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Berezin-type operators on the cotangent bundle of a nilpotent group

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Abstract
We define and study coherent states, a Berezin–Toeplitz quantization and covariant symbols on the product $\Xi := \mathsf{G} \times \mathfrak{g}^\sharp$ between a connected simply connected nilpotent Lie group and the dual of its Lie algebra. The starting point is a Weyl system codifying the natural canonical commutation relations of the system. The formalism is meant to complement the quantization of the cotangent bundle $T^\sharp\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^\sharp$ by pseudo-differential operators, to which it is connected in an explicit way. Some extensions are indicated, concerning τ-quantizations and variable magnetic fields.

Keywords Nilpotent group · Lie algebra · Coherent states · Pseudo-differential operator · Symbol · Berezin quantization

Mathematics Subject Classification Primary 22E25 · 47G30; Secondary 22E45 · 46L65

1 Introduction

Trying to assign global pseudo-differential operators to large classes of locally compact groups G, in [22] second countable, type I unimodular groups have been treated, using operator-valued symbols defined on the product $\mathsf{G} \times \widehat{\mathsf{G}}$, where the unitary dual $\widehat{\mathsf{G}}$ is the family of equivalence classes of irreducible representations of G. Much more can be said in particular cases: (a) for compact Lie groups [27] and (b) for graded nilpotent Lie groups [10]. Actually the number of papers treating these two particular classes in great detail is growing fast. Keeping the same general framework as in [22], in [20] the related Berezin-type quantization has been explored. The operator-valuedness of the

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symbols and the fact that different irreducible representations act in different Hilbert spaces made the theory technically challenging.

In particular cases at least, one hopes for a simpler-looking quantization in terms of scalar-valued symbols. This is possible for connected simply connected nilpotent groups, due to some special properties, allowing finally to define a well-behaved Fourier transformation from functions (or distributions) defined on G to function (or distributions) defined on $\mathfrak{g}^\sharp$, the dual of the Lie algebra $\mathfrak{g}$. There is a drawback, however: this Fourier transformation does not intertwine multiplication with convolution (and this is due to the fact that, while being a diffeomorphism, the exponential map does not have good algebraic properties).

A graded structure on the Lie algebra surely helps. We mention some previous works [2,11–13,18,19,24–26], mainly dedicated to particular types of nilpotent groups or to invariant symbols (depending only on $\xi \in \mathfrak{g}^\sharp$). In [22,23] quantization formulas for the general nilpotent case has been mentioned and connections with the operator-valued calculus on $\mathsf{G} \times \widehat{\mathsf{G}}$ have been indicated. Actually the connection consists in combining together two different partial Fourier transformations. In the case of groups having (generic) square integrable irreducible representations modulo the center, the connection becomes nice and effective, involving Kirillov's theory and the Weyl–Pedersen pseudo-differential calculus on coadjoint orbits [23].

Anyhow, it is natural to introduce and study the generalization of the Berezin–Toeplitz (also called anti-Wick) formalism in the setting of the phase-space $\Xi := \mathsf{G} \times \mathfrak{g}^\sharp \cong T^\sharp\mathsf{G}$. The vector group $\mathsf{G} = \mathbb{R}^n$ is a guiding particular case. In spite of the mentioned connection between the pseudo-differential quantizations on $\mathsf{G} \times \widehat{\mathsf{G}}$ and $\mathsf{G} \times \mathfrak{g}^\sharp$, via a composition of partial Fourier transformations, *the Berezin–Toeplitz formalisms on the two "phase spaces" are not equivalent*. The reason is that their (weak) definitions involve in both cases products of *two* Fourier–Wigner functions (see (11) for instance), and one of the two Fourier transformations behaves badly with respect to multiplication. So there is no isomorphism between the objects from the present article and the analog ones from [20].

After fixing in Sect. 2 some notations and conventions about groups and Hilbert spaces, in Sect. 3 we proceed to describe the basic operators acting in $\mathcal{H} := L^2(\mathsf{G})$ that will be the building blocks of our theory. They can be understood as global or as infinitesimal operations verifying the canonical commutation relations inherent to the pair $(\mathsf{G}, \mathfrak{g}^\sharp)$.

From such building blocks, in Sect. 4 we construct the Weyl system, a highly non-commutative version of the usual one (phase-space shifts) in $\mathbb{R}^n$. Due to the complexity of the canonical commutation relations, it is not even a projective representation of the group $\mathsf{G} \times \mathfrak{g}^\sharp$. So one cannot invoke directly results and techniques from the existing theory in group-form. The "matrix coefficients" of this Weyl system lead to a Fourier–Wigner transform, coherent states, the Bargmann transform, reproducing kernel Hilbert spaces, etc. (We use a certain terminology, especially by analogy with the $\mathbb{R}^n$-case; but even in this commutative case there are so many different denominations. So we do not expect all the readers to be satisfied with our choices.)

In Sect. 5 one defines the Berezin quantization and study its basic properties. It is positive-preserving, it sends L^p spaces of symbols into Schatten–von Neumann classes of order p on $L^2(\mathsf{G})$ and gets a Toeplitz form in the Bargmann representations.

Some simple examples are included. Other more refined results are postponed to a future publication, mainly because they need first to establish a suitable coorbit (and modulation space) theory on $\mathsf{G} \times \mathfrak{g}^\sharp$.

The matrix elements of a bounded operator between coherent states define the covariant (lower) symbol. It is studied in Sect. 6. Among others, it provides some lower bounds for certain Schatten–von Neumann norms. Kernels of regular operators may be expressed in terms of the covariant symbols and the coherent states. Hopefully, this will be used in a future paper to prove a Beals-type criterion for pseudo-differential operators with scalar-valued symbols on $T^\sharp\mathsf{G}$.

Then we compute the pseudo-differential symbol of a Berezin operator; the correspondence is no longer given by a convolution, as in the standard case.

In a final section, we briefly indicate two extensions. First we treat τ-quantizations related to ordering issues. We show how this may be implemented at the level of the basic objects. Then we describe what happens when a variable magnetic field is also present. For pseudo-differential operators this has been done in [3]. Here we put into evidence the changes needed in the Berezin theory.

Up to our knowledge, the results in this article are not contained in the existing literature. In particular, projective group representation methods do not apply. However, the constructions and proofs are inspired by other, different situations. We were mainly guided by the book [29]. The related but not isomorphic theory from [20] has also been valuable. The literature on coherent states, Berezin type (or localization) operators and related topics is huge; we only cite some references [1,4–7,14,15,28–30]. As said above, modulation spaces will be studied in such a framework subsequently and this will bring to our attention the expanding literature on time–frequency methods. Actions in L^p-spaces will also be investigated.

2 Framework

The scalar products in a Hilbert space are linear in the first variable. For a given (complex, separable) Hilbert space $\mathcal{H}$, one denotes by $\mathbb{B}(\mathcal{H})$ the C^*-algebra of all linear bounded operators in $\mathcal{H}$ and by $\mathbb{B}^p(\mathcal{H})$ the bi-sided *-ideal of all Schatten–von Neumann operators of exponent $p \geq 1$. In particular $\mathbb{K}(\mathcal{H}) \equiv \mathbb{B}^\infty(\mathcal{H})$ is the C^*-algebra of all the compact operators in $\mathcal{H}$. The unitary elements of $\mathbb{B}(\mathcal{H})$ form the group $\mathbb{U}(\mathcal{H})$.

Let G be a connected simply connected nilpotent Lie group with unit e, Haar measure dx and unitary dual $\widehat{\mathsf{G}}$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^\sharp$ its dual. If $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^\sharp$ we set $\langle X \mid \xi \rangle := \xi(X)$. We also denote by $\exp : \mathfrak{g} \to \mathsf{G}$ the exponential map, which is a diffeomorphism. Its inverse is denoted by $\log : \mathsf{G} \to \mathfrak{g}$. Under these diffeomorphisms the Haar measure on G corresponds to a Haar measure dX on $\mathfrak{g}$ (normalized accordingly). For each $p \in [1, \infty]$, one has an isomorphism

$$L^p(\mathsf{G}) \xrightarrow{\mathrm{Exp}} L^p(\mathfrak{g}), \;\; \mathrm{Exp}(u) := u \circ \exp$$

with inverse

$$L^p(\mathfrak{g}) \xrightarrow{\text{Log}} L^p(\mathsf{G}), \ \operatorname{Log}(v) := v \circ \log .$$

The Schwartz spaces $\mathcal{S}(\mathsf{G})$ and $\mathcal{S}(\mathfrak{g})$ are defined as in [9, A.2]; they are isomorphic Fréchet spaces.

