Mean-Variance Portfolio Selection With the Ordered Weighted Average

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***Abstract*—Portfolio selection is the theory that studies the process of selecting the optimal proportion of different assets. The first approach was introduced by Harry Markowitz and was based on a mean-variance framework. This paper introduces the ordered weighted average (OWA) in the mean-variance model. The main idea is to replace the classical mean and variance by the OWA operator. By doing so, the new model is able to study different degrees of optimism and pessimism in the analysis being able to develop an approach that considers the decision makers attitude in the selection process. This paper also suggests a new framework for dealing with the attitudinal character of the decision maker based on the numerical values of the available arguments. The main advantage of this method is the ability to adapt to many situations offering a more complete representation of the available data from the most pessimistic situation to the most optimistic one. An illustrative with fictitious data and a real example are studied.**

***Index Terms*—Mean, ordered weighted average (OWA), portfolio selection, variance.**

I. Introduction

Portfolio theory is a field of financial economics that aims to maximize the expected return of a portfolio taking into account a degree of portfolio risk or minimize the expected risk considering a specific amount of expected return. The purpose is to find a set of investment assets that collectively have a lower risk than any individual asset. The first approach was suggested by Markowitz and was based on a mean-variance portfolio selection framework [1]. This approach provided good results finding the optimal portfolio but needed too many calculations according to the technology available in the 1950s and 1960s. Thus, it was not easy to find optimal results with this model. This weakness was solved by Sharpe [2] with the development of a new approach that simplified Markowitzs model significantly. He saw that the majority of the investment assets were affected by similar conditions. Thus, he suggested a diagonal model based on regression methods that could obtain a solution with a significantly lower number of calculations than Markowitz's model needed to do.

Over the last years, Markowitz's framework has received increasing attention by the scientific community due to the development of computers that now can easily make a lot of calculations in a short period of time. A key example showing the growing importance of this approach is the publication of a special issue in the European Journal of Operational Research celebrating the 60th anniversary of Markowitzs theory [3]. This issue published a wide range of reviews and contributions regarding the newest developments in the field [4], [5]. Additionally, note that during the last years, many new extensions and generalizations are being suggested in the literature in a wide range of journals and conferences [6]–[8].

A key problem in Markowitz's mean-variance approach is the aggregation process of the mean and the variance with the arithmetic mean or the weighted average. These two averaging aggregation operators are the most common ones, but it is possible to use other ones for doing so [9]–[12]. A very well-known is the ordered weighted average (OWA) that aggregates the data taking into account the degree of optimism and pessimism of the decision maker [13], [14]. Many authors have improved the OWA operator under a wide range of perspectives [15]. Yager and Filev [16] developed the induced OWA operator which is a more general framework that uses order inducing variables in the reordering process of the information. Fodor *et al.* [17] suggested the quasi-arithmetic OWA operator by using quasi-arithmetic means in the aggregation process. Merigó and Gil-Lafuente subsequently [18] presented the induced generalized OWA operator as a unification of the induced and quasi-arithmetic frameworks. Other studies have analyzed the unification between the probability and the OWA operator [19], [20] and the integration between the weighted average and the OWA operator [21]–[23]. More recently, the unification between the OWA operator, the probability and the weighted average in the same formulation [24] has been developed.

The use of the OWA operator in portfolio selection has been considered in a heuristic model that combines the OWA operator with a data envelopment analysis (DEA) approach [25] and the cross-efficiency evaluation which is also based on DEA methodology [26]. The objective of this study is to develop a mean-variance portfolio approach by using the OWA operator in the aggregation of the expected returns and risks. The main idea is to replace the mean and the variance used in Markowitzs model by the OWA operator and the variance-OWA (Var-OWA) [27]. Observe that the OWA and the Var-OWA are generalizations of the traditional mean and variance that take into account the degree of optimism or pessimism of the decision maker and any situation from the most pessimistic to the most optimistic one. The reason for generalizing the simple or weighted average

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by the OWA operator is because in many situations, the importance of the data is not known. Thus, a different approach is needed to aggregate the information such as the OWA operator that uses the attitudinal character of the decision maker. The OWA aggregates the information by under or overestimating the information according to the specific attitude of the decision maker. Additionally, it also considers any result that can occur from the minimum to the maximum providing a complete representation of the problem that does not loose information in the analysis.

The classical model of Markowitz is revised by introducing OWA operators in the mean and the variance. The paper analyzes several important properties of the conceptual implications of this framework including the measures for the characterization of the OWA operator [13] and related extensions suggested in this paper. Many families of OWA operators are studied in order to consider particular positions that the decision maker may adopt. Illustrative examples regarding the new approach are also presented in order to numerically understand the use of the OWA operator in Markowitzs framework. These exercises reveal that attitudinal characteristics affect the efficient frontier, leading investors to polarize their portfolio choices in favor of either one asset (optimism) or the minimum-risk portfolio (pessimism). Simulations also suggest that a minimum-risk portfolio is the same in both Markowitz and OWA frontiers, a result that is robust to different classes of investor's attitudinal characteristics.

The remainder of the paper is structured as follows. Section II reviews the preliminaries regarding the OWA operator and Markowitz's portfolio model. Section III studies how to implement the OWA in the mean-variance portfolio approach. Section IV presents an illustrative example and a real example of the new methodology, and Section V discusses the key findings and results of the paper.

II. Preliminaries

A. OWA Operator

The OWA is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum [13]. In decision making under uncertainty, it is very useful for taking decisions with a certain degree of optimism or pessimism. It generalizes the classical methods into a single formulation including the optimistic, pessimistic, Laplace, and Hurwicz criterion. It can be defined as follows.

Definition 1: An OWA operator of dimension n is a mapping OWA: $\mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that

$$\text{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the a_i.

An important issue to consider when dealing with the OWA operator is the reordering process. In definition 1, the reordering has been presented in a descending way although it is also possible to consider an ascending order by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the descending OWA and w^*_{n-j+1} the jth weight of the ascending OWA operator.[1] Moreover, it is also possible to adapt the ordering of the arguments to the weights and vice versa [28], [29]. Note that the OWA is commutative, monotonic, bounded, and idempotent.

In order to characterize the weighting vector of an OWA aggregation, Yager [13] suggested the *degree of orness* and the *entropy of dispersion*. The degree of orness measures the tendency of the weights to the minimum or to the maximum. It is formulated as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{2}$$

As we can see, if $\alpha(W) = 1$, the weighting vector uses the maximum and if $\alpha(W) = 0$, the minimum. The more weights are located at the top, the higher it is α and vice versa.

The entropy of dispersion is an extension of the Shannon entropy when dealing with OWA operators. It is expressed as

$$H(W) = -\left(\sum_{j=1}^{n} w_j \ln w_j\right). \tag{3}$$

It can be observed that the highest entropy is found with the arithmetic mean ($H(W) = \ln n$) and the lowest one when selecting only one result such as the minimum or the maximum because in this case the entropy is 0.

B. Portfolio Selection With the Markowitz Approach

Consider a portfolio formed by m individual assets so that r_i^k represents asset k's return at state of nature i for all $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. States of nature are distributed according to the probability vector $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ so that $\pi_i \in [0, 1]$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \pi_i = 1$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be the vector of wealth proportions invested in each individual asset of the portfolio so that $x_k \in [0, 1]$ for all $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} x_k = 1$.[2] The mean return of asset k is then computed as

$$E(r^k) = \sum_{i=1}^{n} \pi_i r_i^k$$

and the mean return portfolio is hence given by

$$E(r^p; \mathbf{x}) = \sum_{k=1}^{m} x_k E(r^k) \tag{4}$$

as the expectation operator E is linear.

Moreover, the covariance between assets j and k is given by

$$\begin{aligned} \text{COV}(r^j, r^k) &= \sum_{i=1}^{n} E(r_i^j - E(r^j))E(r_i^k - E(r^k)) \\ &= \sum_{i=1}^{n} \pi_i (r_i^j - E(r^j))(r_i^k - E(r^k)) \end{aligned}$$

[1] The possibility of presenting OWA in both an ascending or descending order is relevant for portfolio choice applications, as discussed in Section III.

