Analysis on Extensions of Multi-expert Decision Making Model with Respect to OWA-Based Aggregation Processes

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Abstract. In this paper, an analysis on extensions of multi-expert decision making model based on ordered weighted averaging (OWA) operators is presented. The focus is on the aggregation of criteria and the aggregation of individual judgment of experts. First, soft majority concept based on induced OWA (IOWA) and generalized quantifiers to aggregate the experts' judgments is analyzed, in which concentrated on both classical and alternative schemes of decision making model. Secondly, analysis on the weighting methods related to unification of weighted average (WA) and OWA is conducted. An alternative weighting technique is proposed which is termed as alternative OWA-WA (AOWAWA) operator. The multi-expert decision making model then is developed based on both aggregation processes and a comparison is made to see the effect of different schemes for the fusion of soft majority opinions of experts and distinct weighting techniques in aggregating the criteria. A numerical example in the selection of investment strategy is provided for the comparison purpose.

Keywords: Multi-expert decision making · OWA operator · IOWA operator
Weighting methods · Soft majority concept

1 Introduction

In the past, various multi-criteria decision making methods have been developed as tools for modeling human decision making and reasoning, see [4, 5]. The methods have effectively used in numerous applications to deal with the rating, ranking and selection of option(s). In complex decision making, normally a group of experts or decision makers involved in which each of them offset and/or support the others for the comprehensive decision. Since then, the expansion of such models to multi-expert decision making have been extensively focused.

Central to the decision making problems, aggregation process play a crucial role in deriving the final decision, either to aggregate the criteria with respect to each option or

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to aggregate the final agreement of individual experts. Weighted average (WA) and ordered weighted averaging (OWA) operators are generally employed as aggregation processes in decision making models. OWA operators [17, 18, 19] provide a parameterized class of aggregation operators which can be ranged from minimum to maximum and average as normal case. In contrast to the WA which represents the reliability of information sources or criteria, the weights in OWA reflect the importance of values with respect to ordering. The OWA operators can be explained as applying the concept of fuzzy set theory to modify the basic aggregation process used in decision making, precisely, using generalized quantifiers [7, 8, 26] for soft aggregation processes. In addition, the induced OWA (IOWA) operators [22] as its extension deal with the problem which involve pair of values, for example, the additional parameters used to induce the argument values to be aggregated. The OWA and IOWA are useful in the case of the need to consider the attitudinal character of experts, for instance, the behavior of experts regarding the proportion of criteria to consider. Analogously, with respect to group decision making, the soft majority agreement among experts can be implemented using the IOWA operators, which synthesizes the opinions of the majority (such as semantics "most") of the experts. In this case, a majority opinion refers to consensual judgment of a majority of experts who have similar opinions.

With respect to that, the purpose of this paper is on analyzing the multi-expert decision making model based on these two aggregation processes, i.e., aggregation of criteria and aggregation of experts' judgments. At first, the soft majority concept models for aggregating the experts' judgments based on IOWA operators and linguistic quantifier are reviewed, particularly the method as proposed in [14] and its extension as proposed in [2]. The difference between the two majority concept models can be divided into: (i) on assigning the weights for the experts, (ii) the measures used in calculating the support between experts (proximity metric), and (iii) the approach in deriving the support between experts, either based on options (classical scheme) or criteria (alternative scheme). Pasi and Yager [14] proposed the method in case of weights between experts are considered as identical (homogeneous group decision making) and used the support function based on distance measure to compute the overall level of agreement between experts. Besides, the support between experts is calculated with respect to the final result of options of each expert. On the other hand, Bordogna and Sterlacchini [2] extended this idea to include the case of where the experts are assigned with different weights (heterogeneous group decision making) and utilized similarity measure based on Minkowski OWA to calculate the overall support between experts. Moreover, the approach used to calculate the support between experts is based on the similarity measure with respect to each criterion instead of on each option. In this paper, for the purpose of comparison, some modifications have been made to both methods, include an extension of the Pasi and Yager's method from classical scheme to alternative scheme. On contrary, the Bordogna and Sterlacchini's method has been modified to deal with classical scheme. Hence, two additional methods with the existing two original methods are compared as to examine the effect of the approaches on decision scheme used.

Secondly, the weighting methods which stipulate decision strategies for the compensation of criteria in making the decision are studied. Specifically, we analyze some of the methods in deriving the weights based on the unification of WA and OWA, such as, methods for including importances using combination of 'or-and' operators [18], linguistic quantifier [23], fuzzy system modeling [24], weighted OWA (WOWA) [15], OWAWA [10], hybrid WA (HWA) [16] and immediate WA (IWA) [9]. In addition, we propose an alternative OWAWA (AOWAWA) operator which combines the characteristics of IWA and OWAWA using the idea of geometric means. As comparison, the multi-expert decision making model with respect to Bordogna and Sterlacchini's approach on alternative scheme is used as to observe the results of distinct weighting techniques in aggregation of criteria.

