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Decision Support

A study of general and security Stackelberg game formulations

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ABSTRACT

In this paper, we analyze different mathematical formulations for general Stackelberg games (GSGs) and Stackelberg security games (SSGs). We consider GSGs in which a single leader commits to a utility maximizing strategy knowing that p possible followers optimize their own utility taking the leader's strategy into account. SSGs are a type of GSG that arise in security applications where the strategies of the leader consist of protecting a subset of targets and the strategies of the p followers consist of attacking a single target. We compare existing mixed integer linear programming (MILP) formulations for GSGs, ranking them according to the tightness of their linear programming (LP) relaxations. We show that SSG formulations are projections of GSG formulations and exploit this link to derive a new SSG MILP formulation that (i) has the tightest LP relaxation known among SSG MILP formulations and (ii) has an LP relaxation that coincides with the convex hull of feasible solutions in the case of a single follower. We present computational experiments empirically comparing the difficulty of solving the formulations in the general and security settings. The new SSG MILP formulation remains computationally efficient as problem size increases.

1. Introduction

Stackelberg games model situations where players strive to optimize their individual objectives in a single sequential encounter. These models assume a player, referred to as the leader, can commit to a strategy that optimizes his utility function and then players that respond to the leader's decision, referred to as followers, take this decision into account when deciding how to optimize their own utility functions. Stackelberg games were introduced to model market competition (von Stackelberg, 2011) and have been used in diverse applications since, such as traffic equilibrium (Krichene, Reilly, Amin, & Bayen, 2014), network toll setting (Labbé, Marcotte, & Savard, 1998), preventing election control (Yin, Vorobeychik, An, & Hazon, 2018), and defense (Brown, Carlyle, Salmerón, & Wood, 2006; Jiang & Liu, 2018).

In this work, we consider normal form Stackelberg games with finite sets of actions for the leader and followers. We refer to these as general Stackebletg games (GSG). The utility functions of GSGs are described by matrices, where each combination of actions for the leader and follower gives a reward value for each participant. Selecting a single action corresponds to a pure strategy, while a mixed strategy corresponds to a probability distribution over the set of actions for the player. Therefore, the utility functions for GSGs are bilinear functions of the players' mixed strategies.

Stackelberg games can be expressed as bilevel optimization problems, where the top level represents the leader's decision problem and includes the followers' responses as the optimal solution to the second level problem (Colson, Marcotte, & Savard, 2007). Mixed integer formulations of GSGs have been devised by incorporating bilinear functions and linearizing second level optimality conditions using integer variables (Bard, 1998). The manner in which the bilinear objectives and second level problem optimality conditions are linearized leads to the different mixed integer linear programming (MILP) formulations considered in this work. For instance, using big M constraints to linearize both the leader's objective and the second level optimality conditions gives rise to the (D2) formulation (Kiekintveld et al., 2009). The (DOBSS) formulation considers a single big M constraint but introduces new variables representing the product of the leader and follower strategies, Paruchuri et al. (2008). Finally, (MIP-p-G) is a formulation without big M constraints (Yin & Tambe, 2012). Which of these MILP formulations of the bilevel stackelberg game problem is more convenient for computational efficiency is an underlying question of this work. When the leader in a GSG faces a single follower, the problem can be solved in polynomial time, see Conitzer and Sandholm (2006). The authors also show that if there are multiple

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followers, then the problem is NP-hard. A solution for the multiple followers problem can be obtained by using the algorithm for the single follower instance on a Harsanyi transformation of the problem, Harsanyi and Selten (1972), which combines the multiple adversaries into a single adversary with exponentially many actions. Solution methods based on mixed integer formulations of the multiple follower problem have been presented, for example, by Jain, Kiekintveld, and Tambe (2011) and Yang, Jiang, Tambe, and Ordóñez (2013).

Recent work has applied Stackelberg games in security settings where a leader has a limited budget to protect a set of targets while a follower aims to attack a single target. In this domain, the payoff matrices are structured with only two payoff values for every participant depending on whether or not the defender strategy protects the target attacked. We refer to problems that have this structure as Stackelberg security games (SSGs), which are introduced in detail in Section 2. Some SSG applications have included assigning Federal Air Marshals to transatlantic flights (Jain et al., 2010), determining randomized port and waterways patrols for the U.S. Coast Guard (Shieh et al., 2012), preventing fare evasion in public transport systems (Yin et al., 2012), protecting endangered wildlife (Yang, Ford, Tambe, & Lemieux, 2014), and patrolling applications to protect networked infrastructure (Karwowski & Mańdziuk, 2019; Li, Qiao, Deng, & Wu, 2019). The SSG models considered are closely related to the interdiction games literature, McMasters and Mustin (1970), specifically when there is a fortification step. Such fortification-interdiction problems are multi-level optimization problems where a defender decides a limited fortification of a network, so that an interdictor (attacker) blocks a number of edges in the network and an operator tries to maximize flow or minimize a path over the network. If the optimal operation response can be subsumed in the interdictor's decision problem, then the problem has the structure of a Stackelberg security game. There are many variants and extensions of such fortification-interdiction games that allow multiple/sequential interdictions and problem specific formulations and algorithms. For instance, a class of generalized interdiction problems and an optimization-based heuristic to solve them are studied in Fischetti, Monaci, and Sinnl (2018). For additional material on fortification-interdiction games see the reviews in Smith and Lim (2008), Snyder et al. (2016) and Fischetti, Ljubic, Monaci, and Sinnl (2019). To the best of our knowledge, however, there is no polyhedral study of different mixed integer optimization formulations that arise due to the bilevel nature of the interaction between the defender and the attacker.

In this paper we focus on the polyhedral analysis of different mixed integer formulations for GSGs and SSGs. In particular, we provide the following four key contributions. First, we provide an exhaustive comparative study of existing MILP formulations for Stackelberg games. Starting from the natural bilevel representation of Stackelberg games, we use well-known integer programming techniques such as Fourier-Motzkin elimination (Dantzig & Eaves, 1973) and Reformulation Linearization Technique (Sherali & Adams, 1994) to derive known MILP formulations. Our study leads to a ranking of these MILP formulations in terms of the strength of their linear programming (LP) relaxations. Second, we explicitly show the relation between the GSG and the SSG formulations by using variable projections from the polyhedra of their LP relaxations. This allows us to extend our study of GSG formulations to the security setting, leading to a comparison of SSG MILP formulations. Third, we derive (SDOBSS$_{q,y,s}$) and (MIP-p-S$_{q,y}$), two new SSG MILP formulations. We show that (MIP-p-S$_{q,y}$) is the MILP formulation with the tightest linear relaxation among SSG formulations. We further show that if we restrict (MIP-p-S$_{q,y}$) to a single attacker type, its LP relaxation provides a complete linear description of the convex hull of its feasible solutions. Fourth, we provide computational experiments that compare solution times of the MILP formulations in both settings. Our experiments show that the formulations with the tightest LP relaxations have faster solution times as problem size increases. In particular (MIP-p-S$_{q,y}$) scales better than competing formulations, being able to tackle larger-sized instances.

The remainder of this paper is organized as follows. In Section 2, we define general and security Stackelberg games. In Section 3, we derive GSG formulations from the literature. We provide theoretical results comparing the formulations presented. In Section 4, we describe and analyze computational experiments for the formulations in Section 3. In Section 5, we present SSG formulations using projections, in the appropriate space of variables, of the formulations in Section 3, and derive (SDOBSS$_{q,y,s}$) and (MIP-p-S$_{q,y}$), new MILP formulations for SSGs. We then extend our theoretical comparisons of the general formulations to the security formulations. In Section 6, we describe and analyze the computational experiments for the security formulations. We conclude with some closing remarks in Section 7.

2. Notation and definition of the problem

In this section, we provide a formal definition of the two types of problems we study.

2.1. General Stackelberg games – GSGs

Let K be the set of p followers. We denote by I the set of leader pure strategies and by J the set of follower pure strategies. The leader has a known probability of facing follower $k \in K$, denoted by $\pi^k \in [0, 1]$. We denote the n-dimensional simplex by $\mathbb{S}^n = \{a \in [0,1]^n : \sum_{h=1}^n a_h = 1\}$. A mixed strategy for the leader consists in a vector $x \in \mathbb{S}^{|I|}$ such that for $i \in I$, x_i is the probability with which the leader plays pure strategy i. Analogously, a mixed strategy for a follower $k \in K$ is a vector $q^k \in \mathbb{S}^{|J|}$ such that, q^k_j is the probability with which follower k replies with pure strategy $j \in J$. The rewards or payoffs for the leader and each follower, resulting from their choice of strategy, are encoded in a different matrix for each follower. These payoff matrices are denoted by (R^k, C^k), where $R^k \in \mathbb{R}^{|I|\times|J|}$ is the leader's reward matrix when facing follower $k \in K$ and $C^k \in \mathbb{R}^{|I|\times|J|}$ is the reward matrix for follower k. The expected reward of the leader and follower k, respectively, can be expressed as follows:

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\pi^k R^k_{ij}x_i q^k_j, \tag{1}$$

$$\sum_{i\in I}\sum_{j\in J} C^k_{ij}x_i q^k_j, \quad \forall k \in K. \tag{2}$$

For all $k \in K$, we define the function $\mathcal{B}^k : \mathbb{S}^{|I|} \longrightarrow \mathbb{S}^{|J|}$ as the function that, given the leader's mixed strategy x, returns a best response q^k for each follower k. The solution concept used in these games is the Strong Stackelberg Equilibrium (SSE), introduced in Leitman (1978) and defined below.

Definition 1. A profile of mixed strategies $(x, \{\mathcal{B}^k(x)\}_{k\in K})$ form an SSE if:

1. The leader always plays a payoff-maximizing strategy:

   $$x^T R^k \mathcal{B}^k(x) \geq x'^T R^k \mathcal{B}^k(x') \qquad \forall x' \in \mathbb{S}^{|I|}, \forall k \in K.$$

2. Each follower always plays a best-response, $\mathcal{B}^k(x) \in F^k(x)$, where $\forall k \in K$,

   $$F^k(x) = \arg\max_{q^k}\{x^T C^k q^k : q^k \in \mathbb{S}^{|J|}\}$$

   is the set of best responses for each follower.

3. Each follower breaks ties optimally in favor of the leader:

$$x^T R^k \mathcal{B}^k(x) \geq x^T R^k q^k \qquad \forall q^k \in F^k(x).$$

An SSE assumes that the follower breaks ties in favor of the leader by choosing, when indifferent between different follower strategies, the strategy that maximizes the payoff of the leader. An SSE is in practice always achievable as the leader can always induce one by selecting a sub-optimal mixed strategy arbitrarily close to the equilibrium, causing the follower to prefer the desired strategy (von Stackelberg, 2011).

Proposition 1below, shows that we can restrict the follower's best response only to pure strategies, as done in Paruchuri et al. (2008), without influencing the SSE solution concept.

