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EVOLUTION ALGEBRAS, AUTOMORPHISMS, AND GRAPHS

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Abstract. The affine group scheme of automorphisms of an evolution algebra $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$ is shown to lie in an exact sequence $1 \to \mathbf{D} \to \mathbf{Aut}(\mathcal{E}) \to \mathsf{S}$, where $\mathbf{D}$, diagonalizable, and S, constant, depend solely on the directed graph associated to $\mathcal{E}$.

As a consequence, the Lie algebra of derivations $\mathrm{Der}(\mathcal{E})$ (with $\mathcal{E}^2 = \mathcal{E}$) is shown to be trivial if the characteristic of the ground field is 0 or 2, and to be abelian, with a precise description, otherwise.

1. Introduction

Evolution algebras were introduced in 2006 by Tian and Vojtechovsky (see [10]) and present many connections with other fields like graph theory, group theory or Markov chains, to mention a few (see Tian's monograph [9]). They have received considerable attention in the last years (see [1] and the references therein).

In this paper, all the algebras considered will be defined over a ground field $\mathbb{F}$, of arbitrary characteristic, and their dimension will be finite. An algebra is just a vector space $\mathcal{A}$ endowed with a bilinear map (the multiplication) $\mathcal{A} \times \mathcal{A} \to \mathcal{A}, (x, y) \mapsto xy$.

Definition 1.1. An *evolution algebra* is an algebra $\mathcal{E}$ endowed with a basis $B = \{v_1, v_2, \ldots, v_n\}$, called a *natural basis*, such that $v_i v_j = 0$ for any $1 \leq i \neq j \leq n$.

Given any evolution algebra $\mathcal{E}$ with natural basis $B = \{v_1, v_2, \ldots, v_n\}$, and multiplication determined by $v_i^2 = \sum_{i=1}^n \alpha_{ij} v_j$ $(\alpha_{ij} \in \mathbb{F})$, an associated (directed) graph $\Gamma = \Gamma(\mathcal{E}, B)$ is defined in [6]. The set of vertices V of Γ is just the natural basis, and the set of edges $E \subseteq V \times V$ consists of those pairs (v_i, v_j) with $\alpha_{ij} \neq 0$, that is, $(v_i, v_j) \in E$ if v_j appears in v_i^2 with nonzero coefficient.

The graph $\Gamma = \Gamma(\mathcal{E}, B)$ is used in [6, 7] to get new results on these algebras and to provide new natural proofs of some known results.

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In particular, it is shown in [6] that the group of automorphisms $\mathrm{Aut}(\mathcal{E})$ is finite if $\mathcal{E}^2 = \mathcal{E}$ (or equivalently the matrix (α_{ij}) is regular). Over an infinite field $\mathbb{F}$, the regular matrices form a Zariski open, and hence dense, set in $\mathrm{Mat}_n(\mathbb{F})$. So, in a way, we have that $\mathrm{Aut}(\mathcal{E})$ is finite for "almost all" evolution algebras.

Over fields of positive characteristic, or over nonalgebraically closed fields of characteristic 0, the affine group scheme of automorphisms $\mathbf{Aut}(\mathcal{E})$ contains much more information than $\mathrm{Aut}(\mathcal{E})$ including, in particular, the information on the Lie algebra of derivations $\mathrm{Der}(\mathcal{E})$.

Here we follow the functorial approach to affine group schemes (see for instance [11]). An affine group scheme is a representable group-valued functor defined on the category $\mathrm{Alg}_{\mathbb{F}}$ of unital commutative, associative algebras. Thus, given an evolution algebra $\mathcal{E}$, $\mathbf{Aut}(\mathcal{E})$ is the functor $\mathrm{Alg}_{\mathbb{F}} \longrightarrow \mathrm{Grp}$ that takes any object R in $\mathrm{Alg}_{\mathbb{F}}$ to the group $\mathrm{Aut}(\mathcal{E}_R)$ of automorphisms, as an R-algebra, of $\mathcal{E}_R := \mathcal{E} \otimes_{\mathbb{F}} R$. The action on morphisms is the natural one.

The Lie algebra $\mathrm{Lie}(\mathbf{Aut}(\mathcal{E}))$ is canonically isomorphic to the Lie algebra of derivations $\mathrm{Der}(\mathcal{E}) = \{\delta \in \mathrm{End}_{\mathbb{F}}(\mathcal{E}) \mid \delta(xy) = \delta(x)y + x\delta(y) \text{ for any } x, y \in \mathcal{E}\}$ (see [5, Example A.43]).

Now, the fact that $\mathrm{Aut}(\mathcal{E})$ is finite if $\mathcal{E}^2 = \mathcal{E}$ [6, Theorem 4.3] shows, in particular, that $\mathrm{Aut}(\mathcal{E}_{\mathbb{F}_{\mathrm{alg}}})$ is finite, where $\mathbb{F}_{\mathrm{alg}}$ is an algebraic closure of $\mathbb{F}$, and hence the affine group scheme $\mathbf{Aut}(\mathcal{E})$ is finite, that is, the Hopf algebra that represents it is finite dimensional.

If the characteristic of the ground field $\mathbb{F}$ is 0, then any finite affine group scheme is étale, and hence the Lie algebra is trivial. Therefore [6, Theorem 4.8] implies $\mathrm{Der}(\mathcal{E}) = 0$ if $\mathcal{E}^2 = \mathcal{E}$ and $\mathrm{char}(\mathbb{F}) = 0$. (This result over $\mathbb{C}$ has been proven in [2, Theorem 2.6] in a different way).

However, as some examples in [3] show, this is no longer true if $\mathrm{char}(\mathbb{F}) > 0$.

