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"Equilibrium in financial markets with endogenous portfolio constraints and non-ordered preferences"

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Abstract

In this paper we include the possibility of individuals preferences to be non-ordered (i.e. the axioms of completeness and transitivity are not necessarily satisfied) on financial models with endogenous constraints on asset trading depending on prices. We explore three different assumptions that will allow us to show the existence of equilibrium by inducing upper bounds on asset prices that not all agents can access.

1 Introduction

It has always been of interest to what extent the usual hypotheses on consumer preferences can be lifted without compromising the result of existence of equilibrium in an exchange economy. This line of research has gained more interest since it has been shown that consumers do not behave rationally. Loomes and Taylor (1992) and Loewenstein and Prelec (1992) identify situations where transitivity and completeness fail to be satisfied, Philips (1989) gives a more philosophical approach, where he tries to separate rationality from transitivity.

The general result of existence of a Walrasian General Equilibrium that includes the possibility that preferences do not come from an ordered relation is due to Gale and Mas-Colell (1975). He and Yannelis (2016) go further on weakening the assumptions on preferences of the general equilibrium model to include interdependent and price-dependent preferences.

Bewley (2002) studies the case of Knightian uncertainty, agents are faced to make decisions over unknown probability distributions, where they maximize the outcome of the worst-case scenario. This is represented by agents whose utility functions are of the form $u(x) = \min_{\pi \in \Delta} \mathbb{E}_{\pi} x$, where Δ is a set of probability distributions over the possible states of nature and $\mathbb{E}_{\pi}$ is the expected value operator with respect to π. This scenario leads to the possibility that two different bundles may not be compared, thus, to incompleteness on preferences.

On the financial market we can see that access to credit or investment maybe restricted. There are regulatory or institutional considerations that may cause endogenous differentiation in access to commodity or asset markets. We may observe several trading restrictions: collateral requirements, consumption quotas or income-based access to funding. Recently, the focus of theoretical models has been the study of financial restrictions dependent on endogenous variables. In this line of work, Cass et al. (2001) and Carosi et al. (2009) include dependencies on consumption prices, Hoelle et al. (2016) considers dependencies on wealth and in Cea-Echenique and Torres-Martínez (2016a) portfolio constraints depend on prices of commodities and assets.

In this dissertation we extend three models of general equilibrium with endogenous restrictions on financial participation to include the possibility that preferences do not come from an ordered relation, we prove the existence of equilibrium in three different settings. In our first approach, we suppose that payments on segmented contracts can be fully hedged by payments in contracts which all individuals have access, extending the model exposed in Cea-Echenique and Torres-Martínez (2016a), this will allow to establish bounds on segmented contracts prices using the prices of unsegmented contracts.

In our second approach, we explore an hypothesis of essentiality of commodities, where indifference curves through endowments do not intersect the boundary of the consumption set, the existence of equilibrium under traditional assumptions on preferences is due to Faias and Torres-Martínez (2017). In this case we also extend the model to include restrictions on investment, even when preferences may be represented by a utility function. And in our third approach, which is based on Seghir and Torres-Martínez (2011), we suppose that there are agents for which for any reduction on second period consumption there is an amount commodities on the first period that generates a consumption bundle preferred to the first one.

In the next section we present the abstract economy an state equilibrium conditions. Section 3 starts defining the basic assumptions that allow to prove existence on classic models and the three further assumptions that will guarantee equilibrium existence in our context. Then, we present the proof of the existence of equilibrium under this setting. We end with three examples that satisfy only one of the cases of further assumptions.

2 Model

We will consider a two period economy, where 0 will denote the first period and $s \in S$ the state at the second period, with S a finite set. Also, we will denote $\mathcal{S} = \{0\} \cup S$. There is a finite set $\mathcal{L}$ of perfectly divisible commodities, which are traded at prices $p = (p_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$.[1]

The set of financial contracts available for trade at the first period will be denoted as $\mathcal{J}$. We will suppose it is finite and that promises make payments depending on the realization of uncertainty. The vector of asset prices will be denoted as $q = (q_j)_{j\in\mathcal{J}} \in \mathbb{R}_+^{\mathcal{J}}$ and $R_j(p) = (R_{s,j}(p))_{s\in S} \in \mathbb{R}_+^S$ will be the vector of payments respective to the asset $j \in \mathcal{J}$. We define as $\Pi \subset \left(\mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}}\right) \times \left(\mathbb{R}_+^{\mathcal{L}}\right)^S$ the space of commodity and asset prices $((p_0, q), (p_s)_{s\in S})$ satisfying $||p_0||_\Sigma \leq 1$ and $||p_s||_\Sigma = 1 \; \forall s \in S$ and the set of commodity prices $\mathbb{P} := \{(p_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}} \times \left(\mathbb{R}_+^{\mathcal{L}}\right)^S \; : \; ||p_0||_\Sigma \leq 1 \wedge ||p_s||_\Sigma = 1 \; \forall s \in S\}$. Also, let $\mathbb{E} := \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}}$ be the space of consumption and portfolio allocations.

Consumers belong to the set $\mathcal{I}$ which is finite and their preferences are represented by a correspondence $\mathbb{U}^i$, $i \in \mathcal{I}$ from $\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$ to $\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$. For each $x \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$, the set $\mathbb{U}^i(x)$ represents all strictly preferred consumption bundles to x, thus we suppose preferences are irreflexive, i.e. $x \notin \mathbb{U}^i(x), \; \forall x \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$. Agent i has initial endowments $w^i = (w_s^i)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$.

