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FOUNDATIONS

Aggregation operators with moving averages

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Abstract

A moving average is an average that aggregates a subset of variables from the set and moves across the sample. It is widely used in time-series forecasting. This paper studies the use of moving averages in some representative aggregation operators. The ordered weighted averaging weighted moving averaging (OWAWMA) operator is introduced. It is a new approach based on the use of the moving average in a unified model between the weighted average and the ordered weighted average. Its main advantage is that it provides a parameterized family of moving aggregation operators between the moving minimum and the moving maximum. Moreover, it also includes the weighted moving average and the ordered weighted moving average as particular cases. This approach is further extended by using generalized aggregation operators, obtaining the generalized OWAWMA operator. The construction of interval and fuzzy numbers with these operators obtaining the concept of moving interval number and moving fuzzy number is also studied. The paper ends analyzing the applicability of this new approach in some key statistical concepts such as the variance and the covariance and with a numerical example regarding sales forecasting.

Keywords Weighted average · OWA operator · Moving average · Aggregation operators

1 Introduction

In the literature, there are a wide range of aggregation operators (Beliakov et al. 2007; Grabisch et al. 2011; Yu 2015). The moving average is a well-known aggregation operator that moves toward a sample (Elliot et al. 2006; Evans 2002). From a more general context, it is possible to consider moving aggregation operators that can be used in the analysis providing tools for assessing the information in a dynamic way. Another very useful technique is the ordered weighted averaging (OWA) operator (Yager 1988). It is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. Since its introduction, it has been used in a lot of problems (Emrouznejad and Marra 2014; Kacprzyk et al. 2019; He et al. 2017). Focusing on moving averages, Yager (2008) studied the use of the OWA operator as a moving average. He analyzed the usefulness of this approach in time-series forecasting. Later, Merigó and Yager (2013) developed several extensions and generalizations by using induced aggregation operators, quasi-arithmetic means and distance measures. Recently, Yager (2013) has also studied different methodologies when dealing with exponential smoothing approaches. Leon-Castro et al. (2018a; b) present several extensions with induced and heavy aggregation operators.

The OWA operator can be generalized by using generalized aggregation operators (Beliakov et al. 2007; Merigó and Gil-Lafuente 2009). Thus, the generalized OWA (GOWA) operator (Yager 2004) by using generalized means and the quasi-arithmetic OWA (quasi-OWA) operator is formed (Fodor et al. 1995). Since their appearance,

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they have been studied by a lot of authors (He et al. 2017). Merigó and Gil-Lafuente suggested the use of induced aggregation operators (Merigó and Gil-Lafuente 2009). Zhao et al. (2010) studied the use of intuitionistic fuzzy sets (Traneva et al. 2018). Zhou and Chen (2010 and Zhou et al. (2015) introduced logarithmic aggregation operators under this framework and Alfaro-Garcia et al. (2018) with distance measures. Other authors have considered the use of OWA operators in majority processes (Karanik et al. 2016; Peláez and Doña 2006), Bonferroni means (Blanco-Mesa et al. 2016, 2018), prioritized aggregations (Avilés-Ochoa et al. 2018) and other related techniques (Cabrerizo et al. 2017; Morente-Molinera et al. 2019; Ureña et al. 2019).

However, in order to better assess real-world problems, it is necessary to consider other concepts in the analysis such as the use of weighted averages that represent the subjective importance of the information being studied. In the literature, there are several aggregation operators for dealing with the weighted average and the OWA operator in the same formulation. For example, we could mention the hybrid average (Xu and Da 2003), the weighted OWA (WOWA) operator (Torra 1997) and the immediate weights (Merigó 2011). Recently, Merigó (2011) has suggested a new aggregation operator for dealing with these problems. He called it the ordered weighted averaging weighted averaging (OWAWA) operator. Its main advantage is that it can unify the weighted average and the OWA operator in the same formulation and it considers the degree of importance that each concept has in the aggregation. The OWAWA operator can also be generalized by using generalized and quasi-arithmetic means obtaining the generalized OWAWA (GOWAWA) operator and the quasi-arithmetic OWAWA (quasi-OWAWA) operator.

The aim of this paper is to present new moving averages based on the use of the OWAWA operator. We introduce the ordered weighted averaging weighted moving averaging (OWAWMA) operator. It is a moving aggregation operator that unifies the weighted moving average (WMA) and the ordered weighted moving averaging (OWMA) operator in the same formulation and considers the degree of importance that each concept has in the aggregation. Moreover, it provides a parameterized family of moving aggregation operators between the moving minimum and the moving maximum. Thus, it is possible to analyze the information in a dynamic way and consider the subjective information and the degree of or-ness (degree of optimism) of the aggregation. Some of its main properties and particular cases are studied.

This approach is further extended by using generalized aggregation operators obtaining the generalized OWAWMA (GOWAWMA) operator and the quasi-arithmetic OWAWMA (quasi-OWAWMA) operator. Their main advantage is that they include a wide range of particular cases including the generalized weighted moving average (GWMA), the generalized ordered weighted moving average (GOWMA) operator, the geometric OWAWMA (OWGAWMA) operator and the quadratic OWAWMA (OWQAWMA) operator. Therefore, the information can be represented in a more complete and flexible way because it can adapt to the specific needs of the complex environment considered.

The applicability of this approach is also studied, and we see that it is very broad because all the previous studies that use the moving average can be revised and extended with this new approach. Its main advantage is that it can provide a more realistic analysis of the problem because we can consider the information in a dynamic way. In order to understand numerically the new approach, a simple numerical example is presented in sales forecasting where we analyze the sales of a product in North America, Europe and Asia.

This paper is organized as follows: Section 2 briefly reviews some basic preliminaries. Section 3 presents the OWAWMA operator. Section 4 introduces the quasi-OWAWMA operator and Sect. 5 the construction of interval and fuzzy numbers. Section 6 discusses the applicability of the new approach and presents a simple illustrative example. Section 7 summarizes the main findings of the paper.

2 Preliminaries

A moving average is a usual average that moves toward some part of the whole sample (available or to be obtained in the future). More generally, moving averaging aggregation operators are those aggregation operators that use moving averages in the aggregation process (Merigó and Yager 2013). This approach allows considering the results of part of the sample and making changes toward the partial sample selected. The moving average is a very popular tool in time-series smoothing (Yager 2008). For the definition of the weighted moving average (WMA), see, for example, Elliot et al. (2006); Merigó and Yager (2013).

Note that if $w_i = 1/m$ for all i, the WMA becomes the simple moving average. Note that in the literature there are a wide range of moving averaging techniques but in this paper the focus is on these two types and those suggested by Yager (2008) and Merigó and Yager (2013). Yager (Yager 2008) introduced the use of the OWA operator as a moving average. Thus, he provided a parameterized family of moving aggregation operators between the moving minimum and the moving maximum. The ordered weighted moving average (OWMA) can be formulated as follows:

$$\mathrm{OWMA}(a_{1+t}, a_{2+t}, \ldots, a_{m+t}) = \sum_{j=1+t}^{m+t} w_j b_j, \quad (1)$$

where b_j is the jth largest of the a_i, $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0, 1]$, m is the total number of arguments considered from the whole sample and t indicates the movement done in the average from the initial analysis.

A fundamental issue when analyzing these moving aggregation operators is the analysis of the weighting vector used in the aggregation. Following the time-series literature (Elliot et al. 2006; Evans 2002), Yager (2008) stated that some key weighting vectors were those that were decaying such as linear decaying weights, squarely decaying weights and inverse sum weights.

The ordered weighted averaging weighted average (OWAWA) operator (Merigó 2011) is an aggregation operator that uses the weighted average and the OWA operator in the same formulation taking into account the importance that each of them has in the aggregation. For the definition of the OWAWA operator, see, for example, Merigó (2011).