[2] Short sale is thus not allowed.

TABLE I
QUADRATIC AND PARAMETRIC PROGRAMMING IN MARKOWITZ PORTFOLIO SELECTION APPROACH

	Program 1	Program 2
Objective function	$\max_{\mathbf{x}} E(r^p;\mathbf{x})$	$\min_{\mathbf{x}} V(r^p;\mathbf{x})$
Parametric constraints	$V(r^p;\mathbf{x}) = \overline{V}$	$E(r^p;\mathbf{x}) = \overline{E}$
Budget constraints	$\sum_{k=1}^{m} x_k = 1$	$\sum_{k=1}^{m} x_k = 1$
Non negativity	For all $x_k \in [0,1]$	For all $x_k \in [0,1]$

$\overline{V}$ and $\overline{E}$ represent a given level of portfolio variance and portfolio mean return, respectively.

and the variance of the portfolio can then be computed as follows:

$$V(r^p;\mathbf{x}) = \sum_{j=1}^{m}\sum_{k=1}^{m} x_j x_k \mathrm{COV}(r^j, r^k) \tag{5}$$

given the linearity of the operator E.

A key aspect in the Markowitz methodology is to characterize the *efficient frontier*, which collects all the pairs that yield the maximum mean return portfolio for a given level of risk (the portfolio variance or standard deviation), or alternatively, all the pairs representing the minimum portfolio risk for a given level of mean return portfolio. From this duality, two practical methodologies emerge to construct the efficient frontier: a quadratic or a parametric programming model. Both of them are summarized in Table I.

An important result coming from portfolio choice analysis is the investor's wealth allocation that allows him to bear the minimum level of risk, i.e., the so-called *minimum-variance portfolio*. This is a relevant result, especially for a too risk-averse investor who wants to minimize the variability of his position irrespective of whether this portfolio yields quite a low expected return. In formal terms, let us define $\overline{\mathbf{x}}$, the minimum-variance portfolio, as follows:

$$\overline{\mathbf{x}} = \arg\min_{\mathbf{x}} V(r^p;\mathbf{x}) \tag{6}$$

where $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)$ is so that

$$\sum_{k=1}^{m} \overline{x}_k = 1 \text{ for all } \overline{x}_k \in [0,1].$$

III. OWA OPERATOR IN PORTFOLIO SELECTION

Markowitz's approach is based on the use of the mean and the variance. These techniques are usually studied with an arithmetic mean or a weighted average (or probability). However, the degree of uncertainty is usually more complex and it is necessary to represent the information in a deeper way. In this case, more general aggregation operators are needed in order to assess the information properly. A practical technique for doing so is the OWA operator because it provides a parameterized group of aggregation operators between the minimum and the maximum. Moreover, it is able to represent the *attitudinal* character of the decision maker in the specific problem considered. Therefore, the main advantage of this approach is that it represents the problem in a more complete way because it can consider any scenario, varying from the most pessimistic to the most optimistic one and select the situation that is in closest accordance with the investor's attitudes.

In order to revise Markowitz's approach with the OWA operator, it is necessary to change the formulas used to compute the portfolio's mean return and risk, that is, (4) and (5) of Section II. Note that this is suggested for situations with high levels of uncertainty where probabilistic information is not available [8]. In particular, when

1) The return of any asset is uncertain and cannot be assessed with probabilities. Therefore, the expected value cannot be used. In these uncertain environments, the Mean-OWA operator may become an alternative method for aggregating the information of the assets.
2) The risk of any asset cannot be measured with the usual variance because probabilities are unknown. However, it is possible to represent it with the Var-OWA [20], [30], [31].

A. Asset Return and Risk Using OWA

One of the key ideas of this approach is to introduce a model that can adapt better to uncertain environments because Markowitz's approach is usually focused on risky environments. Note that with the expected value it is possible to consider subjective and objective probabilities but it is not possible to assess situations without any type of probabilities. An alternative for dealing with these situations is the OWA operator that aggregates the information according to the attitudinal character of the investor.

Definition 2: Let r_i be an asset's return at state of nature i for $i = 1, 2, \ldots, n$. The Mean-OWA operator can then be represented as follows:

$$\begin{aligned} E_{\mathrm{OWA}} &\equiv \mathrm{OWA}(r_1, r_2, \ldots, r_n) \\ &= \sum_{j=1}^{n} w_j q_j \end{aligned} \tag{7}$$

where q_j is the jth largest of the r_i.

Next, let us look into the other main perspective when dealing with portfolio selection. The analysis of risk in Markowitz's approach is based on the use of the variance measure. In this paper, we have suggested using the Var-OWA as a measure of risk. The main advantage is that this formulation is more general than the classical variance because it provides a parameterized family of variances between the minimum and the maximum one. Thus, it gives a better representation of the problem and selects the specific result that is in closest accordance with the attitude of the decision maker. Following Yager [27], and [20], [32], an asset variance when using the OWA operator can be formulated as follows.

Definition 3: Given an asset with expected returns $(r_1, r_2, \ldots, r_n)$, let us define s_i as

$$s_i = (r_i - E_{\mathrm{OWA}})^2$$

for $i = 1, 2, \ldots, n$, where E_{OWA} is defined according to (7). The Var-OWA can then be defined as follows:

$$V_{\mathrm{OWA}} \equiv \mathrm{OWA}(s_1, s_2, \ldots, s_n) = \sum_{j=1}^{n} w_j t_j \tag{8}$$

where t_j is the jth smallest of the s_i.

It can be seen that, here, we use an ascending order because it is usually assumed that a lower risk represents a better result and, thus, it should appear first in the aggregation. Similarly to the case of expected returns, with the OWA approach we can consider any risk from the minimum to the maximum one by using $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$ (minimum variance) and $w_n = 1$ and $w_j = 0$ for all $j \neq n$ (maximum variance).

B. Portfolio Mean Return and Risk Using OWA

Now consider a portfolio in which m individual assets can be combined so that r_i^k represents asset k's return at state of nature i for all $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be the vector of wealth proportions invested in each individual asset of the portfolio so that $x_k \in [0, 1]$ for all $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} x_k = 1$.[3] Moreover, let us define r_i^p, the portfolio's return at state of nature i, as

$$r_i^p = \sum_{k=1}^{m} x_k r_i^k$$

for all $i = 1, 2, \ldots, n$. The portfolio mean-OWA is then given by

$$E_{\mathrm{OWA}}(r^p; \mathbf{x}) = E_{\mathrm{OWA}}(r_1^p, r_2^p, \ldots, r_n^p) \tag{9}$$

or equivalently, using definition given by (7), it becomes

$$E_{\mathrm{OWA}}(r^p; \mathbf{x}) = \mathrm{OWA}(r_1^p, r_2^p, \ldots, r_n^p) = \sum_{j=1}^{n} w_j q_j^p \tag{10}$$

where q_j^p is the jth largest of the r_i^p.[4]

In addition, by applying (9) or (10), we can define s_i^p as

$$s_i^p = (r_i^p - E_{\mathrm{OWA}}(r^p; \mathbf{x}))^2$$

for $i = 1, 2, \ldots, n$. The portfolio Var-OWA can then be formulated as follows:

$$V_{\mathrm{OWA}}(r^p; \mathbf{x}) = V_{\mathrm{OWA}}(r_1^p, r_2^p, \ldots, r_n^p)$$

[3]Short sale is thus not allowed.