For each agent i there is a correspondence $\Phi^i : \Pi \twoheadrightarrow \mathbb{E}$ that determines her individual trading constraints, these may be endogenous and can depend on prices, investment and consumption. Given prices $(p, q) \in \Pi$, each agent i chooses a consumption bundle $x^i = (x_s^i)_{s\in\mathcal{S}}$ and a portfolio $z^i = (z_j^i)_{j\in\mathcal{J}}$ in her choice set $\mathcal{C}^i(p, q)$, which is characterized by

[1] We will consider only perishable goods but it is easy to generalize our results to durable goods with linear transformation technologies that depend on the state of nature at the second period.

$$\mathcal{C}^i(p,q) := \Big\{(x^i,z^i) \in \Phi^i(p,q) \ : p_0 x_0^i + q z^i \leq p_0 w_0^i,$$
$$p_s x_s^i \leq p_s w_s^i + \sum_{j \in \mathcal{J}} R_{s,j}(p) z_j^i, \quad \forall s \in S.\Big\}$$

Equilibrium

A vector $((\overline{p},\overline{q}),(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}) \in \Pi \times \mathbb{E}^{\mathcal{I}}$, where $(\overline{x}^i,\overline{z}^i) \in \mathcal{C}^i(\overline{p},\overline{q})\ \forall i \in \mathcal{I}$, is a competitive equilibrium for the economy with endogenous trading constraints when the following conditions hold:

1. Each agent $i \in \mathcal{I}$ maximizes her preferences,

$$x \in \mathbb{U}^i(\overline{x}^i) \Rightarrow (x,z) \notin \mathcal{C}^i(\overline{p},\overline{q}),\ \forall z \in \mathbb{R}^{\mathcal{J}}.$$

2. Individuals' plans are market feasible,

$$\sum_{i \in \mathcal{I}} (\overline{x}^i - w^i, \overline{z}^i) = 0.$$

3 Basic Assumptions

In this section we will impose assumptions on the preferences, initial endowments, financial promises and individuals trading constraints that will allow us to prove the existence of equilibrium. We will start with hypotheses on the correspondence $\mathbb{U}^i$ equivalent to traditional assumptions on utility functions that allow the use of traditional fixed-point tecniques to prove the existence of equilibrium.

Assumption (A1)

(a) For any $i \in \mathcal{I}$, $\mathbb{U}^i$ has an open graph, their values are nonempty and for every $s \in \mathcal{S}$, every $(x_k)_{k\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$ and $\epsilon > 0$ there exists $y_s \in \mathbb{R}_+^{\mathcal{L}}$ such that, $||y_s - x_s|| < \epsilon$ and $((x_k)_{k\neq s}, y_s) \in \mathbb{U}^i((x_k)_{k\in\mathcal{S}})$.

(b) For each $(s,l) \in \mathcal{S}\times\mathcal{L}$ there exists $i \in \mathcal{I}$ such that for all $\epsilon > 0$, $x + \epsilon e_{s,l} \in \mathbb{U}^i(x)$, where $e_{s,l}$ is the allocation composed by just one unit of (s,l). In addition, $w^i \gg 0$, $\forall i \in \mathcal{I}$.

(c) Asset payments are continuous functions of prices satisfying $R_j(p) \neq 0$, $\forall j \in \mathcal{J}$, $\forall p \gg 0$.

Assumption (A1)(a) is equivalent to continuity and local non-satiability of preferences. Assumption (A1)(b) means that for each state and commodity there exists an agent whose preference is better off with more consumption of that good in that state, i.e. her utility is increasing in $x_{s,l}$, along with that we also suppose endowments are interior. And Assumption (A1)(c) we suppose that payments do not vary abruptly with prices and that there is no promise on financial contracts that pays zero on every state independently of prices.

Now we will establish basic properties on the trading constraint correspondences.

Assumption (A2)

(a) The correspondences $\{\Phi^i\}_{i\in\mathcal{I}}$ are lower hemicontinous with convex values and closed graph relative to $\Pi \times \mathbb{E}$.

(b) Agents are not burden to trade assets, i.e., $(0,0) \in \bigcap_{(p,q)\in\Pi} \Phi^i(p,q),\ \forall i \in \mathcal{I}$.

(c) There are no constraints on consumption, i.e., $\Phi(p,q) + \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \{0\} \subset \Phi^i(p,q)$, $\forall (p,q) \in \Pi,\ \forall i \in \mathcal{I}$.

Assumption (A2)(a) ensure us that trading constraints behavior does not compromise the continuity of individual demands, every consumption plan that is feasible at prices p can be approximated by consumption plans at prices near p, convex combinations of trading admissible plans are also admissible and any convergent sequence of prices and trading admissible plans has a trading admissible limit. Under Assumption (A2)(b) there is no obligation to trade assets nor restrictions on consumption. And even though under Assumption (A2)(c) there is no restriction on consumption, we allow for restrictions on investment.

We define the correspondence of *attainable allocations* $\Omega : \Pi \twoheadrightarrow \mathbb{E}^{\mathcal{I}}$, as the set of market feasible allocations that satisfy agents' budget and trading constraints associated with each price level, i.e.,

$$\Omega(p,q) := \left\{ ((x^i,z^i))_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathcal{C}^i(p,q) \ : \ \sum_{i\in\mathcal{I}} (x^i,z^i) = \sum_{i\in\mathcal{I}} (w^i,0) \right\}.$$

Notice that, any element of $\Omega(p,q)$ satisfies budget constraints with equality.

A contract j is *unsegmented* if for every vector of prices $(p,q) \in \Pi$ there exists $\delta > 0$ such that $-\delta e_j \in \cap_{i\in\mathcal{I}} \Phi^i(p,q)$, where e_j is the allocation composed by just one unit of j. Let $\mathcal{J}_a$ be the set of unsegmented contracts and $\mathcal{J}_b := \mathcal{J} \setminus \mathcal{J}_a$ the set of segmented contracts.

Assumption (A3)

(a) For any compact set $\mathbb{P}' \subset \mathbb{P}$, $\bigcup_{(p,q)\in\mathbb{P}'\times\mathbb{R}_+^{\mathcal{J}}:(p,q)\gg 0} \Omega(p,q)$ is bounded.

(b) $(x^i,z^i) - \alpha e_k \in \Phi^i(p,q),\ \forall k \in \mathcal{J}_b,\ \forall \alpha \in [0, \max\{z_k^i, 0\}]$.