Generalized aggregation operators (Beliakov et al. 2007; Merigó and Gil-Lafuente 2009) are those aggregation operators that use the generalized or the quasi-arithmetic mean in its formulation. Thus, they are able to provide a wide range of particular cases including geometric and quadratic means. Note that this paper focuses on the quasi-arithmetic mean because it includes the generalized mean as a particular case. For the definition of the weighted quasi-arithmetic average (quasi-WA), see, for example, Beliakov et al. (2007). In addition, it is also possible to use OWA operators forming the ordered weighted quasi-arithmetic mean (quasi-OWA) (Beliakov et al. 2007; Fodor et al. 1995). Furthermore, the OWAWA operator can also be generalized by using generalized aggregation operators forming the generalized OWAWA (GOWAWA) operator and the quasi-arithmetic OWAWA (quasi-OWAWA) operator (Merigó et al. 2016). The quasi-OWAWA is defined as follows.

Definition 1 A quasi-OWAWA operator of dimension n is a mapping QOWAWA: $R^n \rightarrow R$ that has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, and a weighting vector V that affects the WA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\mathrm{QOWAWA}(a_1, \ldots, a_n) = \beta g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right) + (1 - \beta) h^{-1}\left(\sum_{i=1}^{n} v_i h(a_i)\right), \quad (2)$$

where b_j is the jth largest of the a_i, $\beta \in [0, 1]$ and $g(b)$ and $g(h)$ are strictly continuous monotone functions.

Note that if $\beta = 1$, we get the quasi-OWA operator with all its particular cases and if $\beta = 0$ the quasi-WA operator with all its particular cases. Moreover, if $g(b) = h(a) = 2$, we get the quadratic OWAWA (OWQAWA) operator and if $g(b) = h(a) \rightarrow 0$ the geometric OWAWA (OWGAWA) operator. Furthermore, note that it is possible to use different functions or values in $g(b)$ and $h(a)$ for the OWA and the weighted average. Thus, for example, we could form the ordered weighted quadratic averaging weighted cubic averaging (OWQAWCA) operator by using $g(b) = 2$ in the OWA aggregation and $h(a) = 3$ in the weighted average aggregation.

3 Moving averages and OWAWA operators

By using OWAWA operators in the moving average (Merigó and Yager 2013), it is possible to use weighted moving averages (WMA) and ordered weighted moving averages (OWMA) in the same formulation and consider the degree of importance of each concept in the formulation. We call this new approach the ordered weighted averaging weighted moving averaging (OWAWMA) operator. It is an aggregation operator that provides a parameterized family of moving aggregation operators between the minimum and the maximum. It is able to consider the subjective importance of the available information and the degree of or-ness (degree of optimism) of the aggregation. Note that the degree of optimism represents the attitude of the decision maker regarding the potential results of an uncertain or future event. It can be defined as follows.

Definition 2 An OWAWMA operator of dimension m is a mapping OWAWMA: $R^m \rightarrow R$ that has an associated weighting vector W of dimension m with $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\mathrm{OWAWMA}(a_{1+t}, a_{2+t}, \ldots, a_{h+t}, \ldots, a_{m+t}) = \sum_{j=1+t}^{m+t} \hat{v}_j b_j, \quad (3)$$

where b_j is the jth largest argument of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1+t}^{m+t} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0, 1]$, v_j is the weight v_i ordered according to b_j, that is, according to the jth largest of the a_i, m is the total number of arguments considered from the whole sample and t indicates the movement done in the average from the initial analysis.

Note that this formulation can also be expressed separating the part that affects the OWA aggregation and the part that affects the weighted average as follows:

$$\mathrm{OWAWMA}(a_{1+t},\ldots,a_{m+t}) = \beta \sum_{j=1+t}^{m+t} w_j b_j + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i, \quad (4)$$

where b_j is the jth largest argument of the a_i, $\beta \in [0, 1]$, m is the total number of arguments considered from the whole sample and t indicates the movement done in the average from the initial analysis.

Example 1 Assume that we want to forecast a variable (for example, the prize of a product) from period 5 to 9 using the last four periods as shown in Table 1.

Assume that the weighting vector of the OWA is $W = (0.1, 0.2, 0.3, 0.4)$ and the weighting vector of the weighted average $V = (0.2, 0.2, 0.3, 0.3)$. The OWA aggregation has a degree of importance of 40% and the weighted average 60%. Note that the OWA weights represent the attitude of the decision maker that in this example is underestimating the information and the weights of the weighted average indicate the importance that the previous periods have in the aggregation. Thus, the following forecasts can be developed by using the OWAWMA operator. The results are shown in Table 2.

As we can see, each particular type of OWAWMA operator may give different results because they represent different attitudes with respect to the forecast. Thus, the decision maker gets a more complete representation of the problem so he can see the different scenarios that may occur and select the one that is in accordance with his interests. Obviously, the more cases considered, the more complete the information is. The objective of this table is to represent some basic cases to see how a decision maker can understand the uncertain or imprecise information of the problem he is considering. Also observe that Eqs. (9) and (10) can be seen from the point of view of the expected value forming the moving expected value (MEV).

From a generalized perspective of the reordering step, it is possible to distinguish between the descending OWAWMA (DOWAWMA) and the ascending OWAWMA (AOWAWMA) operator. Note that it is also possible to consider a more general reordering process by using $\hat{v}_j = \hat{v}_{n-j+1}$ and buoyancy measures (Yager 1993).

Table 1 Set of arguments for 8 periods

	1	2	3	4	5	6	7	8
P	23	26	29	25	27	24	27	28

Table 2 Forecasting by using different types of OWAWMA operators

	5	6	7	8	9
MA	25.75	26.75	26.25	25.75	26.5
WMA	26	26.6	26.1	25.7	26.7
OWMA	24.8	26.1	25.4	25.2	25.9
OWAWMA	25.52	26.4	25.82	25.5	26.3
Max-WMA	27.2	27.56	27.26	26.22	27.2
Min-WMA	24.8	25.96	25.26	25.02	25.6
Max	29	29	29	27	28
Min	23	25	24	24	24

Note that if the weighting vector is not normalized, i.e., $\hat{V} = \sum_{j=1}^{n} \hat{v}_j \neq 1$, the OWAWMA operator can be expressed as:

$$\mathrm{OWAWMA}(a_1,\ldots,a_n) = \frac{\beta}{W} \sum_{j=1+t}^{m+t} w_j b_j + \frac{(1-\beta)}{V} \sum_{i=1+t}^{m+t} v_i a_i. \quad (5)$$

The OWAWMA operator can be further generalized with mixture operators (Merigó 2011). We call it the mixture OWAWMA (MOWAWMA) operator.

Definition 3 A MOWAWMA operator of dimension m is a mapping f: $R^m \to R$ that has associated a vector of weighting functions f_i, f_j: $I \to]0, \propto[$ and s: $R^l \to R$, such that:

$$f\left(s_y(a_{1+t}),\ldots,s_y(a_{m+t})\right) = \beta \frac{\sum_{j=1+t}^{m+t} f_j(s_y(b_j)) s_y(b_j)}{\sum_{j=1+t}^{m+t} f_j(s_y(b_j))} + (1-\beta) \frac{\sum_{i=1+t}^{m+t} f_i(s_y(a_i)) s_y(a_i)}{\sum_{i=1+t}^{m+t} f_i(s_y(a_i))}, \quad (6)$$

where $s_y(b_j)$ is the jth largest of the $s_y(a_i)$, y indicates that each argument is formed by using a different function where $s_y(b_j)$ is the jth largest of the $s_y(a_i)$, a_i is the argument variable, m is the total number of arguments considered from the whole sample and t indicates the movement done in the average from the initial analysis.