For $X, Y \in \mathfrak{g}$ we set

$$\begin{aligned} X \bullet Y &:= \log[\exp(X)\exp(Y)] \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) + \cdots \end{aligned}$$

It is a group composition law on $\mathfrak{g}$, given by a polynomial expression in X, Y (the Baker–Campbel–Hausdorff formula). The unit element is 0 and $X^\bullet \equiv -X$ is the inverse of X with respect to $\bullet$.

There is a Fourier transformation, given by the duality $\big(\mathfrak{g}, \mathfrak{g}^\sharp\big)$, defined essentially by

$$\big(\mathcal{F}h\big)(\xi) := \int_{\mathfrak{g}} e^{-i\langle X|\xi\rangle} h(X)\, dX.$$

It is a linear topological isomorphism $\mathcal{F} : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g}^\sharp)$ and a unitary map $\mathcal{F} : L^2(\mathfrak{g}) \to L^2(\mathfrak{g}^\sharp)$. Composing with the isomorphisms Exp and Log one gets Fourier transformations

$$\begin{aligned} &\mathscr{F} := \mathcal{F} \circ \operatorname{Exp} : \mathcal{S}(\mathsf{G}) \to \mathcal{S}(\mathfrak{g}^\sharp), \quad \mathscr{F}^{-1} := \operatorname{Log} \circ \mathcal{F}^{-1} : \mathcal{S}(\mathfrak{g}^\sharp) \to \mathcal{S}(\mathsf{G}), \\ &(\mathscr{F}u)(\xi) = \int_{\mathfrak{g}} e^{-i\langle X|\xi\rangle} u(\exp X) dX = \int_{\mathsf{G}} e^{-i\langle \log x|\xi\rangle} u(x) dx, \\ &\big(\mathscr{F}^{-1}w\big)(x) = \int_{\mathfrak{g}^\sharp} e^{i\langle \log x|\xi\rangle} w(\xi) d\xi. \end{aligned}$$

These maps also define unitary isomorphisms of the corresponding L^2-spaces.

3 Canonical commutation relations

One has the (strongly continuous) unitary representation $\mathrm{M} : (\mathfrak{g}^\sharp, +) \to \mathbb{U}\big[L^2(\mathsf{G})\big]$ given by

$$[\mathrm{M}_\zeta(u)](x) := e^{i\langle \log x|\zeta\rangle} u(x).$$

If we denote by $\mathsf{Mult}(\psi)$ the operator of multiplication by functions ψ defined on G, one has

$$\mathsf{M}_\zeta = \mathsf{Mult}(\varepsilon_\zeta) = \mathsf{Mult}\big(e^{i\lambda_\zeta}\big),$$

where we introduced the function

$$\lambda_\zeta : \mathsf{G} \to \mathbb{R}, \quad \lambda_\zeta(x) := \langle \log x \,|\, \zeta \rangle$$

and its imaginary exponential

$$\varepsilon_\zeta : \mathsf{G} \to \mathbb{T} \subset \mathbb{C}, \quad \varepsilon_\zeta(x) := e^{i\langle \log x \mid \zeta\rangle}. \tag{1}$$

For each $\zeta \in \mathfrak{g}^\sharp$ one has a 1-parameter subgroup

$$\mathbb{R} \ni t \to \mathsf{M}_{t\zeta} = e^{it\Lambda_\zeta} \in \mathbb{U}\big[L^2(\mathsf{G})\big],$$

with infinitesimal generator

$$\Lambda_\zeta := \mathsf{Mult}\big(\lambda_\zeta\big) = \mathsf{Mult}(\langle \log(\cdot) \mid \zeta\rangle).$$

One also has *the left and the right unitary representations*

$$\mathsf{L}, \mathsf{R} : (\mathsf{G}, \cdot) \to \mathbb{U}\big[L^2(\mathsf{G})\big],$$

defined by

$$\big[\mathsf{L}_z(u)\big](x) := u\big(z^{-1}x\big), \quad \big[\mathsf{R}_z(u)\big](x) := u(xz).$$

For fixed $Z \in \mathfrak{g}$, there are 1-parameter subgroups

$$\mathbb{R} \ni t \to \mathsf{L}_{\exp(tZ)} = e^{it(iD^{\mathsf{L}}_Z)}, \quad \mathbb{R} \ni t \to \mathsf{R}_{\exp(tZ)} = e^{it(-iD^{\mathsf{R}}_Z)},$$

where

$$\big[D^{\mathsf{L}}_Z(u)\big](x) := \frac{d}{dt}\Big|_{t=0} u\big(\exp[tZ]x\big), \quad \big[D^{\mathsf{R}}_Z(u)\big](x) := \frac{d}{dt}\Big|_{t=0} u\big(x\exp[tZ]\big).$$

Note the “multiplication relations”

$$\mathsf{L}_y\mathsf{L}_z = \mathsf{L}_{yz}, \quad \mathsf{M}_\eta\mathsf{M}_\zeta = \mathsf{M}_{\eta+\zeta}, \quad \mathsf{L}_z\mathsf{M}_\zeta = e^{i\langle \log(z^{-1}\cdot) - \log(\cdot) \mid \zeta\rangle}\mathsf{M}_\zeta\mathsf{L}_z.$$

and the “commutation relations” (on the Schwartz space $\mathcal{S}(\mathsf{G})$, for instance)

$$\begin{aligned} &\big[D^{\mathsf{L}}_Y, D^{\mathsf{L}}_Z\big] = D^{\mathsf{L}}_{[Y,Z]}, \quad \big[D^{\mathsf{R}}_Y, D^{\mathsf{R}}_Z\big] = D^{\mathsf{R}}_{[Y,Z]}, \\ &\big[D^{\mathsf{L}}_Z, \Lambda_\zeta\big] = \mathsf{Mult}\big(D^{\mathsf{L}}_Z\lambda_\zeta\big), \quad \big[D^{\mathsf{R}}_Z, \Lambda_\zeta\big] = \mathsf{Mult}\big(D^{\mathsf{R}}_Z\lambda_\zeta\big). \end{aligned} \tag{2}$$

For concreteness, let us set $\mathsf{ad}_X(Z) := [X, Z]$ and compute (in the BCH formula, only the terms that are linear in tZ contribute):

$$\begin{aligned} \big(D^{\mathsf{L}}_Z\lambda_\zeta\big)(\exp X) &= \frac{d}{dt}\Big|_{t=0} \langle \log(\exp[tZ]\exp X) \mid \zeta\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle\, [tZ] \bullet X \mid \zeta\rangle \\ &= \Big\langle Z - \frac{1}{2}\mathsf{ad}_X(Z) + \frac{1}{12}\mathsf{ad}^2_X(Z) + \cdots \Bigm| \zeta\Big\rangle. \end{aligned} \tag{3}$$

The sum is finite. Let us define *the infinitesimal coadjoint action*

$$\gamma : \mathsf{G} \to \mathsf{Aut}(\mathfrak{g}^\sharp), \quad \gamma_x(\zeta) \equiv \mathsf{ad}^\sharp_{-\log x}(\zeta) := \zeta \circ \mathsf{ad}_{-\log x}.$$

Then (3) may be rewritten

$$\big(D^{\mathsf{L}}_Z \lambda_\zeta\big)(x) = \Big\langle Z \,\big|\, \zeta + \frac{1}{2}\gamma_x(\zeta) + \frac{1}{12}\gamma_x^2(\zeta) + \cdots \Big\rangle.$$

This is a function of x, which becomes a constant $\langle Z \,|\, \zeta\rangle$ precisely when the group G is Abelian. There is a similar formula for $D^{\mathsf{R}}_Z \lambda_\zeta$.

4 Weyl systems, the Fourier–Wigner transform and coherent states

Definition 4.1 For $(z, \zeta) \in \mathsf{G} \times \mathfrak{g}^\sharp$ one defines a unitary operator $\mathsf{W}(z, \zeta) := \mathsf{M}_\zeta \mathsf{L}_z$ in $L^2(\mathsf{G})$ by

$$[\mathsf{W}(z, \zeta)u](x) := e^{i\langle \log x | \zeta\rangle} u(z^{-1}x), \tag{4}$$

with adjoint

$$\big[\mathsf{W}(z, \zeta)^* u\big](y) := e^{-i\langle \log(zy) | \zeta\rangle} u(zy).$$

This extends the notion of *Weyl system* (or *time–frequency shifts*) from the case $\mathsf{G} = \mathbb{R}^n$. These operators also act as isomorphisms of the Schwartz space $\mathcal{S}(\mathsf{G})$ and can be extended to isomorphisms of the space $\mathcal{S}'(\mathsf{G})$ of tempered distributions. Note that they also define isometries in any $L^p(\mathsf{G})$ space.

