. That is, object-by-object individual rationality holds. □

The prices that should be charged in such trade mechanisms retain the spirit of a Vickrey auction: buyers have to pay for the value of the competition they face for each object. We refer to this as the *lower bound on prices imposed by competition amongst buyers*. Also if a buyer $i$ misreports his valuation vector, the prices he pays stay the same (unless the efficient allocation changes after the misreport). But in the network setting the "second price" does not depend only on buyers' valuations but also on the allocation and the network structure. The competition faced by a buyer $i$ at seller $\alpha$ changes with the allocation, as captured by the set $A(i, \alpha, x)$. An interesting feature is that buyers never pay more than their lowest satisfied valuation for each of the objects they get.[11]

The question that remains is the existence of trade mechanisms characterized in Proposition 1. The answer to this question crucially depends on the network structure, as shown in the next two examples and subsequently in Theorem 1.

**Example 1.** *A trade mechanism and the unique set of bilateral trade stable prices.* The situation is depicted in Fig. 1(a) which shows a four buyers-four sellers network without cycles. Let $v$ be such that $v_{i_1} = (3, \varepsilon, \varepsilon, \varepsilon)$, $v_{i_2} = (10, 9, 8, \varepsilon)$, $v_{i_3} = (7, 6, \varepsilon, \varepsilon)$, and $v_{i_4} = (5, \varepsilon, \varepsilon, \varepsilon)$, for $\varepsilon$ arbitrarily small.

The mechanism picks the unique efficient allocation $x = (i_2, i_2, i_2, i_3)$. Since $l_{\alpha_2} = \{i_2\}$, by Proposition 1, $p_2 = 0$. We construct the appropriate sets $A(i, \alpha, x)$. For $i_2$ and $\alpha_1$, $A(i_2, \alpha_1, x) =$

[11] An efficient, strategy-proof and individually rational trade mechanism does not necessarily satisfy object-by-object individual rationality. To see this, consider a network given as $l_{i_1} = \{\alpha_1\}, l_{i_2} = \{\alpha_1, \alpha_2\}, l_{i_3} = \{\alpha_2\}$. The trade mechanism is given by a Vickrey auction but payments are such that whenever $i_2$ buys two objects, seller $\alpha_2$ receive the entire payment coming from $i_2$ (and in all other cases, a seller just receive the payment made by the buyer who gets the object from him). Suppose the valuation profile is $v = ((5, \varepsilon), (7, 6), (2, \varepsilon))$. Then $x = (i_2, i_2)$ and $p = (0, 7)$. Hence object-by-object individual rationality is violated since $p_{\alpha_2} > v_{2i_2}$. Obviously, this mechanism violates bilateral trade stability.

$\{i_1\}$. By Proposition 1, $p_{\alpha_1} = \max_{j \in A(i_2,\alpha_1,x)} v_{(d_j(x)+1)j} = 3$, the highest valuation of $i_1$. Next, we have $A(i_2, \alpha_3, x) = \{i_3, i_4\}$. Although $i_4$ is not connected to $\alpha_3$, he is a (indirect) competitor because both $i_3, i_4 \in l_{\alpha_4}$ and $x_{\alpha_4} = i_3$: we can give $\alpha_3$ to $i_3$ instead of $i_2$ and $\alpha_4$ to $i_4$ instead of $i_3$. Note that since there are no cycles, $A(i_2, \alpha_1, x) \cap A(i_2, \alpha_3, x) = \emptyset$. By Proposition 1 again, we obtain that $p_{\alpha_3} = 6$. Finally, $A(i_3, \alpha_4, x) = \{i_4\}$ giving $p_{\alpha_4} = 5$. We obtain $p = (3, 0, 6, 5)$.