The outline of the paper is as follows. In Sect. 2 the definitions of OWA, IOWA and Minkowski OWA distance operators are presented. In Sect. 3 the aggregation techniques for soft majority concept is discussed and then Sect. 4 reviews the weighting methods based on WA and OWA. In Sect. 5, multi-expert decision making model based on different schemes and weighting techniques of aggregation processes are outlined. A numerical example in a selection of investment strategy is provided in Sect. 6. The paper then is summed up with a conclusion in the Sect. 7.

2 Preliminaries

This section provides some definitions and basic concepts related to OWA and IOWA operators and their generalizations that will be used throughout the paper.

2.1 OWA Operator

Definition 1 [18]**.** An OWA operator of dimension n is mapping $OWA: R^n \to R$ that has an associated weighting vector W of dimension n, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWA(a_1, \ldots, a_n) = \sum\nolimits_{j=1}^{n} w_j a_{\sigma(j)} \tag{1}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$.

The OWA operators are all meet commutative, monotonic, bounded and idempotent properties. Given that a function $Q: [0, 1] \to [0, 1]$ as a regular monotonically non-decreasing fuzzy quantifier and it satisfies: (i) $Q(0) = 0$, (ii) $Q(1) = 1$, (iii) $a > b$ implies $Q(a) \geq Q(b)$, then the associated OWA weights can be derived using this function such in the next definition.

Definition 2 [18]**.** Let Q be a non-decreasing fuzzy quantifier, then a mapping $OWA: R^n \to R$ is an ordered weighted average (OWA) operator of dimension n if:

$$OWA_Q(a_1, a_2, \ldots, a_n) = \sum\nolimits_{j=1}^{n} \omega_j a_{\sigma(j)}, \tag{2}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\omega_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)$, being a monotonic non-decreasing function.

2.2 IOWA Operator

Definition 3 [22]**.** An IOWA operator of dimension n is mapping $IOWA : R^n \rightarrow R$ that has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$IOWA(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum\nolimits_{j=1}^{n} w_j a_{\sigma(j)} \tag{3}$$

where the notion $\sigma(j)$ denotes the inputs $\langle u_j, a_j \rangle$ of the order-inducing variable u_j and argument variable a_j reordered such that $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \ldots \geq u_{\sigma(n)}$ and the convention that if z of the are tied, i.e., $u_{\sigma(j)} = u_{\sigma(j+1)} = \ldots = u_{\sigma(j+z-1)}$, then, the value $a_{\sigma(j)}$ is given as follow [8, 20]:

$$a_{\sigma(j)} = \frac{1}{z} \sum_{k=\sigma(j)}^{\sigma(j+z-1)} a_k \tag{4}$$

The IOWA operators are all meet commutative, monotonic, bounded and idempotent properties.

2.3 Minkowski OWA Distance

Definition 4 [11]**.** A Minkowski OWAD operator of dimension n is a mapping $MOWAD : R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ with $w_j \in [0, 1]$ and the distance between two sets A and B is given as follows:

$$MOWAD(d_1, d_2, \ldots, d_n) = \left(\sum\nolimits_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda} \right)^{1/\lambda}, \tag{5}$$

where $d_{\sigma(j)}$ denotes the components of $D = (d_1, d_2, \ldots, d_n)$ being ordered in non-increasing order $d_{\sigma(1)} \geq d_{\sigma(2)} \geq \ldots \geq d_{\sigma(n)}$, and d_j is the individual distance between A and B, such that $d_j = |a_j - b_j|$ with λ is a parameter in a range $\lambda \in (-\infty, \infty)$.

The MOWAD operators are all meet commutative, monotonic, bounded and idempotent properties. By setting different values for the norm parameter λ, some special distance measures can be derived. For example, if $\lambda = 1$, then the Manhattan OWA distance can be obtained, $\lambda = 2$ then the Euclidean OWA distance can be acquired, $\lambda = \infty$ then Tchebycheff OWA is derived, etc. Equivalently, OWA and IOWA can be generalized using the same formulation, see [12, 20, 25].

3 Aggregation Methods for Soft Majority Concept

In this section, the methods for aggregating the soft majority opinion of individual experts are presented. The method by Pasi and Yager [14] as well as its extension, Bordogna and Sterlacchini [2] are studied. The extension of both methods then are made and applied in a multi-expert decision making model for the analysis purpose. Before that, the general framework of decision making schemes in which the basis of Pasi and Yager's method and also Bordogna and Sterlacchini's method are presented, i.e., a classical scheme and an alternative scheme of decision making process.

3.1 Multi-expert Decision Making Schemes

In general, the method as proposed in [14] is mainly based on the classical scheme where the result of consensus measure is determined according to the support on each option of individual experts. While the method in [2] is based on the alternative scheme in which the majority opinion particularly focuses on each specific criterion.