Proposition 1. *For any leader strategy x and any $k \in K$, there is a best response to the kth follower's problem that is given by a vector $q^k \in \{0, 1\}^{|J|}$ such that $\sum_{j\in J} q^k_j = 1$.*

Proof. Assume that $B^k(x) = \bar{q}^k \notin \{0, 1\}^{|J|}$. We show that any canonical vector e^{jk} such that $\bar{q}^k_j > 0$, is also a best response vector, *i.e.*, $e^{jk} \in F^k(x)$ and $x^T R^k e^{jk} \geq x^T R^k q^k$ for all $q^k \in F^k(x)$. Since $\bar{q}^k = \sum_{j\in J} \bar{q}^k_j e^{jk}$, with $e^{jk} \in \mathbb{S}^{|J|}$, and $x^T C^k e^{jk} \leq x^T C^k \bar{q}^k$ for all $j \in J$, we have that $x^T C^k \bar{q}^k = \sum_{j\in J} \bar{q}^k_j (x^T C^k e^{jk}) \leq \sum_{j\in J} \bar{q}^k_j (x^T C^k \bar{q}^k) = x^T C^k \bar{q}^k$. This implies that for any $\bar{q}^k_j > 0$ we have $x^T C^k e^{jk} = x^T C^k \bar{q}^k$, giving $e^{jk} \in F^k(x)$. A similar argument shows that for any j such that $\bar{q}^k_j > 0$ we have $x^T R^k e^{jk} = x^T R^k \bar{q}^k$; Hence, e^{jk} is a best response vector. □

In mathematical optimization, Stackelberg games are formulated as bilevel programming (BP) problems (Bracken & McGill, 1973). In BP, the optimization problems have two levels where the top level problem considers some variables that are the optimal solution to another, second level optimization problem. Important BP surveys are those by Kolstad (1985), Savard (1989), Anandalingam and Friesz (1992) and Labbé and Violin (2016). In our setting, the first level problem corresponds to the leader's decision problem and the nested problem corresponds to the follower's decision problem. The following model, (BIL-p-G$_{x,q}$), is a bilevel program for the general Stackelberg game problem:

$$(\text{BIL-}p\text{-G}_{x,q}) \quad \text{Max}_{x,q} \quad \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R^k_{ij} x_i q^k_j \tag{3}$$

$$\text{s.t.} \quad x \in \mathbb{S}^{|I|} \tag{4}$$

$$q^k \in \arg\max_{r^k} \left\{ \sum_{i\in I}\sum_{j\in J} C^k_{ij} x_i r^k_j \right\} \qquad \forall k \in K, \tag{5}$$

$$r^k_j \in \{0, 1\} \qquad \forall j \in J, \forall k \in K, \tag{6}$$

$$\sum_{j\in J} r^k_j = 1 \qquad \forall k \in K. \tag{7}$$

The objective function maximizes the leader's expected reward. Condition (4) characterizes the mixed strategies considered by the leader. The second level problem defined by (5)–(7) indicates that the follower maximizes his own payoff by giving a best response with a pure strategy to the leader's mixed strategy. Recall that such a pure strategy always exists as shown in Proposition 1. If there are multiple optimal strategies for the follower, the main level problem selects the one that benefits the objective of the leader.

2.2. Stackelberg security games – SSGs

In a Stackelberg security game (SSG) the defender allocates security resources to protect a subset of targets. Let J be the set of n targets that could be attacked and assume there are security resources to protect up to $m < n$ of these targets. The set I of defender pure strategies is composed by all $\sum_{i=1}^{m} \binom{n}{i}$ subsets of at most m targets of J that the defender can protect simultaneously. With a slight abuse of notation, we refer to $i \in I$ in this context as both the index running through the set of defender pure strategies I and as $i \subset J$ the corresponding subset of J with at most m targets that are protected by security resources. Similar to GSGs, the elements $j \in J$ constitute the pure strategies of each attacker, which for SSG represents the single target attacked by the follower. In SSGs, payoffs for the players only depend on whether the target attacked is protected or not. This means that many of the strategies have identical payoffs. The authors in Kiekintveld et al. (2009) use this fact to construct a compact representation of the payoffs.

Table 1
Payoff structure in an SSG when target j is attacked by an attacker k.

	Protected	Unprotected
Defender	$D^k(j\|p)$	$D^k(j\|u)$
Attacker	$A^k(j\|p)$	$A^k(j\|u)$

We denote by D^k the utility of the defender when facing an attacker $k \in K$ and by A^k the utility of attacker k. Associated with each target and each player there are two payoffs depending on whether or not the target is protected, see Table 1. Kiekintveld et al. (2009) take advantage of the aforementioned compact representation to define a protection vector c whose components, c_j, represent the frequency with which target j is protected. The components of the vector c satisfy

$$c_j = \sum_{i\in I: j\in i} x_i \qquad \forall j \in J, \tag{8}$$

i.e., the frequency with which target j is protected is expressed as the sum of all probabilities of the strategies that protect that target. Variables q^k_j indicate whether an attacker k strikes a target j.

The defender's and attacker k's expected rewards, are, respectively:

$$\sum_{j\in J}\sum_{k\in K} \pi^k q^k_j \{c_j D^k(j|p) + (1 - c_j) D^k(j|u)\}, \tag{9}$$

$$\sum_{j\in J} q^k_j \{c_j A^k(j|p) + (1 - c_j) A^k(j|u)\}, \qquad \forall k \in K. \tag{10}$$

As with GSGs, such a game can be modeled by means of bilevel programming.

(BIL-p-S$_{x,c,q}$)

$$\text{Max} \quad \sum_{j\in J}\sum_{k\in K} \pi^k q^k_j \{c_j D^k(j|p) + (1 - c_j) D^k(j|u)\}$$

$$\text{s.t.} \quad (4), (8),$$

$$q^k \in \arg\max_{r^k} \left\{ \sum_{j\in J} r^k_j (c_j A^k(j|p) + (1 - c_j) A^k(j|u) \right\} \quad \forall k \in K,$$

$$r^k_j \in \{0, 1\} \quad \forall j \in J, \forall k \in K,$$

$$\sum_{j\in J} r^k_j = 1 \quad \forall k \in K.$$

The objective function maximizes the defender's expected reward. Constraints (4) and (8) characterize the exponentially many mixed strategies considered by the defender and relate them to the frequencies with which targets are protected. The remaining constraints constitute the second level optimization problem which ensures that the attacker maximizes his profit by attacking a single target that is the best response to the defender's selected strategy.

Notice that a more compact formulation – one involving a polynomial number of variables and constraints–can be obtained if projecting out the exponentially many x variables does not lead to exponentially many constraints. This would give a polynomial size formulation involving only the c and the q variables. Given an optimal solution to this compact formulation – an optimal protection vector c and an optimal attack vector q – a probability vector x, solution to this game in extensive form, can be obtained by solving the system of linear inequalities defined by conditions (4) and (8). As this system involves $n+1$ equalities, there exists a solution in which the number of variables x_i with a positive value is not larger than $n+1$, *i.e.*, the output size of an SSG, under extensive form, is polynomial in the input size. See Section 5 for more details.

3. General Stackelberg games – GSGs

In Section 3.1, we present equivalent MILP formulations for the p follower GSG. In Section 3.2 we compare the polyhedra of the LP relaxations for the different formulations.

3.1. General Stackelberg games: single level formulations

Paruchuri et al. (2008) tackle the problem of solving the bilevel formulation presented earlier, (BIL-p-$G_{x,q}$) by using a MILP reformulation. They replace the second level nested optimization problem, described by (5)–(7), by the following set of constraints:

$$\sum_{j\in J} q_j^k = 1 \quad \forall k \in K, \tag{11}$$

$$q_j^k \in \{0,1\} \quad \forall j \in J, \forall k \in K, \tag{12}$$

$$0 \le (s^k - \sum_{i\in I} C_{ij}^k x_i) \le (1-q_j^k)\cdot M \quad \forall j \in J, \forall k \in K, \tag{13}$$

where $s^k \in \mathbb{R}$ for all $k \in K$ and M is an arbitrarily large positive constant. The two inequalities in constraints (13) ensure that $q_j^k = 1$ only for a pure strategy that maximizes the follower's payoff. The problem defined by (3) and (4) and (11)–(13) is referred to as ($\text{QUAD}_{x,q,s}$). It is possible to eliminate the nonlinearity in the objective function of (BIL-p-$G_{x,q}$) by adding additional variables that represent the product between x and q. To be more precise, use $z_{ij}^k = x_i q_j^k$ for all $i \in I$, $j \in J$ and $k \in K$. This gives rise to formulation ($\text{DOBSS}_{q,z,s}$) introduced in Paruchuri et al. (2008):

$$(\text{DOBSS}_{q,z,s}) \quad \text{Max} \quad \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k z_{ij}^k$$

$$\text{s.t.} \quad (11), (12),$$

$$\sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 \quad \forall i \in I, \forall k \in K, \tag{14}$$

$$\sum_{i\in I} z_{ij}^k = q_j^k \quad \forall j \in J, \forall k \in K, \tag{15}$$

$$z_{ij}^k \ge 0 \quad \forall i \in I, \quad \forall j \in J, \forall k \in K, \tag{16}$$

$$0 \le s^k - \sum_{i\in I}\sum_{j'\in J} C_{ij}^k z_{ij'}^k \le (1-q_j^k)\cdot M \quad \forall j \in J, \forall k \in K, \tag{17}$$

$$s \in \mathbb{R}^{|K|}.$$

Alternatively the quadratic term in the objective of (BIL-p-$G_{x,q}$) can be addressed by adding $|K|$ new variables and introducing a second family of constraints involving a big M constant. This gives rise to formulation ($\text{D2}_{x,q,s,f}$) below (a DOBSS variant with 2 big M constraints that appears in (Kiekintveld et al., 2009):

$$(\text{D2}_{x,q,s,f}) \quad \text{Max} \quad \sum_{k\in K} \pi^k f^k \tag{18}$$

$$\text{s.t.} \quad (4), (11)-(13),$$

$$f^k \le \sum_{i\in I} R_{ij}^k x_i + (1-q_j^k)\cdot M \quad \forall j \in J, \forall k \in K, \tag{19}$$

$$s, f \in \mathbb{R}^{|K|} \quad \forall k \in K.$$

Additionally, we project the real variables s^k in constraints (13) and (17) out by using Fourier–Motzkin elimination (Dantzig & Eaves, 1973). This gives rise to constraints:

$$\sum_{i\in I} (C_{ij}^k - C_{i\ell}^k) x_i \le (1-q_\ell^k)\cdot M \quad \forall j, \ell \in J, \forall k \in K, \tag{20}$$

$$\sum_{i\in I}\sum_{j'\in J} (C_{ij}^k - C_{i\ell}^k) z_{ij'}^k \le (1-q_\ell^k)\cdot M \quad \forall j, \ell \in J, \forall k \in K. \tag{21}$$

Replacing (13) by (20) in ($\text{D2}_{x,q,s,f}$) and (17) by (21) in ($\text{DOBSS}_{q,z,s}$) yields ($\text{D2}_{x,q,f}$) and ($\text{DOBSS}_{q,z}$). We analyze the behavior of these last two new formulations compared to that of ($\text{D2}_{x,q,s,f}$) and ($\text{DOBSS}_{q,z,s}$) to see if removing variables s at the expense of adding constraints is worthwile.

Another equivalent MILP formulation for the p-follower GSG can be obtained by replacing constraints (17) with the following set of constraints:

$$\sum_{i\in I} (C_{ij}^k - C_{i\ell}^k) z_{ij}^k \ge 0 \quad \forall j, \ell \in J, \forall k \in K. \tag{22}$$

These constraints are derived by multiplying constraints (20) by q_ℓ^k, reorganizing and replacing the nonlinear terms $x_i q_j^k$ by z_{ij}^k. This leads to (MIP-p-$G_{q,z}$):

$$(\text{MIP-}p\text{-}G_{q,z}) \quad \text{Max} \quad \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k z_{ij}^k$$

$$\text{s.t.} \quad (11), (12), (14)-(16), (22).$$

The linear relaxation of (MIP-p-$G_{q,z}$) appears in Yin and Tambe (2012). The MILP formulation is a p-follower extension to the single follower formulation (MIP-1-$G_{q,z}$), due to Conitzer and Korzhyk (2011). Formal proofs that the formulations seen thus far are equivalent MILP formulations, *i.e.*, that they are valid for the p-follower GSG, appear in Paruchuri et al. (2008) for ($\text{DOBSS}_{q,z,s}$) and Paruchuri et al. (2008) and Kiekintveld et al. (2009) for ($\text{D2}_{x,q,s,f}$). These proofs show that each of them is equivalent to ($\text{QUAD}_{x,q,s}$). The equivalence of ($\text{DOBSS}_{z,q}$) and ($\text{D2}_{x,q,f}$) is obtained from the Fourier-Motzkin elimination procedure (Dantzig & Eaves, 1973). The equivalence proof for (MIP-p-$G_{q,z}$) is analogous to the proof used to show the equivalence for ($\text{DOBSS}_{q,z,s}$) and is omitted here.