Another way to mimic this construction is to think of each seller running a separate second price auction. Since a buyer's sets of competitors at $x$ never intersect, it is as if the buyers are participating in independent auctions. Given the efficient allocation $x$, let each buyer $i$ bid $v_{d_i(x)i}$ for each $\alpha$ such that $x_\alpha = i$, and $v_{(d_i(x)+1)i}$ for each $\alpha$ such that $x_\alpha \neq i$. Since these second price auctions are independent, a buyer $i$ has no incentive to change neither his bids of $v_{d_i(x)i}$ nor his bids of $v_{(d_i(x)+1)i}$. By the downward sloping demand assumption, the other valuations of buyer $i$ play no decisive role.

A third way to construct the price vector $p$ is the following. The total payment $t(v)$ made in any generalized Vickrey auction is such that $t(v) = (0, 9, 5, 0)$ at $x$. Bilateral trade stability on its own and the definition of a trade mechanism require $p_{\alpha_1} \geq 3$, $p_{\alpha_2} \geq 0$, $p_{\alpha_3} \geq 6$ and $p_{\alpha_4} \geq 5$. There is a unique way to divide $t(v)$ which is consistent with bilateral trade stability, and this occurs at the lower bound of admissible prices for each seller. We thus obtain that $p = (3, 0, 6, 5)$. $\diamond$

**Example 2.** *Impossibility result: a network with cycle.* The situation is depicted in Fig. 1(b) which shows a four buyers-four sellers network with one cycle. Let $v$ be such that $v_{i_1} = (10, 9, \varepsilon, \varepsilon)$, $v_{i_2} = (6, 1, \varepsilon, \varepsilon)$, $v_{i_3} = (7, 1, \varepsilon, \varepsilon)$, and $v_{i_4} = (8, \varepsilon, \varepsilon, \varepsilon)$, for $\varepsilon$ arbitrarily small.

The mechanism picks the unique efficient allocation $x = (i_1, i_1, i_3, i_4)$. We first have that $A(i_1, \alpha_1, x) = \{i_2, i_3, i_4\}$. By Proposition 1, $p_{\alpha_1} = 6 = v_{1i_2}$. Next, $A(i_1, \alpha_2, x) = \{i_2, i_3, i_4\}$ so that $p_{\alpha_2} = 6$. In the same fashion, one can check that $p_{\alpha_3} = p_{\alpha_4} = 6$. Note that (i) $p$ is solely determined by the presence of $i_2$ who gets no object at $x$, and (ii) $A(i_1, \alpha_1, x) \cap A(i_1, \alpha_2, x) \neq \emptyset$. Consider the manipulation $v'_1 = (10, \varepsilon, \varepsilon, \varepsilon)$, i.e. buyer $i_1$ decreases his demand. The allocation following the manipulation is $x' = (i_1, i_2, i_3, i_4)$. As a result, $i_2$'s first valuation no longer determines prices. By the downward sloping demand assumption, this softens the competition amongst buyers. There is a series of price changes which propagates through the network following $i_1$'s manipulation. Note that $A(i_1, \alpha_1, x') = A(i_1, \alpha_1, x) = \{i_2, i_3, i_4\}$, hence $p'_{\alpha_1} = 1$ and buyer $i_1$'s manipulation is profitable.[12] $\diamond$

Another way to see the tension between strategy-proofness and bilateral trade stability is to compare, for a given buyer $i$, the social opportunity cost of $i$ obtaining the goods against the lower bound on prices imposed by competition amongst buyers, $v_i(x) - (v_{\max}(G) - v_{\max}(G \setminus i)) \leq \sum_{\alpha:\, x_\alpha = i} \max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j}$. Suppose as in Example 2 that a buyer $i$ receives only two objects at an allocation $x$ from sellers $\alpha_1, \alpha_2$ such that $A(i, \alpha_1, x) = A(i, \alpha_2, x)$. Suppose there exists a unique buyer $j$ who is driving the prices paid by buyer $i$ at $x$, i.e. $\{j\} = \mathit{argmax}_{k \in A(i,\alpha_t,x)} \{v_{(d_k(x)+1)k}\}$ for both $\alpha_1$ and $\alpha_2$. Moreover suppose $j$ is also a "third highest" bidder for both objects at $x$, i.e. $v_{(d_j(x)+2)j} \geq \mathit{max}_{k \in A(i,\alpha_t,x) \setminus \{j\}} v_{(d_k(x)+1)k}$, for $t = 1, 2$. Now the above comparison between social