The classical scheme of group decision making process can be divided into two stages of aggregation process, namely internal and external aggregations. The internal aggregation involves the fusion of criteria for each expert, either full or partial compensation. At this stage, the ranking of alternatives for each expert is derived. As regard to this ranking, then in the external aggregation, the soft majority concept is implemented to find the final ranking which reflects the majority opinion of individual experts. Note that the fusion of experts' judgments in this case is focused on each option as proposed in [14].

On the other hand, for the alternative approach, instead of dealing with internal aggregation at the first step, where the individual ranking of options of each expert is derived, this method initiated with the external aggregation to fuse the majority opinion with respect to each criterion. At this stage, the new decision matrix which represents the soft majority of experts is obtained. Then, the internal aggregation to fuse the criteria is performed with the flexibility to compensate the criteria for the final decision.

3.2 The Method Based on Pasi and Yager's Approach

In the following, a brief description of the aforementioned methods is conferred. Two fundamental steps in each method are on determining the inducing variable and deriving the associated weights of experts. The methodology used to obtain the majority opinion based on Pasi and Yager [14] can be expressed as follows:

Suppose that a collection of individual opinion of h experts $(h = 1, 2, \ldots, k)$ is given as the vector $P_i^h = \left(p_i^1, p_i^2, \ldots, p_i^k\right)$ with respect to each option i, $(i = 1, 2, \ldots, m)$. For a simple notation, p_h can be used instead of p_i^h since each option is evaluated independently using the same formulation. For a single option, the similarity of each expert can be calculated using the support function as follows:

$$supp\left(p_l, p_g\right) = \begin{cases} 1 & if\ \left|p_l - p_g\right| < \beta, \\ 0 & otherwise. \end{cases} \tag{6}$$

The support function represents the similarity or dissimilarity between expert l with respect to all the other experts g, such that $l, g \in h$. Then the overall support for each individual expert l can be given as:

$$u_l = \sum\nolimits_{\substack{g=1 \\ g \neq l}}^{k} supp\left(p_l, p_g\right), \tag{7}$$

where $u_l, l \in h = (1, 2, \ldots k)$ constitute the values of order inducing variable $U = \left(u_{\sigma(1)}, \ldots, u_{\sigma(k)}\right)$ which ordered as $u_{\sigma(1)} \leq u_{\sigma(2)} \leq \ldots \leq u_{\sigma(k)}$. Note that, here the values of inducing variable are reordered in non-decreasing order instead of non-increasing order as in the original IOWA, such in Eq. (3). This type of ordering reflects the conformity of quantifier 'most' as to model the majority concept, see [14] for clarification.

In consequence, to compute the weights of the weighting vector, define the values t_l based on an adjustment of the values u_l, such that: $t_l = u_l + 1$ (the similarity of p_l with itself, similarity value equal to one). The t_l values are in non-decreasing order, $t_1 \leq \ldots \leq t_k$. On the basis of t_l values, the weights of the weighting vector are computed as follows:

$$w_l = \frac{Q(t_l/k)}{\sum_{l=1}^{k} Q(t_l/k)}. \tag{8}$$

The value $Q(t_l/k)$ denotes the degree to which a given member of the considered set of values represents the majority. The quantifier Q based on membership function for semantics "most" of experts can be given as follows:

$$Q(r) = \begin{cases} 1 & if\ r \geq 0.8, \\ 2r - 0.6 & if\ 0.3 < r < 0.8, \\ 0 & if\ r \leq 0.3, \end{cases} \tag{9}$$

where $r = t_l/k$. As can be seen, the weight of experts here is derived based on the arithmetic mean (AM) where each expert is considered as having an equal degree of importance or trust, e.g., reflect the average of the most of the similar values. Then, the final evaluation is determined using the IOWA operators such in Eq. (3). Note that, here the IOWA is based on the non-decreasing of inputs u_l, p_l, as well as weights w_l as to comply with the concept of majority opinions.

However, in some cases, the values of the vector $P_i^h = \left(p_i^1, p_i^2, \ldots, p_i^k\right)$ derived after the internal aggregation process are very close to each other due to, for example, the normalization process. This case then leads to the values of $\left|p_l - p_g\right|$ less differentiable and cause a difficulty in assigning a value for β. Hence, in this paper, a slight modification has been made to cope with this problem. The support function in Eq. (6) then can be modified as follows:

$$supp(p_l, p_g) = \begin{cases} 1 & if \ \frac{|p_l - p_g|}{\max_l |p_l - p_g|} < \beta, \\ 0 & otherwise. \end{cases} \quad (10)$$