Lemma 4.2 *For* $(z, \zeta), (y, \eta) \in \mathsf{G} \times \mathfrak{g}^\sharp$ *one has*

$$\mathsf{W}(z, \zeta)\,\mathsf{W}(y, \eta) = \Gamma\big[(z, \zeta), (y, \eta)\big]\,\mathsf{W}(zy, \zeta + \eta),$$

where $\Gamma\big[(z, \zeta), (y, \eta)\big]$ *is the operator of multiplication by the function*

$$x \mapsto \gamma\big[(z, \zeta), (y, \eta); x\big] = \exp\big\{ -i\langle \log x - \log(z^{-1}x) \mid \eta\rangle\big\}.$$

Thus the Weyl system is very far from being a projective representation.

Definition 4.3 For $u, v \in \mathcal{H} := L^2(\mathsf{G})$ one sets $\mathscr{W}_{u,v} \equiv \mathscr{W}_{u\otimes\overline{v}} : \mathsf{G} \times \mathfrak{g}^\sharp \to \mathbb{C}$ by

$$\mathscr{W}_{u,v}(z, \zeta) := \langle \mathsf{W}(z, \zeta)u, v\rangle = \int_{\mathsf{G}} e^{i\langle \log y | \zeta\rangle} u(z^{-1}y)\overline{v(y)}dy. \tag{5}$$

and call it the Fourier–Wigner transform.

Lemma 4.4 *The Fourier–Wigner transform extends to a unitary map*

$$\mathscr{W} : \mathcal{H} \otimes \overline{\mathcal{H}} \cong L^2(\mathsf{G} \times \mathsf{G}) \to L^2(\mathsf{G} \times \mathfrak{g}^\sharp).$$

It also defines isomorphisms

$$\mathscr{W}:\mathcal{S}(\mathsf{G})\,\overline{\otimes}\,\mathcal{S}(\mathsf{G})\cong\mathcal{S}(\mathsf{G}\times\mathsf{G})\to\mathcal{S}(\mathsf{G}\times\mathfrak{g}^\sharp).$$
$$\mathscr{W}:\mathcal{S}'(\mathsf{G})\,\overline{\otimes}\,\mathcal{S}'(\mathsf{G})\cong\mathcal{S}'(\mathsf{G}\times\mathsf{G})\to\mathcal{S}'(\mathsf{G}\times\mathfrak{g}^\sharp).$$

Proof The map $\mathscr{W}=(\mathrm{id}\otimes\mathscr{F}^{-1})\circ\mathrm{C}$ is composed of a partial Fourier transform and a unitary change of variables $(x,y)\to\mathrm{C}(x,y):=\big(x^{-1}y,y\big)$, and this leads easy to a proof of all the assertions. □

In particular, one has *the orthogonality relations*:

$$\big\langle\mathscr{W}_{u,v},\mathscr{W}_{u',v'}\big\rangle_{L^2(\mathsf{G}\times\mathfrak{g}^\sharp)}=\langle u,u'\rangle\langle v',v\rangle. \tag{6}$$

Definition 4.5 For some fixed L^2-normalized $\omega\in\mathcal{S}(\mathsf{G})\subset L^2(\mathsf{G})\equiv\mathcal{H}$ and for every $(z,\zeta)\in\mathsf{G}\times\mathfrak{g}^\sharp$, we define the coherent state $\omega_{z,\zeta}:=\mathsf{W}(z,\zeta)^*\omega$; explicitly

$$\omega_{z,\zeta}(x)=e^{-i\langle\log(zx)|\zeta\rangle}\omega(zx).$$

The associated rank one projector is given by

$$\Omega_{z,\zeta}(u):=\big\langle u,\omega_{z,\zeta}\big\rangle\omega_{z,\zeta}=\mathscr{W}_{u,\omega}(z,\zeta)\,\omega_{z,\zeta},\quad\forall\,u\in\mathcal{H}. \tag{7}$$

It is an integral operator with kernel $\varpi_{z,\zeta}:=\omega_{z,\zeta}\otimes\overline{\omega_{z,\zeta}}\in S(\mathsf{G}\times\mathsf{G})$ [or in $L^2(\mathsf{G}\times\mathsf{G})$ more generally for ω in $L^2(\mathsf{G})$].

The canonical (or modulation) mapping associated to the vector ω (or *the generalized Bargmann transformation*)

$$\mathscr{B}_\omega:\mathcal{H}\to L^2(\mathsf{G}\times\mathfrak{g}^\sharp),\quad\mathscr{B}_\omega(u):=\mathscr{W}_{u,\omega}$$

is an isometry with adjoint

$$\mathscr{B}_\omega^\dagger:L^2(\mathsf{G}\times\mathfrak{g}^\sharp)\to\mathcal{H},\quad\mathscr{B}_\omega^\dagger(h):=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}h(z,\zeta)\,\omega_{z,\zeta}\,dzd\zeta.$$

The isometry condition may be seen as *an inversion formula*:

$$u=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}\big\langle u,\omega_{z,\zeta}\big\rangle\omega_{z,\zeta}\,dzd\zeta. \tag{8}$$

The final projection $\mathscr{P}_\omega:=\mathscr{B}_\omega\mathscr{B}_\omega^\dagger$ is an integral operator with kernel

$$p_\omega(z,\zeta;z',\zeta'):=\big\langle\omega_{z,\zeta},\omega_{z',\zeta'}\big\rangle. \tag{9}$$

One also have *the reproducing formula* $\mathscr{B}_\omega(u)=\big(\mathscr{B}_\omega\mathscr{B}_\omega^\dagger\mathscr{B}_\omega\big)(u)=\mathscr{P}_\omega[\mathscr{B}_\omega(u)]$, i.e.

$$[\mathscr{B}_\omega(u)]\,(x,\xi)=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}\langle\omega_{x,\xi},\omega_{z,\zeta}\rangle\,[\mathscr{B}_\omega(u)](z,\zeta)\,dzd\zeta.$$

Thus $\mathscr{P}_\omega\big[L^2(\mathsf{G}\times\mathfrak{g}^\sharp)\big]$ is a reproducing kernel Hilbert space with reproducing kernel p_ω, composed of bounded continuous functions on $\mathsf{G}\times\mathfrak{g}^\sharp$.

5 The Berezin quantization

Occasionally, we are going to use notations as $\mathcal{X}:=(x,\xi)$, $\mathcal{Y}:=(y,\eta)$, $\mathcal{Z}:=(z,\zeta)\in\Xi:=\mathsf{G}\times\mathfrak{g}^\sharp$, with product measure $d\mathcal{X}:=dxd\xi$. Actually, both types of notations will be used alternatively. We denote by $\langle\cdot,\cdot\rangle_{(\Xi)}$ both the $L^2(\Xi)$-inner product and the various related duality forms (as $\mathcal{S}(\Xi)\times\mathcal{S}'(\Xi)\to\mathbb{C}$ for example). The precise meaning will be specified or will be obvious from the context.

Definition 5.1 Let $\omega\in\mathcal{S}(\mathsf{G})$ be a fixed L^2-normalized vector. We define formally the operator in $L^2(\mathsf{G})$

$$\mathsf{Ber}_\omega(f):=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}f(x,\xi)\,\Omega_{x,\xi}\,dxd\xi=\int_\Xi f(\mathcal{X})\,\Omega_{\mathcal{X}}\,d\mathcal{X},\tag{10}$$

where $\Omega_{\mathcal{X}}$ is defined in (7), and call it the Berezin operator associated to the symbol f and the vector ω.