[12] Note that buyer $i_1$'s manipulation also benefits the other buyers: $A(i_2, \alpha_2, x') = \{i_1, i_3, i_4\}$ and thus $p'_{\alpha_2} = 1$; $A(i_3, \alpha_3, x') = \{i_1, i_2, i_4\}$ and so $p'_{\alpha_3} = 1$; and $A(i_4, \alpha_4, x') = \{i_1, i_2, i_3\}$ with $p'_{\alpha_4} = 1$.

opportunity cost of giving both objects to buyer $i$ and the minimum prices to satisfy bilateral trade stability is $v_{(d_j(x)+1)j} + v_{(d_j(x)+2)j} \leq 2v_{(d_j(x)+1)j}$. A necessary condition for a profitable demand reduction is that this inequality is strict. This is indeed the case in Example 2 where the social opportunity cost is 7 while competition imposes a sum of payments equal to 12 for buyer $i_1$.[13]

Before we go to the main theorem, let us introduce one last definition. For a buyer $i$, a seller $\alpha \in l_i$ and an allocation $x \in X$ such that $x_\alpha \neq i$ we define the following set,

$$\begin{aligned} Z(i,\alpha,x) \equiv \{j \in B \setminus \{i\} \mid \exists x' \in X \text{ such that } x'_\alpha = i, d_k(x') = d_k(x) \text{ for all } k \neq i, j, \\ d_i(x') = d_i(x) + 1,\ d_j(x') = d_j(x) - 1\} \end{aligned}$$

The sets $A(i,\alpha,x)$, defined for $x_\alpha = i$, and $Z(i,\alpha,x)$, defined for $x_\alpha \neq i$, are the two sides of the same coin. The latter is the set of all buyers $j$ from whom we can take one object and give $\alpha$'s object to buyer $i$ via a *Pigou–Dalton* transfer. The essential change from the initial allocation $x$ is that buyer $i$ receives one more object while buyer $j$ receives one less. We can interpret $Z(i,\alpha,x)$ as the set of buyers against whom buyer $i$ is bidding for the object of seller $\alpha$ and fails to obtain it. It is possible that $j \in Z(i,\alpha,x)$ while $j \notin l_\alpha$. We interpret $Z(i,\alpha,x)$ as the *set of (direct and indirect) competitors* of buyer $i$ for seller $\alpha$'s unit, against whom buyer $i$ lost.

**Theorem 1.** *Fix a network $G$. There exists an efficient, strategy-proof, bilateral trade stable and individually rational trade mechanism $h$ if and only if $G$ is cycle-free.*

**Proof. If part:** Let $h$ be a trade mechanism such that for each $v \in \mathbb{R}^{NM}_{++}$, $h(v) = (p,x) \in \mathbb{R}^N_+ \times X$ and $x$ is an efficient allocation. Given $v \in \mathbb{R}^{NM}_{++}$, if there are several efficient allocations, let $h$ choose the one which maximizes the number of objects obtained by the buyers who are labeled with a smaller index.[14] Let us now describe prices. Given $v \in \mathbb{R}^{NM}_{++}$, for any $\alpha \in S$ such that $x_\alpha = i$, if $A(i,\alpha,x) = \emptyset$, let $p_\alpha = 0$, otherwise let $p_\alpha = \underset{j\in A(i,\alpha,x)}{max} v_{(d_j(x)+1)j}$. Since $G$ is cycle-free, at any allocation $x \in X$ for $\alpha, \beta \in \ell_i(G)$ such that $\alpha \neq \beta$, if $x_\alpha = x_\beta = i$ then $A(i,\alpha,x) \cap A(i,\beta,x) = \emptyset$ and if $x_\alpha \neq i$, $x_\beta \neq i$ then $Z(i,\alpha,x) \cap Z(i,\beta,x) = \emptyset$.