[4]Contrary to the expected value, the OWA is *not* a linear operator (see a counter-example in the Appendix). Thus, we cannot proceed as (4) in the Markowitz approach, as in general it is verified that

$$E_{\mathrm{OWA}}(r^p; \mathbf{x}) \neq \sum_{k=1}^{m} x_k E_{\mathrm{OWA}}(r^k).$$

TABLE II
QUADRATIC AND PARAMETRIC PROGRAMMING IN PORTFOLIO SELECTION USING OWA

	Program 1	Program 2
Objective function	$\max_{\mathbf{x}} E_{\mathrm{OWA}}(r^p; \mathbf{x})$	$\min_{\mathbf{x}} V_{\mathrm{OWA}}(r^p; \mathbf{x})$
Parametric constraints	$V_{\mathrm{OWA}}(r^p; \mathbf{x}) = \overline{V}$	$E_{\mathrm{OWA}}(r^p; \mathbf{x}) = \overline{E}$
Budget constraints	$\sum_{k=1}^{m} x_k = 1$	$\sum_{k=1}^{m} x_k = 1$
Non negativity	For all $x_k \in [0, 1]$	For all $x_k \in [0, 1]$

$\overline{V}$ and $\overline{E}$ represent a given level of portfolio Var-OWA and portfolio mean-OWA return, respectively.

and it can be calculated as

$$V_{\mathrm{OWA}}(r^p; \mathbf{x}) = \mathrm{OWA}(s_1^p, s_2^p, \ldots, s_n^p) = \sum_{j=1}^{n} w_j t_j^p$$

where t_j^p is the jth smallest of the s_i^p.[5]

Once this initial information is calculated, the rest of the approach follows the Markowitz methodology where a quadratic or parametric programming model is used (see Table II).

As in the Markowitz approach, we define $\overline{\mathbf{x}}_{\mathrm{OWA}}$, the minimum-Var-OWA portfolio, as follows:

$$\overline{\mathbf{x}}_{\mathrm{OWA}} = \arg\min_{\mathbf{x}} V_{\mathrm{OWA}}(r^p; \mathbf{x}) \tag{11}$$

where $\overline{\mathbf{x}}_{\mathrm{OWA}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)$ is so that

$$\sum_{k=1}^{m} \overline{x}_k = 1 \text{ for all } \overline{x}_k \in [0, 1].$$

C. Investor's Criteria for Mean-OWA

Note that with the OWA operator, the expected returns are studied considering any scenario from the minimum to the maximum one. Thus, the decision maker does not lose any information in this initial stage. Once he selects a specific attitude, he opts for a specific result and decision although he still knows any extreme situation that can occur in the problem. This can be proved analyzing some key particular cases of the OWA aggregation including the minimum, the maximum, and the arithmetic mean:

1) If $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, the OWA becomes the maximum (or optimistic criterion).
2) If $w_n = 1$ and $w_j = 0$ for all $j \neq n$, the minimum is formed (or pessimistic criterion).
3) If $w_j = 1/n$ for all j, it becomes the arithmetic mean (Laplace criterion).

[5]The nonlinearity of the OWA operator also means that we cannot proceed as (5) in the Markowitz approach, as in general it is verified that

$$V_{\mathrm{OWA}}(r^p; \mathbf{x}) \neq \sum_{j=1}^{m} \sum_{k=1}^{m} x_j x_k \mathrm{COV}_{\mathrm{OWA}}(r^j, r^k)$$

where $\mathrm{COV}_{\mathrm{OWA}}$ is given by the CoVar-OWA [32]. This implies that neither one can apply a variant OWA of the correlation coefficient [32], which is a key concept in the Markowitz methodology.

By looking at the maximum and the minimum, it is proved that the OWA operator accomplishes the boundary condition

$$\min\{a_i\} \leq \text{OWA}(a_1, a_2, \ldots, a_n) \leq \max\{a_i\}. \quad (12)$$

Some other interesting particular cases of the OWA operator are the following:

1) Hurwicz criterion: If $w_1 = \beta$, $w_n = 1 - \beta$ (with $\beta \in [0, 1]$), and $w_j = 0$, for all $j \neq 1, n$.
2) Step-OWA: $w_k = 1$ and $w_j = 0$ for all $j \neq k$. Note that if $k = 1$, we get the maximum and if $k = n$, the minimum.
3) Median-OWA: If n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0$ for all others.
4) Olympic-OWA: When $w_1 = w_n = 0$ and for all others $w_j = 1/(n-2)$. Note that if $n = 3$, Olympic-OWA becomes the Step-OWA and the Median-OWA operator.

For further reading on other particular types of OWA operators see [33] and [29].

Another important issue to consider is the attitudinal character used by the decision maker in the selection process. For doing so in the OWA operator, Yager [13] suggested the degree of orness presented in (2) that in decision making problems can be seen as a measure for representing the degree of optimism or pessimism of the decision maker. This technique is useful in many situations. However, sometimes it is necessary to use another approach that can adapt to the specific values of the arguments. This occurs because the arguments are frequently distributed in a heterogeneous way and this must be considered in order to calculate the attitudinal character of the decision maker. In order to do this, let us suggest the following formulation:

$$\alpha^*(W) = \sum_{j=1}^{n} w_j \left(\frac{b_j - b_n}{b_1 - b_n} \right) \quad (13)$$

where b_j is the jth largest of the a_i and b_1 and b_n are the largest and smallest arguments, respectively. The main advantage of this formula is the possibility of representing the specific characteristics of the results considered in the problem. It may be observed that the value of $(b_1 - b_j)/(b_1 - b_n)$ for b_1 is always 1 and for b_n, 0. The difference between this formula and Yager's approach is for the central values. For example, if we have a weighting vector so that $W = (1/3, 1/3, 1/3)$, the initial position of the decision maker must be neutral. However, it is not the same to use this weighting vector in a homogenous distribution of arguments than a heterogeneous one. In a homogenous one, it would be reasonable to conclude that the decision maker is using a neutral attitude. However, a heterogeneous one that would not be the case. Let us look into a numerical example. Let us compare the following set of arguments: $A = (100, 0, 200)$ and $B = (0, 900, 1000)$. The OWA aggregation would produce the following results:

$$\text{OWA}(A) = 200 \times \frac{1}{3} + 100 \times \frac{1}{3} + 0 \times \frac{1}{3} = 100.0$$

$$\text{OWA}(B) = 1000 \times \frac{1}{3} + 900 \times \frac{1}{3} + 0 \times \frac{1}{3} = 633.3.$$

As we can see, the aggregated result of A reaches a central value which is consistent with the neutral position shown in the weighting vector. Therefore, in this case the neutral attitude is confirmed. However, for the aggregation of B, this is not proved because the result is placed closer to the top rather than in a central position.

Yager's measure would indicate that both aggregations are using a neutral attitude of 0.5 because α is calculated as follows:

$$\alpha(W) = \frac{3-1}{3-1} \times \frac{1}{3} + \frac{3-2}{3-1} \times \frac{1}{3} + \frac{3-3}{3-1} \times \frac{1}{3} = 0.5.$$

However, with the new measure suggested in this paper, the result would be

For A:

$$\alpha^*(W) = \frac{200-0}{200-0} \times \frac{1}{3} + \frac{100-0}{200-0} \times \frac{1}{3} + \frac{0-0}{200-0} \times \frac{1}{3} = 0.5.$$

For B:

$$\alpha^*(W) = \frac{1000-0}{1000-0} \times \frac{1}{3} + \frac{900-0}{1000-0} \times \frac{1}{3} + \frac{0-0}{1000-0} \times \frac{1}{3} = 0.633.$$

The first set of arguments still provides a neutral result because the arguments are distributed in a homogeneous way. The differences appear in the second set of arguments because now two of the three arguments are closer to the top so the aggregation tends to produce a result that is closer to the top. In this particular example, it is seen that the result is located at a degree of 63.3% which is clearly above the neutral result of 50%. This is important because when aggregating the information, the attitudinal character of the decision maker is showing the pessimistic or optimistic beliefs regarding the expected results. Thus, a neutral position should provide a result that is close to the central results. In other words, the main advantage of the measure suggested here is that it considers the value of the arguments being able to represent the attitudinal character of the decision maker according to these values.