The goal of the present paper is to show that given any evolution algebra $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$, there is an exact sequence (7)

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \longrightarrow \mathbf{Aut}(\mathcal{E}) \longrightarrow \mathsf{Aut}(\Gamma)$$

where $\mathsf{Aut}(\Gamma)$ is the constant group scheme attached to the group of automorphism of the graph associated to $\mathcal{E}$ in [6], while $\mathbf{Diag}(\Gamma)$ is a finite diagonalizable group scheme defined in terms solely of Γ. That is the elements in the exact sequence, except $\mathbf{Aut}(\mathcal{E})$ itself, depend only on Γ (!!).

An affine group scheme is diagonalizable if it is a subscheme of a torus [11, §2.2] or, equivalently, if the representing Hopf algebra is the gruop algebra of a finitely generated abelian group. In our situation, $\mathbf{Diag}(\Gamma)$ turns out to be a product of schemes of roots of unity $\boldsymbol{\mu}_N$ ($N \in \mathbb{N}$), where $\boldsymbol{\mu}_N(R) = \{r \in R \mid r^N = 1\}$ for any R in $\mathrm{Alg}_{\mathbb{F}}$, which is represented by the quotient $\mathbb{F}[x]/(x^N - 1)$, that is, the group algebra of the cyclic group of order N.

On the other hand, given a finite group G, the associated constant group scheme G is the group scheme represented by $\mathbb{F}^G := \mathrm{Maps}(G, \mathbb{F}) = \bigoplus_{g\in G} \mathbb{F}\epsilon_g$, where

$$\epsilon_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise,} \end{cases}$$

(see [11, §2.4]). For any R in $\mathrm{Alg}_{\mathbb{F}}$ without proper idempotents, $\mathsf{G}(R)$ is (isomorphic to) the group G.

Note that $\mathbb{F}^G \simeq \mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}$ is the cartesian product of $|G|$ copies of $\mathbb{F}$. In particular, $\mathbb{F}^G$ is a separable algebra and hence G is étale.

The paper is structured as follows. Section 2 will be devoted to define and study the diagonalizable affine group scheme $\mathbf{Diag}(\Gamma)$ associated to any graph. For connected Γ, $\mathbf{Diag}(\Gamma)$ is either trivial or isomorphic to $\boldsymbol{\mu}_N$ for some natural number N, given by the so called *balance* of Γ. Section 3 will deal with the group of automorphisms of a graph. Its main result: Theorem 3.2, gives the exact sequence (7) mentioned above. This exact sequence induces a short exact sequence (8) which does not split in general. Finally Section 4 is devoted to describe the Lie algebra of derivations of any evolution algebra $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$. The description is a direct consequence of our results on the affine group scheme $\mathbf{Aut}(\mathcal{E})$. It turns out that $\mathrm{Der}(\mathcal{E})$ depends only on the graph.

2. The diagonal group of a graph

All the graphs considered in this paper are directed graphs. These are pairs $\Gamma = (V, E)$, consisting of a finite set of vertices V and a set of edges (or arrows) $E \subseteq V \times V$.

Given such a graph, we need some definitions

- A *path* is a sequence $\gamma = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ where $n \geq 0$, $v_0, \dots, v_n \in V$, $e_1, \dots, e_n \in E$, and for each $i = 1, \dots, n$, either $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$.

  We define the *balance* of the path γ as the integer

$$\begin{aligned} \mathrm{b}(\gamma) = \Bigl|\{i \mid 1 \leq i \leq n \text{ and } e_i = (v_{i-1}, v_i)\}\Bigr| \\ - \Bigl|\{i \mid 1 \leq i \leq n \text{ and } e_i = (v_i, v_{i-1})\}\Bigr|. \end{aligned}$$

  that is, $\mathrm{b}(\gamma)$ is obtained by adding 1 if the edge e_i goes in the "right" direction (from v_0 to v_n) and -1 if the edge e_i goes in the "wrong" direction, and summing over i.

  The *balance* of Γ is defined as the greatest common divisor of the absolute values of the balances of the cycles in Γ:

$$\mathrm{b}(\Gamma) = \gcd\{|\mathrm{b}(\gamma)| : \gamma \text{ cycle in } \Gamma\}.$$

- A *cycle* is a path $\gamma = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ with $v_0 = v_n$.

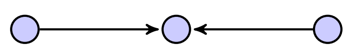
- The *indegree* of a vertex v is the natural number (or 0)
$$\deg^-(v) = \Big|\{w \in V \mid (w,v) \in E\}\Big|,$$
while the *outdegree* is
$$\deg^+(v) = \Big|\{w \in V \mid (v,w) \in E\}\Big|.$$
The vertex v is said to be a *source* if $\deg^-(v) = 0$, and a *sink* if $\deg^+(v) = 0$.
- Γ is said to be *connected* if the underlying undirected graph is connected, that is, if for every $v, w \in V$ there exists a path
$$\gamma = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$$
with $v_0 = v$ and $v_n = w$. Any graph Γ is the "disjoint union" of its *connected components*

Definition 2.1. The *diagonal group* of a graph $\Gamma = (V, E)$ is the (diagonalizable) affine group scheme $\mathbf{Diag}(\Gamma)$ given by
$$\mathbf{Diag}(\Gamma)(R) = \{\varphi : V \longrightarrow R^\times \mid \forall (v,w) \in E,\ \varphi(w) = \varphi(v)^2\},$$
with the natural morphisms.

Note that $\mathbf{Diag}(\Gamma)$ is a subgroup scheme of the torus $(\mathbf{G}_m)^{|V|}$

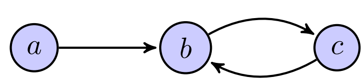
Let us see a few examples.