This mechanism is efficient by definition. Since, when $A(i,\alpha,x) = \emptyset$, $p_\alpha = 0$, otherwise $p_\alpha = \underset{j\in A(i,\alpha,x)}{max} v_{(d_j(x)+1)j}$, the mechanism is object-by-object individually rational, hence individually rational and also bilateral trade stable. Next, we show that $h$ is strategy-proof.

Consider buyer $i$ with valuation $v_i$. Since the outcome of $h(v) = (p,x)$ is efficient then

$$\min_{\alpha\in\ell_i(G)|x_\alpha\neq i} \left[\min_{j\in Z(i,\alpha,x)} v_{d_j(x)j}\right] \geq v_{(d_i(x)+1)i} \text{ and } v_{d_i(x)i} \geq \max_{x_\alpha=i}\left[\max_{j\in A(i,\alpha,x)} v_{(d_j(x)+1)j}\right]$$

Here the right hand side is the maximum of maximal unsatisfied valuations among all the buyers against whom buyer $i$ wins an object and the left hand side is the minimum of minimal satisfied valuations among all the buyers against whom buyer $i$ loses. By efficiency, the minimal satisfied valuation of buyer $i$ must be greater than the unsatisfied valuations of those against whom he

[13] This necessary condition is not sufficient. For instance if we replace in Example 2 the valuations of buyer $i_2$ by $v_{i_2} = (4.5, 1, \varepsilon, \varepsilon)$, the social opportunity cost of obtaining objects for buyer $i_1$ lies below the lower bound imposed by competition, yet buyer $i_1$ cannot profitably manipulate by demand reduction.

[14] Since mechanism $h$ is a generalized Vickrey auction, the buyers would obtain the same utility regardless of the efficient allocation chosen.

wins and the maximal unsatisfied valuation of buyer $i$ must be smaller than the minimal satisfied valuations of those against whom he loses.

Suppose $i$ reports $v'_i \neq v_i$. If we have,

$$\min_{\alpha \in \ell_i(G)|x_\alpha \neq i} \left[ \min_{j \in Z(i,\alpha,x)} v_{d_j(x)j} \right] > v'_{(d_i(x)+1)i} \text{ and } v'_{d_i(x)i} > \max_{x_\alpha = i} \left[ \max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j} \right]$$

then by Lemma 1, at any efficient allocation $x'$, the two above inequalities imply that $i$ receive $d_i(x') = d_i(x)$. Hence, $x$ remains an efficient allocation at $(v'_i, v_{-i})$ and the mechanism will choose $x$ to maximize the number of objects obtained by the buyers who are labeled with a smaller index. By strategy-proofness, $i$ pays the same total amount at $h(v'_i, v_{-i})$. Therefore $i$'s utility is unchanged when going from $(v_i, v_{-i})$ to $(v'_i, v_{-i})$. We consider now two cases.

**Case 1:** $v'_{(d_i(x)+1)i} > \min_{\alpha \in \ell_i(G)|x_\alpha \neq i} \left[ \min_{j \in Z(i,\alpha,x)} v_{d_j(x)j} \right] \equiv T.$

At any efficient allocation $i$ receives at least one additional object at $(v'_i, v_{-i})$. If $i$ receives exactly one more object, he will pay an extra $T$ for any $v'_{(d_i(x)+1)i} > T$. Suppose not. If $i$ gets one more object and pays an extra $T' > T$. Then for any $T < v'_{(d_i(x)+1)i} < T'$ it would be profitable for $i$ to declare $v_i$ when his real valuation is $v'_i$, a contradiction with strategy-proofness. If $i$ pays an extra $T' < T$, then for a valuation $\hat{v}_i$ such that $T > \hat{v}_{(d_i(x)+1)i} > T'$ and $\hat{v}_{ti} = v_{ti}$ for $t \neq d_i(x) + 1$, it would be profitable for $i$ to declare $v'_i$ when his real valuation is $\hat{v}_i$, a contradiction with strategy-proofness.

If he receives only one more object and $v_{(d_i(x)+1)i} = T$ then $i$ obtains the same utility. Otherwise, he is worse off because he is paying more than his real valuation for an extra object.