The choice of the measures to characterize the weighting vector $\hat{V}$ (V and W) is another interesting issue. Following a similar methodology as for the OWA operator (Yager 1988, 1993) and the OWAWA operator (Merigó 2011), we can formulate the attitudinal character (degree of or-ness), the entropy of dispersion, the divergence of the weighting vector and the balance operator.

If we extend the analysis of the or-ness—and-ness measure to the OWAWMA operator, it is possible to use two different interpretations as mentioned in Sect. 2.2. The first one consists in assuming that the weighted average is neutral independently of its weights. Thus, its or-ness should always be 0.5 obtaining the following expression:

$$\alpha(\hat{V}) = \beta \sum_{j=1+t}^{m+t} w_j \left(\frac{m+t-j}{m-1} \right) + (1-\beta) \times 0.5. \quad (7)$$

However, from a mathematical point of view, the or-ness of the weighted average can be studied according to the tendency of the aggregation to the minimum or to the maximum. In this case, we get the following formulation for the degree of or-ness:

$$\alpha(\hat{V}) = \beta \sum_{j=1+t}^{m+t} w_j \left(\frac{m+t-j}{m-1} \right) + (1-\beta) \sum_{j=1+t}^{m+t} v_j \left(\frac{m+t-j}{m-1} \right). \quad (8)$$

Note that v_j is the v_i reordered according to the jth largest of the a_i. As we can see, if $\beta = 1$, we get the or-ness measure of the OWMA operator and if $\beta = 0$ we obtain the or-ness measure of the WMA operator. It is straightforward to calculate the and-ness measure by using the dual. That is, $Andness\,(\hat{V}) = 1 - \alpha(\hat{V})$.

In the following, we present some interesting results obtained with this new or-ness—and-ness measure. For the optimistic (or maximum) criteria in the OWMA, we get the following:

$$\alpha(\hat{V}) = \beta + (1-\beta) \sum_{j=1+t}^{m+t} v_j \frac{m+t-j}{m-1}. \quad (9)$$

Note that we can refer to this situation as the maximum weighted moving average (Max-WMA) or weighted moving maximum. For the pessimistic (or minimum) criteria, we obtain:

$$\alpha(\hat{V}) = (1-\beta) \sum_{j=1+t}^{m+t} v_j \frac{m+t-j}{m-1}. \quad (10)$$

We can refer to this situation as the minimum weighted moving average (Min-WMA) or weighted moving minimum. With the arithmetic mean in the OWMA case (arithmetic weighted moving average (A-WMA)), we get:

$$\alpha(\hat{V}) = 0.5\beta + (1-\beta) \sum_{j=1+t}^{m+t} v_j \frac{m+t-j}{m-1}, \quad (11)$$

and in the WMA (arithmetic-OWMA), we obtain:

$$\alpha(\hat{V}) = \beta \sum_{j=1+t}^{m+t} w_j \left(\frac{m+t-j}{m-1} \right) + 0.5(1-\beta). \quad (12)$$

The entropy of dispersion (Yager 1988) measures the amount of information being used in the aggregation. If we extend it to the OWAWMA operator, we get the following:

$$H(\hat{V}) = -\left(\beta \sum_{j=1+t}^{m+t} w_j \ln(w_j) + (1-\beta) \sum_{i=1+t}^{m+t} v_i \ln(v_i) \right). \quad (13)$$

Note that v_i is the ith weight of the WA aggregation. As we can see, if $\beta = 1$, we get the entropy of dispersion of the OWMA operator [very similar to the Yager entropy (1988)], and if $\beta = 0$, the entropy of dispersion of the WMA [very similar to the Shannon entropy (1948)].

The divergence of W (Yager 2002) measures the divergence of the weights against the degree of or-ness measure. If we extend the divergence of W to the OWAWMA operator, we get the following divergence of $\hat{V}$:

$$\mathrm{Div}(\hat{V}) = \beta \left(\sum_{j=1+t}^{m+t} w_j \left(\frac{m+t-j}{m-1} - \alpha(W) \right)^2 \right) + (1-\beta) \left(\sum_{j=1+t}^{m+t} v_j \left(\frac{m+t-j}{m-1} - \alpha(V) \right)^2 \right). \quad (14)$$

If $\beta = 1$, we get the OWMA divergence, and if $\beta = 0$, the WMA divergence. The highest divergence is found when using the arithmetic mean, while the minimum divergence when putting all the weight in one argument as it is done with the step-OWA (Yager 1993).

The balance operator (Yager 1996) measures the balance of the weights against the or-ness or the and-ness. If we extend the balance operator to the OWAWMA operator, we get the following expression:

$$\mathrm{Bal}(\hat{V}) = \beta \left(\sum_{j=1+t}^{m+t} \left(\frac{n+1-2j}{n-1} \right) w_j \right) + (1-\beta) \left(\sum_{j=1+t}^{m+t} v_j \left(\frac{n+1-2j}{n-1} \right) \right). \quad (15)$$

If $\beta = 1$, we get the classic balance operator of the OWMA operator, and if $\beta = 0$, we obtain the balance operator of the WMA. As we can see, Bal $(V) \in [-1, 1]$.

The OWAWMA is monotonic, bounded and idempotent (Merigó 2011). It is not commutative because the OWAWMA operator includes the weighted average that is not commutative.

Theorem 1 (Boundary condition). *Assume f is the OWAWMA operator, then*:

$$Min\{a_i\} \leq f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) \leq Max\{a_i\}. \quad (16)$$

Proof It is trivial and thus omitted.

Note that the bounded property presented in Theorem 1 is the extreme case where we only use the OWMA operator in the aggregation of the OWAWMA operator. However, the usual boundary conditions used in the OWAWMA operator are more specific because they mix the OWA and the WA so the results usually cannot be so high or so low as in the OWA case.

Theorem 2 (Semi-boundary conditions). *Assume f is the OWAWMA operator, then*:

$$\beta \times \text{Min}\{a_i\} + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i \leq f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) \leq \beta \times \text{Max}\{a_i\} + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i. \quad (17)$$

Proof Let $\max\{a_i\} = c$, and $\min\{a_i\} = d$, then

$$f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) = \beta \sum_{j=1+t}^{m+t} w_j b_j + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i \leq \beta \sum_{j=1+t}^{m+t} w_j c + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i = \beta c \sum_{j=1+t}^{m+t} w_j + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i, \quad (18)$$

$$f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) = \beta \sum_{j=1+t}^{m+t} w_j b_j + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i \geq \beta \sum_{j=1+t}^{m+t} w_j d + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i = \beta d \sum_{j=1+t}^{m+t} w_j + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i. \quad (19)$$

Since $\sum_{j=1+t}^{m+t} w_j = 1$, we get

$$f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) \leq \beta c + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i, \quad (20)$$

$$f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) \geq \beta d + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i. \quad (21)$$

Therefore,

$$\beta \times \text{Min}\{a_i\} + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i \leq f(a_{1+t}, a_{2+t}, \ldots, a_{n+t}) \leq \beta \times \text{Max}\{a_i\} + (1-\beta) \times \sum_{i=1+t}^{m+t} v_i a_i.$$

■

Finally, let us study several families of OWAWMA operators. Thus, we are able to provide a more complete picture of the aggregation process. However, note that each family is just a particular case useful in some special situations according to the interests of the analysis.

First, we consider the two main cases of the OWAWMA operator found by analyzing the coefficient β. Basically, if $\beta = 0$, we get the WMA, and if $\beta = 1$, the OWMA operator. Note that when β increases, more importance is given to the OWMA operator, and when β decreases, vice versa.

Next, let us analyze different manifestations of the weighting vector in the OWAWMA operator. The maximum-WMA is found when $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$ and if $\beta = 1$ the usual moving maximum (M-Max). The minimum-WMA is formed when $w_n = 1$ and $w_j = 0$ for all $j \neq n$ and if $\beta = 1$, the usual moving minimum (M-Min).