Note that this measure represents the degree of optimism. Alternatively, it is also possible to consider the degree of pessimism by using the dual. That is, *degree of optimism* = 1− *degree of pessimism*. Therefore, in the previous example, A would have a degree of optimism and pessimism of 0.5. However, B would have a degree of optimism of 0.633 and pessimism of 0.367.

Next, let us study the results produced by some key particular cases of the OWA operator with this measure. With the optimistic criterion we always obtain 1 because

$$\alpha^*(W) = \frac{b_1 - b_n}{b_1 - b_n} \times 1 = 1 \times 1 = 1.$$

And with the pessimistic criterion 0 because

$$\alpha^*(W) = \frac{b_n - b_n}{b_1 - b_n} \times 1 = 0 \times 1 = 0.$$

The result found with Hurwicz criterion is

$$\alpha^*(W) = \frac{b_1 - b_n}{b_1 - b_n} \times \beta + \frac{b_n - b_n}{b_1 - b_n} \times (1 - \beta)$$
$$= 1 \times \beta + 0 \times (1 - \beta) = \beta.$$

Thus, it is proved that the degree of optimism of Hurwicz coincides with the degree of optimism given with this measure. The difference is that Hurwicz criterion only considers the minimum and the maximum while this approach can consider any argument of the aggregation.

The Laplace criterion gives a different result with this method

$$\alpha^*(W) = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{b_j - b_1}{b_1 - b_n} \right). \tag{14}$$

This result clearly shows the differences between Yager's approach and the measure suggested in this paper. When the arguments are distributed in a homogeneous way, the results will be the same (0.5). However, when the arguments are not distributed in the same way, the results will be different. Basically, Yager's measure indicates the attitudinal character of the decision maker without considering (or quantifying) the results that may occur while in this approach the attitudinal character of the decision maker considers the expected results.

The attitudinal character found with the step-OWA operator is

$$\alpha^*(W) = \frac{b_k - b_n}{b_1 - b_n} \times 1 = \frac{b_k - b_n}{b_1 - b_n} \tag{15}$$

so that $\alpha^*(W) \in [0, 1]$. Note that if $k = 1$ we get the maximum and $\alpha^*(W) = 1$. If $k = n$ we get the minimum and $\alpha^*(W) = 0$.

For the median-OWA operator, we have to distinguish between two situations. If n is odd ($w_{(n+1)/2} = 1$), we get

$$\alpha^*(W) = \frac{b_{(n+1)/2} - b_n}{b_1 - b_n} \times 1 = \frac{b_{(n+1)/2} - b_n}{b_1 - b_n}. \tag{16}$$

And if n is even, we have to aggregate the two central values, for example, with an arithmetic mean ($w_{n/2} = w_{(n/2)+1} = 0.5$)

$$\alpha^*(W) = \frac{b_{n/2} - b_n}{b_1 - b_n} \times 0.5 + \frac{b_{(n/2)+1} - b_n}{b_1 - b_n} \times 0.5.$$

In the olympic-OWA operator, this measure would have the following expression:

$$\alpha^*(W) = \frac{1}{n-2} \sum_{j=2}^{n-1} \left(\frac{b_j - b_n}{b_1 - b_n} \right). \tag{17}$$

Finally, it is also worth noting that there are other measures for characterizing the weighting vector such as the entropy of dispersion shown in (3), the balance operator [27] and the divergence of W [34]. The balance operator could be extended to the framework of the new measure suggested in this paper as follows:

$$\text{Balance}(W) = \sum_{j=1}^{n} w_j \left(\frac{2b_j - b_1 - b_n}{b_1 - b_n} \right). \tag{18}$$

Equivalently to Yager's approach, the balance measure of the maximum aggregation would be 1 and the minimum -1. Note that it is possible to develop a dual of the balance focused on ascending aggregations by using: Dual Balance$(W) = -$Balance(W).

For the divergence of W, the suggested new formulation would be

$$\text{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{b_j - b_n}{b_1 - b_n} - \alpha(W) \right)^2. \tag{19}$$

D. Investor's Criteria for Var-OWA

The measures for characterizing the weighting vector can also be used here, although we should consider *ascending* orders. Therefore, in the context of portfolio selection, Yager's degree of orness would be

$$\alpha(W_{\text{asc}}) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} \right).$$

Observably, for the minimum variance $\alpha(W) = 1$ and for the maximum variance $\alpha(W) = 0$. For the new measure suggested in this paper, and based on Definition (3), the degree of optimism could be expressed as

$$\alpha^*(W_{\text{asc}}) = \sum_{j=1}^{n} w_j \left(\frac{t_j - t_n}{t_1 - t_n} \right). \tag{20}$$

Note that for the minimum variance we get 1, as it represents the situation with the highest degree of optimism and vice versa for the maximum variance. The reason for this is because in portfolio choice the variance has to be connected with the returns, assuming that an optimistic investor looks for scenarios with high returns and low degree of risk.

Following a similar methodology as for the returns, we could study equivalent particular cases but focused on the variance. The balance and the divergence could be also studied in this framework taking into account that now the focus is on ascending orders.

IV. Numerical Example

In this section, we present the results of numerical simulations in order to illustrate the following phenomena:

1) The ordering effect of OWA over the portfolio efficient frontier. combined ordering and weighting effect of OWA over this frontier,
2) A comparison between the traditional Markowitz and the OWA portfolio efficient frontiers.
3) How the OWA frontier changes with different attitudinal profiles.

To illustrate all these results, we perform two classes of exercises: 1) a numerical exercise with fictitious data, and 2) a numerical exercise with real data.

A. Numerical Example With Fictitious Data

To perform the first class of numerical exercise, we assume that the investor has available two risky assets in which to assign his wealth, A and B, with the return profile presented in Table III.

TABLE III
ASSET RETURN PROFILES

	S_1	S_2	S_3
r^A	0.05	0.10	0.15
r^B	0.30	0.20	0.10

S_i denotes state of nature i.

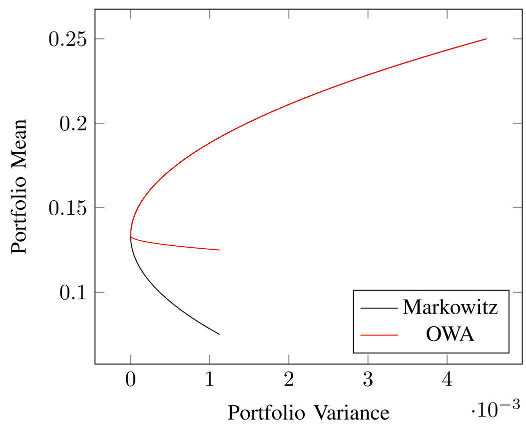

Fig. 1. *Ordering effect.* Frontiers represent all the *portfolio mean-portfolio variance* pairs generated by wealth allocations $(x, 1-x)$ over assets A and B, respectively, such that $x \in [0,1]$. Markowitzian frontier (black line) uses $E(\cdot)$ and $V(\cdot)$, based on probabilistic vector Π, as mean and variance; OWA frontier (red line) uses $E_{\text{OWA}}(\cdot)$ and $V_{\text{OWA}}(\cdot)$, based on weighting vector W, as mean and variance. Efficient frontiers are made up by all the *portfolio mean-portfolio variance* pairs that represent the highest mean for a given variance. To study ordering effect of OWA, we assume $\Pi = W$, and in particular, that $\Pi = W = (0.6, 0.3, 0.1)$. Numerical results point out that both efficient frontiers are identical.