**Case 2:** $v'_{d_i(x)i} < \max_{x_\alpha = i} \left[ \max_{j \in A(i,\alpha,x)} v_{(d_j(x)+1)j} \right] \equiv \tau.$

At any efficient allocation $i$ receives at least one object less at $(v'_i, v_{-i})$ and pays $\tau$ less than before.[15] If he receives only one less object and $v_{d_i(x)i} = \tau$ then $i$ obtains the same utility. Otherwise, he is worse off because he is losing an object for which his marginal payment was less than its value.

**Only if part:** Let $G$ be a network with cycles. We will construct a profile of valuations at which no mechanism can satisfy strategy-proofness, efficiency, bilateral trade stability and individual rationality simultaneously. The valuation profile will give an efficient allocation $x$ such that (a) there is a buyer $i$ who gets two objects $\alpha$ and $\beta$ with $A(i, \alpha, x) \cap A(i, \beta, x) \neq \emptyset$ and (b) there is a buyer $j \in A(i, \alpha, x) \cap A(i, \beta, x)$ who determines the prices for both $\alpha$ and $\beta$. The conditions (a) and (b) are necessary for a profitable manipulation by $i$ via demand reduction.

For a contradiction assume there is such a mechanism $h$ such that $h(v) = (p, x)$.

**Case 1** Suppose that there exists a cycle with two sellers $\alpha_1, \alpha_2$ and two buyers $i_1, i_2$.

Let $n_1$ be the number of sellers connected to $i_1$ and $n_2$ be the number of sellers connected to $i_2$. Let all the valuations of all buyers except $i_1, i_2$ be smaller than 1, that is $v_{1j} < 1$ for all $j$

15 The proof that $i$'s payment decreases exactly by $\tau$ follows an argument similar to the one in Case 1.

except $i_1$ and $i_2$. Let the valuation vector for $i_1$ be such that the first $n_1 - 2$ values are greater than the highest valuations of all other buyers, that is $v_{(n_1-2)i_1} > v_{1j}$ for any buyer $j$.

Let $n_3 \geq 2$ be the number of sellers which are connected both to $i_1$ and $i_2$. Let the valuation vector for $i_2$ be such that the first $n_2 - n_3$ values are strictly greater than 4. As for the rest of the valuations of $i_1$ and $i_2$, let $v_{(n_1-1)i_1} = 6$, $v_{n_1 i_1} = 5$, $v_{(n_2-n_3+1)i_2} = 4$, $v_{(n_2-n_3+2)i_2} = 1$.

By efficiency, buyer $i_1$ receives the objects from all $n_1$ sellers he is connected to. By bilateral trade stability, $i_1$ must pay at least 4 to both sellers $\alpha_1$ and $\alpha_2$—because this is the highest unsatisfied valuation of buyer $i_2$. Hence, buyer $i_1$ pays at least 8 for the two units he gets from $\alpha_1$ and $\alpha_2$. Notice that he receives value 11 from the two units. Instead, if buyer $i_1$ declares $v_{n_1 i_1} < 1$, then he would receive only one object from $\alpha_1$ or $\alpha_2$; he would also pay 1 and receive value 6 from the object.[16] This manipulation is beneficial to buyer $i_1$ and strategy-proofness is violated.

**Case 2** Suppose there exists no cycle with two sellers $\alpha_1, \alpha_2$ and two buyers $i_1, i_2$.

There exists a cycle $\pi$ in $G$ formed by $t$ distinct buyers and $t$ distinct sellers, for $t > 2$. Any two buyers in $\pi$ share at most one seller.[17]

For any buyer $j$ outside the cycle $\pi$ let $v_{1j} < v_{ri}$ for all $r > 1$ and for all buyers $i$ in cycle $\pi$. This means that at any efficient allocation all the sellers connected to a buyer in $\pi$ will give their objects to buyers in $\pi$. By this construction, the valuations of the buyers outside $\pi$ will not affect the efficient allocation of objects inside $\pi$.