Paruchuri et al. (2008) state that the big M constants used are arbitrarily large. To be as computationally competitive as possible, we provide the tightest value for each big M constant in the formulations discussed thus far.

Proposition 2. *The tightest values for the positive constants M are:*

1. *In (19),* $M = \max_{i\in I}\{\max_{\ell\in J} R_{i\ell}^k - R_{ij}^k\}\ \forall j \in J, \forall k \in K.$
2. *In (13) and (17),* $M = \max_{i\in I}\{\max_{\ell\in J} C_{i\ell}^k - C_{ij}^k\}\ \forall j \in J, \forall k \in K.$
3. *In (20) and (21),* $M = \max_{i\in I}\{C_{ij}^k - C_{i\ell}^k\}, \forall j, \ell \in J, \forall k \in K.$

3.2. Comparison of the formulations

Given a formulation F, we denote by $\overline{\text{F}}$ its linear (continuous) relaxation and by $\mathcal{P}(\overline{\text{F}})$ the polyhedral feasible region of $\overline{\text{F}}$. Further, let $Q = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : Ax + Bz \le d\}$. Then the projection of Q into the x-space, denoted $Proj_x Q$, is the polyhedron given by $Proj_x Q = \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^m$ for which $(x, z) \in Q\}$, see Pochet and Wolsey (2006).

First, we introduce an additional formulation which we denote by ($\text{DOBSS}_{x,q,z,s,f}$). This formulation is equivalent to ($\text{DOBBS}_{q,z,s}$), in the sense that the values of their LP relaxations coincide. In this formulation, we introduce variables f^k for all $k \in K$ to rewrite the objective function so that it matches the objective function of ($\text{D2}_{x,q,s,f}$). We also add variables x_i for all $i \in I$ by rewriting (14) as $\sum_{j\in J} z_{ij}^k = x_i$ for all $i \in I$ and all $k \in K$. Using this last condition, we can simplify (17)–(13). The formulation ($\text{DOBSS}_{x,q,z,s,f}$) is as follows.

$$(\text{DOBSS}_{x,q,z,s,f}) \quad \text{Max} \quad \sum_{k\in K} \pi^k f^k$$

$$\text{s.t.} \quad (11) - (13), (15), (16),$$

$$f^k = \sum_{i\in I}\sum_{j\in J} R_{ij}^k z_{ij}^k \quad \forall k \in K, \tag{23}$$

$$\sum_{j\in J} z_{ij}^k = x_i \quad \forall i \in I, \forall k \in K, \tag{24}$$

$$s \in \mathbb{R}^{|K|}.$$

Further, note that from the Fourier Motzkin elimination procedure we have that

$$\mathcal{P}(\overline{\text{D2}_{x,q,f}}) = Proj_{x,q,f}\mathcal{P}(\overline{\text{D2}_{x,q,s,f}}) \text{ and,}$$

$$\mathcal{P}(\overline{\text{DOBSS}_{q,z}}) = Proj_{q,z}\mathcal{P}(\overline{\text{DOBSS}_{q,z,s}}).$$

Proposition 3. *$Proj_{x,q,s,f}\mathcal{P}(\overline{\text{DOBSS}_{x,q,z,s,f}}) \subseteq \mathcal{P}(\overline{\text{D2}_{x,q,s,f}})$. Further, there exist instances for which the inclusion is strict.*

Proof. Note that all the constraints of $\mathcal{P}(\overline{\text{D2}_{x,q,s,f}})$ can be found in the description of $\mathcal{P}(\overline{\text{DOBSS}_{x,q,z,s,f}})$ except for constraints (4) and (19). Constraints (4) are implied by constraints (11), (15), (16) and (24).

Further, the projection of $\mathcal{P}(\overline{\text{DOBSS}_{x,q,z,s,f}})$ on the (x, q, s, f)-space can be obtained by applying Farkas' Lemma Farkas (1902). Constraints (15), (16), (23) and (24) are the only ones involving variables z_{ij}^k and are separable by $k \in K$. For a fixed $k \in K$ the projection is given by:

$$A^k = \{(x, q, f) : \alpha f^k + \sum_{i\in I}\beta_i x_i + \sum_{j\in J}\gamma_j q_j^k \geq 0 \ \forall(\alpha, \gamma, \beta) :$$

$$\alpha R_{ij}^k + \beta_i + \gamma_j \geq 0 \ \forall i \in I, \forall j \in J\} \tag{25}$$

For a fixed $j \in J$, define $\alpha = -1$, $\beta_i = R_{ij}^k$ for all $i \in I$, $\gamma_j = 0$ and $\gamma_\ell = \max_{i\in I}(R_{i\ell}^k - R_{ij}^k)$ for all $\ell \in J$ with $\ell \neq j$. This definition of the parameters satisfies $\alpha R_{ij}^k + \beta_i + \gamma_j \geq 0$ for all $i \in I$, $j \in J$. Substituting these parameters in the generic constraints of A^k yields

$$f^k \leq \sum_{i\in I} R_{ij}^k x_i + \sum_{\ell\in J:\ell\neq j}\max_{i\in I}(R_{i\ell}^k - R_{ij}^k)q_\ell^k \qquad \forall j \in J, \forall k \in K. \tag{26}$$

Constraints (26) imply constraints (19) for the tight value of M provided in Proposition 2 since for all $j \in J$ and $k \in K$,

$$\sum_{\ell\in J:\ell\neq j}\max_{i\in I}(R_{i\ell}^k - R_{ij}^k)q_\ell^k \leq \max_{i\in I}\left\{\max_{\ell\in J} R_{i\ell}^k - R_{ij}^k\right\}$$

$$\sum_{\ell\in J:\ell\neq j} q_\ell^k = \max_{i\in I}\left\{\max_{\ell\in J} R_{i\ell}^k - R_{ij}^k\right\}(1 - q_j^k).$$

This proves the inclusion. To show that the inclusion may be strict, consider the following example where $|I| = |J| = 3$ and $|K| = 1$. Let the payoff matrix for the game be

$$(R, C) = \begin{pmatrix} (1,0) & (0,0) & (0,0) \\ (0,0) & (1,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}$$

and consider the point defined by $x = (1, 0, 0)^t$, $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$, $s = 10$ and $f = 2/3$. Such a point is feasible for $(\overline{\text{D2}_{x,q,s,f}})$ but violates constraints (26) for $j = 2$ and is therefore infeasible for $Proj_{x,q,s,f}\mathcal{P}(\overline{\text{DOBSS}_{x,q,z,s,f}})$. □

Next, we compare the polyhedra $\mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z}})$ and $Proj_{q,z}\mathcal{P}(\overline{\text{DOBSS}_{q,z,s}})$.

Theorem 1. *$\mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z}}) \subseteq \mathcal{P}(\overline{\text{DOBSS}_{q,z}}) = Proj_{q,z}\mathcal{P}(\overline{\text{DOBSS}_{q,z,s}})$. Further, there exist instances for which the inclusion is strict.*

Proof. The description of $\mathcal{P}(\overline{\text{DOBSS}_{q,z}})$ differs from that of $\mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z}})$ by only one set of constraints: (21) must hold instead of (22). Hence, the remainder of the proof consists in showing that (21) are implied by (11), (14)–(16), (22) and the nonnegativity of the q variables. The LHS of (21) can be rewritten as:

$$\sum_{i\in I}(C_{ij}^k - C_{i\ell}^k)z_{i\ell}^k + \sum_{i\in I}\sum_{j'\in J:j'\neq\ell}(C_{ij}^k - C_{i\ell}^k)z_{ij'}^k$$

$$\leq \sum_{i\in I}\sum_{j'\in J:j'\neq\ell}(C_{ij}^k - C_{i\ell}^k)z_{ij'}^k, \text{ using (22),}$$

$$\leq \max_{i\in I}\{C_{ij}^k - C_{i\ell}^k\}\sum_{j'\in J:j'\neq\ell}\sum_{i\in I} z_{ij'}^k$$

$$\leq M\sum_{j'\in J:j'\neq\ell} q_{j'}^k, \text{ given Proposition 2 and (15)}$$

$$= M(1 - q_\ell^k), \text{ by (11).}$$

To show that the inclusion may be strict consider the p-follower GSG between a leader and a fixed follower $k \in K$ where the payoff bimatrix is:

$$(R^k, C^k) = \begin{pmatrix} (0,1) & (1,0) \\ (0,0) & (0,0) \end{pmatrix}$$

The point with coordinates $x = (1/2, 1/2)^t$, $q^k = (1/2, 1/2)^t$ and

$$z^k = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

has an objective value of 1/4 and is feasible in $\mathcal{P}(\overline{\text{DOBSS}_{q,z}})$. However it is not a feasible point in $\mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z}})$ as it does not verify constraints (22) when $j = 2$ and $\ell = 1$. □

From an interpretation point of view, ($\text{MIP-}p\text{-G}_{q,z}$) can be seen as the result of applying Reformulation Linearization Technique (RLT) Sherali and Adams (1994) to ($\text{DOBSS}_{q,z}$). Indeed, by multiplying both sides of constraints (20) by variable q_ℓ^k and noticing that $q_\ell^k(1 - q_\ell^k) = 0$ since q is binary, one obtains $\sum_{i\in I}(C_{ij}^k - C_{i\ell}^k)x_i q_\ell^k \leq 0$ which, once linearized by introducing variables $z_{i\ell}^k$, yields (22).

For a given formulation F, we denote its optimal value by $v(\text{F})$ and the optimal value of its LP relaxation by $v(\overline{\text{F}})$. Since ($\text{D2}_{x,q,s,f}$) and ($\text{DOBSS}_{x,q,s,f}$) and ($\text{DOBSS}_{q,z}$) and ($\text{MIP-}p\text{-G}_{q,z}$) have the same objective function, the following corollary holds.

Corollary 1. *$v(\overline{\text{MIP-}p\text{-G}_{q,z}}) \leq v(\overline{\text{DOBSS}_{q,z}}) = v(\overline{\text{DOBSS}_{x,q,s,f}}) \leq v(\overline{\text{D2}_{x,q,s,f}})$.*

Finally, when (MIP-p-G) is restricted to a single follower type, Conitzer and Korzhyk (2011) showed that the integrality costraints are redundant, *i.e.*, the remaining constraints in (MIP-1-G) provide a complete linear description of the convex hull of feasible solutions.