3.3 The Method Based on Bordogna and Sterlacchini's Approach

In the following, the method based on Bordogna and Sterlacchini [2] is presented. Contrary to the previous method, here, the majority opinion of experts with respect to each specific criterion is conducted for every option. Suppose that a collection of individual opinion of h experts is given as vector $P_i^h = (p_i^1, p_i^2, \ldots, p_i^k)$ for each option $i, (i = 1, 2, \ldots, m)$. In this method, instead of using the support function, they used the Minkowski OWA based similarity measure to obtain the $Q_{coherence}$ for inducing variable. The $Q_{coherence}$ of each expert can be defined as follows:

$$u_l = Q_{coherence}(P_l, P_h) = MOWA(s_1, \ldots, s_k) = \left(\sum_{h=1}^{k} \omega_h s_h^\lambda \right)^{1/\lambda}, \quad (11)$$

where ω_h are the ordered weights with the inclusion of importances of experts (or trust scores of experts, $t_h, h = 1, 2, \ldots, k$), such that $\omega_h, t_h \in [0, 1]$ with $\left(\sum_{h=1}^{k} \omega_h = \sum_{h=1}^{k} t_h = 1 \right)$ and $s_l = 1 - |p_l - p_h|$ as similarity measure between expert l with respect to all the other experts h (includes itself), such that $l \in h$. The norm parameter $\lambda \in (-\infty, \infty)$ provides a generalization of the model.

Then, the order inducing vector can be given as:

$$U = (u_1, \ldots, u_k) = (Q_{coherence}(P_1, P_h), \ldots, Q_{coherence}(P_k, P_h)), \quad (12)$$

Moreover, Q as generalized quantifiers can take any semantics to modify the weights of experts (or trust degrees) for different strategies or behaviors. When $Q(x) = x$, then $Q_{coherence}$ is reduced to:

$$u_l = coherence(P_l, P_h) = \left(\sum_{h=1}^{k} t_h s_h^\lambda \right)^{1/\lambda}, \quad (13)$$

which is the weighted average of trust degrees with similarity measure of experts.

This can be explained as the generalization of trust degrees, where in [14] the trust, t_h are considered as equal, while here they can be extended to WA and OWA weights.

Subsequently, the weights of weighting vector for the IOWA operator can be deriving using the following formula:

$$m_h = \frac{argmin_h(u_1 \cdot t_1, \ldots, u_k \cdot t_k)}{\sum_{h=1}^{k} argmin_i(u_1 \cdot t_1, \ldots, u_k \cdot t_k)}, \tag{14}$$

where m_h are ordered in non-decreasing order. Further, given the quantifier Q with semantics "most" as Eq. (9), the weighting vector $W = (\omega_1, \ldots, \omega_k)$ can computed as:

$$\omega_h = \frac{Q(m_h)}{\sum_{h=1}^{k} Q(m_h)}. \tag{15}$$

Next, the overall aggregation process is computed using the IOWA operator with non-decreasing inputs $\langle u_l, p_l \rangle$. Similarly, here, a simple modification can be made to the similarity measure to cope with the small difference between the values as follows:

$$s(p_l, p_g) = 1 - \left(\frac{|p_l - p_g|}{\max_l |p_l - p_g|} \right). \tag{16}$$

4 The Methods Based on Unification of WA and OWA

In this section, the method for deriving the associated weights for aggregation of criteria is discussed. In particular, the weighting methods based on unification of WA and OWA are reviewed. In addition to the previously proposed methods in the literature, an alternative weighting technique called as AOWAWA operator is suggested. The analysis on some functions that generalizes WA and OWA operators which was done in [9] i.e., WOWA, HWA, OWAWA and IWA, then is extended to include some other functions like OWA-OA, OWA-FSM, and the proposed AOWAWA.

4.1 The Existing Methods

Prior to the definition of unification of WA and OWA as weighting methods, the general definition of WA and OWA weights are given.

Definition 5. A weighting vector $V = (v_1, v_2, \ldots, v_n)$ is a weighting vector of dimension n if and only if $v_j \in [0, 1]$ and $\sum_j v_j = 1$.

Definition 6. Let P be a weighting vector of dimension n, then a mapping $WA : R^n \rightarrow R$ is a weighted average of dimension n if $WA_P(a_1, a_2, \ldots, a_n) = \sum_j p_j a_j$. The WA are monotonic, idempotent and bounded, but it is not commutative [1, 6].

Definition 7 [20]**.** Let W be a weighting vector of dimension n, then a mapping $OWA_W : R^n \rightarrow R$ is an ordered weighted averaging (OWA) operator of dimension n if:

$$OWA_W(a_1, a_2, \ldots, a_n) = \sum_j w_j a_{\sigma(j)}, \tag{17}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$.

There are a number of methods proposed in the literature for obtaining the OWA weights, e.g., linguistic quantifier [18] such in Eq. (2), maximum entropy OWA [13], etc. For the overview of methods for determining OWA weights, see [26]. Next, some of the unification methods of WA and OWA are given.

Definition 8 [18]**.** Let P and W be two weighting vectors of dimension n., then a mapping $OWA : R^n \to R$ is an OWA operator of dimension n if:

$$OWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum\nolimits_j w_j a_{\sigma(j)}, \tag{18}$$

where $a_{\sigma(j)}$ denotes the components of $\check{A} = (\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$ being ordered in non-increasing order $\check{a}_{\sigma(1)} \geq \check{a}_{\sigma(2)} \geq \ldots \geq \check{a}_{\sigma(n)}$ such that $\check{a}_j = H(a_j, p_j) = (p_j \vee \bar{\alpha}) \cdot (a_j)^{p_j \vee \alpha}$ and $\alpha = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j$ is the orness measure and $\bar{\alpha} = 1 - \alpha$ is its complement.