Example 2.2.

$\Gamma :$ 

then
$$\mathbf{Diag}(\Gamma) \simeq \big(\mathbf{G}_m \times \mathbf{G}_m\big)/\{(\mu_1, \mu_2) \mid \mu_1^2 = \mu_2^2\} \simeq \mathbf{G}_m \times \boldsymbol{\mu}_2.$$

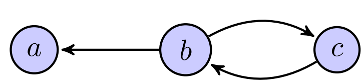
Example 2.3.

$\Gamma :$ 

Γ has no sinks.

If $\varphi \in \mathbf{Diag}(\Gamma)(R)$ and $\varphi(a) = \mu$ ($\in R^\times$), then $\varphi(b) = \mu^2$, $\varphi(c) = \mu^4$, and $\varphi(b) = \varphi(c)^2$, that is $\mu^2 = \mu^8$, so $\mu^6 = 1$. Hence $\mathbf{Diag}(\Gamma) \simeq \boldsymbol{\mu}_6$.

Example 2.4.

$\Gamma :$ 

Γ has no sources.

Again, if $\varphi \in \mathbf{Diag}(\Gamma)(R)$ and $\varphi(c) = \mu$, then $\varphi(b) = \mu^2$, $\varphi(a) = \mu^4$, and $\varphi(c) = \varphi(b)^2$, that is, $\mu = \mu^4$, so $\mu^3 = 1$. Hence $\mathbf{Diag}(\Gamma) \simeq \boldsymbol{\mu}_3$.

From the definitions, we get at once the next result:

Proposition 2.5. *Let $\Gamma = (V, E)$ be a graph with connected components $\Gamma_i = (V_i, E_i)$, $i = 1, \dots, n$ (so that $V = V_1 \dot\cup \cdots \dot\cup V_n$). Then*

$$\mathbf{Diag}(\Gamma) \simeq \mathbf{Diag}(\Gamma_1) \times \cdots \times \mathbf{Diag}(\Gamma_n).$$

If $m = 2s+1$ is an odd natural number the square map

$$\begin{aligned} \boldsymbol{\mu}_m(R) &\longrightarrow \boldsymbol{\mu}_m(R) \\ r &\mapsto r^2, \end{aligned}$$

is a group automorphism for any R in $\mathrm{Alg}_{\mathbb{F}}$, with inverse $r \longrightarrow r^{\frac{1}{2}} := r^{s+1}$. Therefore, expressions like $r^{2^{-3}}$ make sense: $r^{2^{-3}} = ((r^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$.

Lemma 2.6. *Let $\Gamma = (V, E)$ be a graph, $\gamma = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ be a path in Γ. Let $\varphi \in \mathbf{Diag}(\Gamma)(R)$ for R in $\mathrm{Alg}_{\mathbb{F}}$, such that $\varphi(v_i) \in \boldsymbol{\mu}_{m_i}(R)$ with m_i odd for any $i = 0, \dots, n$. Then $\varphi(v_n) = \varphi(v_0)^{2^{\mathrm{b}(\gamma)}}$.*

Proof. Imagine that $\gamma = (v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ with $e_1 = (v_1, v_0)$, $e_2 = (v_1, v_2)$, and $e_3 = (v_3, v_2)$, so $\mathrm{b}(\gamma) = -1$.

$$\begin{tikzcd} v_0 & v_1 \arrow[l] \arrow[r] & v_2 & v_3 \arrow[l] \end{tikzcd}$$

Then

- As $e_1 = (v_1, v_0) \in E$, $\varphi(v_0) = \varphi(v_1)^2$, so $\varphi(v_1) = \varphi(v_0)^{\frac{1}{2}} = \varphi(v_0)^{2^{-1}}$.
- As $e_2 = (v_1, v_2) \in E$, $\varphi(v_2) = \varphi(v_1)^2$, so $\varphi(v_2) = (\varphi(v_0)^{2^{-1}})^2 = \varphi(v_0)$.
- As $e_3 = (v_3, v_2) \in E$, $\varphi(v_2) = \varphi(v_3)^2$, so $\varphi(v_3) = \varphi(v_2)^{-1} = \varphi(v_1)^{2^{-1}} = \varphi(v_0)^{2^{\mathrm{b}(\gamma)}}$.

The general argument follows the same lines. □

Our next result determines the diagonal group of connected graphs without sources. Note that the graphs attached to evolution algebras $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$ have no sources.

Theorem 2.7. *Let $\Gamma = (V, E)$ be a connected graph with no sources. Then $\mathbf{Diag}(\Gamma) \simeq \boldsymbol{\mu}_N$ where $N = 2^{\mathrm{b}(\Gamma)} - 1$.*

Proof. First, the arguments in the proof of [6, Theorem 4.8] show that for any R in $\mathrm{Alg}_{\mathbb{F}}$, any $\varphi \in \mathbf{Diag}(\Gamma)(R)$, and any vector $v \in V$, $\varphi(v) \in \boldsymbol{\mu}_{2^s-1}(R)$ for some natural number s.

Fix a vertex $a \in V$, and consider the restriction homomorphism

$$\begin{aligned} \Phi_a : \mathbf{Diag}(\Gamma) &\longrightarrow \mathbf{G}_m \\ \varphi &\mapsto \varphi(a). \end{aligned}$$

We will follow several steps:

- Φ_a is one-to-one.