This should be taken in weak sense: taking (5) and (7) into account, for any $u,v\in L^2(\mathsf{G})$ one gets

$$\begin{aligned}\big\langle\mathsf{Ber}_\omega(f)u,v\big\rangle&:=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}f(x,\xi)\big\langle\Omega_{x,\xi}(u),v\big\rangle dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp}f(x,\xi)\,\mathscr{W}_{u,\omega}(x,\xi)\,\overline{\mathscr{W}_{v,\omega}(x,\xi)}\,dxd\xi\\&=\big\langle f,\overline{\mathscr{W}_{u,\omega}}\,\mathscr{W}_{v,\omega}\big\rangle_{(\Xi)}.\end{aligned}\tag{11}$$

This allows many different (but compatible) interpretations, based on the properties of the Fourier–Wigner transform. For instance, if $u,v\in\mathcal{S}(\mathsf{G})$, the last term of (11) makes sense for $f\in\mathcal{S}'(\mathsf{G}\times\mathfrak{g}^\sharp)$ and one gets a linear continuous operator

$$\mathsf{Ber}_\omega(f):\mathcal{S}(\mathsf{G})\to\mathcal{S}'(\mathsf{G}),\quad f\in\mathcal{S}'(\mathsf{G}\times\mathfrak{g}^\sharp).$$

For similar reasons, $\mathsf{Ber}_\omega(f):\mathcal{S}'(\mathsf{G})\to\mathcal{S}(\mathsf{G})$ is well-defined, linear and continuous if $f\in\mathcal{S}(\mathsf{G}\times\mathfrak{g}^\sharp)$. If $u,v\in L^2(\mathsf{G})$, then $\mathscr{W}_{u,\omega},\mathscr{W}_{v,\omega}\in L^2(\Xi)$, thus $\mathscr{W}_{u,\omega}\overline{\mathscr{W}_{v,\omega}}\in L^1(\Xi)$ and one gets

$$\mathsf{Ber}_\omega(f)\in\mathbb{B}\big[L^2(\mathsf{G})\big],\quad f\in L^\infty(\mathsf{G}\times\mathfrak{g}^\sharp).$$

It is obvious that $\mathsf{Ber}_\omega(f)^*=\mathsf{Ber}_\omega(\overline{f})$ and that $\mathsf{Ber}_\omega(f)$ *is a positive operator in* $L^2(\mathsf{G})$ *if* $f\in L^\infty(\Xi)$ *is (almost everywhere) positive*. By the orthogonality relations (6) one may write

$$\big\langle\mathsf{Ber}_\omega(1)u,v\big\rangle=\int_\Xi\mathscr{W}_{v,\omega}(\mathcal{X})^*\,\mathscr{W}_{u,\omega}(\mathcal{X})d\mathcal{X}=\big\langle\mathscr{W}_{u,\omega}(\mathcal{X}),\mathscr{W}_{v,\omega}(\mathcal{X})\big\rangle_{(\Xi)}=\langle u,v\rangle,$$

implying that $\mathsf{Ber}_\omega(1) = 1_{L^2(\mathsf{G})}$.

We gather other important properties of the Berezin operators in connection with the Schatten–von Neumann classes in the next result.

Theorem 5.2 *For every $s \in [1, \infty]$ one has a linear bounded map $\mathsf{Ber}_\omega : L^s(\Xi) \to \mathbb{B}^s\big[L^2(\mathsf{G})\big]$ satisfying*

$$\| \mathsf{Ber}_\omega(f) \|_{\mathbb{B}^s[L^2(\mathsf{G})]} \leq 4^{1/s} \, \| f \|_{L^s(\Xi)} \, . \tag{12}$$

In particular, if $f \in L^1(\Xi)$, then $\mathsf{Ber}_\omega(f)$ is a trace-class operator with

$$\mathrm{Tr}\big[\mathsf{Ber}_\omega(f)\big] = \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} f(x, \xi) \, dx d\xi.$$

Proof For $s = \infty$ we write using the definitions, an obvious $L^1 - L^\infty$ estimate, the Cauchy–Schwartz inequality and the orthogonality relation

$$\begin{aligned}
\| \mathsf{Ber}_\omega(f) \|_{\mathbb{B}[L^2(\mathsf{G})]} &= \sup_{\|u\|=1=\|v\|} \big|\big\langle \mathsf{Ber}_\omega(f)u, v\big\rangle\big| \\
&= \sup_{\|u\|=1=\|v\|} \big|\big\langle f, \overline{\mathscr{W}_{u,\omega}} \mathscr{W}_{v,\omega}\big\rangle_{(\Xi)}\big| \\
&\leq \| f \|_{L^\infty} \sup_{\|u\|=1=\|v\|} \| \overline{\mathscr{W}_{u,w}} \mathscr{W}_{v,\omega} \|_{L^1} \\
&\leq \| f \|_{L^\infty} \sup_{\|u\|=1} \| \mathscr{W}_{u,\omega} \|_{L^2} \sup_{\|v\|=1} \| \mathscr{W}_{v,\omega} \|_{L^2} \\
&= \| f \|_{L^\infty} \, .
\end{aligned}$$

There is a version of the computation above showing that $\mathsf{Ber}_\omega(f)$ is in fact also bounded if $f \in L^s(\Xi)$. It is based on complex interpolation, the Hölder inequality and improved properties of the Fourier–Wigner transformation, having as starting point the simple estimate

$$|\mathscr{W}_{u,v}(\mathcal{X})| = |\langle \mathsf{W}(\mathcal{X})u, v\rangle| \leq \| u \| \| v \|, \quad \forall \, \mathcal{X} \in \Xi \, .$$

But one needs the finer result (12), in terms of Schatten–von Neumann classes.

We deal first with the trace class properties of the Berezin operator, assuming that $f \in L^1(\Xi)$ is positive. The connected simply connected nilpotent group G is second countable, so the Hilbert space $L^2(\mathsf{G})$ is separable. If $\{w_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathsf{G})$, one has by (11), (7) and the Parseval identity

$$\begin{aligned}
\mathrm{Tr}\big[\mathsf{Ber}_\omega(f)\big] &= \sum_k \big\langle \mathsf{Ber}_\omega(f)w_k, w_k\big\rangle \\
&= \sum_k \int_\Xi f(\mathcal{X}) \big\langle \Omega_{\mathcal{X}}(w_k), w_k\big\rangle d\mathcal{X}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \int_{\Xi} f(\mathcal{X}) \langle \omega_{\mathcal{X}}, w_k \rangle \langle w_k, \omega_{\mathcal{X}} \rangle \, d\mathcal{X} \\
&= \int_{\Xi} f(\mathcal{X}) \sum_k \langle \omega_{\mathcal{X}}, w_k \rangle \langle w_k, \omega_{\mathcal{X}} \rangle \, d\mathcal{X} \\
&= \int_{\Xi} f(\mathcal{X}) \langle \omega_{\mathcal{X}}, \omega_{\mathcal{X}} \rangle \, d\mathcal{X} \\
&= \int_{\Xi} f(\mathcal{X}) \, d\mathcal{X}.
\end{aligned}
$$

If f is positive, we already know that $\mathsf{Ber}_\omega(f)$ is also positive and its trace norm is computed above:

$$
\| \mathsf{Ber}_\omega(f) \|_{\mathbb{B}^1[L^2(\mathsf{G})]} = \mathrm{Tr}\big[\mathsf{Ber}_\omega(f)\big] = \| f \|_{L^1} \, .
$$

One obtains the $s = 1$ case of (12) for general f by writing $f = \mathsf{Re}[f]_+ - \mathsf{Re}[f]_- + i\mathsf{Im}[f]_+ - i\mathsf{Im}[f]_-$.

The general case in (12) then follows by interpolation of order $\theta = 1/s$ from the cases $s = 1$ and $s = \infty$, because one has $\Big[L^\infty(\Xi), L^1(\Xi)\Big]_{1/s} = L^s(\Xi)$ and $\Big[\mathbb{B}\big[L^2(\mathsf{G})\big], \mathbb{B}^1\big[L^2(\mathsf{G})\big]\Big]_{1/s} = \mathbb{B}^s\big[L^2(\mathsf{G})\big]$. □

One can improve the constant $4^{1/s}$ to 1 by imitating arguments from [29, Ch. 14] or from [8].

Besides the compactness results following directly from Theorem 5.2, one also gets by approximation

Corollary 5.3 *If $f \in C_0(\mathsf{G} \times \mathfrak{g}^\sharp)$, then $\mathsf{Ber}_\omega(f)$ is a compact operator in $L^2(\mathsf{G})$.*

Proof This is true for continuous compactly supported functions, by the result above, and then follows for every continuous function small at infinity, by uniform approximation and the case $s = \infty$ of (12). □

Example 5.4 Theorem 5.2 supplies plenty of compact Berezin operators with symbols not belonging to $L^\infty(\Xi)$. In addition, we have $\Omega_{\mathcal{X}} = \mathsf{Ber}_\omega(\delta_{\mathcal{X}})$ [if (10) seems too formal, one can easily compute with (11)], and this is a rank one projection defined by a distribution.

Example 5.5 For $f := \varphi \otimes 1$, where $\varphi : \mathsf{G} \to \mathbb{C}$, a short computation shows that $\mathsf{Ber}_\omega(\varphi \otimes 1)$ is the operator of multiplication by the function $x \to \big(\check{\varphi} \star |\omega|^2\big)(x) := \int_{\mathsf{G}} \varphi(z) |\omega(zx)|^2 dz$, where $\check{\varphi}(y) := \varphi(y^{-1})$.