This method is based on 'or-and' lattice operator and for the sake of simplicity, in this paper it can be termed as OWA-OA. Note that if $\alpha = 0$, then it is a pure 'and' operator, given as $a_j = a_j^{p_j}$. Since $w_n = 1$, then $D(x) = \underset{j=1,\ldots,n}{Min} A_j(x)^{p_j}, A_j(x) = a_j$. Conversely, if $\alpha = 1$, then it is a pure 'or' operator, given as $a_j = p_j a_j$. Since $w_1 = 1$, then $D(x) = \underset{j=1,\ldots,n}{Max} p_j A_j(x),\ A_j(x) = a_j$. The OWA-OA operators are all meet commutative, monotonic, bounded and idempotent properties. But, OWA-OA operators do not satisfy $O_p^\eta = F_p$ and $O_\eta^w = F^w$.

Definition 9 [24]**.** Let P and W be two weighting vectors of dimension n, then a mapping $OWA : R^n \to R$ is an OWA operator of dimension n if:

$$OWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum\nolimits_i w_j a_{\sigma(j)}, \tag{19}$$

where $a_{\sigma(j)}$ denotes the components of $\hat{A} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$ being ordered in non-increasing order $\hat{a}_{\sigma(1)} \geq \hat{a}_{\sigma(2)} \geq \ldots \geq \hat{a}_{\sigma(n)}$ given that $\hat{a}_j = H(a_j, p_j) = \bar{\alpha}\bar{p}_j + p_j a_j$ and $\bar{\alpha} = 1 - \alpha$, that is the orness measure $\alpha = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j$. This method is based on fuzzy system modeling and can be termed as OWA-FSM. The OWA-FSM operators are all meet commutative, monotonic, bounded and idempotent properties. But, OWA-FSM operators do not satisfy $M_p^\eta = F_p$ and $M_\eta^w = F^w$.

Definition 10 [15]**.** Let P and W be two weighting vectors of dimension n, then a mapping $WOWA : R^n \to R$ is a weighted ordered weighted averaging (WOWA) operator of dimension n if:

$$WOWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum\nolimits_j \omega_j a_{\sigma(j)}, \tag{20}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\omega_j = w^*\left(\sum_{k \leq j} p_{\sigma(j)}\right) - w^*\left(\sum_{k \leq j} p_{\sigma(j)}\right)$ with w^* being a monotonic non-decreasing function that interpolates the points $\left((j/n), \sum_{k \leq j} w_j\right)$ together with the point $(0, 0)$. The function w^* required to be a straight line when the points can be interpolated in this way.

WOWA operators satisfy $W_p^\eta = F_p$ and $W_\eta^w = F^w$. Moreover, they are monotonic, idempotent, and bounded [16]. In a similar way that for the OWA operator, the WOWA operator can be defined using a fuzzy quantifier instead of having the weighting vector w. This definition is similar to the Yager's definition of OWA using importances [23].

Definition 11 [23]**.** Let Q be a non-decreasing fuzzy quantifier, let p be a weighting vector of dimension n, then a mapping $OWA : R^n \to R$ is an OWA operator of dimension n if:

$$OWA_{P,Q}(a_1, a_2, \ldots, a_n) = \sum_j \omega_j a_{\sigma(j)}, \tag{21}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\omega_j = Q\left(\sum_{k \leq j} p_{\sigma(j)}\right) - Q\left(\sum_{k \leq j} p_{\sigma(j)}\right)$.

This operator generalizes the weighted mean and the OWA operator: when $p = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ the operator reduces to the OWA operator and when $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ the operator reduces to the WA.

Definition 12 [16]**.** Let P and W be two weighting vectors of dimension n, then a mapping $HA : R^n \to R$ is a hybrid averaging (HA) operator of dimension n if:

$$HA_{P,W}(a_1, a_2, \ldots, a_n) = \sum_j w_j a_{\sigma(j)}, \tag{22}$$

where $a_{\sigma(j)}$ denotes the components of $\check{A} = (\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$ being ordered in non-increasing order $\check{a}_{\sigma(1)} \geq \check{a}_{\sigma(2)} \geq \ldots \geq \check{a}_{\sigma(n)}$ given that $\check{a}_j = np_j a_j$ and n is the balancing coefficient.

HWA operator generalizes both OWA and WA operators and reflects the importance degrees of both the given argument and the ordered position of the argument. HWA operators satisfy $H_p^\eta = F_p$ and $H_\eta^w = F^w$. Moreover, they are monotonic [9].