Remark 1 The arithmetic moving average-WMA (AMA-WMA) is obtained when $w_j = 1/m$ for all j, and it can be formulated as follows:

$$\text{AMA - WMA}(a_{1+t}, a_{2+t}, \ldots, a_{m+t}) = \frac{1}{m}\beta a_i + (1-\beta) \sum_{i=1+t}^{m+t} v_i a_i. \quad (22)$$

If $v_i = 1/m$, for all i, then we get the arithmetic moving average-OWMA (AMA-OWMA). The AMA-OWMA operator can be formulated as follows:

$$\text{AMA - OWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta \sum_{j=1+t}^{m+t} w_j b_j + (1-\beta) \frac{1}{m} a_i. \quad (23)$$

Note that if $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, the AMA-OWMA operator becomes the AMA-Max that it is also known in the literature as the or-like S-OWMA operator and that if $w_n = 1$ and $w_j = 0$ for all $j \neq m + t$, it becomes the AMA-Min that is known as the and-like S-OWMA operator (extended from static aggregations (Yager 1993)). Finally, if $v_i = 1/m$, for all i and $w_j = 1/m$ for all j, the OWAWMA operator becomes the simple moving average.

Some other aggregations that could be formed following the OWA literature (Merigó 2011; Yager 1993) are the median-OWAWMA, the weighted median-OWAWMA, the centered-OWAWMA and the Olympic-OWAWMA. The previous families satisfy that the weighting vector of the OWA and the WA uses the same structure in order to be the same family. However, we may find that the OWA uses a different family than the WA. For example, the OWA can use a median aggregation, while the WA uses an Olympic one. And so on. Furthermore, note that here we are assuming that the WA follows a similar pattern than the OWA. However, we could also simply analyze the OWA operator and leave the WA as a normal aggregation. In other words, we analyze the classical families of OWA operators of the weighting vector w_j (Yager 1993).

4 The generalized OWAWMA operator

The OWAWMA operator can be generalized by using generalized aggregation operators such as the generalized mean and the quasi-arithmetic mean (Beliakov et al. 2007; Merigó and Gil-Lafuente 2009). By using the quasi-arithmetic mean, we get the quasi-arithmetic OWAWMA (quasi-OWAWMA) operator. It is defined as follows.

Definition 4 A quasi-OWAWMA operator of dimension m is a mapping QOWAWMA: $R^m \rightarrow R$ that has an associated weighting vector W of dimension m with $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0, 1]$, and a weighting vector V with $\sum_{i=1+t}^{m+t} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\mathrm{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta g^{-1}\left(\sum_{j=1+t}^{m+t} w_j g(b_j)\right) + (1-\beta)h^{-1}\left(\sum_{i=1+t}^{m+t} v_i h(a_i)\right), \tag{24}$$

where b_j is the jth largest argument of the a_i, $\beta \in [0, 1]$, m is the total number of arguments considered from the whole sample, t indicates the movement done in the average from the initial analysis and g and h are a strictly continuous monotonic functions.

Observe that if $g(b) = b^\lambda$ and $h(a) = a^\lambda$, the quasi-OWAWMA operator becomes the GOWAWMA operator. Moreover, it is possible to distinguish between descending (quasi-DOWAWMA) and ascending (quasi-AOWAWMA) orders. Furthermore, when $\beta = 1$, we get the quasi-OWMA operator and the quasi-WMA when $\beta = 0$. The more of β located to the top, the more importance we give to the quasi-OWMA operator and vice versa.

Note that in this case we could also study similar properties than the OWAWMA operator such as the use of mixture operators and a wide range of measures for characterizing the weighting vector. For example, the orness measure can be defined as follows:

$$\alpha(\hat{V}) = \beta g^{-1}\left(\sum_{j=1+t}^{m+t} w_j g\left(\frac{m+t-j}{m-1}\right)\right) + (1-\beta)g^{-1}\left(\sum_{j=1+t}^{m+t} v_j g\left(\frac{m+t-j}{m-1}\right)\right). \tag{25}$$

Some other interesting particular cases are found by analyzing the strictly continuous monotonic function g. For example, if $g(b) = b$ and $h(a) = a$, we get the OWAWMA operator as shown in Eq. (10). If $g(b) = b^2$ and $h(a) = a^2$, we get the ordered weighted quadratic moving averaging weighted quadratic moving averaging (OWQMAWQMA) operator. That is:

$$QOWAWMA(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\sum_{j=1+t}^{m+t} w_j b_j^2\right)^{1/2} + (1-\beta)\left(\sum_{i=1+t}^{m+t} v_i a_i^2\right)^{1/2}. \tag{26}$$

If $g(b) = b^3$ and $h(a) = a^3$, we get the ordered weighted cubic moving averaging weighted cubic moving averaging (OWCMAWCMA) operator:

$$\mathrm{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\sum_{j=1+t}^{m+t} w_j b_j^3\right)^{1/3} + (1-\beta)\left(\sum_{i=1+t}^{m+t} v_i a_i^3\right)^{1/3}. \tag{27}$$

If $g(b) \rightarrow b^0$ and $h(a) \rightarrow a^0$, we get the ordered weighted geometric moving averaging weighted geometric moving averaging (OWGMAWGMA) operator:

$$\mathrm{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\prod_{j=1+t}^{m+t} b_j^{w_j}\right) + (1-\beta)\left(\prod_{i=1+t}^{m+t} a_i^{v_i}\right). \tag{28}$$

If $g(b) = b^{-1}$ and $h(a) = a^{-1}$, we get the ordered weighted harmonic moving averaging weighted harmonic moving averaging (OWHMAWHMA) operator:

$$\mathrm{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\frac{1}{\sum_{j=1+t}^{m+t} \frac{w_j}{b_j}}\right) + (1-\beta)\left(\frac{1}{\sum_{i=1+t}^{m+t} \frac{v_i}{a_i}}\right). \tag{29}$$

Moreover, if $g(b) = b^\infty$ and $h(a) = a^\infty$, the moving maximum, and if $g(b) = b^{-\infty}$ and $h(a) = a^{-\infty}$, the moving minimum.

Furthermore, it is possible to consider situations where $g(b)$ and $h(a)$ are not equal. For example, if $g(b) = b^2$ and

$h(a) = a^3$, we get the ordered weighted quadratic moving averaging weighted cubic moving averaging (OWQ-MAWCMA) operator:

$$\text{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\sum_{j=1+t}^{m+t} w_j b_j^2\right)^{1/2} + (1-\beta)\left(\sum_{i=1+t}^{m+t} v_i a_i^3\right)^{1/3}. \quad (30)$$

Another example is if $g(b) = b$ and $h(a) = a^2$, then we get the ordered weighted moving averaging weighted quadratic moving averaging (OWMAWQMA) operator. That is:

$$\text{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\sum_{j=1+t}^{m+t} w_j b_j + (1-\beta)\left(\sum_{i=1+t}^{m+t} v_i a_i^2\right)^{1/2}. \quad (31)$$

Similarly, we could also study $g(b) = b^2$ and $h(a) = a$. This case is the ordered weighted quadratic moving averaging weighted moving averaging (OWQMAWMA) operator:

$$\text{QOWAWMA}(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\sum_{j=1+t}^{m+t} w_j b_j^2\right)^{1/2} + (1-\beta)\sum_{i=1+t}^{m+t} v_i a_i. \quad (32)$$