Example 5.6 For $f := 1 \otimes \psi$, where $\psi : \mathfrak{g}^\sharp \to \mathbb{C}$, a short computation shows that $\mathsf{Ber}_\omega(1 \otimes \psi)$ is an integral operator with kernel

$$
\big[h_\omega(\psi)\big](x, y) := \int_{\mathsf{G}} \tilde{\psi} \, [\log(zx) - \log(zy)] \, \omega(zx) \overline{\omega(zy)} \, dz,
$$

written in terms of the usual Fourier transform $\tilde{\psi}$ of ψ attached to the duality $(\mathfrak{g}, \mathfrak{g}^\sharp)$.

Proposition 5.7 *For any $z \in \mathsf{G}$ and (for example) $f \in L^\infty(\mathsf{G} \times \mathfrak{g}^\sharp)$, one has*

$$\mathsf{L}_z^* \mathsf{Ber}_\omega(f) \mathsf{L}_z = \mathsf{Ber}_\omega\big[f(\cdot z^{-1}, \cdot)\big].$$

Proof One computes

$$\begin{aligned}
\mathsf{L}_z^* \mathsf{Ber}_\omega(f) \mathsf{L}_z u &= \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} f(x,\xi)\, \mathsf{L}_z^* \Omega_{x,\xi} \mathsf{L}_z u \, dx d\xi \\
&= \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} f(x,\xi)\, \big\langle u, \mathsf{L}_z^* \mathsf{W}(x,\xi)^* \omega \big\rangle \mathsf{L}_z^* \mathsf{W}(x,\xi)^* \omega \, dx d\xi \\
&= \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} f(x,\xi)\, \big\langle u, \mathsf{W}(xz,\xi)^* \omega \big\rangle \mathsf{W}(xz,\xi)^* \omega \, dx d\xi \\
&= \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} f(x,\xi)\, \Omega_{xz,\xi}(u) \, dx d\xi,
\end{aligned}$$

and then a change of variables leads to the result. □

The formula for $\mathsf{W}(z,\zeta)^* \mathsf{Ber}_\omega(f) \mathsf{W}(z,\zeta)$ is rather involved, due to Lemma 4.2.

We provide now a Toeplitz-like form of the operator $\mathsf{Tp}_\omega(f) := \mathscr{B}_\omega \circ \mathsf{Ber}_\omega(f) \circ \mathscr{B}_\omega^\dagger$ living in $L^2\big(\mathsf{G} \times \mathfrak{g}^\sharp\big)$.

Proposition 5.8 *One has*

$$\mathsf{Tp}_\omega(f) = \mathscr{P}_\omega \circ \mathsf{Mult}(f) \circ \mathscr{P}_\omega, \tag{13}$$

where $\mathsf{Mult}(f)$ is the point-wise multiplication by $f \in L^\infty(\Xi)$.

Proof Clearly (13) is equivalent to $\mathsf{Ber}_\omega(f) = \mathscr{B}_\omega^\dagger \circ \mathsf{Mult}(f) \circ \mathscr{B}_\omega$. For $u, v \in L^2(\mathsf{G})$ we have

$$\begin{aligned}
\big\langle \mathsf{Ber}_\omega(f) u, v \big\rangle &= \big\langle f, \overline{\mathscr{B}_\omega(u)}\, \mathscr{B}_\omega(v) \big\rangle_{(\Xi)} \\
&= \int_\Xi \big[\mathscr{B}_\omega(u)\big](\mathcal{X}) f(\mathcal{X}) \overline{\big[\mathscr{B}_\omega(v)\big](\mathcal{X})} \, d\mathcal{X} \\
&= \int_\Xi \Big(\mathsf{Mult}(f)\big[\mathscr{B}_\omega(u)\big]\Big)(\mathcal{X})\, \overline{\big[\mathscr{B}_\omega(v)\big](\mathcal{X})} \, d\mathcal{X} \\
&= \Big\langle \mathsf{Mult}(f)\big[\mathscr{B}_\omega(u)\big], \mathscr{B}_\omega(v) \Big\rangle_{(\Xi)} \\
&= \Big\langle \big[\mathscr{B}_\omega^\dagger \circ \mathsf{Mult}(f) \circ \mathscr{B}_\omega\big] u, v \Big\rangle_{(\Xi)}
\end{aligned}$$

and the Proposition is proved. □

It follows immediately that $\mathsf{Tp}_\omega(f)$ is an integral operator with kernel

$$\big[t_\omega(f)\big](\mathcal{X}, \mathcal{Y}) := \int_\Xi f(\mathcal{Z}) \langle \omega_{\mathcal{X}}, \omega_{\mathcal{Z}} \rangle \langle \omega_{\mathcal{Z}}, \omega_{\mathcal{Y}} \rangle d\mathcal{Z}. \tag{14}$$

6 The covariant symbol and the Berezin transform

Definition 6.1 The covariant symbol $\mathrm{cov}_\omega(T) : \Xi \times \Xi \to \mathbb{C}$ of an operator $T \in \mathbb{B}\big[L^2(\mathsf{G})\big]$ is

$$\big[\mathrm{cov}_\omega(T)\big](\mathcal{X}, \mathcal{X}') := \big\langle T\omega_{\mathcal{X}}, \omega_{\mathcal{X}'}\big\rangle = \big\langle \mathsf{W}(\mathcal{X}')T\mathsf{W}(\mathcal{X})^*\omega, \omega\big\rangle.$$

For the diagonal version we are also going to use the notation

$$\big[\mathrm{Cov}_\omega(T)\big](\mathcal{X}) := \big[\mathrm{cov}_\omega(T)\big](\mathcal{X}, \mathcal{X}) = \mathrm{Tr}\big[T\,\Omega_{\mathcal{X}}\big].$$

Clearly $\mathrm{cov}_\omega : \mathbb{B}\big[L^2(\mathsf{G})\big] \to L^\infty(\Xi \times \Xi)$ *is a linear contraction and any* $\mathrm{cov}_\omega(T)$ *is actually a continuous function*. Under further requirements on ω, it might have further regularity properties. For instance, if $\omega \in \mathcal{S}(\mathsf{G})$ (as we usually assume), then $\mathrm{cov}_\omega(T)$ is smooth, with bounded derivatives.

Recall the composition of integral kernels

$$(F \,\Box\, G)(\mathcal{X}, \mathcal{Y}) := \int_\Xi F(\mathcal{X}, \mathcal{Z})G(\mathcal{Z}, \mathcal{Y})d\mathcal{Z}$$

and the adjoint

$$F^{\Box}(\mathcal{X}, \mathcal{Y}) = \overline{F(\mathcal{Y}, \mathcal{X})}.$$

Proposition 6.2 *One has*

$$\mathrm{cov}_\omega(ST) = \mathrm{cov}_\omega(T) \,\Box\, \mathrm{cov}_\omega(S), \quad \mathrm{cov}_\omega(T^*) = \mathrm{cov}_\omega(T)^{\Box}.$$

Proof By using the definitions and the inversion formula (8) one gets

$$\begin{aligned}
\big[\mathrm{cov}_\omega(ST)\big](\mathcal{X}, \mathcal{Y}) &= \big\langle T\omega_{\mathcal{X}}, S^*\omega_{\mathcal{Y}}\big\rangle \\
&= \int_\Xi \big\langle T\omega_{\mathcal{X}}, \omega_{\mathcal{Z}}\big\rangle\big\langle \omega_{\mathcal{Z}}, S^*\omega_{\mathcal{Y}}\big\rangle d\mathcal{Z} \\
&= \int_\Xi \big\langle T\omega_{\mathcal{X}}, \omega_{\mathcal{Z}}\big\rangle\big\langle S\omega_{\mathcal{Z}}, \omega_{\mathcal{Y}}\big\rangle d\mathcal{Z} \\
&= \int_\Xi \big[\mathrm{cov}_\omega(T)\big](\mathcal{X}, \mathcal{Z})\big[\mathrm{cov}_\omega(S)\big](\mathcal{Z}, \mathcal{Y})d\mathcal{Z} \\
&= \big[\mathrm{cov}_\omega(T) \,\Box\, \mathrm{cov}_\omega(S)\big](\mathcal{X}, \mathcal{Y}).
\end{aligned}$$

The formula for the adjoint is obvious. Taking diagonal values one gets

$$\mathrm{Cov}_\omega(T^*) = \overline{\mathrm{Cov}_\omega(T)}.$$

□

The diagonal covariant symbol provides lower bounds for the operator trace norm.