Fix any buyer in $\pi$ and label it as buyer $i_1$, and label one of its links in $\pi$ as seller $\alpha_1$. Label the other connection of seller $\alpha_1$ in $\pi$ as buyer $i_2$ and label the other connection of buyer $i_2$ in $\pi$ as seller $\alpha_2$. Iteratively, label the rest of the buyers and sellers in $\pi$ accordingly. Hence buyer $i_1$ is connected to sellers $\alpha_1$ and $\alpha_t$ in $\pi$. From now on the subscripts will refer to this ordering of the buyers inside the cycle.

For a buyer $i$ in $\pi$, let $f_i$ be the number of sellers outside $\pi$ which are connected to $i$. Let $v_{f_{i_t} i_t} > v_{1 i_z}$ for $t < z$.

We still did not construct the valuations which will determine the allocation of objects inside the cycle $\pi$. For buyer $i_1$, let $v_{(f_{i_1}+2)i_1} > v_{(f_i+1)i}$ for all buyers $i$ in $\pi$ other than buyer $i_1$. Hence in any efficient allocation $x$ buyer $i_1$ will get both of the objects from the sellers $\alpha_1$ and $\alpha_t$. At such an allocation $x$, three conditions should be met for a demand reduction to be profitable: (a) buyer $i_1$'s set of competitors for $\alpha_1$ and $\alpha_t$ should intersect, i.e. $A(i_1, \alpha_1, x) \cap A(i_1, \alpha_t, x) \neq \emptyset$, (b) there should be a unique buyer $i$ who determines the prices of both $\alpha_1$ and $\alpha_t$ at allocation $x$ (and note that $i$ is not necessarily $i_1$), i.e. there exists $i \in A(i_1, \alpha_1, x) \cap A(i_1, \alpha_t, x)$ such that $\{i\} = \underset{j \in A(i_1, \alpha_1, x)}{argmax} \{v_{(d_j(x)+1)j}\} = \underset{j \in A(i_1, \alpha_t, x)}{argmax} \{v_{(d_j(x)+1)j}\}$, and (c) giving up one object to such a buyer $i$ softens the competition sufficiently for the other object $i_1$ gets, i.e. $2v_{(d_i(x)+1)i} > v_{(f_{i_1}+2)i_1} + v_{(d_i(x)+2)i}$.[18] Next, we continue constructing the valuation profile so that these three conditions are met.

[16] Buyer $i_1$ pays at least 1 by bilateral trade stability, and at most 1 by strategy-proofness.

[17] If two buyers shared more than one seller, then two such sellers and the two buyers would form a cycle, contradicting that there are no cycles with two sellers $\alpha_1, \alpha_2$ and two buyers $i_1, i_2$.

[18] If $|\underset{j \in A(i_1, \alpha_1, x)}{argmax} \{v_{(d_j(x)+1)j}\} \cup \underset{j \in A(i_1, \alpha_t, x)}{argmax} \{v_{(d_j(x)+1)j}\}| \neq 1$ then $p_{\alpha_1} + p_{\alpha_t}$ equals the social opportunity cost of giving these objects to $i_1$. If $i_1$ gives up one of the objects via demand reduction, the price of the other object would not change. Hence there would be no profitable manipulation.

For buyer $i_2$, let $v_{(f_{i_2}+1)i_2} > v_{(f_i+1)i}$ for all buyers $i$ different from buyers $i_1$ and $i_2$. Hence in any efficient allocation buyer $i_2$ will get the object from seller $\alpha_2$. We will define the rest of the valuations $v_{(f_i+1)i}$ such that in any efficient allocation buyer $i_r$ will get the object from seller $\alpha_r$, except buyer $i_t$. Buyer $i_t$ will not get any objects from the sellers in $\pi$ at any efficient allocation.

Also, for all buyers $i_z$ in $\pi$ other than $i_1$ and for all buyers $i_r$ in $\pi$ such that $z < r$, we let $v_{(f_{i_z}+2)i_z} < v_{(f_{i_r}+1)i_r}$. Given these valuations, in cycle $\pi$, at any efficient allocation the highest unsatisfied valuation is of buyer $i_t$ and is equal to $v_{(f_{i_t}+1)i_t}$. Moreover let $v_{(f_{i_t}+2)i_t} > v_{(f_i+2)i}$ for any buyer $i$ different than $i_1$.