Furthermore, to isolate the effect coming exclusively from the ordering of outcomes carried out by OWA operator, we must first assume that the probabilistic vector Π and the weighting vector W are identical. Notice that this assumption implies that

$$E_{\text{OWA}}(r^B) = E(r^B)$$

as in the case of asset B the order of states of nature coincides with the descending order of results induced by the OWA operator. In particular, suppose that $\Pi = W = (0.6, 0.3, 0.1)$ so that it is verified that

$$E_{\text{OWA}}(r^A) = 0.125 > 0.075 = E(r^A)$$

and,

$$E_{\text{OWA}}(r^B) = 0.25 = E(r^B).$$

As can be seen in Fig. 1, both frontiers are identical along the efficient interval. In our example, this interval is described for all wealth combinations $\mathbf{x} = (x_A, x_B) \equiv (x, 1-x)$ so that $x \leq \overline{x} \simeq 0.67$. Indeed, the overlapping of both frontiers occurs as in this interval the return profile of asset B (with no distinguishable ordering OWA effect) has sufficient weight in the portfolio to neutralize the ordering effect of OWA on asset A's return profile. Above this cutoff $\overline{x}$, asset A has now sufficient weight in the portfolio, and thus, the greater degree of optimism introduced by OWA over $E_{\text{OWA}}(r^A)$ also increases the E_{OWA} of the entire portfolio (the red line is above the black line), without increasing the risk measure. As discussed below, notice however that $\overline{x}$ characterizes the wealth allocation allowing the investor to attain the minimum-variance portfolio in both Markowitz and OWA frontiers, as it corresponds to the adaptation of definitions given by expressions (6) and (11) when $m = 2$. We have thus to disregard the region in which $x > \overline{x}$, and therefore, both efficient frontiers proceed to being identical.[6]

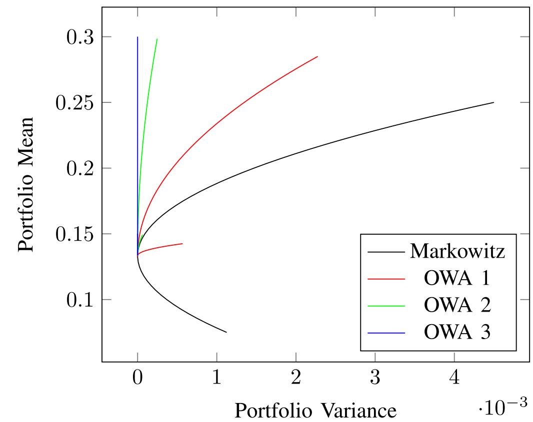

Fig. 2. *Optimistic investor.* Markowitz efficient frontier for $\Pi = (0.6, 0.3, 0.1)$ and OWA frontier for $W = (0.9, 0.05, 0.05)$ (OWA 1), $W = (0.99, 0.005, 0.005)$ (OWA 2), and $W = (1, 0, 0)$ (OWA 3). The slope of the efficient frontier increases as vector W varies from a less optimistic (OWA 1) to a fully optimistic attitude (OWA 3), increasing thus the polarization degree of the investor's portfolio decision toward asset B.

1) Optimistic Investor: To explore the role played by optimism, we consider weighting vectors with two characteristics: 1) they assign the highest weight over the best return, and 2) this weight is close to 1. Specifically, we examine three cases for W: (0.9, 0.05, 0.05), (0.99, 0.005, 0.005), and (1, 0, 0). Regarding the probabilistic vector, we assume in all the cases that $\Pi = (0.6, 0.3, 0.1)$. Our numerical exercises show that in the three cases analyzed the following properties emerge:

a) *Dominance of OWA Frontier*: Results of simulations drawn in Fig. 2 show that

$$E_{\text{OWA}}(r^p) > E(r^p)$$
$$V_{\text{OWA}}(r^p) < V(r^p)$$

for all $x \in [0,1]$. This suggests that an optimistic investor encounters a better return-risk profile for all investment strategies. In turn, this dominance of OWA frontier implies that an optimistic investor with reward-risk preferences will expect, for a given level of either return or risk, a higher utility than a Markowitzian investor.[7] In addition, this dominance gets exacerbated as the optimism degree

[6] Although it could appear from Fig. 1 that this minimum variance is zero, it is indeed not the case: it is very low but positive.

[7] With the exception of a too risk averse investor that selects a minimum risk portfolio, who would get the same utility level irrespective if he is an OWA or Markowitzian investor (see the analysis presented below).

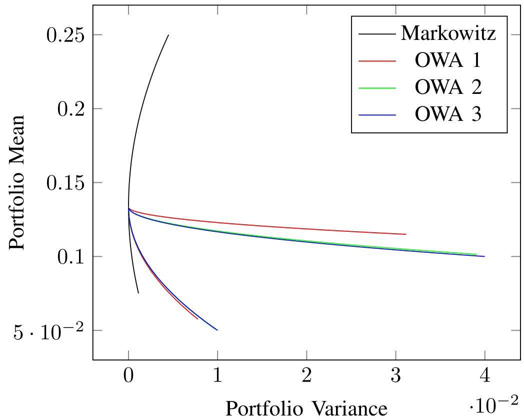

Fig. 3. *Pessimistic investor.* Markowitz efficient frontier for $\Pi = (0.6, 0.3, 0.1)$ and OWA frontier for $W = (0.05, 0.05, 0.9)$ (OWA 1), $W = (0.005, 0.005, 0.99)$ (OWA 2), and $W = (0, 0, 1)$ (OWA 3). In all OWA cases, the efficient frontier has a negative slope, making optimal for the investor to always choose the minimum-risk portfolio.

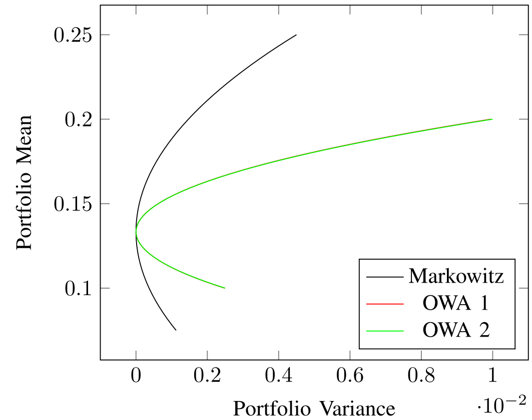

Fig. 4. *Moderate investor.* Markowitz efficient frontier for $\Pi = (0.6, 0.3, 0.1)$ and OWA frontier for $W = (0.005, 0.99, 0.005)$ (OWA 1) and $W = (0, 1, 0)$ (OWA 2). The slope of the efficient frontier decreases as vector W moves from a less moderate (OWA 1) to a fully moderate attitude (OWA 2), increasing the bias of the investor's portfolio decision toward the minimum-risk portfolio.

increases, leading to an increasing polarization in the investment decision in favor of asset B. For instance, in the extreme optimism case $W = (1, 0, 0)$, the OWA efficient frontier always involves a minimum-risk portfolio (in fact zero variance), and leads a reward-risk utility maximizing investor to choose $x^* = 0$ irrespective of his risk-aversion degree.

b) *Minimum-Risk Portfolio*: As in our example there are just two assets, we adapt expression (6) to define $\overline{x}$, the minimum-variance portfolio, as follows:

$$\overline{x} = \arg\min_{x\in[0,1]} V(r^p; x)$$

and expression (11) to define $\overline{x}_{\text{OWA}}$, the minimum-Var-OWA as follows:

$$\overline{x}_{\text{OWA}} = \arg\min_{x\in[0,1]} V_{\text{OWA}}(r^p; x).$$

Simulations for the three cases examined point out that[8]

$$\overline{x}_{\text{OWA}} = \overline{x} \tag{21}$$

which implies that a more optimistic attitude on the part of the investor plays no role in portfolio choice when he is too risk averse and looks for a minimum-risk portfolio.