Proposition 6.3 *If* $T \in \mathbb{B}^1\big[L^2(\mathsf{G})\big]$ *then*

$$\| \operatorname{Cov}_\omega(T) \|_{L^1(\mathsf{G})} \leq \| T \|_{\mathbb{B}^1} \ .$$

Proof The trace-class operator T admits the strongly convergent representation

$$T = \sum_{k=1}^{\infty} s_k(T) \langle \cdot, \varphi_k \rangle \psi_k,$$

in terms of the (positive) singular values of T and two orthonormal families. For every $\mathcal{X} \in \Xi$ we have

$$\begin{aligned} \big|\big[\operatorname{Cov}_\omega(T)\big](\mathcal{X})\big| &= \big|\langle T\omega_{\mathcal{X}}, \omega_{\mathcal{X}} \rangle\big| \\ &= \Big| \sum_{k=1}^{\infty} s_k(T) \langle \omega_{\mathcal{X}}, \varphi_k \rangle \langle \psi_k, \omega_{\mathcal{X}} \rangle \Big| \\ &\leq \frac{1}{2} \sum_{k=1}^{\infty} s_k(T) \Big(|\langle \omega_{\mathcal{X}}, \varphi_k \rangle|^2 + |\langle \psi_k, \omega_{\mathcal{X}} \rangle|^2 \Big), \end{aligned}$$

implying

$$\| \operatorname{Cov}_\omega(T) \|_{L^1(\mathsf{G})} \leq \frac{1}{2} \sum_{k=1}^{\infty} s_k(T) \left(\int_\Xi |\langle \omega_{\mathcal{X}}, \varphi_k \rangle|^2 d\mathcal{X} + \int_\Xi |\langle \psi_k, \omega_{\mathcal{X}} \rangle|^2 d\mathcal{X} \right).$$

By the inversion formulas and by the normalization of the vectors, the two integrals equal 1, and the remaining factor is the trace norm of the operator. □

By interpolation one readily gets

Corollary 6.4 *If* $T \in \mathbb{B}^p\big[L^2(\mathsf{G})\big]$, *with* $p \in [1, \infty]$, *then*

$$\| \operatorname{Cov}_\omega(T) \|_{L^p(\mathsf{G})} \leq \| T \|_{\mathbb{B}^p} \ .$$

Proposition 6.5 *If T is a compact operator,* $\operatorname{Cov}_\omega(T) \in C_0(\Xi)$, *i. e. it is a continuous function converging to zero at infinity.*

Proof Continuity has already been mentioned. One still has to show that

$$\lim_{\mathcal{X} \to \infty} \langle T\mathsf{W}(\mathcal{X})\omega, \mathsf{W}(\mathcal{X})\omega \rangle = 0.$$

The operator T being compact, it turns weak convergence into norm convergence. Also using the density of $\mathcal{S}(\mathsf{G})$ in $L^2(\mathsf{G})$ and the unitarity of the Weyl system, we are thus reduced to showing that

$$\mathscr{W}_{\omega,v}(\mathcal{X}) = \langle \mathsf{W}(\mathcal{X})\omega, v \rangle \underset{\mathcal{X} \to \infty}{\longrightarrow} 0, \quad \forall\, v \in S(\mathsf{G}).$$

This is obvious from the fact that $\omega \in \mathcal{S}(\mathsf{G})$ and that $\mathscr{W}$ is the composition between a change of variables and a partial Fourier transform. If ω is only square integrable, one can still finish the proof by density and approximation. □

Proposition 6.6 *For every $f \in L^1(\mathsf{G} \times \mathfrak{g}^\sharp)$ one has in terms of the kernel (14) of the Toeplitz operator*

$$\mathrm{cov}_\omega\big(\mathsf{Ber}_\omega(f)\big) = t_\omega(f), \tag{15}$$

with the particular case (the Berezin transform)

$$\big[\mathsf{BT}_\omega(f)\big](\mathcal{X}) := \big[\mathrm{Cov}_\omega\big(\mathsf{Ber}_\omega(f)\big)\big](\mathcal{X}) = \int_\Xi f(\mathcal{Z})\,\big|\langle\omega_\mathcal{X}, \omega_\mathcal{Z}\rangle\big|^2 d\mathcal{Z}, \tag{16}$$

and

$$\int_\Xi \big[\mathsf{BT}_\omega(f)\big](\mathcal{X})\,d\mathcal{X} = \int_\Xi f(\mathcal{X})\,d\mathcal{X}. \tag{17}$$

Proof Checking (15) [and thus (16)] is an easy direct verification. Then proving (17) relies on the formula

$$\int_\Xi |\langle\omega_\mathcal{X}, \omega_\mathcal{Z}\rangle|^2 d\mathcal{X} = 1, \quad \forall\,\mathcal{Z} \in \Xi. \tag{18}$$

Recalling the kernel (9) of the *projection* $\mathscr{P}_{(\omega)}$ and the normalization of ω, (18) becomes obvious. One may also use (8) directly. □

Let us say that the operator $T \in \mathbb{B}\big[L^2(\mathsf{G})\big]$ is *regularizing* if it extends to a continuous operator $T : \mathcal{S}'(\mathsf{G}) \to \mathcal{S}(\mathsf{G})$; then it will have a kernel belonging to $\mathcal{S}(\mathsf{G} \times \mathsf{G})$. This kernel may be expressed in terms of the covariant symbol and the coherent states.

Proposition 6.7 *The kernel $K_T : \mathsf{G} \times \mathsf{G} \to \mathbb{C}$ of the regularizing operator T is given through the formula*

$$K_T(x, y) = \int_\Xi \int_\Xi \big[\mathrm{cov}_\omega(T)\big](\mathcal{Z}, \mathcal{Z}')\,\omega_{\mathcal{Z}'}(x)\,\overline{\omega_\mathcal{Z}}(y)\,d\mathcal{Z}\,d\mathcal{Z}'.$$

Proof Computing for $u \in \mathcal{S}(\mathsf{G})$ (for instance), we are going to use the inversion formula twice:

$$\begin{aligned}
(Tu)(x) &= \int_\Xi \big\langle Tu, \omega_{\mathcal{Z}'}\big\rangle\,\omega_{\mathcal{Z}'}(x)\,d\mathcal{Z}' \\
&= \int_\Xi \big\langle u, T^*\omega_{\mathcal{Z}'}\big\rangle\,\omega_{\mathcal{Z}'}(x)\,d\mathcal{Z}' \\
&= \int_\Xi \Big\langle u, \int_\Xi \langle T^*\omega_{\mathcal{Z}'}, \omega_\mathcal{Z}\big\rangle\,\omega_\mathcal{Z}\,d\mathcal{Z}\Big\rangle\,\omega_{\mathcal{Z}'}(x)\,d\mathcal{Z}' \\
&= \int_\Xi \int_\Xi \overline{\langle T^*\omega_{\mathcal{Z}'}, \omega_\mathcal{Z}\big\rangle}\big\langle u, \omega_\mathcal{Z}\big\rangle\,\omega_{\mathcal{Z}'}(x)\,d\mathcal{Z}d\mathcal{Z}'
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathsf{G}} \int_{\Xi} \int_{\Xi} \big\langle T\omega_{\mathcal{Z}}, \omega_{\mathcal{Z}'} \big\rangle \omega_{\mathcal{Z}'}(x)\, \overline{\omega_{\mathcal{Z}}(y)}\, u(y) dy\, d\mathcal{Z}\, d\mathcal{Z}' \\
&= \int_{\mathsf{G}} K_T(x, y) u(y) dy = \big[\mathsf{Int}(K_T) u\big](x).
\end{aligned}
$$

□

7 Connection with pseudo-differential operators

One defines *the pseudo-differential operator with symbol* $a : \mathsf{G} \times \mathfrak{g}^\sharp \to \mathbb{C}$ by the formula

$$\big[\mathsf{Op}(a)u\big](x) := \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} e^{i\langle \log(xy^{-1}) | \xi\rangle} a(x, \xi) u(y)\, dy d\xi. \tag{19}$$

It is an integral operator with kernel $\mathsf{K}_a(x, y) := \int_{\mathfrak{g}^\sharp} e^{i\langle \log(xy^{-1})|\xi\rangle} a(x, \xi) d\xi$. The structure of this kernel (obtained from the symbol a by a partial Fourier transform and a change of variables) allows various types of interpretation of the formula (19) and leads to the properties of the quantization Op, that we do not discuss here in detail. Examining this kernel, one sees for instance that (19) defines a unitary mapping $\mathsf{Op} : L^2\big(\mathsf{G} \times \mathfrak{g}^\sharp\big) \to \mathbb{B}^2\big[L^2(\mathsf{G})\big]$. Versions involving Schwartz spaces are also easy to obtain. Note that the Weyl system (4) can be recuperated as

$$\mathsf{W}(z, \zeta) = \mathsf{Op}(\epsilon_{z,\zeta}), \quad \text{where} \quad \epsilon_{z,\zeta}(x, \xi) := e^{i\langle \log x|\zeta\rangle} e^{-i\langle \log z|\xi\rangle}$$

[see also (1)] and that the Fourier–Wigner transform (5) may also be involved in the definition of Op. If a only depends on x then Op is a multiplication operator, while if a only depends on ξ, Op becomes a (left) convolution operator.