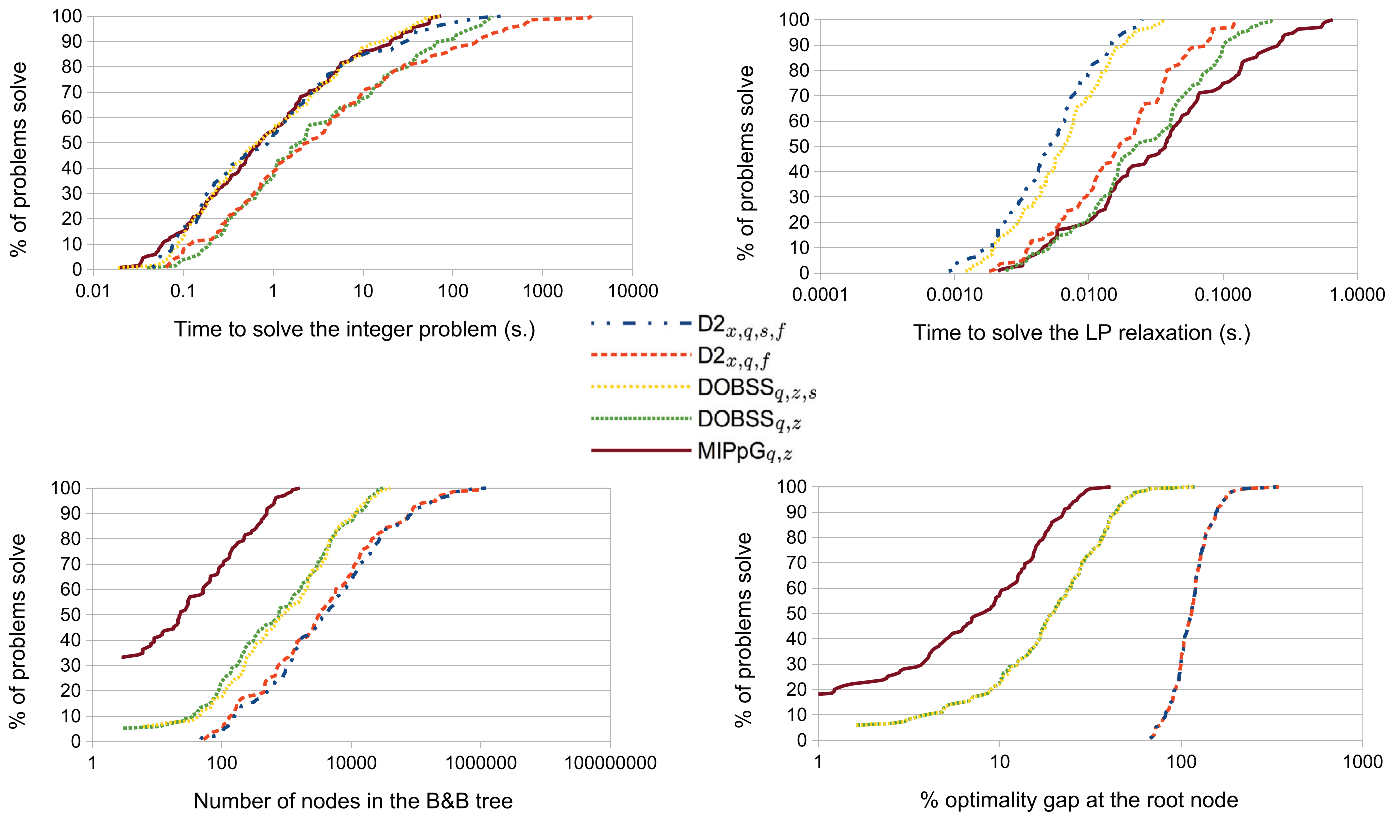

Fig. 1. GSGs: $|I| \in \{10, 20, 30\}$, $|J| \in \{10, 20, 30\}$, $|K| \in \{2, 4, 6\}$ – without variability.

4. Computational experiments for GSGs

Here, we present computational experiments for the formulations in Section 3. The machine used for these experiments is an Intel Core i7-4930K CPU, 3.40 gigahertz, equipped with 64 gigabytes of RAM, 6 cores, 12 threads and running the Ubuntu operating system release 12.10 (kernel Linux 3.5.0-41-generic). The experiments were coded in the programming language Python and GUROBI version 6.5.1 was the optimization solver used with a 3 hour solution time limit.

The instances solved in the computational experiments are randomly generated. We consider two different ways of randomly generating the payoff matrices for the leader and the different follower types. First, we consider matrices where all the elements are randomly generated between 0 and 10 and second, we consider matrices where 90% of the values are between 0 and 10 but we allow for 10% of the data to deviate between 0 and 100. In the first case we say that there is no variability in the payoff matrices, in the sense that all the data is uniformly distributed, whereas in the second case, we refer to the payoff matrices as matrices with variability.

A general Stackelberg game instance is defined by three parameters: $|I|$, the number of leader pure strategies, $|J|$, the number of follower pure strategies and $|K|$, the number of follower types. For the purpose of these experiments, we have considered instances where $|I| \in \{10, 20, 30\}$, $|J| \in \{10, 20, 30\}$ and $|K| \in \{2, 4, 6\}$. For each instance size, 5 instances are generated without variability in the payoff matrices and 5 are generated with variability. In total, we consider 135 instances without variability and 135 instances with variability.

Performance profiles summarize our results, with respect to the following 4 measures: total running time employed to solve the integer problem, running time employed to solve the linear relaxation of the integer problem, total number of nodes explored in the branch and bound (B&B) tree and percentage optimality gap at the root node. The percentage optimality gap at the root node is calculated by comparing the optimal values of the formulation and of its LP relaxation: $\frac{v(\bar{F})-v(F)}{v(F)} \cdot 100$. A performance profile graph plots the total percentage of problems solved for each value of these measures.

We study the behavior of ($D2_{x,q,s,f}$), ($D2_{x,q,f}$), ($DOBSS_{q,z,s}$), ($DOBSS_{q,z}$) and (MIP-p-$G_{q,z}$). Figs. 1 and 2 compare the performance profiles when the payoff matrices are generated without variability and with variability, respectively.

We observe that the instances where variability is introduced in the payoff matrices solve faster than those where no variability is considered. When there is no variability, ($DOBSS_{q,z,s}$) and (MIP-p-$G_{q,z}$) are the two most competitive formulations. ($D2_{x,q,s,f}$) can also be solved efficiently for the mid-range instances but slows down for the more difficult instances. Introducing variability in the payoff matrices, however, leads to a dominance of (MIP-p-$G_{q,z}$) with ($DOBSS_{q,z,s}$) coming in a close second and ($D2_{x,q,s}$) becoming non-competitive for these instances. Regarding the time spent solving the linear relaxation of the problems, formulation (MIP-p-$G_{q,z}$) is the hardest to solve due to the fact that is has the most variables and constraints, $\mathcal{O}(|K||J|^2)$. On the other hand, ($D2_{x,q,s,f}$), with $\mathcal{O}(|K||J|)$ variables and constraints, is the fastest. With respect to the number of nodes and gap percentage, our theoretical findings are corroborated: (MIP-p-$G_{q,z}$) is the tightest formulation and therefore uses the fewest nodes. This is even more the case when variability is introduced.

Table 2 summarizes the mean percentage optimality gap at the root node obtained across the instances solved. Finally, note that the formulations obtained through Fourier-Motzkin, ($D2_{x,q,f}$) and ($DOBSS_{q,z}$), explore slightly less nodes in the B&B tree than their counterparts, ($D2_{x,q,s,f}$) and ($DOBSS_{q,z,s}$), but because of the increase

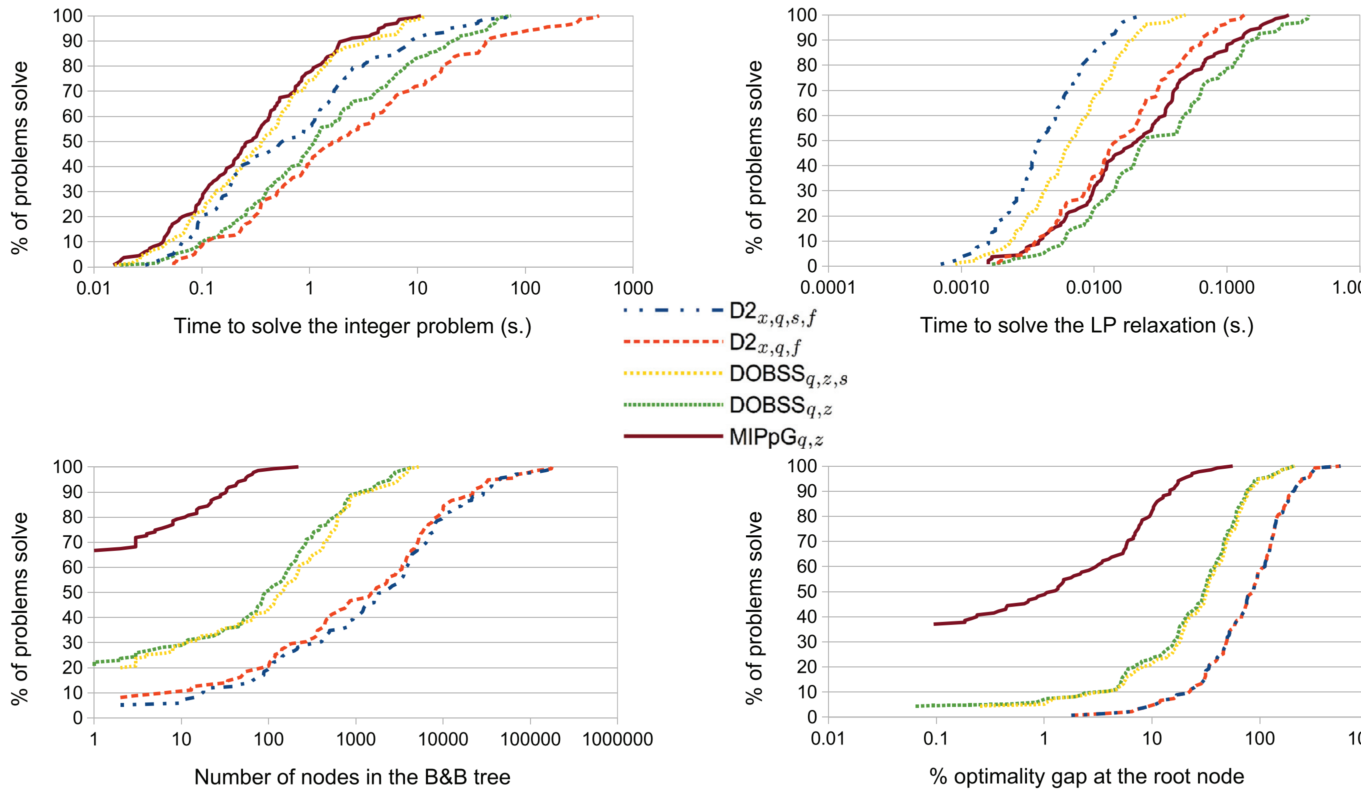

Fig. 2. GSGs: $|I| \in \{10, 20, 30\}$, $|J| \in \{10, 20, 30\}$, $|K| \in \{2, 4, 6\}$ – with variability.

Table 2
Mean percentage optimality gap at the root node recorded for GSG formulations.

	($D2_{x,q,s,f}$)	($DOBSS_{q,z,s}$)	($MIP\text{-}p\text{-}G_{q,z}$)
Mean % opt. gap (no variability)	117.68	23.01	9.94
Mean % opt. gap (with variability)	103.44	40.74	5.17
Total mean % opt. gap	110.56	31.88	7.56

in the number of constraints, the time to solve each linear relaxation increases. This increases the overall solution time of the Fourier–Motzkin formulations.