Definition 13 [9]**.** Let P and W be two weighting vectors of dimension n, then a mapping $IWA : R^n \to R$ is an immediate weighted averaging (IWA) operator of dimension n if:

$$IWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum_j \pi_j a_{\sigma(j)}, \tag{23}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\pi_j = w_j p_j / \sum_{j=1}^{n} w_j p_j$.

As can be seen, the IWA is a manipulation of immediate probability [3, 11, 21] by using the WA instead of probability distribution. IWA operators satisfy $I_p^{\eta} = F_p$ and $I_{\eta}^{w} = F^{w}$[9].

Definition 14 [11]**.** Let P and W be two weighting vectors of dimension n, then a mapping $OWAWA : R^n \rightarrow R$ is an ordered weighted averaging-weighted average (OWAWA) operator of dimension n if:

$$OWAWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum\nolimits_j \varphi_j a_{\sigma(j)}, \tag{24}$$

where $a_{\sigma(j)}$ denotes the components of $A = (a_1, a_2, \ldots, a_n)$ being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\varphi_j = \beta w_j + (1-\beta) p_{\sigma(j)}$ with $\beta \in [0, 1]$.

OWAWA operator is all meet monotonic, idempotent, bounded properties. Moreover the value returned by the OWAWA operator lies between the values returned by the WA and OWA, and coincides with them when both are equal. But, OWAWA operators do not satisfy $N_p^{\eta} = F_p$ and $N_{\eta}^{w} = F^{w}$.

In addition, by taking the advantages of IWA and OWAWA operators, a new weighting method can be derived as in the next sub-section.

4.2 The Proposed Alternative OWAWA Operator

Definition 15. Let P and W be two weighting vectors of dimension n, then a mapping $AOWAWA : R^n \rightarrow R$ is an alternative ordered weighted averaging-weighted average (AOWAWA) operator of dimension n if:

$$AOWAWA_{P,W}(a_1, a_2, \ldots, a_n) = \sum\nolimits_j \hat{\varphi}_j a_{\sigma(j)}, \tag{25}$$

where $a_{\sigma(j)}$ denotes the components of A being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \ldots \geq a_{\sigma(n)}$ and $\hat{\varphi}_j = w_j^{\beta} * p_{\sigma(j)}^{(1-\beta)} / \sum\limits_{j=1}^{n} w_j^{\beta} \cdot p_{\sigma(j)}^{(1-\beta)}$ with $\beta \in [0, 1]$, by convention $(0^0 = 0)$.

The AOWAWA operator is monotonic, bounded, idempotent. However, it is not commutative because the AOWAWA operator includes the WA. In addition, AOWAWA operators do not satisfy $A_p^{\eta} = F_p$ and $A_{\eta}^{w} = F^{w}$.

Theorem 1 (Monotonicity)**.** Assume f is the AOWAWA operator, let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two sets of arguments. If $a_j \geq b_j$, $\forall j \in \{1, 2, \ldots, n\}$, then:

$$f(a_1, a_2, \ldots, a_n) \geq f(b_1, b_2, \ldots, b_n).$$

Proof. It is straightforward and thus omitted.

Theorem 2 (Idempotency)**.** Assume f is the AOWAWA operator, if $a_j = a$, $\forall j \in \{1, 2, \ldots, n\}$, then:

$$f(a_1, a_2, \ldots, a_n) = a.$$

Proof. It is straightforward and thus omitted.

Theorem 3 (Bounded)**.** Assume f is the AOWAWA operator, then:

$$Min\{a_j\} \leq f(a_1, a_2, \ldots, a_n) \leq Max\{a_j\}.$$

Proof. It is straightforward and thus omitted.

5 Multi-expert Decision Making Model Based on Different Schemes of Aggregation Processes

In this section, some multi-expert decision making models based on classical and alternative aggregation schemes are presented. First, the majority concept of Pasi and Yager's method which is originally based on classical aggregation scheme is extended to the alternative scheme. Here, the multi-expert decision making model using Pasi and Yager's method with respect to classical scheme is stated as MEDM-PY I and for alternative scheme is denoted as MEDM-PY II. Secondly, Bordogna and Sterlacchini's method which is based on alternative scheme is modified to the case of classical method. Here, the MEDM-BS I represents decision making model using the alternative scheme and MEDM-BS II denoted as the method based on classical scheme. Moreover, for the aggregation process of criteria, each of the weighting methods based on unification of WA and OWA are implemented for comparison purpose.

5.1 The Proposed Alternative OWAWA Operator

Stage I: Internal aggregation (*Local aggregation*)

Step 1: First, a decision matrix for each expert $D^h, h = 1, 2, \ldots, k$, is constructed as follows:

$$D^h = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \ldots & C_n \end{array} \\ \begin{pmatrix} a_{11}^h & \cdots & a_{1n}^h \\ \vdots & \ddots & \vdots \\ a_{m1}^h & \cdots & a_{mn}^h \end{pmatrix} \end{array}, \tag{26}$$

where A_i indicates the alternative $i(i = 1, 2, \ldots, m)$ and C_j denotes the criterion $j(j = 1, 2, \ldots, n)$, and a_{ij}^h with $a_{ij}^h \in [0, 1]$ denotes the preferences for alternative A_i with respect to criterion C_j.