  Actually, for R in $\mathrm{Alg}_{\mathbb{F}}$, and $\varphi \in \mathbf{Diag}(\Gamma)(R)$, with $\varphi(a) = 1$, by connectedness for any vertex $v \in V$ there is a path $\gamma =$

$(v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ with $v_0 = a$ and $v_n = v$. By Lemma 2.6, $\varphi(v) = \varphi(a)^{2^{\mathrm{b}(\gamma)}} = 1^{2^{\mathrm{b}(\gamma)}} = 1$.

- For any R in $\mathrm{Alg}_{\mathbb{F}}$, and $\varphi \in \mathbf{Diag}(\Gamma)(R)$, $\varphi(a) \in \boldsymbol{\mu}_N(R)$.

  Indeed, by the previous argument, for any $v \in V$, $\varphi(v) = \varphi(a)^{2^{\mathrm{b}(\gamma)}}$ for any path γ connecting a and v. As the order of $\varphi(a)$ is odd, $\varphi(a)$ and $\varphi(v)$ generate the same subgroup of $R^\times$. In particular $\varphi(a)$ and $\varphi(v)$ have the same order.

  Given any cycle $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ in Γ ($v_n = v_0$), we get $\varphi(v_0) = \varphi(v_0)^{2^{\mathrm{b}(\gamma)}}$, or $\varphi(v_0)^{2^{\mathrm{b}(\gamma)}-1} = 1$. Thus the order of $\varphi(a)$ divides $2^{|\mathrm{b}(\gamma)|} - 1$ for any cycle γ. Using that $2^{\gcd(m_1,m_2)} - 1 = \gcd(2^{m_1} - 1, 2^{m_2} - 1)$, our result follows.

- The image of Φ_a is exactly $\boldsymbol{\mu}_N$.

  For any R in $\mathrm{Alg}_{\mathbb{F}}$, and any $\mu \in \boldsymbol{\mu}_N(R)$, define $\varphi : V \longrightarrow R^\times$ as follows: For any $v \in V$, select a path connecting a and v: $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ with $v_0 = a$ and $v_n = v$, and define $\varphi(v) = \mu^{2^{\mathrm{b}(\gamma)}}$. This is well defined, because for any other path $\hat\gamma = (\hat v_0, \hat e_1, \hat v_1, \ldots, \hat v_{n-1}, \hat e_n, \hat v_n)$ connecting $a = \hat v_0$ and $v = \hat v_n$, then

  $$\gamma\hat\gamma^{-1} := (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = \hat v_n, \hat e_n, \hat v_{n-1}, \ldots, \hat e_1, \hat v_0)$$

  is a cycle with balance $\mathrm{b}(\gamma\hat\gamma^{-1}) = \mathrm{b}(\gamma) - \mathrm{b}(\hat\gamma)$ and, therefore, $\mu = \mu^{2^{\mathrm{b}(\gamma)-\mathrm{b}(\hat\gamma)}}$. Hence

  $$\mu^{2^{\mathrm{b}(\hat\gamma)}} = (\mu^{2^{\mathrm{b}(\gamma)-\mathrm{b}(\hat\gamma)}})^{2^{\mathrm{b}(\hat\gamma)}} = \mu^{2^{\mathrm{b}(\gamma)}}.$$

  Finally, $\varphi \in \mathbf{Diag}(\Gamma)(R)$, because for any $e = (v, w) \in E$, if $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ is a path connecting $a = v_0$ and $v = v_n$, then $\gamma' = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n, e, w)$ is a path connecting a and w with $\mathrm{b}(\gamma') = \mathrm{b}(\gamma) + 1$. Hence, $\varphi(w) = \mu^{2^{\mathrm{b}(\gamma')}} = (\mu^{2^{\mathrm{b}(\gamma)}})^2 = \varphi(v)^2$.

□

Corollary 2.8. *Let $\Gamma = (V, E)$ be a connected graph with no sources and with a loop $e = (v, v)$. Then $\mathbf{Diag}(\Gamma) = 1$.*

Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{v_1, \ldots, v_n\}$ and let $\Gamma = \Gamma(\mathcal{E}, B) = (V, E)$ be the attached graph ($V = B$). For any R in $\mathrm{Alg}_{\mathbb{F}}$, and any $\varphi \in \mathbf{Diag}(\Gamma)(R)$, φ induces the linear (diagonal) isomorphism

$$\begin{aligned} \hat\varphi : \mathcal{E}_R &\longrightarrow \mathcal{E}_R \\ v_i &\mapsto \varphi(v_i)v_i. \end{aligned} \tag{1}$$

Let $v_i^2 = \sum_{j=1}^n \alpha_{ij} v_j$ for $i = 1, \ldots, n$, with $\alpha_{ij} \in \mathbb{F}$, then

$$\hat\varphi(v_i^2) = \sum_{j=1}^n \alpha_{ij}\hat\varphi(v_j) = \sum_{j=1}^n \alpha_{ij}\varphi(v_j)v_j,$$

and $\hat\varphi(v_i)^2 = \varphi(v_i)^2 \sum_{j=1}^n \alpha_{ij} v_j$.