5. Stackelberg security games-SSGs

In this section, we derive three SSG formulations: ($ERASER_{c,q,s,f}$), due to Kiekintveld et al. (2009), and ($SDOBSS_{q,y,s}$) and ($MIP\text{-}p\text{-}S_{q,y}$). We derive these formulations by exploring the inherent link between the general setting, considered up to now and the security setting, defined in Section 2.2. In this setting, the defender pure strategies $i \in I$ correspond to the different ways in which up to m targets can be protected simultaneously. With a slight abuse of notation, $i \in I$ refers both to the index running through the set of pure strategies I and to the subset of at most m targets protected by pure strategy $i \in I$. Recall that the payoff matrices of SSGs satisfy:

$$R^k_{ij} = \begin{cases} D^k(j|p) & \text{if } j \in i \\ D^k(j|u) & \text{if } j \notin i \end{cases} \tag{27}$$

$$C^k_{ij} = \begin{cases} A^k(j|p) & \text{if } j \in i \\ A^k(j|u) & \text{if } j \notin i \end{cases} \tag{28}$$

The payoff for the leader that commits to a pure strategy $i \in I$ and a follower of type $k \in K$ responds by selecting strategy $j \in J$ is either a reward if pure strategy $i \in I$ protects attacked target $j \in J$, or, a penalty if strategy i does not protect target j. The same argument explains the link between payoffs for the attackers.

5.1. Stackelberg security games: single level formulations

The first formulation we derive is based on ($D2_{x,q,s,f}$). Consider ($D2_{c,x,q,s,f}$), an extended description of ($D2_{x,q,s,f}$) where we introduce the c variables through constraints (8) (see Section 2.2). We further use relations (27) and (28) to adapt the payoff structure:

$$(D2_{c,x,q,s,f})$$

$$\text{Max} \sum_{k \in K} \pi^k f^k$$

$$\text{s.t.} (4), (8), (11), (12),$$

$$0 \le s^k - A^k(j|p)c_j - A^k(j|u)(1 - c_j) \le (1 - q^k_j) \cdot M \quad \forall j \in J, \forall k \in K, \tag{29}$$

$$f^k \le D^k(j|p)c_j + D^k(j|u)(1 - c_j) + (1 - q^k_j) \cdot M \quad \forall j \in J, \forall k \in K, \tag{30}$$

$$s, f \in \mathbb{R}^K.$$

This extended formulation is equivalent to ($D2_{x,q,s,f}$), because, even though they are defined in different spaces of variables, the value of their LP relaxations coincide.

The formulation above has a large number of non-negative variables since in the security setting, the set I of all defender pure strategies is exponential in the number of targets as it contains all subsets of at most m targets of J that the defender can protect simultaneously. In order to avoid having exponentially many non-negative variables in our formulation, we project out variables x_i, $i \in I$, from the formulation. Note that only constraints (4) and (8) involve said variables.

Proposition 4. *Consider the following two sets:*

$$A = Proj_c\left\{(x, c) \in \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} : (4), (8)\right\}$$

$$B=\left\{c\in\mathbb{R}^{|J|}:\sum_{j\in J}c_j\le m,\ c_j\in[0,1]\ \forall j\in J\right\}$$

Then, $A=B$.

Proof. Observe first that using Farkas' Lemma Farkas (1902):

$$A=\left\{c\in\mathbb{R}^{|J|}:\sum_{j\in J}\alpha_jc_j+\alpha_{|J|+1}\ge 0\ \forall\alpha\in\mathbb{R}^{|J|+1}:\right.$$

$$\left.\sum_{j\in J:j\in i}\alpha_j+\alpha_{|J|+1}\ge 0\ \forall i\in I:|i|\le m\text{ and }\alpha_{|J|+1}\ge 0\right\},$$

Thus $A\subseteq B$. Indeed, the following $2|J|+1$ vectors in $\mathbb{R}^{|J|+1}$:

$$\forall j\in J,\ e^j\in\mathbb{R}^{|J|+1}:e^j_j=1,\ e^j_k=0\ \forall k\in J:k\neq j\text{ and }e^j_{|J|+1}=0,$$

$$\forall j\in J,\ f^j\in\mathbb{R}^{|J|+1}:f^j_j=-1,\ f^j_k=0\ \forall k\in J:k\neq j\text{ and }f^j_{|J|+1}=1\text{ and}$$

$$g\in\mathbb{R}^{|J|+1}:g_j=-1\ \forall j\in J\text{ and }g_{|J|+1}=m,$$

satisfy $\sum_{j\in J:j\in i}\alpha_j+\alpha_{|J|+1}\ge 0$ and $\alpha_{|J|+1}\ge 0$. Additionally, when we substitute the above vectors into the generic constraints defining A, they yield all the constraints defining B.

To show that $A=B$, it remains to show that any other inequality

$$\sum_{j\in J}\alpha_jc_j+\alpha_{|J|+1}\ge 0 \tag{31}$$

such that α satisfies

$$\sum_{j\in J:j\in i}\alpha_j+\alpha_{|J|+1}\ge 0\qquad\forall i\in I:|i|\le m\text{ and }\alpha_{|J|+1}\ge 0, \tag{32}$$

is dominated by some nonnegative linear combination of the constraints defining B.

First, note that we can restrict our attention to constraints such that $\alpha_j\le 0$ for all $j\in J$. If there exists $\hat{j}\in J$ such that $\alpha_{\hat{j}}>0$, since α must satisfy (32) and $|i\setminus\{\hat{j}\}|\le|i|\le m$, it follows that $\bar{\alpha}$ with $\bar{\alpha}_{\hat{j}}=0$ and $\bar{\alpha}_j=\alpha_j$ for all $j\in J\setminus\{\hat{j}\}$ also satisfies (32) and since $c\ge 0$, we have that

$$\sum_{j\in J}\bar{\alpha}_jc_j+\bar{\alpha}_{|J|+1}\le\sum_{j\in J}\alpha_jc_j+\alpha_{|J|+1}.$$

Therefore, the constraint defined by α is dominated by the constraint defined by $\bar{\alpha}$. We thus distinguish two cases of α satisfying (32):

Case 1. $|\{j:\alpha_j<0\}|=k\le m$, and
Case 2. $|\{j:\alpha_j<0\}|=k>m$.

In Case 1, by considering a linear combination of inequalities $c_j\le 1$ for $1\le j\le k$ with respective weights $-\alpha_j\ge 0$, we obtain that:

$$0\le\sum_{j=1}^k\alpha_jc_j-\sum_{j=1}^k\alpha_j\le\sum_{j\in J}\alpha_jc_j+\alpha_{|J|+1},$$

since $\alpha_j=0$ for all $j>k$ and α satisfies (32) for $i=\{1,\dots,k\}$.

For Case 2, assume *w.l.o.g* that $\alpha_1\le\alpha_2\le\dots\le\alpha_k<0$ and $\alpha_j=0$ for all $j>k$. Then, build a linear combination of inequality $\Sigma_{j\in J}c_j\le m$ with weight $-\alpha_m\ge 0$ and inequalities $c_j\le 1$ for $1\le j\le m$ with respective weights $\alpha_m-\alpha_j\ge 0$. The valid inequality thus obtained is:

$$0\le\sum_{j=1}^m\alpha_jc_j+\sum_{j>m}\alpha_mc_j-\sum_{j=1}^m\alpha_j\le\sum_{j\in J}\alpha_jc_j$$
$$-\sum_{j=1}^m\alpha_j,\text{ since }\alpha_j\ge\alpha_m\text{ for all }j>m$$
$$\le\sum_{j\in J}\alpha_jc_j+\alpha_{|J|+1},$$

since α satisfies (32) for $i=\{1,\dots,m\}$. $\square$

Proposition 4 leads to the following formulation based on $(\mathrm{D2}_{c,x,q,s,f})$:

$(\mathrm{ERASER}_{c,q,s,f})$

$$\begin{array}{lll}
\text{Max} & \sum_{k\in K}\pi^kf^k & \\
\text{s.t.} & (11),(12),(29),(30), & \\
 & \sum_{j\in J}c_j\le m, & \\
 & 0\le c_j\le 1 & \forall j\in J,\\
 & s,f\in\mathbb{R}^K. &
\end{array}$$

The above formulation involves a polynomial number of variables and constraints and was presented in Kiekintveld et al. (2009). The next result is also an immediate consequence of Proposition 4.

Corollary 2. *$Proj_{c,q,s,f}\mathcal{P}(\overline{\mathrm{D2}_{c,x,q,s,f}})=\mathcal{P}(\overline{\mathrm{ERASER}_{c,q,s,f}})$.*

We now derive new SSG formulations based on $(\mathrm{DOBSS}_{q,z,s})$ and $(\text{MIP-}p\text{-G}_{q,z})$. We first present extended descriptions of both formulations by considering $y^k_{\ell j}$ variables satisfying:

$$y^k_{\ell j}=\sum_{i\in I:\ell\in i}z^k_{ij}\qquad\forall j,\ell\in J,\forall k\in K. \tag{33}$$

We use (27) and (28) to adapt the payoffs to the security setting leading to:

$(\mathrm{DOBSS}_{q,z,y,s})$

$$\text{Max}\quad\sum_{j\in J}\sum_{k\in K}\{\pi^k(D^k(j|p)y^k_{jj}+D^k(j|u)(q^k_j-y^k_{jj}))\} \tag{34}$$

$$\text{s.t.}\quad(11),(12),(14)-(16),(33),$$
$$0\le s^k-A^k(j|p)\sum_{j'\in J}y^k_{jj'}-$$
$$A^k(j|u)(1-\sum_{j'\in J}y^k_{jj'})\le(1-q^k_j)\cdot M\qquad\forall j\in J,\forall k\in K, \tag{35}$$

$$s\in\mathbb{R}^{|K|}. \tag{36}$$

$$\begin{array}{lll}
(\text{MIP-}p\text{-G}_{q,z,y}) & \text{Max} & \sum_{j\in J}\sum_{k\in K}\pi^k(D^k(j|p)y^k_{jj}+D^k(j|u)(q^k_j-y^k_{jj}))\\
 & \text{s.t.} & (11),(12),(14)-(16),(33),\\
 & & A^k(j|p)y^k_{jj}+A^k(j|u)(q^k_j-y^k_{jj})\\
 & & \quad -A^k(\ell|p)y^k_{\ell j}-A^k(\ell|u)(q^k_j-y^k_{\ell j})\ge 0\\
 & & \forall j,\ell\in J,\forall k\in K.\qquad(37)
\end{array}$$

Further, consider the following constraints:

$$\sum_{j\in J}y^k_{\ell j}=\sum_{j\in J}y^1_{\ell j}\qquad\forall\ell\in J,\forall k\in K, \tag{38}$$

and let us define the following polyhedra C and D:

$$C := \{(q, z, y, s) \in [0,1]^{|K||J|} \times [0,1]^{|K||I||J|} \times [0,1]^{|K||J|^2} \times \mathbb{R}^{|K|} :$$

$$(11), (15), (16), (33), (35), (36), (38)\}$$

$$D := \{(q, z, y) \in [0,1]^{|K||J|} \times [0,1]^{|K||I||J|} \times [0,1]^{|K||J|^2} : (11), (15), (16), (33), (35), (36), (38)\}$$

Lemma 1. *$C \supseteq \mathcal{P}(\overline{\text{DOBSS}_{q,z,y,s}})$ and $D \supseteq \mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z,y}})$*

Proof. Consider constraints (14) and sum over all $i \in I$ such that $\ell \in i$:

$$\sum_{\substack{i \in I:\\ \ell \in i}} \sum_{j \in J} z_{ij}^k = \sum_{\substack{i \in I:\\ \ell \in i}} \sum_{j \in J} z_{ij}^1 \qquad \forall \ell \in J, \forall k \in K. \tag{39}$$

Applying (33)–(39) yields (38) and the result follows. □

We now project the z variables from the larger polyhedra C and D. Said variables only appear in constraints (15), (16) and (33).

Lemma 2. *Consider the following two sets;*

$$\mathcal{X} = Proj_{q,y}\left\{(q, z, y) \in \mathbb{R}^{|K||J|^2+|K||J|+|I||J||K|} : (15), (16), (33)\right\}$$

$$\mathcal{Y} = \{(q, y) \in \mathbb{R}^{|K||J|^2+|K||J|} : \sum_{\ell \in J} y_{\ell j}^k \le m q_j^k \ \forall j \in J, \forall k \in K,$$

$$0 \le y_{\ell j}^k \le q_j^k \ \forall j, \ell \in J, \forall k \in K\}$$

Then, $\mathcal{X} = \mathcal{Y}$.