In a similar way, we could study a lot of other cases by using other formulations in the function g and h. Following Beliakov et al. (2007); Merigó and Gil-Lafuente (2009), we could develop the trigonometric OWAWMA and the radical OWAWMA. The trigonometric OWAWMA operator is found when $g_1(t) = \sin((\pi/2)\, t)$, $g_2(t) = \cos((\pi/2)\, t)$ and $g_3(t) = \tan((\pi/2)\, t)$ are the generating functions. Thus,

$$f(a_{1+t}, \ldots, a_{m+t}) = \beta\frac{2}{\pi}\arcsin\left(\sum_{j=1+t}^{m+t} w_j \sin\left(\frac{\pi}{2}b_j\right)\right) + (1-\beta)\frac{2}{\pi}\arcsin\left(\sum_{i=1+t}^{m+t} v_i \sin\left(\frac{\pi}{2}a_i\right)\right). \quad (33)$$

$$f(a_{1+t}, \ldots, a_{m+t}) = \beta\frac{2}{\pi}\arccos\left(\sum_{j=1+t}^{m+t} w_j \cos\left(\frac{\pi}{2}b_j\right)\right) + (1-\beta)\frac{2}{\pi}\arccos\left(\sum_{i=1+t}^{m+t} v_i \cos\left(\frac{\pi}{2}a_i\right)\right) \quad (34)$$

$$f(a_{1+t}, \ldots, a_{m+t}) = \beta\frac{2}{\pi}\arctan\left(\sum_{j=1+t}^{m+t} w_j \tan\left(\frac{\pi}{2}b_j\right)\right) + (1-\beta)\frac{2}{\pi}\arctan\left(\sum_{i=1+t}^{m+t} v_i \tan\left(\frac{\pi}{2}a_i\right)\right). \quad (35)$$

The radical OWAWMA is found if $\gamma > 0$, $\gamma \neq 1$, and the generating function is $g(t) = \gamma^{1/t}$. Then, the radical OWAWMA operator is:

$$f(a_{1+t}, \ldots, a_{m+t}) = \beta\left(\log_\gamma\left(\sum_{j=1+t}^{m+t} w_j \gamma^{1/b_j}\right)\right)^{-1} + (1-\beta)\left(\log_\gamma\left(\sum_{i=1+t}^{m+t} v_i \gamma^{1/a_i}\right)\right)^{-1} \quad (36)$$

5 Construction of moving interval and fuzzy numbers with the OWAWMA operator

In this section, we analyze how to construct interval numbers and other related structures such as fuzzy numbers by using OWMA and OWAWMA operators.

5.1 Construction of interval numbers with OWMA operators

The OWMA operator provides a parameterized family of moving aggregation operators between the moving minimum and the moving maximum. Thus, with the OWMA operator we can analyze any result inside this interval. This is worth noting because when we want to analyze some problem, we need to collect the data and summarize it in an efficient way so it is easy to understand. A practical way for doing so is by constructing an interval number that at least will include the minimum and the maximum potential result. Moreover, a wide range of internal results can be added depending on our particular interests in the analysis (Merigó 2012). Particularly, we can construct triplets by adding a central value that represents the most expected result and quadruplets by adding an internal interval that shows the most expected result in the form of an interval.

Assume a set of arguments $A = (a_1, a_2, \ldots, a_n)$. For the construction of a 2-tuple interval number (Moore 1966), we simply aggregate the information of the OWMA operator in the following way: $C = [\text{M-Min}\{a_i\}, \text{M-Max}\{a_i\}]$. Thus, we are considering an interval number that considers the lowest and the highest result of the set of arguments A. However, note that in the OWMA operator a dynamic process is being carried out and these bounds may change throughout time. Therefore, we can refer to this interval number as a moving interval number (MIN).

If we construct a triplet, we can use the moving minimum, the moving maximum and the OWMA aggregation that is more in accordance with the interests of the decision maker. In this case, we get a moving triplet: $C = [\text{M-Min}\{a_i\}, \mathit{OWMA}, \text{M-Max}\{a_i\}]$.

In order to consider a quadruplet, two OWMA aggregations are used such that one of the aggregations is closer to the moving minimum and the other one to the moving maximum. That is: $C = [\text{M-Min}\{a_i\}, OWMA_*, OWMA^*, \text{M-Max}\{a_i\}]$, where $OWMA_*$ is the OWMA aggregation closer to the minimum and $OWMA^*$ to the maximum. This type of interval is a moving quadruplet.

Following this methodology, we could develop moving quintuplets, moving sextuplets and so on, by adding more information to the moving interval with the use of other types of OWMA aggregations. Note that this methodology can also be connected to the concept of a box plot (Tukey 1977). Therefore, by using moving averages, we could form moving box plots. Next, let us present a brief numerical example on how to construct interval numbers with the OWA operator.

Example 2 Assume the information of Example 1. Thus, the following moving triplets are formed as shown in Table 3.

Graphically, these results can be represented as shown in Fig. 1.

As we can see, the triplets evolve throughout time because they are not static and may change when we add more information in the aggregation.

Note that we could now consider a wide range of properties and operations with interval numbers following the classical literature (Moore 1966). For example, we could consider the sum, the subtraction, the multiplication and the division of two moving intervals. Observe that a lot of complexities may occur because we may want to operate with moving intervals that consider different periods of time.

Furthermore, more complex structures can also be constructed by using OWMA operators such as moving fuzzy numbers (MFNs). For example, once a moving triplet is constructed with an OWMA aggregation, we can assume that the internal information of the moving triplet can be represented with linear functions. Thus, it is possible to assume that the moving triplet is a moving triangular FN (MTFN) represented in a ternary way (Dubois and Prade 1980; Kaufmann and Gupta 1985). Therefore, we can construct the α-cut representation of the MTFN in the following way: $C_\alpha = [\text{M-Min}\{a_i\} + (\text{OWMA} - \text{M-Min}\{a_i\}) \times \alpha, \text{M-Max}\{a_i\} - (\text{M-Max}\{a_i\} - \text{OWMA}) \times \alpha]$. With a moving quadruplet, a similar analysis can also be developed obtaining a moving trapezoidal FN (MTpFN). In this case, the α-cut representation is formed as follows: $C_\alpha = [\text{M-Min}\{a_i\} + (\text{OWMA}_* - \text{M-Min}\{a_i\}) \times \alpha, \text{M-Max}\{a_i\} - (\text{M-Max}\{a_i\} - \text{OWMA}^*) \times \alpha]$. From this, it is straightforward to construct the membership function of the MTFN and the MTpFN. Figure 2 briefly presents the MTFN formed with a moving triplet.

Table 3 Moving triplets

Time period	M-Min	OWMA	M-Max
5	23	24.8	29
6	25	26.1	29
7	24	25.4	29
8	24	25.2	27
9	24	25.9	28

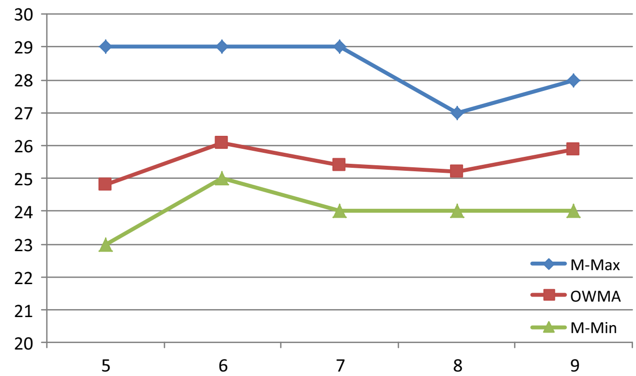

Fig. 1 Moving triplets with the OWMA operator

As we can see, the MTFN is not static so its structure changes over time. Therefore, throughout time we may see an increase in the minimum, in the maximum and so on. Figure 2 shows a general example where the MTFN increases in all its values from period 1 to period 2. Following Example 2, we could form the following MTFNs:

- $C_5 = [23 + 1.8\alpha, 29 - 4.2\alpha]$.
- $C_6 = [25 + 1.1\alpha, 29 - 2.9\alpha]$.
- $C_7 = [24 + 1.4\alpha, 29 - 3.6\alpha]$.
- $C_8 = [24 + 1.2\alpha, 27 - 1.8\alpha]$.