This assumption restricts the trading constraints of each individual. In Assumption (A3)(a) we suppose that for any price level there are endogenous bounds on individuals' variables, as equilibrium may cease to exist with incomplete markets, real assets and unbounded admissible short-sales[2]. Assumption (A3)(b) imposes that long positions in segmented contracts can be reduced without compromising trading admissibility.

Further Assumptions

In this section we will introduce further assumptions that will be essential to prove equilibrium existence as each of them will allow us to find upper bounds on segmented asset prices, we will assume only one of them holds at a time.

Assumption (A4)I – Super-replication

(a) For each compact set $\Pi' \subset \Pi$, there exists $\widehat{z} \in \mathbb{R}_{+}^{\mathcal{J}_a}$ such that,

$$\sum_{j \in \mathcal{J} \setminus \mathcal{J}_a} R_{s,j}(p) \leq \sum_{k \in \mathcal{J}_a} R_{s,k}(p) \widehat{z}_k, \ \forall s \in S, \ \forall p \in \mathbb{P}.$$

This assumption requires payments to be able to be super-replicated by positions on contracts that all agents can short sale.

Assumption (A4)II – Essentiality of Commodities

In this case, in order to avoid generating dependency cycles, we will assume that personalized trading constraints do not depend on asset prices.

(a) $\forall (p,q),(p,q') \in \Pi, \ \Phi^i(p,q) = \Phi^i(p,q'), \ \forall i \in \mathcal{I}.$

Furthermore, we will assume the following property of *compensation of small losses in segmented markets*,

(b) For each $i \in \mathcal{I}$ and $x \in \mathbb{R}_{++}^{\mathcal{L}\times\mathcal{S}}$, there exists $\big(\varepsilon^i(x), \tau^i(x)\big) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\mathcal{L}}$, that are continuous on x and net trades $(\theta_s^i(p,x))_{s\in S} \in \mathbb{R}^{\mathcal{L}\times S}$ for any $p \in \Pi$ with $p \gg 0$ such that

$$p_s \theta_s^i(p,x)) \leq -\varepsilon^i(x) \sum_{k \in \mathcal{J}_b} R_{s,k}(p), \quad \forall s \in S,$$

and $(x_0 + \tau^i(x), (x_s + \theta_s^i(p,x))_{s\in S}) \in \mathbb{U}^i(x)$.

Faias and Torres-Martinez (2017) show that this property always hold in economies with continuous, strictly quasiconcave and locally non-satiable utility functions together with Assumption A1(c).

Along with this, we assume *essentiality of commodities*, that is, the set of preferred bundles to initial endowments does not include zero consumption of any good,

(c) $w^i \in \mathbb{U}^i(y), \ \forall i \in \mathcal{I}, \ \forall y \in \partial\mathbb{R}_{+}^{\mathcal{L}\times\mathcal{S}}$.[3]

[2] This was pointed out by Hart (1975)

[3] Where for any set X, $\partial X = \overline{X} \setminus \mathring{X}$.

By the irreflexibity property of $\mathbb{U}^i$, this also implies that $w^i \notin \partial\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$.

Assumption (A4)III – Impatience on Preferences

(a) There exists a non-empty subset of agents $\mathcal{I}' \subset \mathcal{I}$ such that, given $i \in \mathcal{I}'$ and $(\rho, x^i) \in (0,1) \times \mathbb{R}_{++}^{\mathcal{L}\times\mathcal{S}}$, there exists $\tau^i(\rho, x^i) \in \mathbb{R}^{\mathcal{L}}$ such that,

$$\left(x_0^i + \tau^i(\rho, x^i), (\rho x_s^i)_{s\in S}\right) \in \mathbb{U}^i(x^i).$$

(b) Given $j \in \mathcal{J}_b$, there is $i \in \mathcal{I}'$ and $z^i \in \mathbb{R}_-^{\mathcal{J}}$ such that $z_j^i < 0$ and $(0, z^i) \in \Phi^i(p,q),\ (p,q) \in \Pi$.

(c) For $x, y \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$ such that $x_{l,s} \geq y_{l,s}\ \forall (l,s) \in \mathcal{L}\times\mathcal{S}$ and for some $(l,s),\ x_{l,s} > y_{l,s}$, then $x \in \mathbb{U}^i(y)\ \forall i \in \mathcal{I}$.

(d) If $x \in \mathbb{U}^i(y)$ for some $x, y \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$, then $\mathbb{U}^i(x) \subset \mathbb{U}^i(y)$.

The first of these assumptions requires that for some agents can compensate reduction in utility due to a cut on consumption in the second period with an increase in first period consumption. The second requirement guarantees that for any segmented contract, there exists an agent that can short-sale it without the need to consume or invest in other assets.

We consider two other assumptions on preferences to find upper bounds on prices of segmented assets. So the third assumption is monotonicity for all individuals on all goods and states, and the fourth is transitivity of preferences. The non-ordering of preferences that we still allow is incompleteness.

4 Equilibrium Existence

Theorem 4.1. *Under Assumptions (A1)-(A3) and each of the cases in Assumption (A4) there is a competitive equilibrium.*

Proof. We start by defining the *augmented preference mapping* $\widehat{\mathbb{U}}^i(x)$ as

$$\widehat{\mathbb{U}}^i(x) := \mathbb{U}^i(x) \cup \{x' \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} : x' = \lambda x'' + (1-\lambda)x \ \text{ for } \ 0 < \lambda < 1 \text{ and } \ x'' \in \mathbb{U}^i(x)\}.$$

Notice that the sets $\widehat{\mathbb{U}}^i(x)$ inherits the irreflexivity property from $\mathbb{U}^i(x)$ and has an open graph with non-empty convex values. Also, notice that $x \in \overline{\widehat{\mathbb{U}}^i(x)}\ \forall x \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$. It is easy to see that any equilibrium with the augmented preferences must be an equilibrium with the original preference mapping.