2) Pessimistic Investor (Wald Criterion): To study the effects of pessimism, we consider weighting vectors with two characteristics: 1) they assign the highest weight over the worst return, and 2) this weight is close to 1. In particular, we examine three cases for W: $(0.05, 0.05, 0.9)$, $(0.005, 0.005, 0.99)$, and $(0, 0, 1)$. As in the optimistic case, the probabilistic vector is assumed to be $\Pi = (0.6, 0.3, 0.1)$.

a) *Dominance of Markowitz Frontier*: Results of numerical simulations shown in Fig. 3 point out that

$$E_{\text{OWA}}(r^p) < E(r^p)$$
$$V_{\text{OWA}}(r^p) > V(r^p)$$

for all $x \in [0, 1]$. This fact suggests that a pessimistic investor could have a worse return-risk profile for all portfolio strategies. In turn, this implies that a pessimistic investor with reward-risk preferences will always expect a lower utility level than a Markowitzian investor.[9] Furthermore, it can be seen that the efficient OWA frontier exhibits a negative slope. This implies that a pessimistic and reward-risk utility investor will always take an optimal portfolio decision consistent with the minimum OWA risk portfolio (which also attains the maximum OWA-mean return), irrespective of his risk-aversion degree.

b) Minimum-risk portfolio simulations for the three cases examined indicate that the minimum-risk portfolio coincides for both frontiers so that (21) is also satisfied.

3) Moderate Investor (Olympic Criterion): In order to examine the effects of moderation or aversion to extreme results, we consider weighting vectors with two characteristics: 1) they assign the highest weight to the intermediate return, and 2) this weight is close to 1. We examine two cases for W in particular: $(0.005, 0.99, 0.005)$ and $(0, 1, 0)$. As before, the probabilistic vector is assumed to be $\Pi = (0.6, 0.3, 0.1)$.

a) *Dominance of Markowitz efficient frontier*: Numerical simulations drawn in Fig. 4 point out that

$$E_{\text{OWA}}(r^p) < E(r^p)$$
$$V_{\text{OWA}}(r^p) > V(r^p),$$

for all $x \in [0, \overline{x})$. This result suggests that a moderate investor faces a worse return-risk profile along the efficient portfolio frontier than an investor that uses probabilities $\Pi = (0.6, 0.3, 0.1)$. Thus, a moderate investor with reward-risk preferences will always obtain a lower utility level than his Markowitzian peer for any risk-aversion

[8] This property is also satisfied by the special case of full optimism $W = (1, 0, 0)$, in which $\overline{x}_{\text{OWA}}$ is given by any $x \in [0, 1]$.

[9] Again, with the exception of a too risk averse investor looking for a minimum-risk portfolio.

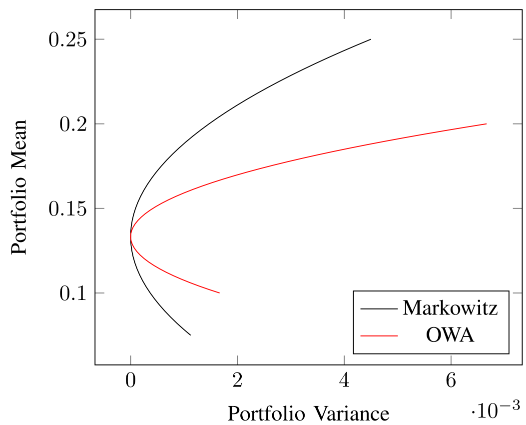

Fig. 5. *Laplace-type investor.* Markowitz efficient frontier for $\Pi = (0.6, 0.3, 0.1)$ and OWA frontier for $W = (1/3, 1/3, 1/3)$. The slope of the Laplace-type efficient frontier is smaller than the Markowitz efficient frontier, biasing the investor's portfolio decision toward the minimum-risk portfolio.

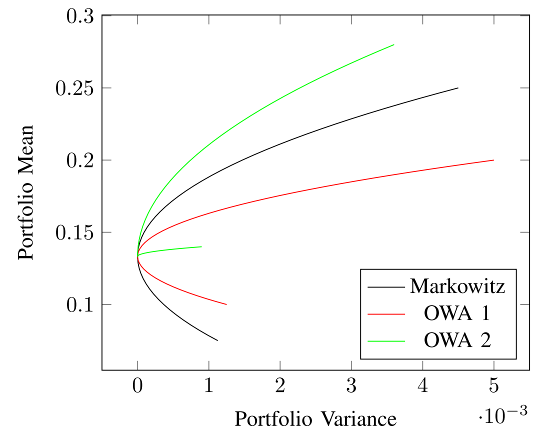

Fig. 6. *Extremist investor.* Markowitz efficient frontier for $\Pi = (0.6, 0.3, 0.1)$ and OWA frontier for $W = (0.5, 0.0, 0.5)$ (OWA 1) and $W = (0.9, 0.0, 0.1)$ (OWA 2). The efficient curve changes along from a balanced extreme-believer (OWA 1) to an optimistic extreme-believer (OWA 2) investor. As compared with the Markowitz frontier, whereas in the first case the investor's portfolio decision is biased toward the minimum-risk portfolio, in the second case this decision is biased toward asset B.

degree.[10] Of course, these results may change if we consider a vector Π assigning high probabilities to states of nature that imply low returns.

b) *Minimum-Risk Portfolio*: Simulations for the two cases studied suggest that both minimum-variance portfolios coincide.

4) Laplace-Type Investor: To explore the effects of an investor with a tendency to a perfectly equitable attitude to results, we consider weighting vectors that distribute weights in a highly balanced way among returns. Specifically, we examine the case $W = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. As before, the probabilistic vector is assumed to be $\Pi = (0.6, 0.3, 0.1)$.

a) *Dominance of Markowitz Efficient Frontier*: The numerical simulation summarized in Fig. 5 reveals that

$$E_{\text{OWA}}(r^p) < E(r^p)$$
$$V_{\text{OWA}}(r^p) > V(r^p)$$

for all $x \in [0, \overline{x})$. Thus, a Laplace-type investor exhibits an efficient portfolio frontier worse than an investor assigning probabilities $\Pi = (0.6, 0.3, 0.1)$. As with the olympic criterion, this dominance of Markowitz frontier may no longer hold if we consider a vector Π with high probabilities for states of nature involving low returns.

b) *Minimum-Risk Portfolio*: The performed simulation suggests that the minimum-risk portfolio is the same for both frontiers.

5) Extremist Investor (Hurwicz Criterion): Finally, we study the effects of an investor with an extremist profile, who may be called an *extrema-believer* investor. To this end, let us consider weighting vectors with two characteristics: 1) they assign the highest weights to extreme returns (the best and the worst), and 2) the sum of these weights is close to 1. Two cases in particular for W are examined: $(0.5, 0, 0.5)$ and $(0.9, 0, 0.1)$. As before, the probabilistic vector is assumed to be $\Pi = (0.6, 0.3, 0.1)$.

[10] With the exception of a too risk-averse investor.

TABLE IV
MINIMUM-RISK PORTFOLIO

Type of Investor	Π or W	$\overline{x}$	$V(r^p; \overline{x})$	$E(r^p; \overline{x})$
Markowitzian	$\Pi = (0.6, 0.3, 0.1)$	0.6735	$4.69e-07$	0.1321
OWA-probabilistic	$W = (0.6, 0.3, 0.1)$	0.6735	$4.69e-07$	0.1332
Optimistic	$W = (1, 0, 0)$	$[0, 1]$	0.00000	$[0.1337, 0.3]$
Pessimistic	$W = (0, 0, 1)$	0.6735	$4.16e-06$	0.1316
Moderate	$W = (0, 1, 0)$	0.6735	$1.04e-06$	0.1327
Laplace-type	$W = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	0.6735	$6.94e-07$	0.1327
Extremist (balanced)	$W = (0.5, 0, 0.5)$	0.6735	$5.21e-07$	0.1327
Extremist (optimistic)	$W = (0.9, 0, 0.1)$	0.6735	$3.75e-07$	0.1335

a) *Ambiguous Comparison Between Efficient Frontiers*: Numerical exercises illustrated in Fig. 6 suggest that there is no a clear cut supremacy of one of both frontiers, as this depends on the chosen weights to compute the OWA operator. While more weight is put on higher returns—a more optimistic extrema-believer investor—the OWA efficient frontier provides a better reward-risk tradeoff than the Markowitz approach is provided by the OWA efficient frontier. The opposite ensues notwithstanding when we consider a more balanced extrema-believer investor; for instance, with $W = (0.5, 0, 0.5)$.

b) *Minimum-Risk Portfolio*: Simulations suggest that in the two numerical examples studied the minimum-variance portfolio coincides for both classes of frontiers.