But if $\alpha_{ij} \neq 0$, then $(v_i, v_j) \in E$, so $\varphi(v_j) = \varphi(v_i)^2$. Hence $\hat{\varphi} \in \mathrm{Aut}(\mathcal{E}_R)$ and we obtain the following result:

Theorem 2.9. *Let $\mathcal{E}$ be an evolution algebra with natural basis B and let $\Gamma = \Gamma(\mathcal{E}, B)$ be the attached graph. Then there is an injective homomorphism $\iota : \mathbf{Diag}(\Gamma) \longrightarrow \mathbf{Aut}(\mathcal{E})$ such that for any R in $\mathrm{Alg}_{\mathbb{F}}$, and any R-point $\varphi \in \mathbf{Diag}(\Gamma)(R)$, $\iota(\varphi) = \hat{\varphi}$ (as in (1)).*

3. Graph Automorphisms

The goal of this section is, given an evolution algebra $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$ with attached graph $\Gamma(\mathcal{E}, B)$ (which is independent, up to isomorphism, of the natural basis B chosen [6, Corollary 4.7]), to show the existence of a natural homomorphism

$$\rho : \mathbf{Aut}(\mathcal{E}) \longrightarrow \mathsf{Aut}(\Gamma) \tag{2}$$

where $\mathsf{Aut}(\Gamma)$ is the constant group scheme attached to the group of automorphisms of Γ, denoted by $\mathrm{Aut}(\Gamma)$. If $B = \{v_1, \dots, v_n\}$ is a natural basis we may identify $\mathrm{Aut}(\Gamma)$ with a subgroup of the symmetric group S_n of degree n:

$$\mathrm{Aut}(\Gamma) \simeq \left\{\sigma \in S_n \mid \forall\, 1 \leq i, j \leq n,\ (v_i, v_j) \in E \Rightarrow (v_{\sigma(i)}, v_{\sigma(j)}) \in E\right\}.$$

If we just look at the rational points in $\mathrm{Aut}(\mathcal{E}) = \mathbf{Aut}(\mathcal{E})(\mathbb{F})$, any $\varphi \in \mathrm{Aut}(\mathcal{E})$ has an attached permutation $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\varphi(v_i) \in \mathbb{F}^{\times} v_{\sigma(i)}$ for any $i = 1, \dots, n$ ([6, Theorem 4.4]). Thus the coordinate matrix of φ relative to B is a monomial matrix (i.e., it has exactly one nonzero entry in each row and column). In order to deal with the group scheme $\mathbf{Aut}(\mathcal{E})$, some extra care must be taken. Let R be in $\mathrm{Alg}_{\mathbb{F}}$, and let $\varphi \in \mathbf{Aut}(\mathcal{E})(R) = \mathrm{Aut}(\mathcal{E}_R)$, with $\varphi(v_i) = \sum_{j=i}^{n} r_{ij} v_j$ for any $i = 1, \dots, n$. Then $r = \det\bigl(r_{ij}\bigr) \in R^{\times}$:

$$r = \sum_{\sigma \in S_n} (-1)^{\sigma} r_{\sigma(1)1} \cdots r_{\sigma(n)n} \in R^{\times}.$$

For any $i \neq j$ we have $0 = \varphi(v_i v_j) = \varphi(v_i)\varphi(v_j) = \sum_{k=1}^{n} r_{ik} r_{jk} v_k^2$.
Because $\mathcal{E}^2 = \mathcal{E}$, $\{v_1^2, \dots, v_n^2\}$ form a basis of $\mathcal{E}$ and hence

$$r_{ik} r_{jk} = 0 \quad \text{for any } 1 \leq i, j \leq n \text{ with } i \neq j. \tag{3}$$

Therefore, for any $\sigma \neq \tau$ in S_n, $(r_{\sigma(1)1} \cdots r_{\sigma(n)n})(r_{\tau(1)1} \cdots r_{\tau(n)n}) = 0$.
For any $\sigma \in S_n$, consider the element

$$e_{\sigma}^{\varphi} = (-1)^{\sigma} r^{-1} r_{\sigma(1)1} \cdots r_{\sigma(n)n}.$$

Then $1 = \sum_{\sigma \in S_n} e_{\sigma}^{\varphi}$, and $e_{\sigma}^{\varphi} e_{\tau}^{\varphi} = 0$ for $\sigma \neq \tau$ in S_n. Therefore, the e_{σ}^{φ}'s are orthogonal idempotent elements, and $R = \bigoplus_{\sigma \in S_n} R e_{\sigma}^{\varphi}$. Moreover, (3) implies

$$r_{ij} e_{\sigma}^{\varphi} = 0 \quad \text{unless } i = \sigma(j), \tag{4}$$

and the coordinate matrix $\bigl(r_{ij}\bigr)$ of φ splits into a sum of monomial matrices over the orthogonal ideals Re^φ_σ. Thus, for instance, with $n=3$ we have:

$$\begin{aligned} e^\varphi_1 &= r_{11}r_{22}r_{33}, & e^\varphi_{(123)} &= r_{21}r_{32}r_{13}, & e^\varphi_{(132)} &= r_{31}r_{12}r_{23},\\ e^\varphi_{(12)} &= -r_{21}r_{12}r_{33}, & e^\varphi_{(23)} &= -r_{11}r_{32}r_{23}, & e^\varphi_{(13)} &= -r_{31}r_{22}r_{13}. \end{aligned}$$

and $A=\bigl(r_{ij}\bigr)=\sum_{\sigma\in S_3}A_\sigma$, with $A_\sigma=e^\varphi_\sigma A\in \mathrm{Mat}_3(Re^\varphi_\sigma)$ a monomial matrix thanks to (4):

$$\begin{aligned} A_1 &= e^\varphi_1\begin{pmatrix} r_{11} & 0 & 0\\ 0 & r_{22} & 0\\ 0 & 0 & r_{33}\end{pmatrix}, & A_{(123)} &= e^\varphi_{(123)}\begin{pmatrix} 0 & 0 & r_{13}\\ r_{21} & 0 & 0\\ 0 & r_{32} & 0\end{pmatrix},\\ A_{(132)} &= e^\varphi_{(132)}\begin{pmatrix} 0 & r_{12} & 0\\ 0 & 0 & r_{23}\\ r_{13} & 0 & 0\end{pmatrix}, & A_{(12)} &= e^\varphi_{(12)}\begin{pmatrix} 0 & r_{12} & 0\\ r_{21} & 0 & 0\\ 0 & 0 & r_{33}\end{pmatrix},\\ A_{(23)} &= e^\varphi_{(23)}\begin{pmatrix} r_{11} & 0 & 0\\ 0 & 0 & r_{23}\\ 0 & r_{32} & 0\end{pmatrix}, & A_{(13)} &= e^\varphi_{(13)}\begin{pmatrix} 0 & 0 & r_{13}\\ 0 & r_{22} & 0\\ r_{31} & 0 & 0\end{pmatrix}. \end{aligned}$$