For $M \in \mathbb{N}$, we define the normalized space of prices $\Pi(M) := \{((p_0, (q_k)_{k\in\mathcal{J}_a}), (q_j)_{j\in\mathcal{J}_b}, (p_s)_{s\in S}) \in \Pi : ||(p_0, (q_k)_{k\in\mathcal{J}_a})||_\Sigma = 1, q_j \in [0, M]\ \forall j \in \mathcal{J}_b\}$, and let $\mathbb{P}_0(M) = \{((p_0, (q_k)_{k\in\mathcal{J}_a}), (q_j)_{j\in\mathcal{J}_b}) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}} : ||(p_0, (q_k)_{k\in\mathcal{J}_a})||_\Sigma = 1, q_j \in [0, M]\ \forall j \in \mathcal{J}_b\}$ and for every $s \in S$, $\mathbb{P}_s := \{p_s \in \mathbb{R}_+^{\mathcal{L}} : ||p_s||_\Sigma = 1\}$, so we have $\Pi(M) = \mathbb{P}_0(M) \times \mathbb{P}_s^S$. In addition we will consider $\mathcal{P} = \{((p_s)_{s\in\mathcal{S}}, (q_j)_{j\in\mathcal{J}_a}) \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}_a} \;:\; ||(p_0, (q_j)_{j\in\mathcal{J}_a})||_\Sigma = 1,\ ||p_s||_\Sigma = 1,\ \forall s \in S\}$.

For each of the cases of Assumption (A4) we will consider a different truncation of the allocation set defined by a correspondence, but the arguments for the existence of equilibrium will be identical. Let us consider a correspondence $\mathbb{K} : \mathbb{R} \twoheadrightarrow \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}}$ such that it has convex and compact values and for any $M \in \mathbb{R}_+$ and $(p,q) \in \Pi(M)$, $\mathbb{K}(M) \cap \mathring{\mathcal{C}}^i(p,q) \neq \emptyset$. For any $(p,q) \in \Pi(M)$, we define

$$\Gamma^i(p,q) := \{(x^i, z^i) : (x^i, z^i) \in \mathbb{K}(M) \cap \mathring{\mathcal{C}}^i(p,q)\}$$

which has an open graph in $\Pi(M) \times \mathbb{K}(M)$. For each $i \in \mathcal{I}$ let $\varphi^i : \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \twoheadrightarrow \mathbb{K}(M)$ be defined as,

$$\varphi^i\left(p, q, (x^i, z^i)_{i\in\mathcal{I}}\right) := \begin{cases} \Gamma^i(p,q) & \text{if } x^i \notin \mathcal{C}^i(p,q), \\ \Gamma^i(p,q) \cap \widehat{\mathbb{U}}^i(x^i) & \text{if } x^i \in \mathcal{C}^i(p,q), \end{cases}$$

which are convex valued and may be empty valued. To see that they have an open graph, take $i \in \mathcal{I}$ and

$$\begin{aligned} A_i &:= \left\{((p,q),(x,z),(y,w)) \in \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \times \mathbb{K}(M) : (x_i, z_i) \notin \mathcal{C}^i(p,q)\right\}, \\ B_i &:= \left\{((p,q),(x,z),(y,w)) \in \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \times \mathbb{K}(M) : (y,w) \in \mathring{\mathcal{C}}^i(p,q)\right\}, \\ C_i &:= \left\{((p,q),(x,z),(y,w)) \in \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \times \mathbb{K}(M) : z \in \mathbb{U}^i(x)\right\}, \end{aligned}$$

which are all open relative to $\Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \times \mathbb{K}(M)$, and so is the graph of φ^i that can be written as $(A_i \cap B_i) \cup (C_i \cap B_i)$. Let $\varphi^0 : \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \twoheadrightarrow \mathbb{P}_0(M)$ be defined as,

$$\varphi^0\left(p, q, (x^i, z^i)_{i\in\mathcal{I}}\right) := \left\{(p_0', q') \in \mathbb{P}_0(M) \;:\; \sum_{i\in\mathcal{I}} p_0' x_0^i + \sum_{i\in\mathcal{I}} q' z^i > \sum_{i\in\mathcal{I}} p_0' w_0^i\right\},$$

and for every $s \in S$ we define $\varphi^s : \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}} \twoheadrightarrow \mathbb{P}_s$ as,

$$\varphi^s\left(p, q, (x^i, z^i)_{i\in\mathcal{I}}\right) := \left\{p_s' \in \mathbb{P}_s \;:\; \sum_{i\in\mathcal{I}} p_s' x_s^i > \sum_{i\in\mathcal{I}} \left[p_s' w_s^i + \sum_{j\in\mathcal{J}} R_{s,j}(p') z_j^i\right]\right\},$$

it is easy to see that these correspondences have open graphs and are convex valued.

Lemma 4.2. *There exists* $\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) \in \Pi(M) \times \mathbb{K}(M)^{\mathcal{I}}$ *such that* $\varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset\ \forall i \in \mathcal{I}$, $\varphi^s\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset\ \forall s \in S$ *and* $\varphi^0\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset$.

Proof: The correspondences φ^i, φ^0 and φ^s satisfy the conditions of the Gale and Mas-Colell Fixed Point Theorem, that is φ^i are convex valued and have an open graph in $\Pi(M) \times \mathbb{K}(M)^{\mathcal{L}} \times \mathbb{K}(M)\ \ \forall i \in \mathcal{I}$, φ^0 is convex valued and has an open graph in $\Pi(M) \times \mathbb{K}(M)^{\mathcal{L}} \times \mathbb{P}_0(M)$ and φ^s are convex valued and have an open graph in $\Pi(M) \times$

$\mathbb{K}(M)^{\mathcal{L}} \times \mathbb{P}_s \ \forall s \in S$. Let $\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ be the fixed point that satisfies $(\overline{x}^i, \overline{z}^i) \in \varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ or $\varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset \ \forall i \in \mathcal{I}$, $(\overline{p}_0, \overline{q}) \in \varphi^0\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ or $\varphi^0\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset$ and $\overline{p}_s \in \varphi^s\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ or $\varphi^s\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset \ \forall s \in S$.