We end this analysis by referring to Table IV, which summarizes our numerical results regarding the minimum-risk portfolio along different frontiers and types of investors.

From a general perspective, numerical exercises suggest that when uncertainty is high, OWA-based methods may explain why certain portfolio decisions, apparently puzzling under the Markowitzian lens, are indeed consistent with a utility maximizing investor that uses ordering and weighting properties of the OWA instead of the classical mean. In the case of an optimistic

TABLE V
MAIN DESCRIPTIVE STATISTICS FOR REAL-DATA ASSETS

	FRI	S&P
Mean	$1.00e-3$	$1.88e-3$
Variance	$0.72e-6$	$615.64e-6$
Median	$0.95e-3$	$4.48e-3$
Minimum	$-1.45e-3$	$-93.60e-3$
Maximum	$3.82e-3$	$119.24e-3$

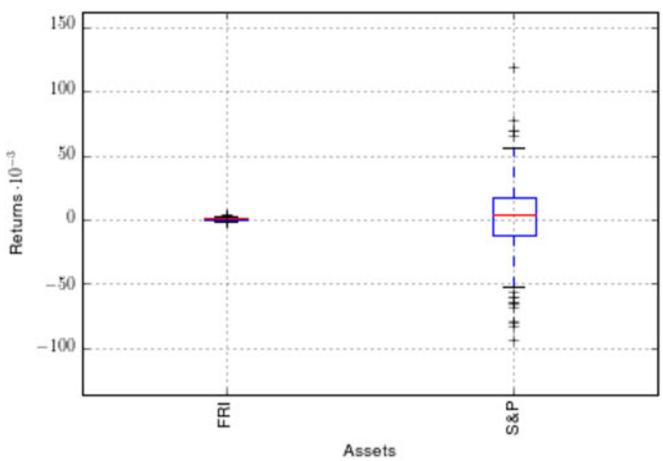

Fig. 7. *Boxplot of returns.* FRI represents a FRI of the Chilean capital market, and S&P is the S&P index.

investor, the OWA-based method may explain wealth allocations which are more polarized toward only one asset (that whose best state of nature yields the highest return of both assets). On the other hand, in the case of a pessimistic investor, the OWA-based method may explain more conservative wealth allocations, and in particular, more biased toward the minimum-risk portfolio.

B. Numerical Example With Real Data

To perform the second class of numerical exercise, we consider two portfolios available for investors operating in the Chilean capital market. These portfolios are: 1) a fixed-rent index (FRI), and 2) the Standard & Poor's index (S&P). Our sample is a historical series of weekly returns for each portfolio during the period between 10.15.2003 and 09.25.2013 ($n = 519$). The main descriptive statistics of both assets are presented in Table V, and a boxplot of the returns is displayed in Fig. 7.

As we will see now, our numerical exercises show that the same properties found with fictitious data also emerge with real data. To exemplify this fact, we only present the results of the exercises conducted with the optimistic and the pessimistic criteria, but simulations with the other criteria are available upon the author's request.

1) Ordering Effect: To isolate the effect coming from the ordering of outcomes carried out by OWA operator, we first assume that weighting vector W is identical to the probabilistic vector Π, which in this case assigns equal probability

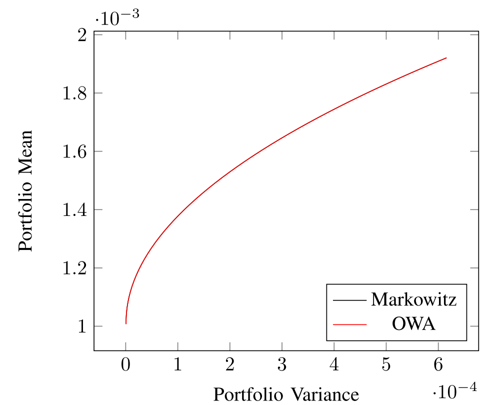

Fig. 8. *Ordering effect with real data.* Frontiers represent all the portfolio mean-portfolio variance pairs generated by wealth allocations $(x, 1-x)$ over assets FRI and S&P, respectively, such that $x \in [0, 1]$. Markowitzian frontier (black line) uses $E(\cdot)$ and $V(\cdot)$, based on probabilistic vector Π, as mean and variance; OWA frontier (red line) uses $E_{\text{OWA}}(\cdot)$ and $V_{\text{OWA}}(\cdot)$, based on weighting vector W, as mean and variance. Efficient frontiers are made up by all the portfolio mean-portfolio variance pairs that represent the highest mean for a given variance. To study ordering effect of OWA, we assume $\Pi = W$, in particular, the Laplace criterion so that an equal probability is assigned to all the returns of the sample. Numerical results indicate that both efficient frontiers are identical.

$(1/n) = (1/519)$ to all the sample results.[11] Thus, this exercise is similar to assuming a vector W consistent with the Laplace criterion. Notice that this assumption implies that

$$E_{\text{OWA}}(r^A) = E(r^A)$$
$$E_{\text{OWA}}(r^B) = E(r^B)$$

as under the Laplace criterion the order of states of nature is irrelevant.

As can be seen in Fig. 8, both frontiers are identical along the efficient interval. In our sample, this interval is described for all wealth combinations $\mathbf{x} = (x_A, x_B) \equiv (x, 1-x)$ because $\overline{x} = 1$. As we will see later, notice that $\overline{x}$ also characterizes the wealth allocation allowing to attain the minimum-variance portfolio in both Markowitz and OWA frontiers.[12] Therefore, with real data we obtain, in qualitatively terms, the same conclusions on the ordering effect found with fictitious data.

2) Optimistic Investor: To explore the effects of optimism, we consider a weighting vector W that assigns a weight of 0.8 to the best 20% returns and a weight of 0.2 to the remaining 80% of returns (a 80/20 rule). This rule is so that 0.8 is equally distributed among the first quintile of returns and 0.2 is equally distributed among the remaining four quintiles of the sample of returns. Regarding the probabilistic vector Π, we assume the Laplace criterion so that it assigns a probability $(1/n) = (1/519)$ to each return of the sample.

a) *Dominance of OWA Frontier*: Results contained in Fig. 9 show that

$$E_{\text{OWA}}(r^p) > E(r^p)$$
$$V_{\text{OWA}}(r^p) < V(r^p)$$

[11]This vector is consistent with the computation of the arithmetic mean.
[12]The minimum variance in Fig. 8 is not zero; it is very low but positive.

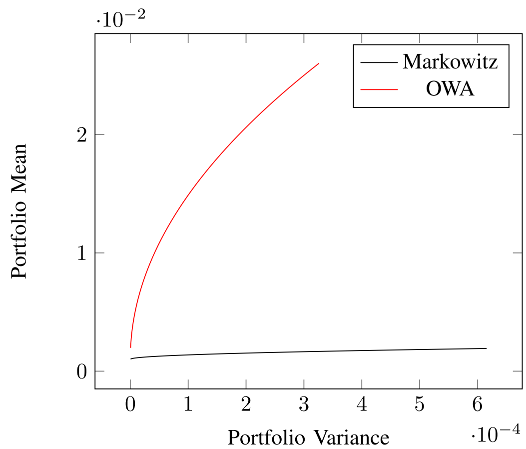

Fig. 9. *Optimistic investor with real data.* Markowitz efficient frontier for Π consistent with Laplace criterion and OWA frontier for W consistent with a $80/20$ rule.

