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Graphs admitting antimagic labeling for arbitrary sets of positive integers

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Abstract

A connected graph $G = (V, E)$ with m edges is called *universal antimagic* if for each set B of m positive integers there is an bijective function $f : E \to B$ such that the function $\tilde{f} : V \to \mathbb{N}$ defined at each vertex v as the sum of all labels of edges incident to v is injective. In this work we prove that several classes of graphs are universal antimagic. Among others, paths, cycles, split graphs, and any graph which contains the complete bipartite graph $K_{2,n}$ as a spanning subgraph.

Keywords: Antimagic graphs, split graphs, complete bipartite graphs

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1 Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. Let B be a set of m positive integers. For a bijective labeling of the edges of G with labels in the set B, we define the function $\tilde{f} : V \to \mathbb{N}$ by setting $\tilde{f}(v)$ to be the sum of labels on edges containing v. If $\tilde{f}$ is an injective function, then we say that f is a B-antimagic labeling of G. If for all sets B of m positive integers there is a B-antimagic labeling of G, then we say that G is *universal* antimagic. Hartsfield and Ringel in 1990 conjectured that any connected graph with at least two edges has an $\{1, \ldots, m\}$-antimagic labeling. This question remains open even restricted to trees [9]. Several classes of graphs have been shown to be antimagic (see [2,3,5,6,13,14,15,11,4,7,10,12]).

The notion of universal antimagic graphs is motivated by the so called weighted-k-antimagic graphs which are graphs with m edges admitting, for any given vertex function w, an edge labeling f with image in $\{1, \ldots, m+k\}$ and such that the function $\tilde{f} + w$ is injective. This notion was introduced in [16] based on previous concepts presented in [8]. Among other results, in [16] it was proved that any graph with maximum degree $n-1$ is weighted-2-antimagic. This motivates us to study universal antimagic graphs which can be seen as the *network* version of antimagic labeling. We highlight that in [8] it was noticed that no path P_i, for each $i \in \{3, 4, 5\}$ has an antimagic labeling using numbers in $\{-1, 0, \ldots, i-3\}$, and a similar observation is valid for the graph $K_{1,n}$. In contrast, in this work, we prove that paths, cycles, split graphs and any graph containing a complete bipartite graph $K_{2,n-2}$ as spanning subgraph are universal antimagic.

We start with the case of paths and cycles which corresponds to connected graphs with maximum degree 2. The proof is easy and it is omitted because of space restrictions.

Proposition 1.1 *Any connected graph G with maximum degree 2 is universal antimagic unless it is a path of length one.*

We can extend the previous result to any graph with maximum degree at most two whose connected components are cycles of any length or paths of odd length at least three.

Proposition 1.2 *Any graph G with maximum degree 2 is universal antimagic if it has no connected component being a path of even length or of length one.*

Proof sketch. We assign the larger labels to the edges in cycles. Among them, we assign the larger ones to one cycle, the second larger to the next cycles and so on. Inside each cycle we proceed as in Proposition 1.1. The remaining connected components are paths of odd length at least three. In this case their parity plays a crucial role, and further analysis is needed. □

A *split-partition* of a connected graph $G = (V, E)$ is a partition $\{S, K, R\}$ of the set V, where S is an independent set and the following properties are satisfied: (1) for each $x \in S$, $N_G(x) \subsetneq K$ and (2) for each $x \in K$, $R \subseteq N_G(x)$, where $N_G(x)$ denotes the set of neighbors of a vertex x in G.

In [3], Barrus proves that any connected graph with at least three vertices, m edges and admitting a split-partition $\{S, K, R\}$, with K a set of pairwise adjacent vertices, is $\{1, \ldots, m\}$-antimagic. It is not hard to see that the proof of this result can be modified to show that graphs admitting a such split-partition are universal antimagic. Therefore, the following result holds.

Theorem 1.3 *Any connected graph with at least three vertices and admitting a split-partition $\{S, K, R\}$, with K a set of pairwise adjacent vertices, is universal antimagic. In particular, split graphs are universal antimagic.*

In [13], Barrus' result was extended to each graph G admitting a split-partition $\{S, K, R\}$, where K induces in G a regular graph. The proof of this result relies on arithmetic relations between the elements of the set $\{1, \ldots, m\}$, which do not hold for general sets of integers. In this work we prove that a graph admitting a split partition with the set K having two elements is universal antimagic. When $S = \emptyset$, this situation corresponds to graphs having the complete bipartite graph $K_{2,n-2}$ as a spanning subgraph. For clarity's sake, we first show that the complete bipartite graph $K_{2,n-2}$ is universal antimagic. We later extend the proof of this result to the following theorem.