Since $\Gamma^i(\overline{p}, \overline{q}) \subset \mathring{\mathcal{C}}^i(\overline{p}, \overline{q})$ and its nonempty, we must have $\overline{x}^i \in \mathcal{C}^i(\overline{p}, \overline{q})$, or else the point would not satisfy the conclusion of the theorem. Because $\widehat{\mathbb{U}}^i(x)$ is irreflexive, we cannot have $\overline{x}^i \in \varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ this implies $\varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset \ \forall i \in \mathcal{I}$.

Likewise, $\overline{x}^i \in \mathcal{C}^i(\overline{p}, \overline{q}) \ \forall i \in \mathcal{I}$ implies that

$$\sum_{i\in\mathcal{I}} \overline{p}_0 \overline{x}^i_0 + \sum_{i\in\mathcal{I}} \overline{q} \overline{z}^i \le \sum_{i\in\mathcal{I}} \overline{p}_0 w^i_0,$$

$$\sum_{i\in\mathcal{I}} \overline{p}_s \overline{x}^i_s \le \sum_{i\in\mathcal{I}} \left[\overline{p}_s w^i_s + \sum_{j\in\mathcal{J}} R_{s,j}(\overline{p}) \overline{z}^i_j\right], \quad \forall s \in S$$

which means $(\overline{p}_0, \overline{q}) \notin \varphi^0\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ and $\overline{p}_s \notin \varphi^s\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) \ \forall s \in S$, so $\varphi^0\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset$ and $\varphi^s\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset$ □

Now we will show that in each of the cases of Assumption (A4) the prices of segmented assets are bounded. In each of the cases we will consider specific correspondences $\mathbb{K}$ that will define bounds on consumption, investment and debt, we will show that for M large enough we have $(\overline{x}^i, \overline{z}^i) \in \mathring{\mathbb{K}}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$, $\forall i \in \mathcal{I}$. Finally, we will provide a proof that the point is effectively an equilibrium for the economy, for any of the three cases.

Superreplication:

Let us consider the truncated allocation set

$$\mathbb{K}(M) := [0, 2\overline{w}(M)]^{\mathcal{L}\times\mathcal{S}} \times [-\overline{\Omega}(M), \#\mathcal{L}\overline{\Omega}(M)]^{\mathcal{J}}$$

where,

$$\overline{\Omega}(M) := 2 \sup_{(p,q)\in\Pi(M):(p,q)\gg 0} \ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)} \sum_{i\in\mathcal{I}} ||z^i||_{\Sigma},$$

and

$$\overline{w}(M) := \left(\#\mathcal{J}\#\mathcal{I}\overline{\Omega}(M) + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}} \sum_{i\in\mathcal{I}} w^i_{s,l}\right)\left(1 + \max_{((p,q),s)\in\Pi\times\mathcal{S}} \sum_{j\in\mathcal{J}} R_{s,j}(p)\right).$$

It is clear that the correspondence $\mathbb{K}$ defined above has compact and convex values, moreover the correspondence Γ^i is not empty for any $M > 0$.

As Assumption (A4)(II.) hold, we can find $\widehat{z}$ that super-replicates the promises of segmented assets. Because Assumption (A1)(a) is equivalent to local non-satiability of preferences, we can use Lemma 3 on Cea-Echenique and Torres-Martínez (2016) to assert that if $\left(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}\right)$ is such that,

$$\overline{p}\gg 0,\quad \sum_{i\in\mathcal{I}}\overline{z}^i_k\le 0,\ \forall k\in\mathcal{J}_a;$$

$$\sum_{i\in\mathcal{I}}\overline{x}^i_{s,l}<2\overline{w}(M),\ \forall(s,l)\in\mathcal{S}\times\mathcal{L}.$$

Then, for $\overline{Q}:=\max\{1,2\max_{k\in\mathcal{J}_a}\widehat{z}_k\}$ and any $j\in\mathcal{J}_b$ we have that,

$$\overline{q}_j>0\ \wedge\ \sum_{i\in\mathcal{I}}\overline{z}^i_j>0\Longrightarrow\overline{q}_j\le\overline{Q}.$$

For any $M>\overline{Q}$, let $\left(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}\right)$ be the fixed point of the correspondences φ^i, φ^0 and φ^s. Since $\varphi^0\left(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}\right)=\emptyset$ we have that

$$p_0\sum_{i\in\mathcal{I}}\left(\overline{x}^i_0-w^i\right)+\sum_{i\in\mathcal{I}}q\overline{z}^i\le 0,\quad\forall(p_0,q)\in\mathbb{P}_0(M).$$

Adding individual's budget constraints and from the definition of $\mathbb{K}(M)$ we get that,

$$p_s\sum_{i\in\mathcal{I}}(\overline{x}^i_s-w^i_s)\le\overline{w}(M),\quad\forall p_s\in\mathbb{P}_s,\ \forall s\in S.$$

So we have

$$\sum_{i\in\mathcal{I}}(\overline{x}^i_0-w^i_0)\le 0,\quad\sum_{i\in\mathcal{I}}\overline{z}^i_k\le 0,\quad\forall k\in\mathcal{J}_a,\tag{1}$$

$$\sum_{i\in\mathcal{I}}\overline{x}^i_{s,l}<2\overline{w}(M),\quad\forall(s,l)\in S\times\mathcal{L}.\tag{2}$$