Moreover, if $\sigma\in S_n$ and $e^\varphi_\sigma\neq 0$, then the monomial matrix

$$A_\sigma = e^\varphi_\sigma\bigl(r_{ij}\bigr)=\sum_{i=1}^n e^\varphi_\sigma r_{\sigma(i)i}E_{\sigma(i)i},$$

where E_{ij} denotes the matrix with 1 in the (ij) slot and 0's elsewhere, correspond to an automorphism of $\mathcal{E}_{Re^\varphi_\sigma}$. This forces $\sigma\in\mathrm{Aut}(\Gamma)$. Therefore,

$$e^\varphi_\sigma\neq 0 \text{ only if } \sigma\in\mathrm{Aut}(\Gamma),\qquad 1=\sum_{\sigma\in\mathrm{Aut}(\Gamma)} e^\varphi_\sigma\,. \tag{5}$$

Recall that the coordinate Hopf algebra of the constant group scheme $\mathsf{Aut}(\Gamma)$ is $\mathbb{F}^{\mathrm{Aut}(\Gamma)}=\mathrm{Maps}\bigl(\mathrm{Aut}(\Gamma),\mathbb{F}\bigr)$, which has a natural basis $\{\epsilon_\sigma\mid\sigma\in\mathrm{Aut}(\Gamma)\}$, with

$$\epsilon_\sigma(\tau)=\begin{cases}1 & \text{if } \sigma=\tau,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathsf{Aut}(\Gamma)(R)$ is identified with $\mathrm{Hom}_{\mathrm{Alg}_\mathbb{F}}(\mathbb{F}^{\mathrm{Aut}(\Gamma)},R)$.

We are ready to define the homomorphism ρ in (2). For R in $\mathrm{Alg}_\mathbb{F}$ and $\varphi\in\mathbf{Aut}(\mathcal{E})(R)=\mathrm{Aut}(\mathcal{E}_R)$, the image of φ under ρ is defined as the element $\rho(\varphi)\in\mathrm{Hom}_{\mathrm{Alg}_\mathbb{F}}(\mathbb{F}^{\mathrm{Aut}(\Gamma)},R)$ given by

$$\begin{aligned}\rho(\varphi):\mathbb{F}^{\mathrm{Aut}(\Gamma)} &\longrightarrow R\\ \epsilon_\sigma &\mapsto e^\varphi_\sigma.\end{aligned}\tag{6}$$

It is trivially checked that this gives a homomorphism $\rho:\mathbf{Aut}(\mathcal{E})\to\mathsf{Aut}(\Gamma)$.

Remark 3.1. Exactly as over $\mathbb{F}$, if R in $\mathrm{Alg}_{\mathbb{F}}$ has no proper idempotents, then $1 = e_\sigma^\varphi$ for a unique $\sigma \in \mathrm{Aut}(\Gamma)$ and the matrix of φ is a monomial matrix attached to σ. In this case $\mathsf{Aut}(\Gamma)(R) \simeq \mathrm{Aut}(\Gamma)$ and $\rho(\varphi)$ is just σ under this identification.

The main result of this section is the following:

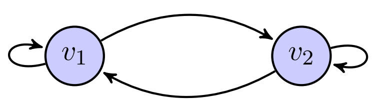
Theorem 3.2. *Let $\mathcal{E}$ be an evolution algebra with $\mathcal{E}^2 = \mathcal{E}$ and natural basis $B = \{v_1, \dots, v_n\}$. Let $\Gamma = \Gamma(\mathcal{E}, B)$ be its associated graph. Then the sequence*

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \xrightarrow{\iota} \mathbf{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathsf{Aut}(\Gamma) \tag{7}$$

is exact.

Proof. $\ker(\rho)(R)$ consists of the automorphisms $\varphi \in \mathbf{Aut}(\mathcal{E})(R) = \mathrm{Aut}(\mathcal{E}_R)$ such that $e_\sigma^\varphi = 0$ for any $1 \neq \sigma \in \mathrm{Aut}(\Gamma)$. Hence $1 = e_1^\varphi$ and φ is diagonal, that is, the elements of B are eigenvectors for φ. These automorphisms are precisely the elements in the image of ι. □

Example 3.3. The homomorphism ρ is not surjective in general. Take, for instance, the evolution algebra $\mathcal{E} = \mathbb{F}v_1 \oplus \mathbb{F}v_2$, with natural basis $B = \{v_1, v_2\}$, and multiplication given by $v_1^2 = v_1 + \alpha v_2$, $v_2^2 = \beta v_1 + v_2$, with $0 \neq \alpha, \beta \in \mathbb{F}$, $\alpha \neq \beta$, $\alpha\beta \neq 1$. Then the associated graph $\Gamma(\mathcal{E}, B)$ is the complete graph



While $\mathrm{Aut}(\Gamma) = C_2$, let us check that $\mathbf{Aut}(\mathcal{E}) = 1$. To do that, it is enough to prove that $\mathrm{Aut}(\mathcal{E}_R) = 1$ for R in $\mathrm{Alg}_{\mathbb{F}}$ without proper idempotents.