Proof. Note that constraints (15), (16) and (33) can be treated independently for each $k \in K$ and each $j \in J$. First consider the case where $q_{\hat{j}}^{\hat{k}} = 0$ for $\hat{j} \in J$ and $\hat{k} \in K$. Constraints (15) then imply that for all $i \in I$, $z_{i\hat{j}}^{\hat{k}} = 0$ and constraints (33) force $y_{\ell\hat{j}}^{\hat{k}} = 0$ for all $\ell \in J$ and the result holds. For all $j \in J$, $k \in K$ such that $q_j^k \neq 0$, consider $x_i = z_{ij}^k / q_j^k$ and $c_\ell = y_{\ell j}^k / q_j^k$ and apply Proposition 4. The result follows. □

Consider $Proj_{q,y,s}C$ and $Proj_{q,y}D$ as the feasible regions of the linear relaxations of two MILP formulations–($\text{SDOBSS}_{q,y,s}$) and (MIP-p-$\text{S}_{q,y}$)–where we maximize the objective function (34) under the additional requirement that the q variables be binary. Hence, we present ($\text{SDOBSS}_{q,y,s}$), a security formulation based on ($\text{DOBSS}_{q,z,y,s}$),

($\text{SDOBSS}_{q,y,s}$)

$$\text{Max} \quad \sum_{j \in J} \sum_{k \in K} \pi^k (D^k(j|p) y_{jj}^k + D^k(j|u)(q_j^k - y_{jj}^k))$$

$$\text{s.t.} \quad (11), (12), (35), (38)$$

$$\sum_{\ell \in J} y_{\ell j}^k \le m q_j^k \qquad \forall j \in J, \forall k \in K, \tag{40}$$

$$0 \le y_{\ell j}^k \le q_j^k \qquad \forall j, \ell \in J, \forall k \in K, \tag{41}$$

$$s \in \mathbb{R}^{|K|}.$$

And we also present (MIP-p-$\text{S}_{q,y}$), a security formulation based on (MIP-p-$\text{G}_{q,z,y}$),

$$(\text{MIP-}p\text{-S}_{q,y}) \quad \text{Max} \quad \sum_{j \in J} \sum_{k \in K} \pi^k (D^k(j|p) y_{jj}^k + D^k(j|u)(q_j^k - y_{jj}^k))$$

$$\text{s.t.} \quad (11), (12), (35), (38)$$

The following corollaries are an immediate consequence of Lemmas 1 and 2.

Corollary 3. *$Proj_{q,y,s}\mathcal{P}(\overline{\text{DOBSS}_{q,z,y,s}}) \subseteq \mathcal{P}(\overline{\text{SDOBSS}_{q,y,s}})$.*

Corollary 4. *$Proj_{q,y}\mathcal{P}(\overline{\text{MIP-}p\text{-G}_{q,z,y}}) \subseteq \mathcal{P}(\overline{\text{MIP-}p\text{-S}_{q,y}})$.*

In addition, note that if we restrict (MIP-p-$\text{G}_{q,z,y}$) to a single type of follower, constraints (14) disappear and one thus obtains the following corollary.

Corollary 5. *$Proj_{q,y}\mathcal{P}(\overline{\text{MIP-1-G}_{q,z,y}}) = \mathcal{P}(\overline{\text{MIP-1-S}_{q,y}})$*

The above corollary immediately leads to the following theorem.

Theorem 2. *($\overline{\text{MIP-1-S}_{q,y}}$) is a linear description of the convex hull of feasible solutions for the Stackelberg security game with a single type of attacker.*

Proof. The result follows from Corollary 5 and from Conitzer and Korzhyk (2011) showing that ($\overline{\text{MIP-1-G}_{q,z}}$) is a linear description for general games. □

As in general games, we use Fourier–Motzkin elimination on constraints (29) and (35) to project out the s variables from formulations ($\text{ERASER}_{c,q,s,f}$) and ($\text{SDOBSS}_{q,y,s}$), respectively. This leads to the following two families of inequalities:

$$(A^k(j|p) - A^k(j|u))c_j + (A^k(\ell|u) - A^k(\ell|p))c_\ell + A^k(j|u) - A^k(\ell|u) \le (1 - q_\ell^k) \cdot M \ \forall j, \ell \in J, \forall k \in K, \tag{42}$$

$$(A^k(j|p) - A^k(j|u)) \sum_{h \in J} y_{jh}^k + (A^k(\ell|u) - A^k(\ell|p)) \sum_{h \in J} y_{\ell h}^k + A^k(j|u) - A^k(\ell|u) \le (1 - q_\ell^k) \cdot M \qquad \forall j, \ell \in J, \forall k \in K, \tag{43}$$

Replacing constraints (29) by (42) in ($\text{ERASER}_{c,q,s,f}$) and (35) by (43) in ($\text{SDOBSS}_{q,y,s}$) leads to ($\text{ERASER}_{c,q,f}$) and ($\text{SDOBSS}_{q,y}$).

In the same spirit as Proposition 2, we present the following proposition, establishing the tightest values for the big M constants in the formulations seen so far:

Proposition 5. *The tightest values for the positive constants M are:*

1. *In (30), $M = \max_{\ell \in J}\{D^k(\ell|p), D^k(\ell|u)\} - \min\{D^k(j|p), D^k(j|u)\}$, $\forall j \in J, k \in K$.*
2. *In (29), (35), $M = \max_{\ell \in J}\{A^k(\ell|p), A^k(\ell|u)\} - \min\{A^k(j|p), A^k(j|u)\}$, $\forall j \in J, k \in K$.*
3. *In (42), (43), $M = \max\{A^k(j|p), A^k(j|u)\} - \min\{A^k(\ell|p), A^k(\ell|u)\}$, $\forall j, \ell \in J, k \in K$.*

5.2. Comparison of the formulations

First, we introduce an additional formulation which we denote by ($\text{SDOBSS}_{c,q,y,s,f}$). This formulation is equivalent to ($\text{SDOBSS}_{q,y,s}$), in the sense that the value of their LP relaxations coincide. In this formulation, we introduce variables f^k for all $k \in K$ to rewrite the objective function so that it matches the objective function of ($\text{ERASER}_{c,q,s,f}$). We also add variables c_ℓ for all $\ell \in J$ and rewrite constraints (38) as $\sum_{j \in J} y_{\ell j}^k = c_\ell$ for all $\ell \in J$ and all $k \in K$. Using this last condition we can simplify (35) to (29). The formulation ($\text{SDOBSS}_{c,q,y,s,f}$) is as follows.

$$(\text{SDOBSS}_{c,q,y,s,f}) \quad \text{Max} \quad \sum_{k \in K} \pi^k f^k$$

$$\text{s.t.} \quad (11), (12), (29), (40), (41),$$

$$f^k = \sum_{j \in J} \{y_{jj}^k (D^k(j|p) - D^k(j|u)) + q_j^k D^k(j|u)\} \quad \forall k \in K \tag{44}$$

$$\sum_{j\in J} y^k_{\ell j} = c_\ell \quad \forall \ell \in J, \forall k \in K, \tag{45}$$
$$s \in \mathbb{R}^{|K|}.$$

Note that

$\mathcal{P}(\overline{\text{ERASER}_{c,q,f}}) = Proj_{c,q,f}\mathcal{P}(\overline{\text{ERASER}_{c,q,s,f}})$ and

$\mathcal{P}(\overline{\text{SDOBSS}_{q,y}}) = Proj_{q,y}\mathcal{P}(\overline{\text{SDOBSS}_{q,y,s}})$.

Proposition 6. *$Proj_{c,q,s,f}\mathcal{P}(\overline{\text{SDOBSS}_{c,q,y,s,f}}) \subseteq \mathcal{P}(\overline{\text{ERASER}_{c,q,s,f}})$. Further, there exist instances for which the inclusion is strict.*

Proof. The projection of $\mathcal{P}(\overline{\text{SDOBSS}_{c,q,y,s,f}})$ onto the (c, q, s, f)-space is obtained by applying Farkas' Lemma. Constraints (40)-(41) and (44)-(45) are the only ones involving variables $y^k_{\ell j}$ and are separable by $k\in K$. For a fixed $k\in K$, the projection is given by:

$$A^k = \{(c,q,f) : \alpha(f^k - \sum_{j\in J} D^k(j|u)q^k_j) + \sum_{\ell\in J}\beta_\ell c_\ell + m\sum_{j\in J}\gamma_j q^k_j + \sum_{j\in J}\sum_{\ell\in J}\delta_{\ell j} q^k_j \geq 0$$

$$\forall(\alpha,\beta,\gamma,\delta) : \gamma,\delta \geq 0,\ \beta_\ell + \gamma_j + \delta_{\ell,j} \geq 0\ \forall \ell, j\in J : \ell \neq j, \text{ and}$$

$$\alpha(D^k(j|c) - D^k(j|u)) + \beta_j + \gamma_j + \delta_{\ell j} \geq 0\ \forall j\in J\} \tag{46}$$

Consider, for each $k\in K$, the following set B^k:

$$B^k = \{(c,q,f) : c_\ell \leq \sum_{j\in J} q^k_j, \quad \forall \ell \in J, \tag{47}$$

$$c_\ell \geq 0, \quad \forall \ell \in J, \tag{48}$$

$$\sum_{\ell\in J} c_\ell \leq m\sum_{j\in J} q^k_j, \tag{49}$$

$$f^k \leq c_j(D^k(j|p) - D^k(j|u)) + \sum_{\ell\in J:\ell\neq j} q^k_\ell D^k(\ell|p) + q^k_j D^k(j|u) \qquad \forall j\in J, \tag{50}$$

$$q^k_j \geq 0 \quad \forall j \in J, \forall k\in K.\}$$

Let us see that $A^k\subseteq B^k$ for all $k\in K$. First note that if we set $\alpha = 0$, the following definitions of the parameters β, γ and δ comply with the conditions in (46):

$$\beta = e^h, \gamma = \{0\}_{j\in J}, \delta = \{0\}_{\ell,j\in J},\ \forall h\in J,$$

$$\beta = -e^\ell, \gamma = \{0\}_{j\in J}, \delta_\ell = \{1\}_{j\in J},\ \forall \ell\in J,$$

$$\beta = \{-1\}_{\ell\in J}, \gamma = \{1\}_{j\in J}, \delta = \{0\}_{\ell,j\in J},$$

$$\beta = \{0\}_{\ell\in J}, \gamma = \{0\}_{j\in J}, \delta_1 = \{e^j\},\ \forall j\in J.$$

Substituting these valid parameters in the generic constraints in A^k, produces all of the constraints in B^k except (50). Further, for a fixed $j\in J$, consider $\alpha = -1$, $\beta_\ell = 0$ and $\gamma_\ell = \frac{1}{m}(D^k(\ell|p) - D^k(\ell|u))$ for all $\ell\in J$ such that $\ell\neq j$, $\beta_j = D^k(j|p) - D^k(j|u)$ and $\gamma_j = 0$. Finally, set $\delta_{\ell j} = 0$ for all $\ell, j\in J$. This definition of parameters is valid as it satisfies the conditions in (46). Substituting in the generic constraints in A^k yields (50).