And $\overline{q}_j=M$ for every $j\in\mathcal{J}_b$ such that $\sum_{i\in\mathcal{I}}\overline{z}^i_k>0,\ \forall k\in\mathcal{J}_b$. Equation (2) implies that $\overline{p}\gg 0$ because of Assumption A1(b). By Lemma 3 on Cea-Echenique and Torres-Martínez (2016) and because $M>\overline{Q}$ we have that $\sum_{i\in\mathcal{I}}\overline{z}^i\le 0$. We conclude that,

$$(\overline{p},\overline{q})\in\mathbb{P}(M),\quad(\overline{p},\overline{q})\gg 0,$$

$$\sum_{i\in\mathcal{I}}\left(\overline{x}^i-(w^i_s)_{s\in\mathcal{S}}\right)=0,\quad\sum_{i\in\mathcal{I}}\overline{z}^i=0,$$

and because of Assumption (A3) we have that $(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}\in\Omega(\overline{p},\overline{q})\cap\mathring{\mathbb{K}}(M)^{\mathcal{I}}$. □

Essentiality of Commodities:

We will define $\Lambda : \mathbb{P} \twoheadrightarrow \mathbb{E}^{\mathcal{I}}$ as

$$\Lambda(p) = \left\{ (x,z) \in \Omega(p,(1,...,1)) \ : \ p_s x_s = p_s w_s^i + \sum_{j\in\mathcal{J}} R_{s,j}(p) z_j^i, \ \forall (i,s) \in \mathcal{I} \times S \right\}.$$

And let $\mathbb{K}^* := [0, 3w^*]^{\mathcal{L}\times\mathcal{S}} \times [-2\Lambda^*, 2\#\mathcal{I}\Lambda^*]^{\mathcal{J}}$, where

$$w^* := 1 + \sum_{i\in\mathcal{I}} ||w^i||_\Sigma + \max_{(p,s)\in\mathbb{P}\times S} \left(\#\mathcal{J}\#\mathcal{I}2\Lambda^* \sum_{j\in\mathcal{J}} R_{s,j}(p) \right),$$

$$\Lambda^* := \sup_{p\in\mathbb{P}:p\gg 0} \ \sup_{a\in\Lambda(p)} ||a||_\Sigma.$$

Λ^* is finite as a consequence of Assumption (A4)(II). We can see that $\mathbb{K}^*$ satisfies convexity and is compact, also, the correspondence Γ^i is not empty.

Lemma 4.3. *Under Assumptions (A1)(a) and (A4)(II.), let $\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ be a fixed point of the correspondences φ^i, φ^s and φ^0 that satisfies $\overline{p} \gg 0$, $\sum_{i\in\mathcal{I}} \overline{x}_0^i \leq \sum_{i\in\mathcal{I}} w_0^i$ and $\max_{s\in S} \sum_{i\in\mathcal{I}} ||\overline{x}_s^i||_\Sigma \leq 2w^*$. Then for any $k \in \mathcal{J}_b$, we have that*

$$\sum_{i\in\mathcal{I}} \overline{z}_k^i > 0 \Rightarrow \overline{q}_k < Q^* := \max_{i\in\mathcal{I}} \max_{x\in[0,3w^*]^{\mathcal{L}\times\mathcal{S}}:x_{s,l}\geq 0.5\rho, \forall (s,l)\in\mathcal{S}\times\mathcal{L}} \frac{||\tau^i(x)||_\Sigma}{\varepsilon^i(x)},$$

where $(\varepsilon^i, \tau^i)_{i\in\mathcal{I}}$ implements the compensation of small losses in segmented markets.

Suppose that there is excess of demand for $k \in \mathcal{J}_b$. Let $h \in \mathcal{I}$ be such that $\overline{z}_k^h > 0$. Because $\overline{p} \gg 0$ and $(\overline{x}_s^h)_{s\in\mathcal{S}} \gg 0$, we can find $(\theta_s^h(\overline{p}, \overline{x}^h))_{s\in S} \in \mathbb{R}^{\mathcal{L}\times S}$ that satisfies

$$\overline{p}_s \theta_s^h(\overline{p}, \overline{x}^h) \leq -\varepsilon^h(\overline{x}^h) \sum_{j\in\mathcal{J}_b} R_{s,j}(\overline{p}), \quad s \in S,$$

such that $\left(\overline{x}^h + (\tau^h(\overline{x}^h), (\theta_s^h(\overline{p}, \overline{x}^h))_{s\in S}\right) \in \widehat{\mathbb{U}}^h(\overline{x}^h)$. Since $\sum_{i\in\mathcal{I}} \overline{x}_0^i \leq \sum_{i\in\mathcal{I}} w_0^i$ and $\max_{s\in S} \sum_{i\in\mathcal{I}} ||\overline{x}_s^i||_\Sigma \leq 2w^*$ we can find $\nu \in (0,1]$ such that $0 < \nu\varepsilon^h(\overline{x}^h) \leq \overline{z}_k^h$, $\overline{x}_0^h + \nu\tau^h(\overline{x})^h \in [0, 3w^*]^{\mathcal{L}}$, and $\overline{x}_s^h + \nu\theta_s^h(\overline{p}, \overline{x}^h) \in [0, 3w^*]^{\mathcal{L}}$, $\forall s \in S$. From the continuity of payoffs, agent h can reduce her position in k from $\overline{z}_k^h$ to $\overline{z}_k^h - \nu\varepsilon^h(\overline{x}^h)$. By the definition of $\widehat{\mathbb{U}}^h$ we have that $\left(\overline{x}^h + \nu(\tau^h(\overline{x}^h), (\theta_s^h(\overline{p}, \overline{x}^h)_{s\in S})\right) \in \widehat{\mathbb{U}}^h(\overline{x}^h)$, the resources she obtains from the reduction, $\overline{q}_k\nu\varepsilon^h(\overline{x}^h)$, cannot be enough to buy the bundle $\nu\tau^h(\overline{x}^h)$. We conclude $\overline{q}_k < Q^*$ because $(\overline{p}, \overline{q}) \in \Pi(M)$. □

Impatience on Preferences:

For $M>0$, let $\mathbb{K}(M):=[0,2\widehat{w}+M]^{\mathcal{L}\times\mathcal{S}}\times\left[-\widehat{\Omega},\#\mathcal{I}\widehat{\Omega}\right]^{\mathcal{J}}$, where

$$\widehat{w}:=\left(\#\mathcal{J}\#\mathcal{I}\widehat{\Omega}+\sum_{(s,l)\in\mathcal{S}\times\mathcal{L}}\sum_{i\in\mathcal{I}}w^i_{s,l}\right)\left(1+\max_{s\in S}\max_{(p,q)\in\mathcal{P}}\sum_{j\in\mathcal{J}}R_{s,j}(p)\right),$$

$$\widehat{\Omega}:=2\sup_{(p,q)\in\mathcal{P}\times\mathbb{R}^{\mathcal{J}_b}_+:(p,q)\gg0}\ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)}\sum_{i\in\mathcal{I}}||z^i||_\Sigma.$$

Assumption A4(III.) guarantees that $\widehat{\Omega}$ is finite. Once again, the correspondences $\mathbb{K}$ and Γ^i satisfy the needed assumptions to guarantee the existence of the fixed point of Lemma 4.2.

Lemma 4.4. *Under Assumptions (A1)-(A3) and (A4)(III.), let $\left(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}\right)$ be a fixed point of the correspondences φ^i, φ^s and φ^0 that satisfies $\overline{x}^i_{s,l}<2\widehat{w}$, $\forall\mathcal{S}\times\mathcal{L}$. Then, there exists $\widehat{Q}>0$ such that, for M large enough, $\overline{q}_j\leq\widehat{Q}$, $\forall j\in\mathcal{J}_b$.*

For $i\in\mathcal{I}'$ take $\rho^i\in(0,1)$ such that $2\widehat{w}\rho^i=0.25\min_{(s,l)\in S\times\mathcal{L}}w^i_{s,l}$. Assumption (A4)(III.)(a), (c) and (d) ensure us that,

$$\left(2\widehat{w}(1,...,1)+\tau^i(\rho^i,2\widehat{w}(1,...,1)),\left(\frac{w^i_s}{2}\right)_{s\in S}\right)\in\widehat{\mathbb{U}}^i(2\widehat{w}(1,...,1))\subset\widehat{\mathbb{U}}^i(\overline{x}^i).$$

For any $j\in\mathcal{J}_b$ we can choose $i(j)\in\mathcal{I}'$ such that there is $z^i\leq0$ with $z^i_j<0$ and $(0,z^i)\in\Phi^i(p,q)$, $\forall(p,q)\in\Pi$. Because $(0,0)\in\Phi^i(p,q)$ and the convexity of $\Phi^i(p,q)$, we have that $(0,\epsilon z^i)\in\Phi^i(\overline{p},\overline{q})$, $\forall\epsilon\in[0,1]$. Since payoffs are continuous we can find $\epsilon^i\in(0,1)$ such that,

$$\epsilon^i\max_{(p,q)\in\mathcal{P}}\max_{s\in S}\sum_{k\in\mathcal{J}}R_{s,k}(p)z^i_k<0.5\min_{(s,l)\in S\times\mathcal{L}}w^i_{s,l}.$$

From assumption (A2)(c) we have that, for every $M>\widehat{M}$ where we consider $\widehat{M}:=\max_{i\in\mathcal{I}'}||\tau^i(\rho^i,2\widehat{w}(1,...,1))||_\Sigma$,

$$\left(\left(2\widehat{w}(1,...,1)+\tau^i(\rho^i,2\widehat{w}(1,...,1)),\left(\frac{w^i_s}{2}\right)_{s\in S}\right),\epsilon^iz^i\right)\in\Phi^i(\overline{p},\overline{q})\cap\mathbb{K}(M).$$

Because $(\overline{x}^i,\overline{z}^i)$ is an optimal choice for agent i in $\mathcal{C}^i(\overline{p},\overline{q})\cap\mathbb{K}(M)$ and $z^i\leq0$, it we have that

$$2\widehat{w}||\overline{p}_0||_\Sigma+\overline{p}_0(\tau^i(\rho^i,2\widehat{w}(1,...,1))-w^i_0)>-\epsilon^i\overline{q}z^i\geq\epsilon^i\overline{q}_j|z^i_j|,$$

it follows that $\overline{q}_j\leq(2\widehat{w}+\widehat{M})/(\epsilon^i|z^i_j|)$. So we can consider

$$\widehat{Q}:=\max_{j\in\mathcal{J}_b}\frac{2\widehat{w}+\widehat{M}}{\epsilon^{i(j)}|z^{i(j)}_j|},$$

where we use the fact that $i=i(j)$ is fixed. □

Lemma 4.5. *Under Assumptions (A1)-(A3) and for M large enough, the fixed point $\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ is a competitive equilibrium for the economy.*

Suppose that for some $i \in \mathcal{I}$ there exists $x \in \widehat{\mathbb{U}}^i(\overline{x}^i)$ such that $(x, z) \in \mathbb{K}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$ for some $z \in \mathbb{R}^{\mathcal{J}}$. Because $\Gamma^i(\overline{p}, \overline{q})$ is nonempty we can find $(x', z') \in \mathbb{K}(M) \cap \mathring{\mathcal{C}}^i(\overline{p}, \overline{q})$. The openness of $\widehat{\mathbb{U}^i}(\overline{x}^i)$ guarantees that there exists $\lambda \in (0, 1)$ such that $(x_\lambda, z_\lambda) = \lambda(x', z') + (1 - \lambda)(x, z) \in \widehat{\mathbb{U}}^i(\overline{x})$. But since $\mathbb{K}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$ is convex we must have $(x_\lambda, z_\lambda) \in \mathbb{K}(M) \cap \mathring{\mathcal{C}}^i(\overline{p}, \overline{q})$, this implies $(x_\lambda, z_\lambda) \in \varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right)$ which contradicts $\varphi^i\left(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}\right) = \emptyset$. So, there is no such $(x, z) \in \mathbb{K}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$.