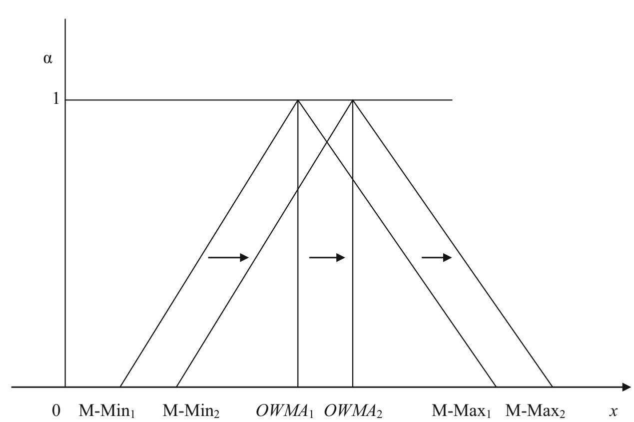

Fig. 2 Two moving triangular fuzzy numbers constructed with the OWMA operator

- $C_9 = [24 + 1.9\alpha, 28 - 2.1\alpha]$.

In a similar way, more complex FNs and more complex structures such as the linguistic variables could be studied. Moreover, we could also construct interval numbers with a wide range of extensions of the OWMA operator such as the use of generalized aggregation operators, induced aggregation operators (Merigó 2011), Choquet integrals (Belles-Sampera et al. 2014), distance measures (Merigó et al. 2018; Zeng et al. 2017) and norms (Merigó et al. 2014).

5.2 Construction with the OWAWMA operator

The construction of moving interval numbers and moving fuzzy numbers can also be developed by using OWAWMA operators. Its main advantage is that it can deal with subjective information and with the attitudinal character of the decision maker in the same formulation and considering the degree of importance that each concept has in the analysis. Note that this analysis could also be developed with other models that deal with the weighted average and the OWA operator in the same formulation such as the weighted OWA (WOWA) operator (Torra 1997), the hybrid average (Xu and Da 2003) and the immediate weights (Merigó 2011). Note that with these models we should develop a previous transformation by using moving averages forming the weighted OWMA (WOWMA) operator, the hybrid moving average (HMA) and the moving immediate weights (MIW).

The interesting issue of using OWAWMA operators is that we can introduce the subjective beliefs of the decision maker in the interval numbers. This leads us to the subjective moving interval number (SMIN) and the subjective moving fuzzy number (SMFN). Their key advantage is that they consider the degree of importance that each argument has in the aggregation and in the intervals.

In the construction of the SMIN with the OWAWMA operator, it is worth noting the construction of subjective moving triplets, subjective moving quadruplets, subjective moving quintuplets and subjective moving sextuplets. The subjective moving triplet and the subjective moving quadruplet follow the same methodology than the OWMA. That is: $C = [\text{M-Min}\{a_i\}, \text{OWAWMA}, \text{M-Max}\{a_i\}]$ and $C = [\text{M-Min}\{a_i\}, \text{OWAWMA}_*, \text{OWAWMA}^*, \text{M-Max}\{a_i\}]$. Note that if $\beta = 1$, the OWAWMA becomes the OWMA operator, and thus, we get the same results than Sect. 6.1. If $\beta = 0$, it becomes the WMA and thus we can construct a moving interval number where we do not use the OWMA but we consider a subjective importance of the arguments. That is: $C = [\text{M-Min}\{a_i\}, \text{WMA}, \text{M-Max}\{a_i\}]$ and $C = [\text{M-Min}\{a_i\}, WMA_*, WMA^*, \text{M-Max}\{a_i\}]$. Note that in Sect. 6.1., it is assumed that we do not know the subjective importance of the arguments and we only focus on the OWMA aggregation.

As it is stated in Theorem 2, with the OWAWMA operator we obtain semi-boundary conditions when we use the WMA with the OWMA bounds. Thus, the bounds can be contracted according to the information given by the WMA. However, it is worth noting that this reduction is only artificial because the result can always move from the moving minimum to the moving maximum. But sometimes the usual bounds are too broad and it is needed to contract them in order to reduce the uncertainty and be able to deal with the imprecision of the available information. In these situations, it becomes useful to consider subjective moving quintuplets and subjective moving sextuplets in the analysis:

- $C = [\text{M-Min}\{a_i\}, \text{M-Min-WA}, \text{OWAWMA}, \text{M-Max-WA}, \text{M-Max}\{a_i\}]$.
- $C = [\text{M-Min}\{a_i\}, \text{M-Min-WA}, \text{OWAWMA}_*, \text{OWAWMA}^*, \text{M-Max-WA}, \text{M-Max}\{a_i\}]$;

where M-Min-WA is the convex combination $\beta \times \text{M-Min}\{a_i\} + (1 - \beta) \times \text{WMA}$, and M-Max-WMA is $\beta \times \text{M-Max}\{a_i\} + (1 - \beta) \times \text{WMA}$.

Example 3 Assume the information given in Example 1. Thus, the following subjective moving quintuplets can be constructed as shown in Table 4.

Graphically, the results shown in Table 4 would look as shown in Fig. 3.

As we can see, the moving minimum and the moving maximum delimitate the extreme results that may occur. The OWAWMA operator is the expected results according to the beliefs and attitudes of the decision maker that are semi-bounded by a maximum and a minimum that take into account the weighted average in the aggregation.

With the OWAWMA operator, it is also possible to construct subjective moving FNs (SMFNs). For subjective moving triplets and subjective moving quadruplets, it follows the same procedure as with the OWMA operator. Thus, we can assume the use of the α-cut representation of the subjective moving TFN (SMTFN) and the subjective moving TpFN (SMTpFN) as follows:

- $C_\alpha = [\text{M-Min}\{a_i\} + (\text{OWAWMA} - \text{M-Min}\{a_i\}) \times \alpha, \text{M-Max}\{a_i\} - (\text{M-Max}\{a_i\} - \text{OWAWMA}) \times \alpha]$.
- $C_\alpha = [\text{M-Min}\{a_i\} + (\text{OWAWMA}_* - \text{M-Min}\{a_i\}) \times \alpha, \text{M-Max}\{a_i\} - (\text{M-Max}\{a_i\} - \text{OWAWMA}^*) \times \alpha]$.

By using subjective moving quintuplets and subjective moving sextuplets, we can also build SMFNs. However, it is more complex to deal with these cases because several linear functions should be introduced and there are several ways for doing so. For example, we can construct an α-cut

Table 4 Subjective moving quintuplets

Time	M-Min	M-Min-WMA	OWAWMA	M-Max-WMA	M-Max
5	23	24.8	25.52	27.2	29
6	25	25.96	26.4	27.56	29
7	24	25.26	25.82	27.26	29
8	24	25.02	25.5	26.22	27
9	24	25.6	26.3	27.2	28

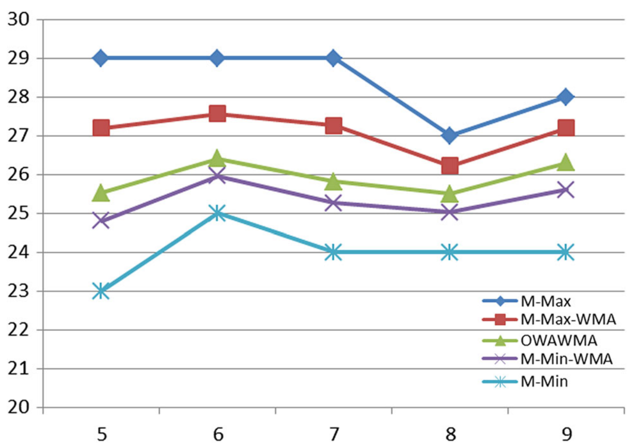

Fig. 3 Moving quintuplets with the OWAWMA operator

representation from the moving minimum and the moving maximum to the OWAWMA and from the M-Min-WMA and the M-Max-WMA to the OWAWMA forming an interval-valued SMFN (IVSMFN). Thus, we get:

- C_α = [M-Min$\{a_i\}$ + (OWAWMA − M-Min$\{a_i\}$) × α, M-Min-WMA + (OWAWMA − M-Min-WMA) × α, M-Max-WMA − (M-Max-WMA − OWAWMA) × α, M-Max$\{a_i\}$ − (M-Max$\{a_i\}$ − OWAWMA) × α].
- With a moving sextuplet, the following representation is obtained:
- C_α = [M-Min$\{a_i\}$ + (OWAWMA$_*$ − M-Min$\{a_i\}$)- × α, M-Min-WMA + (OWAWMA$_*$ − M-Min-WMA) × α, M-Max-WMA − (M-Max-WMA − OWAWMA*) × α, M-Max$\{a_i\}$ − (M-Max$\{a_i\}$ − OWAWMA*) × α].

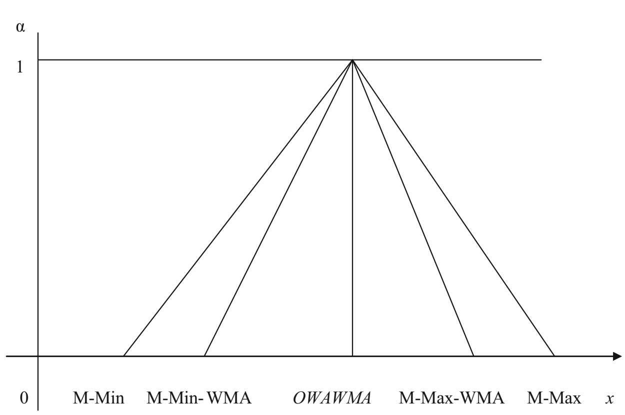

Fig. 4 Moving interval-valued fuzzy number

Note that the interval-valued SMFN formed with the quintuplet can be represented graphically as shown in Fig. 4.

Following Example 3, the following SMTFNs and IVSMFNs can be formed as shown in Table 5 by using the α-cut representation:

Finally, it is worth noting that more complex structures could be developed by using a wide range of families of OWAWMA operators in the analysis. Those presented here represent a general overview of some of the most basic ones.

6 Applicability of the OWAWMA operator and its generalizations

This section briefly describes the applicability of the OWAWMA operator and its different generalizations. The OWAWMA operator can be applied in a wide range of applications. Since it includes the simple moving average, the weighted average and the OWA as particular cases, it can always be reduced to these simple expressions. Therefore, all the previous studies that have used one of them are also applicable with the OWAWMA operator. If there is a need for dealing dynamic information, the OWAWMA operator becomes very useful. But it can be reduced to the simple averages when dealing with simple problems.

Some key areas where it could be implemented are statistics, soft computing, business administration, economics, decision sciences, politics, operational research, engineering and any other discipline that uses statistical methods based on moving averages.

Following (Merigó 2011), we could present some key examples in statistics such as the moving variance, the moving covariance and a moving linear regression. Additionally, it is also possible to develop the multi-person—OWAWMA (MP-OWAWMA) operator and all of its particular cases which are equivalent to those explained in Sect. 3.

Table 5 Construction of SMTFN and IVSMFN

T	SMTFN	IVSMFN
5	[23 + 2.52α, 29 − 3.48α]	[23 + 2.52α, 24.8 + 0.72α, 27.2 − 1.68α, 29 − 3.48α]
6	[25 + 1.4α, 29 − 2.6α]	[25 + 1.4α, 25.96 + 0.44α, 27.56 − 1.16α, 29 − 2.6α]
7	[24 + 1.82α, 29 − 3.18α]	[24 + 1.82α, 25.26 + 0.56α, 27.26 − 1.44α, 29 − 3.18α]
8	[24 + 1.5α, 27 − 1.5α]	[24 + 1.5α, 25.02 + 0.48α, 26.22 − 0.72α, 27 − 1.5α]
9	[24 + 2.3α, 28 − 1.7α]	[24 + 2.3α, 25.6 + 0.7α, 27.2 − 0.9α, 28 − 1.7α]

Next, let us present a simple numerical example regarding how to deal with the aggregation operators presented in the paper. The example presents a sales forecasting problem where the board of directors of a company is analyzing their sales and wants to forecast the future. They are currently selling products in North America, Europe and Asia. Note that the example presented here is a simple one with only three regions. However, it is possible to consider more regions such as Africa and South America, and a smaller analysis by countries or provinces where the company studies the sales in the USA, Canada, UK, France, Germany and so on.

In order to develop the analysis, the company asks a group of three experts to analyze the data in order to form some representative forecasts. The sales of the product considered are established every quarter and follows a cyclical trend. The experts do not know the sales of the data but have some approximations based on subjective data of the last 8 quarters and want to forecast the next eight by always considering the last eight available. Note that each expert has a different opinion when forming the historical sales. The expected sales in North America, Europe and Asia according to each expert are presented in Table 6.

Next, the experts share their opinions and integrate their data into a collective result. For doing so, it is assumed that the three experts are equally important. That is, $U = (1/3, 1/3, 1/3)$. Although the information represents approximations, this is the data that the experts will use for the calculation of the forecasts. The results are presented in Table 7.

Table 6 Approximations of the historical sales of the product according to each expert

Period	1	2	3	4	5	6	7	8
Expert 1								
North America	68	64	73	67	69	62	70	66
Europe	65	72	71	70	66	71	73	70
Asia	49	54	58	56	52	59	56	58
Expert 2								
North America	69	65	73	66	68	62	70	66
Europe	67	71	72	69	66	70	73	70
Asia	48	53	57	58	53	58	56	59
Expert 3								
North America	70	65	73	65	68	62	71	66
Europe	66	72	71	70	66	71	73	70
Asia	48	53	58	57	54	57	56	59

By using the OWAWMA operator and its particular cases, the experts can develop forecasts based on the last eight periods. They forecast the results from today until 8 periods in the future. Note that in Tables 6 and 7, today is period 8. Table 8 presents the forecasts of the three experts. Each of them analyzes the M-Min, M-Max, Min-WMA, Max-WMA, the simple moving average (MA), WMA, OWMA, AMA-WMA, AMA-OWMA and the OWAWMA operator. They assume the following weights: $W = (0.1, 0.1, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1)$ and $V = (0.2, 0.1, 0.1, 0.1, 0.2, 0.1, 0.1, 0.1)$. For this example, the experts believe that the OWA and the weighted average are equally important. That is, both have a weight of 0.5.

By looking to Table 8, the board of directors of the company can get a general idea of the potential sales that may occur in the future taking into account different scenarios from the most pessimistic to the most optimistic one. Usually, the OWAWMA operator should be the final assumption and decision. Therefore, it seems that at the beginning the sales in Asia were quite low but they tend to increase. North America and Europe have similar results although Europe tends to have higher ones because there are more people in this region. However, depending on the scenario considered, the sales may be higher in North America.

The evolution of the results can be studied graphically for the three regions as shown in Fig. 5. Note that for simplicity, only the Min-WMA, the Max-WMA and the OWAWMA operator are considered in Fig. 5.