Theorem 1.4 *For each $n \geq 3$, any connected graph containing the complete bipartite graph $K_{2,n-2}$ as a spanning subgraph is universal antimagic.*

The proof that the complete bipartite graph $K_{2,n-2}$ is universal antimagic splits in two cases. We analyse the first case in a separated lemma; the second one is considered inside the proof itself.

Lemma 1.5 *Given numbers $a > b > c$ and $d > e > g$, there are injective functions $\sigma, \sigma' : \{a, b, c\} \to \{d, e, g\}$ such that the functions $x \to x + \sigma(x)$ and $x \to x + \sigma'(x)$ are injective, $\sigma(a) \neq d$ and $\sigma'(c) \neq g$.*

Proof sketch. We show that, if $(\sigma(a), \sigma(b), \sigma(c))$ represents σ, then (e, d, g), (g, d, e) or (e, g, f) satisfies the conclusion. A similar idea works for σ'. □

Proposition 1.6 *For each $n \geq 3$, $K_{2,n-2}$ is universal antimagic.*

Proof. The case $n \leq 4$ was already considered in Proposition 1.1. We now consider $n \geq 5$. Let $V = \{x, w\} \cup \{v_1, \ldots, v_{n-2}\}$ be the set of the vertices of $G := K_{2,n-2}$ where $\{x, w\}$ is the independent set of size two. Given a set B of $m = 2(n-2)$ positive integers $y_1 > \cdots > y_m$, let f be the edge labeling given by $f(xv_i) = y_i$ and $f(wv_i) = y_{i+n-2}$, for each $i \in \{1, \ldots, n-2\}$. Then, $\tilde{f}(v_i) = y_i + y_{i+n-2}$, for each $i \in \{1, \ldots, n-2\}$, $\tilde{f}(x) = \sum_{i=1}^{n-2} y_i$ and $\tilde{f}(w) = \sum_{i=1}^{n-2} y_{i+n-2}$. Thus,

$$\tilde{f}(x) > \tilde{f}(v_1) > \cdots > \tilde{f}(v_{n-2})$$

and $\tilde{f}(x) > \tilde{f}(w)$. If $\tilde{f}(w) \neq \tilde{f}(v_i)$, for each $i \in \{1, \ldots, n-2\}$, then f is a $B-$antimagic labeling of G. Otherwise, $\tilde{f}(w) = \tilde{f}(v_i)$, for some $i \in \{1, \ldots, n-2\}$.

We shall prove that under some small modifications we can transform f into a B-antimagic labeling f' of G.

We first consider the case $i = 1$. To ease the presentation, we define $a = y_1$, $b = y_2$, $c = y_3$, $d = y_{n-2+1}$, $e = y_{n-2+2}$ and $g = y_{n-2+3}$. We have that $\tilde{f}(w) = a + d$, $a > b > c$ and $d > e > g$. Let σ be the injective function given in Lemma 1.5 when applied to a, b, c, d, e and g. Then $\sigma(a) \neq d$ and the function $x \to x + \sigma(x)$ is injective. We use σ to modify f on edges in the set $\{wv_1, wv_2, wv_3\}$. The modified labeling f' is given by $f'(wv_1) = \sigma(a)$, $f'(wv_2) = \sigma(b)$ and $f'(wv_3) = \sigma(c)$. It satisfies $\tilde{f}(v) = \tilde{f}'(v)$ for all $v \notin \{v_1, v_2, v_3\}$. Moreover, if v_1', v_2', v_3' is the non-increasing order according to $\tilde{f}'$ of the vertices v_1, v_2, v_3, we have

$$\begin{aligned}\tilde{f}'(x) = \tilde{f}(x) > a + d = \tilde{f}'(w) = \tilde{f}(w) > \tilde{f}'(v_1) > \tilde{f}'(v_2') > \tilde{f}'(v_3') \\ > \tilde{f}'(v_4) = \tilde{f}(v_4) > \cdots > \tilde{f}'(v_{n-2}) = \tilde{f}(v_{n-2}).\end{aligned}$$

This completes the proof in the case $i = 1$. It is clear that the case $i = n-2$ can be settled in a symmetric manner by modifying f in the set $\{wv_{n-4}, wv_{n-3}, wv_{n-2}\}$, according to the function σ' given in Lemma 1.5. When $3 \leq i$ or $i \geq n-4$ we can proceed as before by modifying f in the set $\{wv_{i-2}, wv_{i-1}, wv_i\}$ or in the set $\{wv_i, wv_{i+1}, wv_{i+2}\}$, respectively.