Now, take $(y, w) \in \mathcal{C}^i(\overline{p}, \overline{q})$ such that $y \in \widehat{\mathbb{U}}^i(\overline{x}^i)$. Because $(\overline{x}^i, \overline{z}^i) \in \mathring{\mathbb{K}}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$ we can find $\lambda' \in (0, 1)$ such that $\lambda'(y, w) + (1 - \lambda')(\overline{x}^i, \overline{z}^i) \in \mathbb{K}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$. By the definition of $\widehat{\mathbb{U}}^i$ we have $\lambda'(y, w) + (1 - \lambda')(\overline{x}^i, \overline{z}^i) \in \widehat{\mathbb{U}}^i(\overline{x}^i)$, but that contradicts the fact that there is nothing better that $(\overline{x}^i, \overline{z}^i)$ in $\mathbb{K}(M) \cap \mathcal{C}^i(\overline{p}, \overline{q})$. □

5 Examples

In this section we will present three different examples that satisfy only one of cases of Assumption (A4) to illustrate how the different cases play a role in the existence of equilibrium.

5.1 Essentiality

Consider a two period economy with no uncertainty, one commodity in each period and one asset that pays a fixed amount on the second period. Two agents $i \in \{1, 2\}$ with interior endowments and

$$\Phi^1 := \mathbb{R}^2_+ \times [0, \infty[\,,$$
$$\Phi^2 := \mathbb{R}^2_+ \times]-\infty, \infty[\,.$$

And, for each $i \in \mathcal{I}$ preferences are characterized by,

$$\mathbb{U}^i(x_1, x_2) := \left\{ (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1^{\frac{x_1}{1+x_1}} y_2^{1-\frac{x_1}{1+x_1}} > x_1^{\frac{x_1}{1+x_1}} x_2^{1-\frac{x_1}{1+x_1}} \right\}.$$

Proposition 5.1. *There exists an equilibrium in this economy.*

Proof. It is easy to see that the correspondences $\mathbb{U}^i$ have an open graph, are nonempty and are monotonically increasing for each $i \in \mathcal{I}$. They are convex valued because given (x_1, x_2) the function $f(y_1, y_2) = y_1^{\frac{x_1}{1+x_1}} y_2^{1-\frac{x_1}{1+x_1}}$ is concave. Asset payments are continuous on prices and the correspondences Φ^i satisfy Assumptions (A2) and (A3).

They satisfy Assumption (A4)(II.) since for any $(x_1, x_2) \in \partial\mathbb{R}^2_+$, and $(y_1, y_2) \gg 0$ we have,

$$y_1^{\frac{x_1}{1+x_1}} y_2^{1-\frac{x_1}{1+x_1}} > 0 \quad \Longrightarrow \quad (y_1, y_2) \in \mathbb{U}^i(x_1, x_2). \qquad \square$$

Note that Assumption (A4)(I.) is not satisfied since there are no unsegmented contracts to super-replicate the positions on segmented ones. Also, preferences are not transitive, for example $(0.5, 0.8) \in \mathbb{U}^i(0.9, 0.4)$ and $(0.2, 1.4) \in \mathbb{U}^i(0.5, 0.8)$ but $(0.2, 1.4) \notin \mathbb{U}^i(0.9, 0.4)$, so Assumption (A4)(III.)(d) is not satisfied.

5.2 Super-Replication

Now consider an economy with two agents and two possible states of nature at the second period and one commodity. The subjective probability of each state comes from a set Δ which is compact and convex. Two financial contracts, the first one with risk free payment and the second an Arrow security. Endowments are interior and $\forall i \in \{1, 2\}$ preferences are characterized by,

$$\mathbb{U}^i(x_0, x_s) = \frac{x_0}{(1 + x_0)} + \min_{s\in\Delta} E_{s\in\Delta}(x_s).$$

Financial positions are restricted with constant correspondences defined as,

$$\Phi^1 := \mathbb{R}^2_+ \times [-1, 1] \times [0, \infty[\,,$$
$$\Phi^2 := \mathbb{R}^2_+ \times [-1, 1] \times] - \infty, \infty[\,.$$

In this case, it is direct to see that all hypotheses are satisfied. Assumption (A4)(II.)(c) is not satisfied since a reduction of consumption on the first period from x_0 to 0 can be compensated by an increase in $x_0 + \epsilon$ for any $\epsilon > 0$ in each of the states in the second period. Assumption (A4)(III.)(a) is not satisfied because not any reduction on the second period can be compensated by more consumption on the first period.

5.3 Impatience

Let us consider another two period economy with two agents and two states of nature with subjective probability distribution coming from a compact and convex set Δ. One commodity and an Arrow security that pays a fixed positive amount only on one of the possible states of nature and zero on the other. Preferences will be defined for each $i \in \{0, 1\}$ as,

$$\mathbb{U}^i(x_0, x_s) = x_0 + \min_{s\in\Delta} E_{s\in\Delta}(x_s).$$

Restrictions on financial positions are defined again with constant correspondences,

$$\Phi^1 := \mathbb{R}^2_+ \times [0, \infty[\,,$$
$$\Phi^2 := \mathbb{R}^2_+ \times] - \infty, \infty[\,.$$

The hypotheses for the existence of equilibrium are satisfied. Assumption (A4)(I.) is not satisfied as there are no unsegmented contracts. ssumption (A4)(II.)(c) is not satisfied by the same argument as the previous example.

6 Concluding Remarks

We allow for intransitive and incomplete preference relations in a general framework for two period economies with uncertainty and restricted access to financial markets. We prove the existence of equilibrium under three specific set of hypothesis, the possibility to super-replicate payments of segmented assets by trading unsegmented assets, the fact that no consumption bundle that includes zero of any commodity is preferred to interior endowments, and that proportional loses on second period consumption can always be compensated by more consumption on the first period.

For further research, the model could be extended to include interdependent and price-dependent preferences, as in He and Yannelis (2016). Also, the transitivity and strict monotonicity on the impatience assumption could be replaced by weaker hypothesis.

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