The arguments above show that the coordinate matrix relative to $B = \{v_1, v_2\}$ of any $\varphi \in \mathrm{Aut}(\mathcal{E}_R)$ is either

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & r_1 \\ r_2 & 0 \end{pmatrix},$$

with $r_1, r_2 \in R^\times$.

In the first case $\varphi(v_1^2) = r_1 v_1 + \alpha r_2 v_2$, while $\varphi(v_1)^2 = r_1^2(v_1 + \alpha v_2)$, so $r_1^2 = r_1 = r_2$, and hence, due to the absence of proper idempotents, $\varphi = \mathrm{id}$.

In the second case $\varphi(v_1^2) = r_1 v_2 + \alpha r_2 v_1$, while $\varphi(v_1)^2 = r_1^2 v_2^2 = r_1^2(\beta v_1 + v_2)$, so $r_1^2 = r_1$ and $\alpha r_2 = \beta r_1^2$. Hence, $r_1 = 1$, $r_2 = \beta\alpha^{-1} \neq 1$. But $\varphi(v_2^2) = \varphi(v_2)^2$ forces $r_2 = 1$, a contradiction.

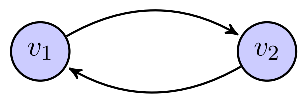
Any subgroup scheme of a constant group scheme is itself a constant group scheme. Hence we have the next consequence:

Corollary 3.4. *Let $\mathcal{E}$ be an evolution algebra with $\mathcal{E}^2 = \mathcal{E}$ and natural basis $B = \{v_1, \dots, v_n\}$. Let $\Gamma = \Gamma(\mathcal{E}, B)$ be its associated graph. Then there is a subgroup H of* $\mathrm{Aut}(\Gamma)$ *and a short exact sequence*

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \xrightarrow{\iota} \mathbf{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathsf{H} \longrightarrow 1, \tag{8}$$

where H *is the constant group scheme associated to* H*.*

Example 3.5. The short exact sequence in Corollary 3.4 does not split in general. Take, for instance the evolution algebra $\mathcal{E} = \mathbb{F}v_1 \oplus \mathbb{F}v_2$ with $v_1^2 = v_2$, $v_2^2 = \alpha v_1$, with $0 \neq \alpha \in \mathbb{F}$. The associated graph is



Then $\mathbf{Diag}(\Gamma) = \boldsymbol{\mu}_3$ (Theorem 2.7) and $\rho : \mathbf{Aut}(\mathcal{E}) \longrightarrow \mathbf{Aut}(\Gamma) \simeq \mathsf{C}_2$ is surjective, as it is so over an algebraic closure $\mathbb{F}_{\text{alg}}$. Indeed, over $\mathbb{F}_{\text{alg}}$ the assignment

$$v_1 \mapsto \alpha^{-1/3} v_2, \quad v_2 \mapsto \alpha^{1/3} v_1, \tag{9}$$

gives an automorphism φ with $\rho(\varphi)$ being the generator of $\operatorname{Aut}(\Gamma)$. Moreover, $\varphi^2 = \text{id}$ and this proves that (8) splits over $\mathbb{F}_{\text{alg}}$.

Let us check that the short exact sequence

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \longrightarrow \mathbf{Aut}(\mathcal{E}) \longrightarrow \mathsf{Aut}(\Gamma) \longrightarrow 1 \tag{10}$$

splits if and only if there is $\mu \in \mathbb{F}$ such that $\alpha = \mu^3$.

Actually, if $\alpha = \mu^3$ the assignment (9) makes sense over $\mathbb{F}$, so the sequence splits. Conversely, if (10) splits, there is an automorphism $\varphi \in \operatorname{Aut}(\mathcal{E})$ with $\varphi^2 = \text{id}$, such that $\varphi(v_1) \in \mathbb{F}^\times v_2$, $\varphi(v_2) \in \mathbb{F}^\times v_1$. With $\varphi(v_1) = \nu v_2$, $\varphi(v_2) = \mu v_1$, we get $\nu = \mu^{-1}$, as $\varphi^2 = \text{id}$, and

$$\mu v_1 = \varphi(v_2) = \varphi(v_1^2) = \varphi(v_1)^2 = \mu^{-2} v_2^2 = \mu^{-2}\alpha v_1,$$

so that $\alpha = \mu^3$. $\square$

4. Derivations

The results of the previous sections allow us to compute easily the Lie algebra of derivations of any evolution algebra $\mathcal{E}$, with $\mathcal{E}^2 = \mathcal{E}$. This Lie algebra depends only on the associated graph!

Theorem 4.1. *Let* $\mathcal{E}$ *be an evolution algebra with* $\mathcal{E}^2 = \mathcal{E}$*. Let* B *be a natural basis and let* $\Gamma = \Gamma(\mathcal{E}, B)$ *be the attached graph. Then:*

1. *If the characteristic of* $\mathbb{F}$ *is* 0 *or* 2*, then* $\operatorname{Der}(\mathcal{E}) = 0$*.*
2. *If the characteristic of* $\mathbb{F}$ *is* $p \neq 0, 2$*, then* $\operatorname{Der}(\mathcal{E})$ *is an abelian Lie algebra whose dimension is the number of connected components* Γ_i *of* Γ *such that the order of* 2 *in* $\mathbb{Z}/p\mathbb{Z}$ *divides the balance* $\mathrm{b}(\Gamma_i)$*.*