Due to space restriction for the remaining case, $n = 5$ and $i = 2$, we shall give only a brief sketch of the proof. In this situation, we have

$$\tilde{f}(x) = a + b + c > \tilde{f}(v_1) = a + d > \tilde{f}(v_2) = b + e > \tilde{f}(v_3) = c + g,$$

and $\tilde{f}(w) = d+e+g = b+e$. Hence, the modified labeling f' given by $f'(wv_1) = e$ and $f'(wv_2) = d$ is a B-antimagic labeling for G, unless $a+e = b+d$. When this latter equality holds, the modified labeling f' given by $f'(wv_2) = g$ and

$f'(wv_3) = e$ is a B-antimagic labeling of G, unless $c + e = b + g$. When $a+e = b+d$ and $c+e = b+g$ the modified labeling f' defined by $f'(wv_1) = g$, $f'(wv_2) = d$ and $f'(wv_3) = e$ is a B-antimagic labeling of G, unless $b+e = a+g$. Moreover, the modified labeling f' defined by $f'(wv_1) = e$, $f'(wv_2) = g$ and $f'(wv_3) = d$ is a B-antimagic labeling of G, unless $b + e = c + d$.

If none of previous modifications work, that is when $d + e + g = b + e = a + g = c + d$, the modified labeling f' defined by $f'(xv_1) = g$ and $f'(wv_1) = a$ is a B-antimagic labeling of G and the proof is completed. □

We now give the proof of the main contribution of this work.

Proof of Theorem 1.4. Let G be a graph with m edges and $n \geq 5$ vertices. Let $\{x, w\}$ be the independent set of size two of the spanning subgraph $K_{2,n-2}$ of G. Let B be any set of m positive integers and let us assume that the elements of B are the integers $0 < y_m < \cdots < y_1$. The case when x and w are adjacent is contained in Theorem 1.3. In fact, we can take $S = \emptyset$, $K = \{x, w\}$ and R the set of the remaining vertices.

We only consider the case where x and w are not adjacent in G. We assign the values $y_m, \ldots, y_{2n-5}$ to the edges in the graph induced by $V \setminus \{x, w\}$. This partial labeling defines partial sums at each vertex. Denote the vertices of $V \setminus \{x, w\}$ by $v_1, \ldots, v_{n-2}$, where the vertices are indexed in the non-increasing order of their partial sums. Let f be the labeling of G obtained from this partial labeling by assigning to the edge xv_i the value y_i, for $i \in [n-2]$, and to each edge wv_i the value y_{n-2+i}. The labeling f satisfies

$$\tilde{f}(v_{n-2}) < \tilde{f}(v_{n-3}) < \cdots < \tilde{f}(v_1) < \tilde{f}(x).$$

If $\tilde{f}(w) \neq \tilde{f}(v_i)$ for every $i \in [n-2]$, then f is a B-antimagic labeling of G. Otherwise, there is a unique index i such that $\tilde{f}(v_i) = \tilde{f}(w)$. As $n - 2 \geq 4$ we can assume that $i + 2 \leq n - 2$ or that $i - 2 \geq 1$. From the proof of Proposition 1.6 we know that these two possibilities are symmetric. Hence, we can assume that $i \geq 3$. As in the proof of Proposition 1.6 we know that there is a local modification f' of f in the set of edges $\{xv_i, xv_{i+1}, xv_{i+2}\}$, such that $\tilde{f}(w) = \tilde{f}(v_i) \notin \{\tilde{f}'(v_i), \tilde{f}'(v_{i+1}), \tilde{f}'(v_{i+2})\}$, the values $\tilde{f}'(v_i), \tilde{f}'(v_{i+1}), \tilde{f}'(v_{i+2})$ are distinct, and they lay in the open interval $(\tilde{f}(v_{i+2}), \tilde{f}(v_i))$. Hence, f' is a B-antimagic labeling of G. □

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