Proposition 7.1 *Suppose (say) that* $f \in \mathcal{S}(\mathsf{G} \times \mathfrak{g}^\sharp)$. *The Berezin operator* $\mathsf{Ber}_\omega(f)$ *is a pseudo-differential operator with symbol*

$$
\begin{aligned}
\big[a_\omega(f)\big](x, \xi) := \int_{\mathsf{G}} \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} & e^{-i\langle \log y|\xi\rangle}\, e^{i\langle \log(zy^{-1}x) - \log(zx)|\zeta\rangle} \\
& f(z, \zeta)\, \omega(zx)\, \overline{\omega(zy^{-1}x)}\, dydzd\zeta.
\end{aligned} \tag{20}
$$

Remark 7.2 In the Abelian case $\mathsf{G} = \mathbb{R}^n$ (20) simply reduces to a convolution:

$$
\begin{aligned}
\big[a_\omega(f)\big](x, \xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z, \zeta)\, e^{-i\langle y|\zeta+\xi\rangle} \omega(z + x)\, \overline{\omega(z - y + x)}\, dydzd\zeta \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} e^{-i\langle y|\eta\rangle} \omega(s)\, \overline{\omega(s - y)}\, dy\Big] f(s - x, \eta - \xi) ds d\eta.
\end{aligned}
$$

Proof As said above, $\mathsf{Op}(a)$ is an integral operator with kernel $\mathsf{K}_a : \mathsf{G} \times \mathsf{G} \to \mathbb{C}$ given by

$$\mathsf{K}_a(x, y) = \int_{\mathfrak{g}^\sharp} e^{i\langle \log(xy^{-1})|\xi\rangle} a(x, \xi) d\xi = \big[\big(\mathrm{id} \otimes \mathscr{F}^{-1}\big) a\big]\big(x, xy^{-1}\big).$$

The symbol may be recovered from the kernel by means of the formula

$$a(x,\xi)=\int_{\mathsf G} e^{-i\langle\log y|\xi\rangle}\mathsf K_a\big(x,y^{-1}x\big)dy. \tag{21}$$

On the other hand, a short computation shows that $\mathsf{Ber}_\omega(f)$ is an integral operator in $L^2(\mathsf G)$ with kernel

$$\kappa_\omega(f):=\int_{\mathsf G}\int_{\mathfrak g^\sharp} f(z,\zeta)\,\omega_{z,\zeta}\otimes\overline{\omega_{z,\zeta}}\,dzd\zeta.$$

Hence one will have $\mathsf{Ber}_\omega(f)=\mathsf{Op}\big[a_\omega(f)\big]$ if and only if

$$\begin{aligned}\big[a_\omega(f)\big](x,\xi)&=\int_{\mathsf G}\int_{\mathsf G}\int_{\mathfrak g^\sharp} e^{-i\langle\log y|\xi\rangle} f(z,\zeta)\,\omega_{z,\zeta}(x)\,\overline{\omega_{z,\zeta}(y^{-1}x)}\,dydzd\zeta\\&=\int_{\mathsf G}\int_{\mathsf G}\int_{\mathfrak g^\sharp} e^{-i\langle\log y|\xi\rangle} f(z,\zeta)\,e^{-i\langle\log(zx)|\zeta\rangle}\omega(zx)\\&\quad e^{i\langle\log(zy^{-1}x)|\zeta\rangle}\,\overline{\omega(zy^{-1}x)}\,dydzd\zeta.\end{aligned}$$

□

Remark 7.3 We can compute the pseudo-differential symbol of the (regularizing) operator T in terms of the covariant symbol and the coherent states, using Proposition 6.7 and formula (21). This means, at least formally, that $T=\mathsf{Op}\big(a_T\big)$, with

$$\begin{aligned}a_T(x,\xi)&=\int_{\mathsf G} e^{-i\langle\log y|\xi\rangle}K_T\big(x,y^{-1}x\big)dy\\&=\int_{\mathsf G}\int_{\mathsf G}\int_{\mathfrak g^\sharp}\int_{\mathsf G}\int_{\mathfrak g^\sharp} e^{-i\langle\log y|\xi\rangle}\big[\mathrm{cov}_\omega(T)\big](z,\zeta;z',\zeta')\\&\quad\omega_{z',\zeta'}(x)\,\overline{\omega_{z,\zeta}\big(y^{-1}x\big)}\,dy\,dzd\zeta\,dz'd\zeta'.\end{aligned}$$

8 Other versions

8.1 τ-quantizations

Let $\tau:\mathsf G\to\mathsf G$ be any continuous map, that does not need to be a group morphism or to commute with inversion. The model is $x\to\tau x$ with $\tau\in[0,1]$ from the Abelian case $\mathsf G=\mathbb R^n$, but even in this simple case one can master much more than scalar transformations. In [22] such a parameter has been used in the global quantization involving the unitary dual $\widehat{\mathsf G}$ of the group. In a final section, for nilpotent groups, it also appeared involved in generalizing the quantization (19) of symbols on $\Xi=\mathsf G\times\mathfrak g^\sharp$, that may be replaced with

$$\big[\mathsf{Op}^{\tau}(a)u\big](x)=\int_{\mathsf{G}}\int_{\mathfrak{g}^{\sharp}}e^{i\langle\log(y^{-1}x)|\xi\rangle}a\big(\tau(xy^{-1})^{-1}x,\xi\big)u(y)\,dyd\xi. \tag{22}$$

Among other results, one can show the formula for the adjoint

$$\mathsf{Op}^{\tau}(a)^{*}=\mathsf{Op}^{\tilde{\tau}}(\overline{a}),\quad\text{where}\quad\tilde{\tau}(x):=\tau(x^{-1})x.$$

Note that $\tau(\cdot)=\mathsf{e}$ corresponds to the identity map $\tilde{\tau}(x)=x$, switching from the left to the right quantization, and vice versa. [For the right quantization $a(y,\xi)$ appears in (22)]. Thus the Hilbert space adjoint corresponds to complex conjugation of symbols if and only if $\tau=\tilde{\tau}$. If $\mathsf{G}=\mathbb{R}^n$ (with addition) the number $\tau=1/2$ solves this and corresponds to the Weyl quantization.

In [22, Sect. 4] the existence problem of such a symmetric parameter τ has been tackled for very general groups. In particular, a natural solution has been found for our nilpotent case, based on the vector structure of the Lie algebra on the fact that the group and the Lie algebra are diffeomorphic. Explicitly, one sets

$$\tau(x):=\int_0^1\exp[s\log x]ds.$$

Keeping τ arbitrary, we briefly (and formally) indicate in the sequel some of the modifications needed in the present paper to accommodate the quantization parameter τ.

Instead of (4), one can start with the family of unitary operators in $L^2(\mathsf{G})$

$$\big[\mathsf{W}^{\tau}(z,\zeta)u\big](x):=e^{i\langle\log[\tau(z)^{-1}x]|\zeta\rangle}u(z^{-1}x),\quad(z,\zeta)\in\mathsf{G}\times\mathfrak{g}^{\sharp},$$

coinciding with those from (4) if $\tau(\cdot):=\mathsf{e}$. We recall the formula $\mathsf{W}(z,\zeta)\equiv\mathsf{W}^{\mathsf{e}}(z,\zeta)=\mathsf{M}_{\zeta}\mathsf{L}_{z}$ (multiplications are placed to the left). For $\tau=\mathrm{id}$ one has the opposite ordering $\mathsf{W}^{\mathrm{id}}(z,\zeta)=\mathsf{L}_{z}\mathsf{M}_{\zeta}$.