Proof. The exact sequence (7) induces an exact sequence (see eq. [Milne, 10d]):

$$0 \longrightarrow \operatorname{Lie}(\mathbf{Diag}(\Gamma)) \xrightarrow{\mathrm{d}\iota} \operatorname{Lie}(\mathbf{Aut}(\mathcal{E})) \xrightarrow{\mathrm{d}\rho} \operatorname{Lie}(\mathsf{Aut}(\Gamma))$$

But $\operatorname{Lie}(\mathsf{Aut}(\Gamma)) = 0$, as $\mathsf{Aut}(\Gamma)$ is a constant group scheme, and hence étale. On the other hand, $\operatorname{Lie}(\mathbf{Aut}(\mathcal{E})) = \operatorname{Der}(\mathcal{E})$ (see [5, Example A.43]), so that $\operatorname{Der}(\mathcal{E})$ is isomorphic to $\operatorname{Lie}(\mathbf{Diag}(\Gamma))$ through the differential of ι.

However, $\mathrm{Lie}(\boldsymbol{\mu}_m)$ is either 0 if $\mathrm{char}(\mathbb{F}) \nmid m$, or it has dimension 1 if $\mathrm{char}(\mathbb{F}) \mid m$ (see [5, Example A42]). Hence Theorem 2.7 gives the results. $\square$

Remark 4.2. As mentioned in the Introduction, the fact that $\mathrm{Der}(\mathcal{E})$ is 0 for any evolution algebra $\mathcal{E}$ with $\mathcal{E}^2 = \mathcal{E}$ over $\mathbb{C}$ has already been proved in [2, Theorem 2.1].

Consider the algebra of dual numbers $\mathbb{F}[\epsilon] = \mathbb{F}1 \oplus \mathbb{F}\epsilon$, with $\epsilon^2 = 0$, and the natural homomorphism $\pi : \mathbb{F}[\epsilon] \longrightarrow \mathbb{F}$ in $\mathrm{Alg}_{\mathbb{F}}$ ($\pi(1) = 1$, $\pi(\epsilon) = 0$). Given a graph $\Gamma = (V, E)$, $\mathrm{Lie}(\mathbf{Diag}(\Gamma))$ is the kernel of the induced group homomorphism $\pi_* : \mathbf{Diag}(\Gamma)(\mathbb{F}[\epsilon]) \longrightarrow \mathbf{Diag}(\Gamma)(\mathbb{F})$. The elements of $\ker \pi_*$ are the maps

$$\begin{aligned} \varphi : V &\longrightarrow \mathbb{F}[\mathcal{E}] \\ v &\mapsto 1 + \delta(v)\epsilon, \end{aligned}$$

for a linear map $\delta : V \longrightarrow \mathbb{F}$, such that, for any $(v, w) \in E$, $\varphi(w) = \varphi(v)^2$, which is equivalent to $\delta(w) = 2\delta(v)$.

Therefore we obtain the following straightforward consequence of Theorems 4.1, 2.7 and 2.9.

Corollary 4.3. *Let $\mathcal{E}$ be an evolution algebra with $\mathcal{E}^2 = \mathcal{E}$ over a field $\mathbb{F}$ of characteristic $p \neq 0, 2$. Let B be a natural basis and let $\Gamma = \Gamma(\mathcal{E}, B)$ be the associated graph. Let $\Gamma_i = (V_i, E_i)$ ($V_i \subseteq B$), $i = 1, \cdots, r$, be the connected components of Γ such that $p \mid 2^{\mathrm{b}(\Gamma_i)} - 1$. For any $i = 1, \ldots, r$, fix an element $v_i \in V_i$. Then a basis of $\mathrm{Lie}(\mathbf{Diag}(\Gamma))$ is given by $\hat{\delta}_1, \cdots, \hat{\delta}_r$, where*

- $\hat{\delta}_i(v) = 0$ *if* $v \notin V_i$,
- $\hat{\delta}_i(v_i) = v_i$,
- $\hat{\delta}_i(w) = 2^{\mathrm{b}(\gamma)} w$ *if* $w \in V_i$ *and* $\gamma = (w_0, e_1, w_1, \ldots, e_n, w_n)$ *is a path connecting* $w_0 = v_i$ *and* $w_n = w$.

Example 4.4. The evolution algebra $\mathcal{E}$ in Example 3.3 has trivial group scheme of automorphisms, so $\mathrm{Der}(\mathcal{E}) = 0$ for any ground field $\mathbb{F}$.

However, for the evolution algebra $\mathcal{E}$ in Example 3.5, we have the short exact sequence in (10), and $\mathbf{Diag}(\Gamma) \simeq \boldsymbol{\mu}_3$. Hence $\mathrm{Der}(\mathcal{E}) = 0$ unless $\mathrm{char}(\mathbb{F}) = 3$. In the later case, $\mathrm{Der}(\mathcal{E})$ is spanned by the map $d : v_1 \mapsto v_1$, $v_2 \mapsto 2v_2 = -v_2$.

Remark 4.5. It must be remarked that for $\alpha = 1$, the evolution algebra $\mathcal{E}$ in Example 3.5 is the two-dimensional split para-Hurwitz algebra, and hence, for arbitrary α ($\neq 0$), $\mathcal{E}$ is a symmetric composition algebra (see [4] and references therein).

As shown in Example 3.5, the short exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_3 \longrightarrow \mathsf{Aut}(\mathcal{E}) \longrightarrow \mathsf{C}_2 \longrightarrow 1$$

splits if and only if $\alpha \in \mathbb{F}^3$, that is, if and only if $\mathcal{E}$ is, up to isomorphism, the split two-dimensional para-Hurwitz algebra.

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