It remains to show that for all $k\in K$, constraints (50) imply (30) for the tight value of M shown in Proposition 5. The implication holds because

$$\sum_{\ell\in J:\ell\neq j} q^k_\ell D^k(\ell|p) \leq \max_{\ell\in J}\{D^k(\ell|p)\} \sum_{\ell\in J:\ell\neq j} q^k_\ell = (1-q^k_j)\max_{\ell\in J}\{D^k(\ell|p)\} \qquad \forall j\in J, \forall k\in K.$$

Hence, $Proj_{c,q,s,f}\mathcal{P}(\overline{\text{SDOBSS}_{c,q,y,s,f}}) \subseteq \mathcal{P}(\overline{\text{ERASER}_{c,q,s,f}})$. To show that the inclusion may be strict, consider the following example where $m = 1$, $|J| = 3$ and $|K| = 1$. Let the reward and penalty matrices for the defender and attacker be $D(\cdot|p) = [1,0,0]$, $D(\cdot|u) = [0,0,0]$, $A(\cdot|p) = [0,0,0]$ and $A(\cdot|u) = [0,0,0]$. Consider the point defined by $q = (\frac{1}{3},\frac{1}{3},\frac{1}{3})^t$, $c = (1,0,0)^t$, $s = 10$ and $f = 2/3$. Such a point is feasible for $(\overline{\text{ERASER}_{c,q,s,f}})$ but violates constraints (50) when $j = 2$ and is therefore infeasible for $Proj_{c,q,f,s}\mathcal{P}(\overline{\text{SDOBSS}_{c,q,y,s,f}})$. □

Based on Theorem 1 we can present the following theorem comparing the polyhedra $\mathcal{P}(\overline{\text{MIP-}p\text{-S}_{q,y}})$ and $Proj_{q,y}\mathcal{P}(\overline{\text{SDOBSS}_{q,y,s}})$:

Theorem 3. *$\mathcal{P}(\overline{\text{MIP-}p\text{-S}_{q,y}}) \subseteq \mathcal{P}(\overline{\text{SDOBSS}_{q,y}}) = Proj_{q,y}\mathcal{P}(\overline{\text{SDOBSS}_{q,y,s}})$.*

Proof. The inclusion is a consequence of Theorem 1, the relations between the payoffs described in (27) and (28) and the relation between the z and y variables described in (33).

To show that the inclusion may be strict, consider the following game. We set $m = 2$, $|J| = 2$ and $|K| = 1$. The reward and penalty payoff matrices for both the defender and the attacker are given by $D(\cdot|p) = [1,0]$, $D(\cdot|u) = [0,0]$, $A(\cdot|p) = [0,0]$ and $A(\cdot|u) = [0,1]$. Additionally, the point with coordinates

$$c^t = (1/2, 1/2),\ q^t = (1/2, 1/2) \text{ and } y^k = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

has an objective value of 1/4 and is a valid feasible solution of $\mathcal{P}(\overline{\text{SDOBSS}_{q,y}})$. However, it is not feasible in $\mathcal{P}(\overline{\text{MIP-}p\text{-S}_{q,y}})$ as it does not verify constraints (37) when $j = 1$ and $\ell = 2$. □

Observe that (MIP-p-S$_{q,y}$) can be obtained by applying RLT Sherali and Adams (1994) to (SDOBSS$_{q,y}$). Multiplying both sides of constraints (42) by variable q^k_ℓ and noticing that $q^k_\ell(1 - q^k_\ell) = 0$, since q^k_ℓ is binary, one obtains constraints that once linearized, by introducing variables $y^k_{\ell j}$, yield (37).

Since (ERASER$_{c,q,s,f}$) and (f-SDOBSS$_{c,q,s,f}$) and (SDOBSS$_{q,y}$) and (MIP-p-S$_{q,y}$) have the same objective function, the following corollary holds.

Corollary 6. $v(\overline{\text{MIP-}p\text{-S}_{q,y}}) \leq v(\overline{\text{SDOBSS}_{q,y}}) = v(\overline{\text{SDOBSS}_{c,q,s,f}}) \leq v(\overline{\text{ERASER}_{c,q,s,f}})$.

6. Computational experiments for SSGs

Our security experiments are run on randomly generated instances. For each instance, four payoff matrices have to be generated that satisfy $D^k(\cdot|p) \geq D^k(\cdot|u)$ and $A^k(\cdot|u) \geq A^k(\cdot|p)$. We consider two ways of generating these matrices. First, we generate matrices where the values for the penalty matrices ($D^k(\cdot|u)$ and $A^k(\cdot|p)$) are randomly generated between 0 and 5 and all values for the reward matrices ($D^k(\cdot|p)$ and $A^k(\cdot|u)$) are randomly generated between 5 and 10. We refer to these as matrices with no variability. Second, we consider an alternative where 90% of the values for the penalty matrices are randomly generated between 0 and 5 (between 5 and 10 for the reward matrices) and 10% of the values for the penalty matrices are randomly generated between 0 and 50 (between 50 and 100 for the reward matrices). We refer to these as matrices with variability. We use a solution limit of 3 hours.

A Stackelberg security game instance is defined by $|J|$, the number of targets, $|K|$ the number of attacker types and m, the number of security resources available to the defender. Recall from the computational experiments for GSGs that using payoff matrices

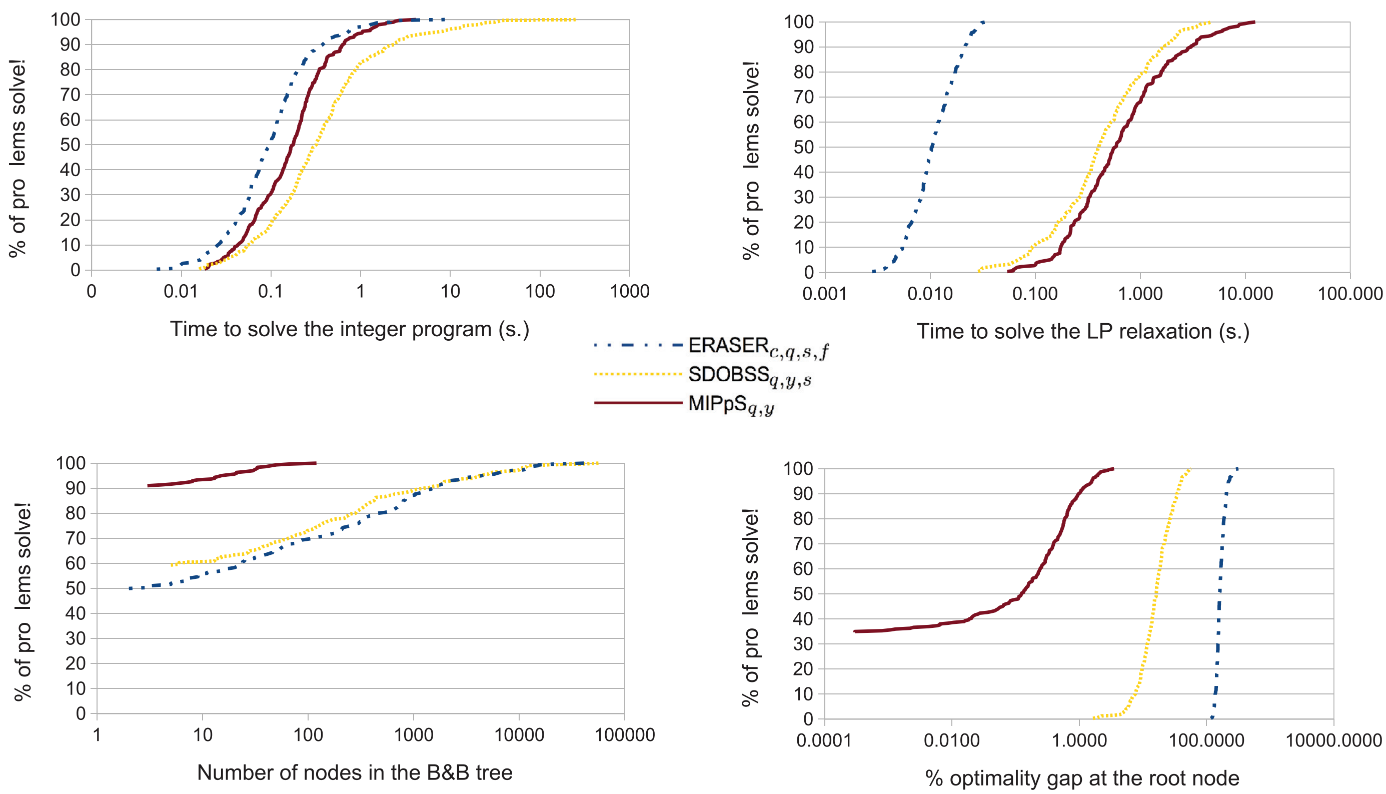

Fig. 3. SSGs: $K = \{6, 8, 10, 12\}$, $J = \{30, 40, 50, 60, 70\}$ – with variability.

with variability, amounts to endowing the game with more structure, thus making it somewhat easier to solve. We have encountered the same phenomenon in SSGs. For games whose payoff matrices have variability, we have considered $J = \{30, 40, 50, 60, 70\}$, $K = \{6, 8, 10, 12\}$ and we have allowed m to be either 25%, 50% or 75% of the number of targets. For games whose payoff matrices do not have variability we have had to be less ambitious in order to solve all instances to optimality within the stipulated time limit and have considered $J = \{10, 20, 30, 40, 50\}$, $K = \{2, 4, 6, 8\}$ while still considering m to be either 25%, 50% or 75% of the number of targets. In either case, for each instance size, we generate 5 random instances as described above. In total, we consider 300 randomly generated instances.

We study the behavior of (ERASER$_{c,q,s,f}$), (SDOBSS$_{q,y,s}$) and (MIP-p-S$_{q,y}$). For the sake of clarity, we no longer consider the Fourier-Motzkin formulations (ERASER$_{c,q,f}$) and (SDOBSS$_{q,y}$). Performance-wise, (ERASER$_{c,q,s,f}$) and (SDOBSS$_{q,y,s}$) compare to their Fourier-Motzkin formulations in a similar way to how (D2$_{x,q,s,f}$) and (DOBSS$_{q,z,s}$) compared to theirs in Section 4 (results not shown). We plot performance profile graphs in Figs. 3 and 4. Note that for the experiments with variability, (ERASER$_{c,q,s,f}$) is the fastest formulation for most of the instances. However, we see that for the more difficult instances, its solution time increases significantly, eventually surpassing the solution time of (MIP-p-S$_{q,y}$). This indicates that for these instances (ERASER$_{c,q,s,f}$) ceases to be competitive and (MIP-p-S$_{q,y}$) is the formulation that solves the fastest. As for the instances whose payoff matrices have no variability, and are thus harder to solve, we observe that (ERASER$_{c,q,s,f}$) outperforms the running time of the other two formulations for 80% of the instances. However, for the most difficult instances, (MIP-p-S$_{q,y}$) is faster than the other two formulations. For the last 5% of the instances, (ERASER$_{c,q,s,f}$) is the worst formulation. In terms of size of the formulations, (ERASER$_{c,q,s,f}$) is the formulation with the least number of constraints and variables: $\mathcal{O}(|J||K|)$. Observe that (MIP-p-S$_{q,y}$) and (SDOBSS$_{q,y,s}$) have $\mathcal{O}(|J|^2|K|)$ constraints and variables. Thus, these formulations have larger LP relaxations and thus take longer time to solve than (ERASER$_{c,q,s,f}$) does. However, Figs. 3 and 4 confirm our theoretical findings: (MIP-p-S$_{q,y}$) has the tightest LP relaxation and this translates into a clear dominance with respect to node usage in the B&B tree.

