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Erratum

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Erratum to: A note on Galois embeddings of abelian varieties

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The original Theorem in the article is revised in this erratum based on a referee's request.

A smooth projective d-dimensional variety A possesses a *Galois embedding* if there exists an embedding $i : A \hookrightarrow \mathbb{P}^n$ and a linear projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^d$ with center disjoint from $i(A)$ such that $\pi \circ i$ is a Galois morphism. In particular, there is a subgroup of automorphisms G of A such that the quotient variety A/G is isomorphic to $\mathbb{P}^d$. The Main Theorem of [1] states the following in the case that A is an abelian variety:

Theorem 0.1. *Let A be an abelian variety of dimension d.*

(1) *If A possesses a Galois embedding into some projective space, then there exists an elliptic curve E such that $A \sim E^d$.*
(2) *If $A = E^d$, then A possesses infinitely many Galois embeddings with Galois groups of arbitrarily large order.*

This theorem is correct, but the proof of (1) relies on a fact stated by Yoshihara in [3] that if G is a group that acts on an abelian variety A, G_0 is the normal subgroup of all translations contained in G, and H is the group of automorphisms $\{f - f(0) : f \in G\}$, then there is an exact sequence

$$0 \to G_0 \to G \to H \to 1$$

such that $A/G \simeq (A/G_0)/H$ and H acts on A/G_0 as a group of automorphisms that fix the origin. The (quite subtle) problem with this statement is that the action of $G/G_0 \simeq H$ does not necessarily fix the origin. For example, if $G_0 = \{0\}$, then there are two actions of H on A: one as G (which is isomorphic to H but does not necessarily fix the origin), and the other natural action of H that does fix the origin. These two actions do not necessarily have the same quotient.

Consider the following example which was brought to my attention by Bronson Lim: Let $A = E^2$ for an elliptic curve E, let $p \in E[2]$ be a 2-torsion point on E, and let G be generated by $\sigma_1 : (x, y) \mapsto (-x, y + p)$ and $\sigma_2 : (x, y) \mapsto (x + p, -y)$. In this case, we have $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$, $G_0 = \{0\}$ and H is generated by $(x, y) \mapsto$

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$(-x, y)$ and $(x, y) \mapsto (x, -y)$. It is easy to see that A/G is singular (if $t \in E$ is such that $2t = p$, then (t, t) is fixed only by $\sigma_1\sigma_2$, which is not a pseudoreflection), but $A/H \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

In what follows, we will present a correct proof of statement (1) of the theorem. Indeed, we will actually prove the following slightly stronger statement:

Theorem 0.2. *Let A be an abelian variety of dimension d, and let G be a group of automorphisms of A such that $A/G \simeq \mathbb{P}^d$. Then there exists an elliptic curve E on A such that A is isogenous to E^d.*

Proof. Since the quotient of A by G is smooth (and not an abelian variety), A must contain an elliptic curve E. This statement was proved in [3] and also in [1] by other means. Let

$$H = \{f - f(0) : f \in G\}$$

be as above; this is a group of automorphisms that fix the identity of A.

Note that since the Picard number of A/G is 1, we have that the fixed locus of G in the Néron-Severi group is isomorphic to the Néron-Severi group of the quotient, which is $\mathbb{Z}$. In particular,

$$(\mathrm{NS}(A) \otimes \mathbb{Q})^G = \mathbb{Q}\pi^*\mathcal{O}_{\mathbb{P}^d}(1).$$

We observe, as stated above, that the actions of G and H do not necessarily give the same quotient. However, since each element in H differs from an element in G by a translation, we have that the actions of G and H on $\mathrm{NS}(A)$ are the same. We now consider the abelian subvariety

$$X = \sum_{\sigma \in H} \sigma(E).$$

If $X = A$, then A is isogenous to E^d and we are done. Assume then that X is a proper abelian subvariety, and consider the norm endomorphism $N_X \in \mathrm{End}(A)$ of X with respect to the polarization $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^d}(1)$. Since X is H-invariant and the polarization is fixed by each element of H, we have that the complementary abelian subvariety of X with respect to $\mathcal{L}$ is also H-invariant. In particular, each element of H commutes with N_X, and therefore, $N_X^*\mathcal{L}$ gives an element fixed by H in $\mathrm{NS}(A)$. Therefore, $N_X^*\mathcal{L}$ is a (rational) multiple of $\mathcal{L}$. However, $\mathcal{L}$ is ample, and if $X \neq A$, then by [2, Theorem 3.3] its top intersection number is 0, a contradiction. Therefore, $X = A$. □

References

[1] Auffarth, R.: A note on Galois embeddings of abelian varieties. Manuscr. Math. **154**(3–4), 279–284 (2017)

[2] Auffarth, R., Lange, H., Rojas, A.M.: A criterion for an abelian variety to be non-simple. J. Pure Appl. Algebra **221**(8), 1906–1925 (2017)

[3] Yoshihara, H.: Galois embedding of algebraic variety and its application to abelian surface. Rend. Sem. Mat. Univ. Padova **117**, 69–85 (2007)