As we can see, the forecasted sales in Asia are much lower than in North America and Europe. The other two regions are quite similar although more sales are expected in Europe although in some specific situations the sales in North America may be higher. According to the data, the forecasts indicate that the average result in Europe is equivalent to the weighted maximum of North America and the weighted minimum of Europe is a bit higher than the average in North America.

Table 7 Collective results for the historical sales

Period	1	2	3	4	5	6	7	8
Collective results								
North America	69.00	64.67	73.00	66.00	68.33	62.00	70.33	66.00
Europe	66.00	71.67	71.33	69.67	66.00	70.67	73.00	70.00
Asia	48.33	53.33	57.67	57.00	53.00	58.00	56.00	58.67

Table 8 Forecasts provided by the collective results of the three experts

Period	9	10	11	12	13	14	15	16
North America								
M-Min	62.00	62.00	62.00	62.00	62.00	62.00	62.00	62.00
M-Max	73.00	73.00	73.00	73.00	73.00	73.00	73.00	73.00
Min-WMA	64.84	64.24	65.16	64.36	64.59	64.16	64.88	64.46
Max-WMA	70.34	69.74	70.66	69.86	70.09	69.66	70.38	69.96
MA	67.42	67.22	67.54	66.85	66.96	66.79	67.39	67.02
WMA	67.67	66.47	68.31	66.71	67.18	66.31	67.76	66.91
OWMA	67.67	67.40	67.67	67.07	67.18	67.04	67.54	67.25
AMA-WMA	67.55	66.85	67.93	66.78	67.07	66.55	67.58	66.97
AMA-OWMA	67.55	67.31	67.61	66.96	67.07	66.92	67.47	67.14
OWAWMA	67.67	66.94	67.99	66.89	67.18	66.68	67.65	67.08
Europe								
M-Min	66.00	66.00	66.00	66.00	66.00	66.00	66.00	66.00
M-Max	73.00	73.00	73.00	73.00	73.00	73.00	73.00	73.00
Min-WMA	67.52	68.19	68.22	67.94	67.72	68.20	68.30	68.00
Max-WMA	71.02	71.69	71.72	71.44	71.22	71.70	71.80	71.50
MA	69.79	70.27	70.09	69.94	69.97	70.46	70.44	70.12
WMA	69.03	70.37	70.44	69.88	69.44	70.39	70.60	70.00
OWMA	70.03	70.44	70.23	70.07	70.11	70.52	70.51	70.26
AMA-WMA	67.90	68.14	68.05	67.97	67.99	68.23	68.22	68.06
AMA-OWMA	69.91	70.36	70.16	70.01	70.04	70.49	70.48	70.19
OWAWMA	69.53	70.41	70.34	69.98	69.78	70.46	70.56	70.13
Asia								
M-Min	48.33	48.33	48.33	48.33	48.33	48.33	48.33	48.33
M-Max	58.67	58.67	58.67	58.67	58.67	58.67	58.67	58.67
Min-WMA	51.33	52.13	52.38	52.42	51.98	52.44	52.29	52.44
Max-WMA	56.50	57.30	57.55	57.59	57.15	57.61	57.46	57.61
MA	55.25	56.11	56.46	56.31	56.23	56.63	56.46	56.51
WMA	54.33	55.93	56.43	56.50	55.62	56.54	56.25	56.54
OWMA	55.67	56.40	56.71	56.52	56.42	56.76	56.64	56.71
AMA-WMA	54.79	56.02	56.45	56.41	55.93	56.59	56.36	56.53
AMA-OWMA	55.46	56.26	56.59	56.42	56.33	56.70	56.55	56.61
OWAWMA	55.00	56.17	56.57	56.51	56.02	56.65	56.45	56.63

7 Conclusions

This work presents the use of the moving average in a wide range of aggregation operators. Special attention is given to the OWAWA operator because it provides a unified framework between the OWA operator and the weighted average. Thus, the article presents the OWAWMA operator. It is a moving aggregation operator that provides a parameterized family of aggregation operators between the moving minimum and the moving maximum. It uses the weighted average and the OWA operator in the same formulation considering the degree of importance of each concept in the formulation. Thus, it is possible to use weighted moving averages and ordered weighted moving averages in the same formulation being able to consider the degree of or-ness (degree of optimism) and the subjective

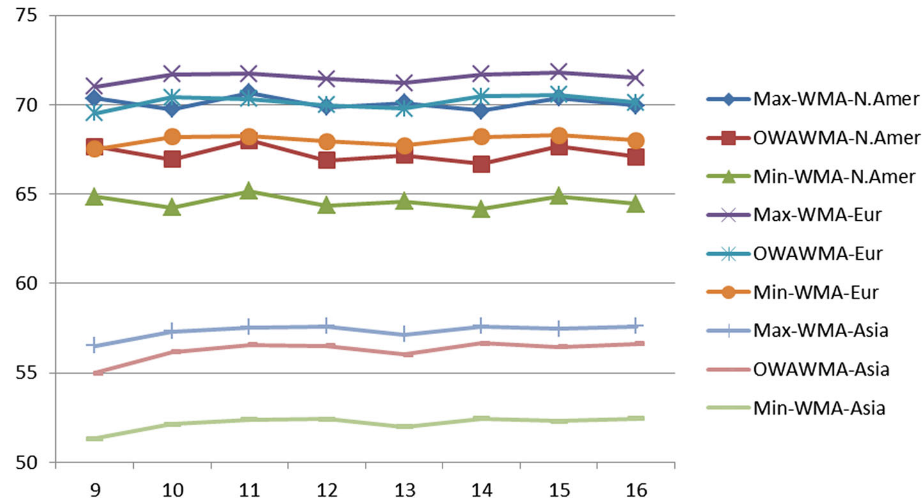

Fig. 5 Sales forecasting evolution for the three regions

information in the same formulation. Moreover, this approach permits to represent better real-world problems because the information can be analyzed in a dynamic and flexible way such as the use of different time series in the analysis. The main advantage of this approach is the possibility of combining subjective data from classical models with the specific attitudinal character that the decision maker has in the problem.

This approach has been further extended by using generalized aggregation operators including the generalized mean and the quasi-arithmetic mean. The quasi-OWAWMA operator has been presented. It generalizes the OWAWMA operator by using quasi-arithmetic means. Its main advantage is that it includes a wide range of particular cases such as the OWGMAWGMA and the OWQ-MAWQMA operator.

Additionally, the paper has presented new techniques for constructing moving interval and fuzzy numbers. The reason is because the OWA aggregation provides a parameterized family of aggregation operators between the minimum and the maximum. Therefore, by using these two extremes we get an interval from where we can analyze central values and develop functions that could form fuzzy numbers. For the case of moving averages, this implies the creation of moving interval and moving fuzzy numbers. The work has presented some representative cases with the OWMA and the OWAWMA operators. Note that with the OWMA operator we are forming attitudinal intervals and fuzzy numbers where the central values are calculated according to the attitude of the decision maker. And with the OWAWMA operator we get subjective attitudinal intervals and subjective attitudinal fuzzy numbers where the central values consider the attitude of the decision maker and some subjective information available in the data.

The study also analyzes briefly the applicability, and we have seen that it is very broad because all the previous studies that use the moving average can be revised and extended with this new approach. Some extensions when dealing with the variance and the covariance have also been presented. Finally, the paper has studied an illustrative example of the main aggregation operators in a multi-expert sales forecasting problem. The example focuses on a comparative process between three regions: USA, Europe and Asia. This approach shows how the results of different intervals can be compared in a dynamical way.

In future research, we expect to develop further extensions to this approach by using more general formulations such as the use of induced aggregation operators, probabilistic information, distance measures and norms. Several applications in real-world problems will be also considered including time-series forecasting, statistics, economics and decision making (Blanco-Mesa et al. 2017).

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Compliance with ethical standards

Conflict of interest The authors declare that they do not have any conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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