Based on our results, we observe a trend that indicates that for difficult instances, particularly in the case of payoff matrices with no variability, one could expect (ERASER$_{c,q,s,f}$) and (SDOBSS$_{q,y,s}$) to perform very poorly compared to (MIP-p-S$_{q,y}$). To analyze this, we consider instances where the payoff matrices have no variability and where $K = \{6, 8, 10, 12\}$, $J = \{30, 40, 50, 60, 70\}$ and m is 25%, 50% and 75% of the targets. We generate 5 random instances for each size. In addition, for practical reasons, we consider a time limit of 30 minutes. The computational results for these instances are shown in Fig. 5. Note that (MIP-p-S$_{q,y}$) is able to solve 95% of the 300 instances within the stipulated time limit, outperforming (SDOBSS$_{q,y,s}$) and (ERASER$_{c,q,s,f}$), which are only able to solve 56% and 45% of the instances, respectively, within the same time frame. For the 45% of instances which can be solved by the three formulations, we observe that (MIP-p-S$_{q,y}$) offers a much tighter percentage optimality gap than the other two formulations. Because of this, the node usage in the B&B tree is significantly smaller in (MIP-p-S$_{q,y}$) compared to (ERASER$_{c,q,s,f}$) and (SDOBSS$_{q,y,s}$). Table 3 records the mean percentage optimality gap at the root node across all the instances for the three formulations under study. Observe that $\overline{\text{(MIP-}p\text{-S}_{q,y})}$ is significantly tighter than the LP relaxations of the other formulations. We may thus conclude that for the payoff matrices without variability, (MIP-p-S$_{q,y}$) is the fastest formulation for the most difficult instances. On the other hand, (ERASER$_{c,q,s,f}$) is the fastest formulation when we endow the security game with further structure by allowing matrices to experience variability. Even then, (ERASER$_{c,q,s,f}$) looses ground to (MIP-p-S$_{q,y}$). This is due to the fact that (MIP-p-S$_{q,y}$) has the

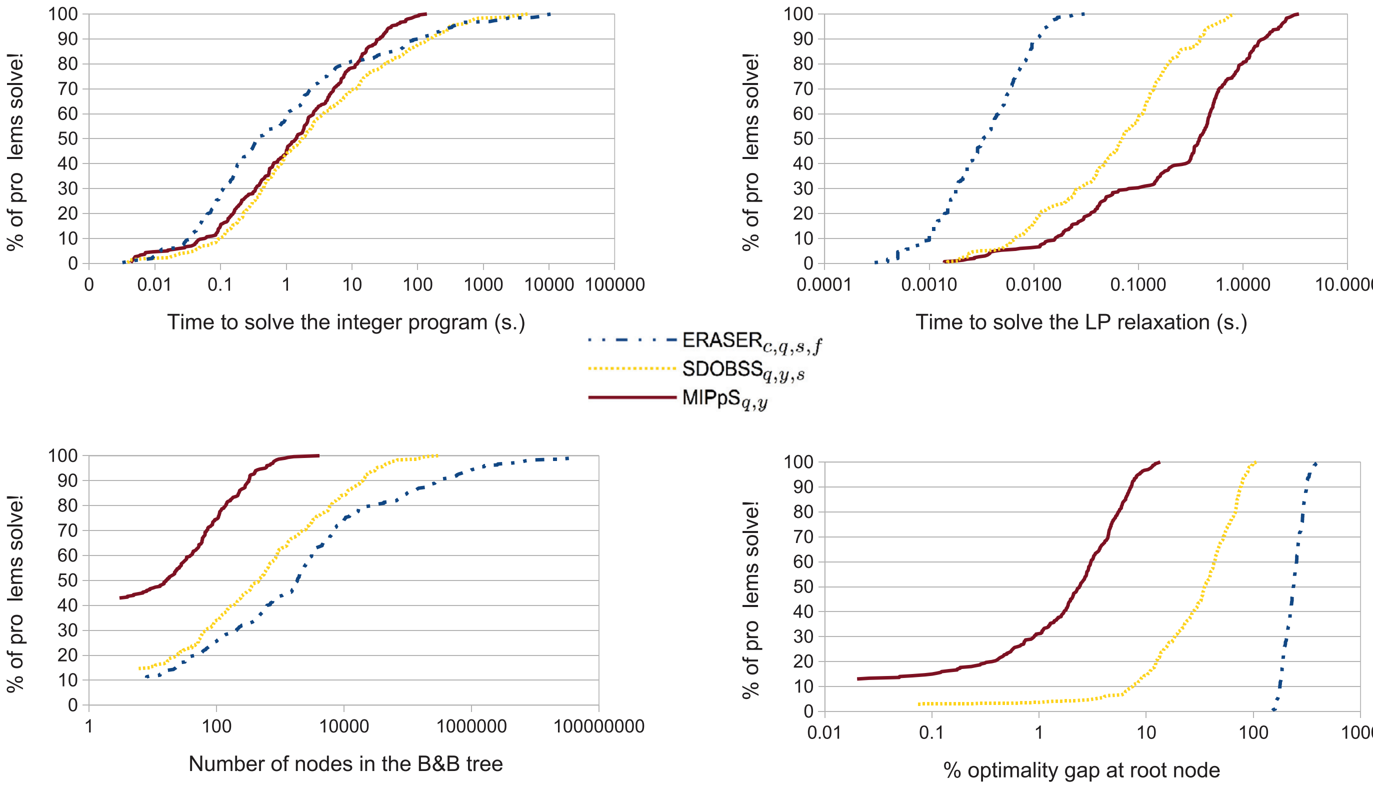

Fig. 4. SSGs: $K = \{2, 4, 6, 8\}, J = \{10, 20, 30, 40, 50\}$ – without variability.

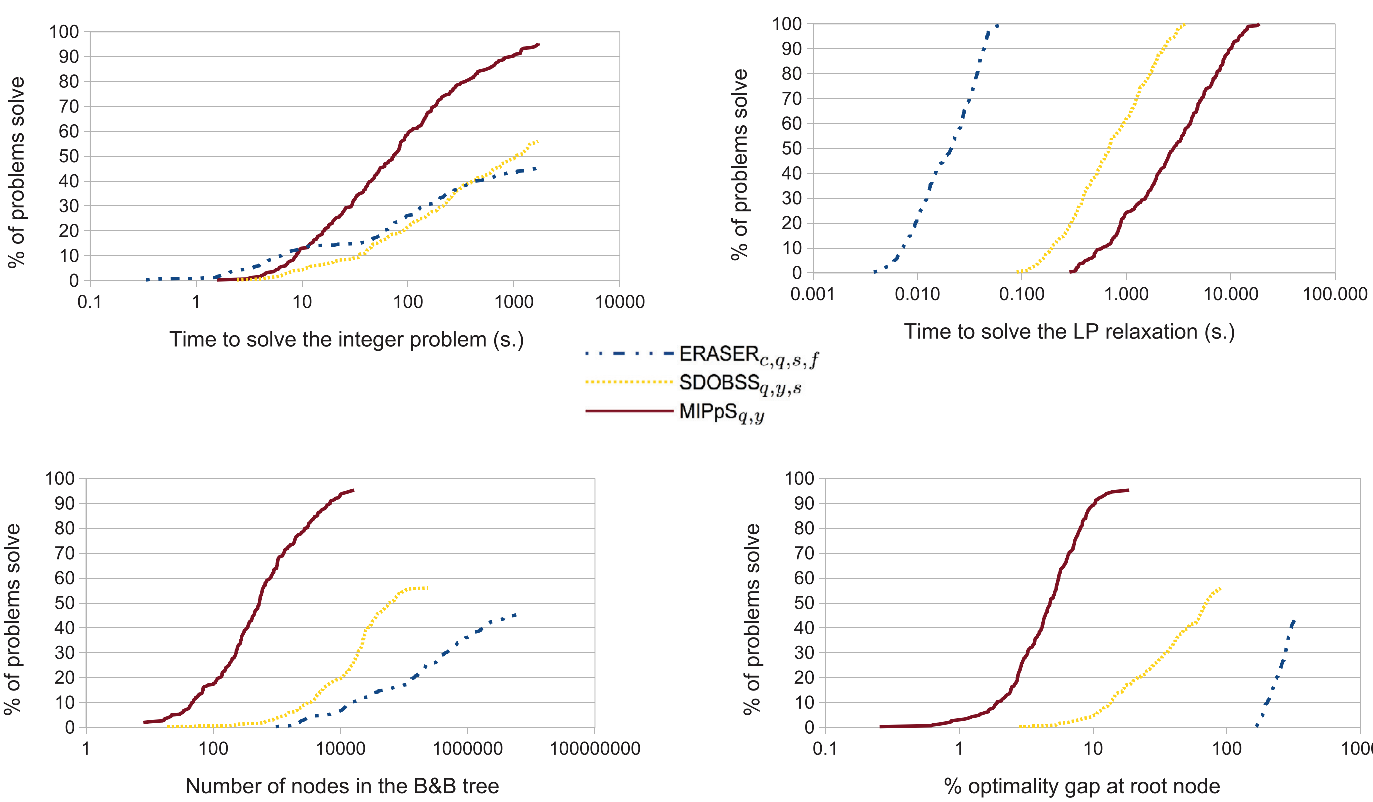

Fig. 5. SSGs: $K = \{6, 8, 10, 12\}, J = \{30, 40, 50, 60, 70\}$ – without variability.

Table 3
Meanpercentage optimality gap at the root node recorded for SSG formulations.

	($\text{ERASER}_{c,q,s,f}$)	($\text{SDOBSS}_{q,y,s}$)	(MIP-p-$S_{q,y}$)
Mean % opt. gap (no variability)	241.26	38.87	3.09
Mean % opt. gap (with variability)	168.37	18.66	0.35
Total mean % opt. gap	204.82	28.76	1.72

tightest LP relaxation. The quality of the upper bound obtained from ($\overline{\text{MIP-}p\text{-}S_{q,y}}$) translates into a smaller B&B tree and this translates into reaching optimality of the integer problem faster in many cases.

7. Conclusions and future work

In this paper, we consider Stackelberg games in two different settings. We first analyze the general Stackelberg setting, which models a hierarchical competitive game between different agents, and the specific Stackelberg security setting, where an agent must secure subsets of targets from attackers.

In the general setting, we have studied known MILP formulations and have ordered them with respect to the strength of their linear relaxations. We have presented a formal theoretical link between GSG formulations and SSG formulations involving the projection of variables. Exploiting this link has allowed us to i) derive two new SSG MILP formulations ($\text{SDOBSS}_{q,y,s}$) and (MIP-p-$S_{q,y}$); and ii) extend our study of GSG formulations to SSG formulations, leading to a ranking of the security formulations with respect to the strength of their linear relaxations, where (MIP-p-S) has been shown to be the strongest SSG formulation. Further, we have shown its single type of attacker restriction, (MIP-1-$S_{q,y}$), to be an ideal formulation.

Our computational studies have shown that (MIP-p-$G_{q,z}$) and (MIP-p-$S_{q,y}$), the tightest formulations in each setting, are highly competitive with respect to solving time. Further, in the case of (MIP-p-S), we have seen it scales significantly better than competing formulations when tackling instances with no variability in their payoff structure. Formulation (MIP-p-S) represents a significant theoretical and practical improvement over previously existing SSG formulations.

However, the obvious bottleneck, at this time, is solving the tighter but larger LP relaxations for (MIP-p-$G_{q,z}$) and (MIP-p-$S_{q,y}$). The main challenge is to provide an efficient way of solving these tight formulations. It is our contention that this can be done by exploiting the inherent problem structure in the Stackelberg paradigm to develop either decomposition or cutting plane approaches.

While this paper focuses on the polyhedral analysis of general normal form Stackelberg games and Stackelberg security games, similar polyhedral analyses could be carried out on specific bilevel security problems in order to develop efficient algorithms for such problems. In particular, extensions to problems that consider multiple attacks by followers, dynamic settings, imperfect information, or non-rational response would be interesting lines of future research.

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