Step 2: Determine the weighting vector for each expert using the unification of WA and OWA. All the weighting methods can be implemented such as in Eqs. (17)–(25). In this step, the attitudinal character of experts reflects the proportion of criteria used under consideration.

Step 3: Aggregate the judgment matrix of each expert by the weighting vector in Step 2. At this stage, each expert derives the ranking/priorities of all alternatives individually (individual experts' judgments).

Stage II: External aggregation (*Global aggregation*)

With respect to the type of aggregation method, the consensus measure for the majority of experts can be calculated as follows:

(A) Pasi and Yager's method: MEDM-PS I (Homogeneous group decision making)

Step 4A: Determine the inducing variable using the Eqs. (6)–(7) or in case of the values are very close to each other, use the modified support function such in Eq. (10).

Step 5A: Calculate the weighting vector which represents the majority of experts using the Eq. (8) based on quantifier "most" such in Eq. (9). In this case, the weights are considered as equal for all experts.

(B) Modified version of Bordogna and Sterlacchini's method: MEDM-BS I (Heterogeneous group decision making process).

Step 4B: Determine the inducing variable using the Eq. (11) or in case of the values are very close to each other use the similarity measure such in Eq. (16).

Step 5B: Calculate the weighting vector using the Eqs. (14)–(15). In this case, the weights of experts or trust degrees are associated to each expert.

5.2 The Proposed Alternative OWAWA Operator

Stage I: External aggregation (*Local aggregation*)

Step 1: By the similar way, a decision matrix for each expert is constructed such in Eq. (26). Then the aggregation of majority of experts can be implemented using one of the methods as follows:

(A) Bordogna and Sterlacchini's method: MEDM-BS II

Step 2A: Determine the inducing variable such in Step 4B of classical scheme. But, instead of aggregate the opinion of experts with respect to each option, in this step, the aggregation process is conducted on each criterion.

Step 3A: Calculate the weighting vector such in Step 5B of classical scheme using the values of inducing variable in Step 2A.

(B) Extension of Pasi and Yager's method: MEDM-PS II

Step 2: Determine the inducing variable such in Step 4A of classical scheme. But, instead of aggregate the opinion of experts with respect to each option, here, the aggregation process is conducted on each criterion.

Step 3: Calculate the weighting vector such in Step 5A of classical scheme using the values of inducing variable in Step 2B.

Stage II: Internal aggregation (*Global aggregation*)

Step 4: Determine the weighting vector using the unification of WA and OWA such in Eqs. (17)–(25).

Step 5: Finally, aggregate the judgment matrix of majority of experts by the derived weighting vector. Here, the proportion of criteria is respected to the attitudinal character of majority of experts.

6 Numerical Example

In the following, a numerical example is presented. An investment selection problem is studied where a group of experts are assigned for the selection of an optimal strategy.

Different cases of multi-expert decision making methods are analyzed, in particular with respect to aggregation process of majority opinions of experts based on different schemes (namely classical and alternative schemes), and also on different weighting methods. Note that with this analysis, the optimal choices will be obtained depend on the scheme and aggregation operator used in each particular case. As can be seen each scheme and aggregation operator leads to different results and decisions.

Assume that a company plans to invest some money in a region. At first, they consider five possible investment options as follows: $A_1 =$ invest in the European market, $A_2 =$ invest in the American market, $A_3 =$ invest in the Asian market, $A_4 =$ invest in the African market, $A_5 =$ do not invest money.

In order to evaluate these investments, the investor has brought together a group of experts E_k. This group considers that each investment option can be described with the following characteristics: $C_1 =$ benefits in the short term, $C_2 =$ benefits in the mid-term, $C_3 =$ benefits in the long term, $C_4 =$ risk of the investment, $C_5 =$ other variables.

The available investment strategies, depending on the characteristic C_i and the option A_i for each expert are shown in Table 1.

The aggregated results of the different approaches are presented in the Table 2 and their rankings are given in Table 3. Should be noted that in this case, all the criteria are set to have equal degrees of importance and the experts' weights are given as 0.3, 0.1, 0.1, 0.4, 0.1 for expert E_1, E_2, E_3, E_4 and E_5, respectively for MEDM-BS I and MEDM-BS II. While for MEDM-PY I and MEDM-PY II the experts' weights are considered as equal.

As can be seen, there is a slight difference between the results which derived from both soft majority aggregation approaches with respect to different decision schemes. The majority opinion of individual experts which calculated based on the classical scheme provided A4, A2, A1, A5 and A3 as ranking for both methods. While the majority opinions computed with respect to alternative scheme gave the ranking of A4, A1, A5, A2 and A3 for both methods. Hence, the results show the effect of different decision schemes in ranking the options.