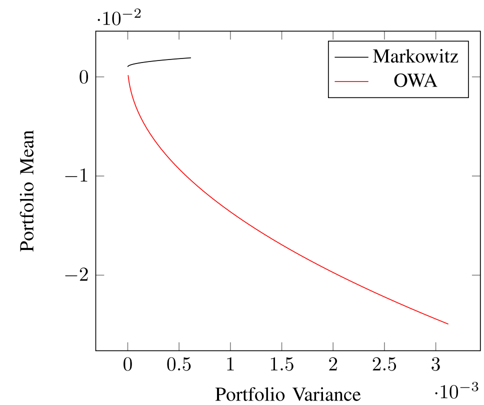

Fig. 10. *Pessimistic investor with real data.* Markowitz efficient frontier for Π consistent with Laplace criterion and OWA frontier for W consistent with a $20/80$ rule. The OWA efficient frontier has a negative slope, making optimal for de investor to choose de minimum-risk portfolio.

for all $x \in [0, 1]$. This confirms that an optimistic investor faces a better return-risk profile for all investment strategies. In turn, this dominance of OWA frontier implies that an optimistic investor with reward-risk preferences will expect, for a given level of either return or risk, a higher utility than a Markowitzian investor.[13] Although we do not present these results here, this dominance gets exacerbated as the optimism degree increases, leading to an increasing polarization in the investment decision in favor of portfolio S&P.[14]

b) *Minimum-Risk Portfolio*: Simulations indicate that the minimum-risk portfolio is the same for both frontiers so that (21) is also satisfied. Thus, a more optimistic attitude on the part of the investor plays no role in portfolio choice when he is too risk averse and looks for a minimum-risk portfolio.

3) Pessimistic Investor (Wald criterion): To study the role played by pessimism, we take a weighting vector W that assigns a weight of 0.8 to the worst 20% returns and a weight of 0.2 to the best 80% of returns (a $20/80$ rule). This rule is so that 0.8 is equally distributed among the last quintile of returns and 0.2 is equally distributed among the first four quintiles of the return sample. Regarding the probabilistic vector Π, we assume the Laplace criterion, and thus, an equal probability $(1/n) = (1/519)$ is assigned to each return of the sample.

a) *Dominance of Markowitz Frontier*: Results of our simulations with real data shown in Fig. 10 indicate that

$$E_{\text{OWA}}(r^p) < E(r^p)$$
$$V_{\text{OWA}}(r^p) > V(r^p)$$

for all $x \in [0, 1]$. This fact suggests that a pessimistic investor faces a worse return-risk profile than a Markowitzian investor for all portfolio strategies.[15] Like with exercises with fictitious data, notice that the efficient OWA frontier also exhibits a negative slope, which implies that it will always be optimal for a pessimistic and reward-risk utility investor to choose the minimum OWA risk portfolio (which also attains the maximum OWA-mean return), irrespective of his risk-aversion degree.

b) *Minimum-Risk Portfolio*: Our exercise with real data suggests that

$$\overline{x}_{\text{OWA}} = \overline{x} = 1$$

which implies that a more pessimistic attitude on the part of the investor plays no role in portfolio choice when he is too risk averse and looks for a minimum-risk portfolio. In such a case, the investor under both frontiers will choose to invest all his wealth in the FRI (portfolio FRI).

In sum, our numerical exercises suggest that, in qualitatively terms, the properties characterized with fictitious data are confirmed with real data.

V. Conclusion

The OWA operator can improve the current portfolio selection approaches based on Markowitz mean-variance framework. The key advantage of the OWA operator is the ability to represent complex scenarios by using the degree of optimism and pessimism of the decision maker. Therefore, the OWA aggregates the data considering the attitudinal character that an individual has in the specific analysis considered. This methodology implicitly assumes that the probabilistic information is unknown mainly because there is a high degree of uncertainty. The paper has implemented the use of the OWA operator in the mean return and the risk of any asset instead of using the classical approaches based on weighted averages. By using the OWA operator, the

[13] The exception is a too risk averse investor that selects the same minimum risk portfolio, irrespective whether he is either an OWA or Markowitzian investor (see the analysis presented below).

[14] In fact, in the extreme optimism case in which the vector W assigns 1 to the best return and zero to all the remaining returns, the OWA efficient frontier involves always a minimum-risk portfolio (zero variance) so that it is optimal for a reward-risk utility maximizing investor to choose $x^* = 0$, irrespective of his level of risk-aversion.

[15] With the exception of a very risk-averse investor interested in a minimum-risk portfolio.

investor obtains a more general perspective of the situation because he can analyze any scenario from the minimum to the maximum and select the specific case closest to his interests. The study considers many families of OWA aggregations in order to see the different results produced by using a wide range of particular cases. The classical methods for decision making under uncertainty have also been studied as particular cases of the OWA operator, including the pessimistic, optimistic, Laplace, and Hurwicz criterion.

The paper has suggested a new way for analyzing the degree of optimism and pessimism that considers the specific values of the arguments. Thus, rather than only considering the weights, it considers the position of the numerical values of the arguments which also conditions the optimism or pessimism of the aggregation. Several numerical examples are studied in order to numerically understand the new approach. These examples are developed considering different types of OWA operators that can be used in the aggregation process including the optimistic, pessimistic, Olympic, Laplace, and Hurwicz criterion. Various results emerge from this numerical exercise. First, the isolated ordering effect of OWA indicates that this operator gives more optimism only on the inefficient portfolio region, and thus, both efficient frontiers (Markowitz and OWA) are identical. Second, the combined ordering and weighting effect of OWA suggests that the minimum-risk portfolio is the same in both types of frontiers. This result is robust to different attitudinal characters of investors. Third, an optimistic investor faces better return-risk profile, so he expects, for a given level of either return nor risk, a higher utility than a Markowitzian investor. This dominance becomes exacerbated as the optimism degree increases, leading to an increasing polarization of portfolio choice in favor of the individual asset with the best return profile. Fourth, a pessimistic investor faces a worse return-risk tradeoff, so he expects a lower utility than a Markowitzian investor. As a consequence, he always selects the minimum risk portfolio irrespective of his risk-aversion degree. Fifth, in the case of moderate, Laplace-type, and extremist investor, the results in general are more ambiguous as they depend on the specific probabilistic and weighting vectors under consideration.

Future research will consider other extensions and generalizations of the OWA operator in Markowitz mean-variance portfolio approach including induced aggregation operators [16], [33], the probabilistic OWA operator [20], geometric operators [35], interval [36] and fuzzy information [37], [38], and multiperson techniques [11]. Additionally, many other portfolio selection models can be studied with the OWA including the Sharpe approach, the CAPM and the APT. Finally, note that the suggested approach may become a starting point to provide alternative explanations to several controversial results in financial economics such as the two-fund separation puzzle.

Appendix A
A Counter-Example of OWA Linearity

It is easy to prove that the OWA operator is nonlinear. Let us suppose an example of an OWA operator defined on $\mathbb{R}^2$, i.e., $\text{OWA}(u) \doteq \langle w, T(u) \rangle$, where w is the vector of ponderation, T the operator which permutes the elements of u in decreasing order, and $\langle \cdot, \cdot \rangle$ the usual scalar product. Without loss of generality we take the fully optimistic case, i.e., $w \doteq (1, 0)$. Now we take two elements $u, v \in \mathbb{R}^2$ and prove that, in this particular case, $\text{OWA}(u + v) \neq \text{OWA}(u) + \text{OWA}(v)$.

For this, it is enough to suppose that u is given by $(u_1, 0)$ with $u_1 > 0$ and v is given by $(0, v_2)$ with $v_2 > 0$. Then $\text{OWA}(u + v) = \langle w, T(u + v) \rangle = \langle (1, 0), T((u_1, v_2)) \rangle$, which is either u_1 if $u_1 \geq v_2$, or v_2 if $v_2 \geq u_1$. On the other hand, $\text{OWA}(u) + \text{OWA}(v) = \langle (1, 0), T((u_1, 0)) \rangle + \langle (1, 0), T((0, v2)) \rangle = u_1 + v_2$, which is strictly higher than $\text{OWA}(u + v)$.

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