Then the τ-Fourier–Wigner transform will be

$$\begin{aligned}\mathscr{W}^{\tau}_{u,v}(z,\zeta):=\big\langle\mathsf{W}^{\tau}(z,\zeta)u\,|\,v\big\rangle&=\int_{\mathsf{G}}e^{i\langle\log[\tau(z)^{-1}y]|\zeta\rangle}u\big(z^{-1}y\big)\overline{v(y)}dy\\&=\int_{\mathsf{G}}e^{i\langle\log x|\zeta\rangle}u(z^{-1}\tau(z)x)\overline{v(\tau(z)x)}dx.\end{aligned}$$

It consists of a partial Fourier transformation composed with a τ-depending change of variable.

Computing the adjoint of $\mathsf{W}^{\tau}(z,\zeta)$ leads to coherent states built upon $\omega\in\mathcal{S}(\mathsf{G})$ and depending on τ :

$$\omega^{\tau}_{z,\zeta}(x):=\big[\mathsf{W}^{\tau}(z,\zeta)^{*}\omega\big](x)=e^{-i\langle\log[\tau(z)^{-1}zx]|\zeta\rangle}\omega(zx).$$

This releases a sequence of τ-analogs of many of the notions and formulas above, with similar properties. For instance, a (slightly formal) expression for the τ-Berezin quantization is

$$\begin{aligned}&\big[\mathsf{Ber}^{\tau}_{\omega}(f)u\big](x)\\&\quad=\int_{\mathsf{G}}\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} e^{i\langle \log[\tau(z)^{-1}zy]-\log[\tau(z)^{-1}zx]|\zeta\rangle} f(z,\zeta)\,\overline{\omega(zy)}\,\omega(zx)\,u(y)\,dydzd\zeta.\end{aligned}$$

The reader may formulate other results in the setting of the τ-Berezin quantization. We only indicate another covariance result, valid for $\tau(x)=x$, that is different from Proposition 5.7 (in a very non-commutative setting ordering issues do matter if one wants simple formulas).

Proposition 8.1 *For any $\zeta\in\mathfrak{g}^\sharp$ and $f\in L^\infty(\mathsf{G}\times\mathfrak{g}^\sharp)$, one has*

$$\mathsf{M}^*_\zeta\mathsf{Ber}^{\mathrm{id}}_\omega(f)\mathsf{M}_\zeta=\mathsf{Ber}^{\mathrm{id}}_\omega\big[f(\cdot,\cdot-\zeta)\big].$$

Proof Using notations from Sect. 3, one has

$$\begin{aligned}\mathsf{M}^*_\zeta\,\mathsf{Ber}^{\mathrm{id}}_\omega(f)\mathsf{M}_\zeta u&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\mathsf{M}^*_\zeta\,\Omega^{\mathrm{id}}_{x,\xi}\mathsf{M}_\zeta u\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\big\langle u,\mathsf{M}^*_\zeta\mathsf{W}^{\mathrm{id}}(x,\xi)^*\omega\big\rangle\mathsf{M}^*_\zeta\mathsf{W}^{\mathrm{id}}(x,\xi)^*\omega\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\big\langle u,\mathsf{M}^*_\zeta\mathsf{M}^*_\xi\mathsf{L}^*_x\omega\big\rangle\mathsf{M}^*_\zeta\mathsf{M}^*_\xi\mathsf{L}^*_x\omega\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\big\langle u,\mathsf{M}^*_{\zeta+\xi}\mathsf{L}^*_x\omega\big\rangle\mathsf{M}^*_{\zeta+\xi}\mathsf{L}^*_x\omega\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\big\langle u,\mathsf{W}^{\mathrm{id}}(x,\zeta+\xi)^*\omega\big\rangle\mathsf{W}^{\mathrm{id}}(x,\zeta+\xi)^*\omega\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\xi)\,\Omega^{\mathrm{id}}_{x,\zeta+\xi}u\,dxd\xi\\&=\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} f(x,\eta-\zeta)\,\Omega^{\mathrm{id}}_{x,\eta}(u)\,dxd\eta.\end{aligned}$$

□

8.2 Magnetic quantization

In the same setting of a connected simply connected nilpotent group G, we consider *a magnetic field* B, i.e. a closed 2-form on G. It can be written as $B=dA$ for some 1-form (*vector potential*). Any other vector potential $\tilde{A}$ satisfying $B=d\tilde{A}$ is related to the first by $\tilde{A}=A+d\psi$, where ψ is a smooth function on G; it would lead to a unitarily equivalent formalism (gauge covariance).

For $x, y \in \mathsf{G}$ one defines the smooth function $[x, y] : \mathbb{R} \to \mathsf{G}$ by

$$[x, y]_s := \exp[(1 - s) \log x + s \log y] = \exp[\log x + s(\log y - \log x)].$$

Its range $[[x, y]] := \big\{ [x, y]_s \mid s \in [0, 1] \big\}$ is the segment in G connecting x to y. The circulation of the 1-form A through the segment $[[x, y]]$ is

$$\Gamma^A[[x, y]] \equiv \int_{[[x,y]]} A := \int_0^1 \big\langle \log y - \log x \,\big|\, A\big([x, y]_s\big)\big\rangle\, ds.$$

This leads to the following magnetic modification of the quantization (19)

$$[\mathsf{Op}^A(a)u](x) = \int_{\mathsf{G}} \int_{\mathfrak{g}^\sharp} e^{i \int_{[[x,y]]} A}\, e^{i\langle \log(xy^{-1})|\xi\rangle} a\big(x, \xi\big) u(y)\, dy d\xi,$$

that has been introduced in [3, Sect. 4]. One finds in [3] more general constructions, consisting in twisting by 2-cocycles pseudo-differential formalisms attached to type I unimodular locally compact groups. The Abelian case $\mathsf{G} = \mathbb{R}^n$ is deeply studied in [16,17,21], mainly in connection with magnetic Schrödinger operators.

At a basic level, the key modification is to replace the left regular representation $\mathsf{L} : \mathsf{G} \to \mathbb{B}[L^2(\mathsf{G})]$ with *the family of left magnetic translations*

$$\big[\mathsf{L}^A_z(u)\big](x) := e^{i \int_{[[x,z^{-1}x]]} A}\, u\big(z^{-1}x\big). \tag{23}$$

They do not even form a projective representation. By using Stokes' Theorem one checks that

$$\mathsf{L}^A_y \mathsf{L}^A_z = \Omega^B(y, z) \mathsf{L}^A_{yz}, \quad \forall\, y, z \in \mathsf{G}, \tag{24}$$

with $\Omega^B(y, z)$ the operator of multiplication by the function $x \to e^{\Gamma^B(x;y,z)}$, where $\Gamma^B(x; y, z)$ is the flux of the magnetic field B through the "triangle" in G with corners $x, y^{-1}x$ and $z^{-1}y^{-1}x$, defined by "segments" of the form $[\![a, b]\!]$ as defined above. So there is a magnetic contribution to the canonical commutation relations.

Consequently, one defines the family of unitary operators $\mathsf{W}^A(z, \zeta) := \mathsf{M}_\zeta \mathsf{L}^A_z$ in $L^2(\mathsf{G})$ (*the magnetic Weyl system*, labeled by $\Xi = \mathsf{G} \times \mathfrak{g}^\sharp$) by

$$\big[\mathsf{W}^A(z, \zeta)u\big](x) := e^{i\langle \log x|\zeta\rangle} e^{i \int_{[[x,z^{-1}x]]} A}\, u(z^{-1}x).$$

This leads to *magnetic coherent states*

$$\omega^A_{z,\zeta}(x) := \big[\mathsf{W}^A(z, \zeta)^* \omega\big](x) = e^{-i\langle \log(zx)|\zeta\rangle} e^{-i \int_{[[zx,x]]} A}\, \omega(zx), \tag{25}$$

and *the magnetic Fourier–Wigner transform*

$$\mathscr{W}^A_{u,v}(z, \zeta) := \big\langle \mathsf{W}^A(z, \zeta)u, v\big\rangle = \int_{\mathsf{G}} e^{i\langle \log y|\zeta\rangle} e^{i \int_{[[y,z^{-1}y]]} A}\, u(z^{-1}y)\, \overline{v(y)}\, dy. \tag{26}$$

The output is *a magnetic Berezin quantization*

$$\left[\mathsf{Ber}_{\omega}^{A}(f)u\right](x) = \int_{\mathsf{G}}\int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp} e^{i\langle \log(zy)-\log(zx)|\zeta\rangle} \exp\left\{ i\left(\int_{[[zy,y]]} A - \int_{[[zx,x]]} A\right)\right\}$$
$$f(z,\zeta)\,\overline{\omega(zy)}\,\omega(zx)\,u(y)\,dydzd\zeta .$$

The reader can easily extend the results of the main body of this article to the magnetic case. The τ-quantizations are also possible in this set up.

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