Table 1. Available investment strategies of each expert, E_h

E_1				
C1	C2	C3	C4	C5
0.7	0.6	0.7	0.6	0.9
0.8	1	0.2	1	0.6
0.6	0.7	0.6	0.6	0.5
0.9	0.6	0.8	1	0.9
0.3	0.7	0.7	0.8	0.9

E_2				
C1	C2	C3	C4	C5
0.6	0.9	1	0.9	0.9
1	0.7	0.1	1	0.8
0.4	0.9	0.8	0.7	0.6
0.9	0.5	0.7	1	0.9
0.7	0.7	0.9	0.9	0.9

E_3				
C1	C2	C3	C4	C5
0.5	0.7	0.9	0.8	0.9
0.9	0.9	0.2	1	0.7
0.8	0.8	0.7	0.7	0.6
0.9	0.5	0.8	1	0.7
0.8	0.7	0.8	0.9	0.8

E_4				
C1	C2	C3	C4	C5
0.4	0.7	0.9	0.8	0.8
0.9	0.7	0.1	0.9	0.6
0.6	0.6	0.5	0.8	0.4
0.7	0.5	0.7	0.7	0.9
0.4	0.6	0.7	0.8	0.9

E_5				
C1	C2	C3	C4	C5
0.5	0.6	0.7	0.6	0.8
0.9	0.8	0.4	0.9	0.5
0.6	0.6	0.5	0.8	0.7
0.8	0.7	0.6	0.9	0.8
0.2	0.6	0.8	0.6	0.8

Table 2. The aggregated results

	MEDM-PY I	MEDM-PY II	MEDM-BS I	MEDM-BS II
A1	0.7143	0.7726	0.7169	0.7989
A2	0.7178	0.6992	0.7200	0.6580
A3	0.6280	0.6361	0.5952	0.6057
A4	0.7886	0.8027	0.7800	0.8000
A5	0.7029	0.7225	0.6800	0.6969

Table 3. The ranking of financial strategies

Method	Ranking
MEDM-PY I	A4 > A 2 > A1 > A5 > A3
MEDM-PY II	A4 > A1 > A5 > A2 > A3
MEDM-BS I	A4 > A 2 > A1 > A5 > A3
MEDM-BS II	A4 > A1 > A5 > A2 > A3

As further analysis, we extend the method of Bordogna and Sterlacchini II to include the unification of WA and OWA weights with different criteria' weights. Tables 4 and 5 show the aggregated results and the final ordering of the financial strategies.

Table 4. The aggregated results with respect to MEDM-BS II model

	OWA (Q)	WOWA (Q)	HA	IWA	OWA WA	AOWAWA	OWA (FSM)	OWA (OA)
A1	0.880	0.764	1.193	0.872	0.845	0.851	0.355	0.255
A2	0.914	0.421	1.097	0.942	0.767	0.853	0.343	0.196
A3	0.678	0.586	0.922	0.685	0.652	0.663	0.301	0.164
A4	0.947	0.687	1.210	0.965	0.868	0.910	0.360	0.211
A5	0.838	0.657	1.066	0.806	0.778	0.785	0.330	0.234

Table 5. The ordering of financial strategies

	Ordering		Ordering
OWA (Q)	A3 > A 2 > A5 > A1 > A4	OWAWA	A2 > A4 > A5 > A1 > A3
WOWA (Q)	A1 > A5 > A4 > A2 > A3	AOWAWA	A3 > A2 > A5 > A1 > A4
HA	A2 > A3 > A5 > A1 > A4	OWA (FSM)	A2 > A3 > A5 > A1 > A4
IWA	A3 > A2 > A5 > A1 > A4	OWA (OA)	A1 > A4 > A5 > A3 > A2

The weighted average, p for each criteria is given as 0.1, 0.2, 0.3, 0.3, 0.1 and the ordering weights, w which represent 'most' of the criteria is given as 0.0044, 0.0356, 0.1200, 0.2844, 0.5556. As can be seen, the proposed AOWAWA weights with $\beta = 0.5$ provided the ranking similar to the IWA weights. While the rest weighting techniques shown slightly different results.

7 Conclusions

In this paper, the analysis on extensions of multi-expert decision making model based on ordered weighted average (OWA) operators is conducted. The focus is on the aggregation processes with respect to criteria and individual judgment of experts. First, the soft majority concept based on induced OWA (IOWA) and linguistic quantifiers to aggregate the experts' judgments is analyzed, in which concentrated on the classical and alternative schemes of decision making model. Then, analysis on the weighting methods related to integration of weighted average (WA) and OWA is conducted. The alternative weighting technique has been proposed which is termed as alternative OWA-WA (AOWAWA) operator. The multi-expert decision making model based on both aggregation processes then has been developed and a comparison is made to see the effect of different weighting techniques in aggregating the criteria and the results of using different schemes for the fusion of soft majority opinions of experts. A numerical example in the selection of investments is provided for the comparison purpose.

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