By bilateral trade stability, seller $\alpha_t$ should receive at least $v_{(f_{i_t}+1)i_t}$ for his object, because this is the highest unsatisfied valuation of buyer $i_t$ who is also connected to $\alpha_t$. Similarly, seller $\alpha_{t-1}$ should also be paid at least $v_{(f_{i_t}+1)i_t}$. Now, since buyer $i_{t-1}$ is connected both to sellers $\alpha_{t-1}$ and $\alpha_{t-2}$, buys from $\alpha_{t-1}$ and pays $v_{(f_{i_t}+1)i_t}$, seller $\alpha_{t-2}$, who sells the object to $i_{t-2}$ should also be paid at least $v_{(f_{i_t}+1)i_t}$, due to bilateral trade stability. Otherwise, seller $\alpha_{t-2}$ could sell the object to buyer $i_{(t-1)}$ at a price closer to $v_{(f_{i_t}+1)i_t}$. Following this argument iteratively we can conclude all sellers in $\pi$ should also be paid at least $v_{(f_{i_t}+1)i_t}$.

By strategy-proofness, each buyer $i$ in $\pi$ pays at most $v_{(f_{i_t}+1)i_t}$ for the objects from the sellers in $\pi$. This is because if such a buyer $i$ pays $p > v_{(f_{i_t}+1)i_t}$, he can claim to have $p > v_{(f_i+1)i} > v_{(f_{i_t}+1)i_t}$ and adjust the rest of his valuation vector as in Lemma 1 to guarantee that he still gets exactly the same object(s), and pays at most $v_{(f_i+1)i}$ for each object by object-by-object individual rationality.

Hence, buyer $i_1$ pays a total of $2v_{(f_{i_t}+1)i_t}$ for the two objects he receives from the sellers in $\pi$. Suppose that buyer $i_1$ reports $v'_{i_1}$ with $v_{(f_{i_1}+2)i_1} < v_{(f_{i_t}+1)i_t}$. Now, by efficiency, each buyer in $\pi$ gets a single object from a seller in $\pi$. By Proposition 1, buyer $i_1$ pays $v_{(f_{i_t}+2)i_t}$ for the object he gets (as this is the highest unsatisfied valuation in the cycle). Following the misreport, the utility of buyer $i_1$ changes by $2v_{(f_{i_t}+1)i_t} - v_{(f_{i_1}+2)i_1} - v_{(f_{i_t}+2)i_t}$, which is positive when $2v_{(f_{i_t}+1)i_t} > v_{(f_{i_1}+2)i_1} + v_{(f_{i_t}+2)i_t}$. This contradicts with strategy-proofness. □

**Remark 1.** For any network with cycles, there always exist valuation profiles for which a profitable demand reduction exists. A key condition is the intersection between sets of competitors of a buyer at a given allocation. However, as highlighted in the proof of Theorem 1, additional conditions on such a valuation profile need to be met for a demand reduction to be profitable. In the online appendix we show that the valuation profiles which allow a profitable manipulation have a positive measure in the space of admissible valuation profiles.[19]

**Remark 2.** For a cycle-free network, the ascending price algorithm originally defined by Kranton and Minehart (2001) for single unit demands can be extended to our multi-units case. In the online appendix of this paper we show how to extend their ascending algorithm to our set-up without modifying the structure of the network.

**Remark 3.** If we simply wanted to sell multiple units in a single auction, Ausubel (2004) provides an efficient and strategy-proof mechanism. Yet when the sellers are acting individually, bilateral trade stability is necessary to guarantee that all sellers participate in the mechanism. Part 2 of Theorem 1 shows that this is not possible when the network is not cycle-free. It also shows that Ausubel's mechanism is not bilateral trade stable.

[19] The online appendix is available at https://sites.google.com/site/rahmiecon/apa.

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