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Stochastic Processes and their Applications 130 (2020) 2296–2311

stochastic processes and their applications

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Asymptotic error distribution for the Euler scheme with locally Lipschitz coefficients

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Received 21 January 2018; received in revised form 20 June 2019; accepted 3 July 2019
Available online 19 July 2019

Abstract

In traditional works on numerical schemes for solving stochastic differential equations (SDEs), the globally Lipschitz assumption is often assumed to ensure different types of convergence. In practice, this is often too strong a condition. Brownian motion driven SDEs used in applications sometimes have coefficients which are only Lipschitz on compact sets, but the paths of the SDE solutions can be arbitrarily large. In this paper, we prove convergence in probability and a weak convergence result under a less restrictive assumption, that is, locally Lipschitz and with no finite time explosion. We prove if a numerical scheme converges in probability uniformly on any compact time set (UCP) with a certain rate under a global Lipschitz condition, then the UCP with the same rate holds when a globally Lipschitz condition is replaced with a locally Lipschitz plus no finite explosion condition. For the Euler scheme, weak convergence of the error process is also established. The main contribution of this paper is the proof of $\sqrt{n}$ weak convergence of the normalized error process and the limit process is also provided. We further study the boundedness of the second moments of the weak limit process and its running supremum under both global Lipschitz and locally Lipschitz conditions.

Keywords: Stochastic differential equation; Locally Lipschitz; Convergence in probability; Euler scheme; Normalized error process; Weak convergence

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1 Supported in part by NSF grants DMS-1612758.
2 Supported by BASAL project AFB170001.

https://doi.org/10.1016/j.spa.2019.07.003
0304-4149/

1. Introduction

In this paper, we consider the numerical solution of a one-dimensional stochastic partial differential equation (SDE) of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ \ 0 \le t \le T,\ \ X_0 = x_0. \tag{1}$$

Here $X_t \in \mathbb{R}$ for each t, $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ are coefficient functions, and W is a one dimensional Brownian motion. We assume the initial value $x_0 \in \mathbb{R}$ is nonrandom. For background information about SDEs, we refer to Chapter 5 of Protter [20], Chapter 9 of Revuz and Yor [22] and Chapter 5 of Karatzas and Shreve [12]. In applications, one often would like to solve (1) numerically, as explicit solution is usually not obtainable. This is often done through Monte Carlo technique which requires heavy computational complexity. Hence in practice, it is advisable to solve (1) with the Euler scheme, rather than a complicated one. (See the survey paper of Talay [25] for a discussion of this issue). Our primary objective is to study uniform convergence in probability and weak convergence of normalized error process of Euler scheme under local Lipschitz and no finite explosion assumption on (1). Noted that the result on uniform convergence in probability in this paper is not restricted to the Euler scheme, but applicable to all numerical schemes satisfying some mild assumptions.

There are vast studies concerning using the Euler scheme to solve Brownian motion driven SDEs with various convergence criteria. A lot of existing works impose conditions on μ and σ in (1), namely the globally Lipschitz condition or linear growth condition. We list some of the works here. For convergence rate of the expectation of functionals, see Talay and Tubaro [27]; for convergence rate of the distribution function, see Bally and Talay [2]; for convergence rate of the density, see Bally and Talay [3]; for error analysis, see Bally and Talay [1]; for Euler scheme with irregular coefficients and Hölder continuous coefficients see Yan [28]; for complete reviews, see Talay [26] and Kloeden and Platen [14]. There are also some works on numerical scheme solving SDEs driven by semimartingales with jumps. The case of SDEs driven by discontinuous semimartingales can be found in Kurtz and Protter [18], they studied weak convergence of the normalized Euler scheme error. The L^p estimates of the Euler scheme error were given by Kohatsu-Higa and Protter [15]. Protter and Talay [21] also studied the Euler scheme for SDE driven by Lévy processes. Protter and Jacod [9] obtained a celebrated result about the asymptotic error distributions for the Euler scheme solving SDEs driven by a vector of semimartingales.

More recent works focused on numerical schemes to solve SDEs with relaxed conditions, namely the locally Lipschitz condition. Under the locally Lipschitz condition, the Euler scheme may diverge strongly or weakly for approximating expectations of functionals. This issue was studied in Hutzenthaler, Jentzen and Kloeden [8]. Convergence in probability for Euler-type scheme in general still holds, see Hutzenthaler and Jentzen [7] and the citations therein. Strong convergence of the Euler scheme requires additional assumptions besides the locally Lipschitz condition, see Higham, Mao and Stuart [6], and Jentzen and Kloeden [11]. However, to our best knowledge, there is barely work on the limit distribution for error process of Euler scheme with locally Lipschitz condition. Under C^1 and linear growth condition on μ, σ in (1), Kurtz and Protter [18] obtained limit distribution for normalized error process of Euler scheme. It remains an open question, whether similar result on error process of Euler scheme is achievable under more general conditions. In this paper, we answer this question to the positive by providing a weak convergence result with locally Lipschitz assumption plus no finite explosion and σ in (1) being bounded away from 0. This paper is not only a mathematical extension but also has wide applications in practice. A lot of SDEs used in finance or physics in form of (1) have coefficients which is not in C^1 and linear growth or global Lipschitz, for example, the constant elasticity model, Scalar stochastic Ginzburg–Landau equation and Stochastic Verhulst equation, etc. By relaxing from C^1 and linear growth to local Lipschitz, we are able to deal with coefficients that may have super linear growth and their derivatives may have very poor smoothness or may not even exist.

This paper consists of four sections. In Section 2 of this paper, we prove if a numerical scheme converges uniformly in probability on compact time interval with a certain rate under globally Lipschitz condition, then the same result holds when replacing globally Lipschitz to local Lipschitz and no finite explosion condition. Euler and Milstein schemes are studied as examples. In Section 3, we prove the normalized error process with normalizing coefficient as $\sqrt{n}$ is relatively compact. Furthermore by proving uniqueness of limiting point, we prove the normalized error process converges weakly. The limit error process is provided in form of SDE. In Section 4, we study the properties of the limit error process under both global Lipschitz and locally Lipschitz conditions.

2. Convergence in probability

Under the globally Lipschitz condition, most of the proposed numerical schemes including the Euler and Milstein schemes have been proved to converge uniformly in probability at a finite time point. Fortunately, the same result can be extended to the locally Lipschitz case if one also adds a no finite time explosion condition. To prove this, we need a localization technique. Let us start with some notation.

We assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_t$, that satisfies the usual hypothesis. We also assume that W is an adapted Brownian Motion defined in Ω.

Notation 2.1. *Given an adapted and right continuous process Z, we denote by*

$$\mathbb{T}^m(Z) = \inf\{t \geq 0 : |Z_t| > m\}.$$

Also, given a stopping time $\mathbb{T}$, we denote by $Z^{\mathbb{T}}$ the stopped process.

On the other hand, we denote by Z^* the process of its running absolute maximum, that is $Z^*_t = sup_{0\leq s\leq t} |Z_s|$.

In what follows, we denote by $X = X(x_0, \mu, \sigma, W)$ the unique solution of the SDE (1), where the coefficients $\mu,\ \sigma$ are assumed regular enough to have a unique strong solution (for example locally Lipschitz).

For every $m \geq 1$ consider $\mu^{(m)}$ a continuous modification of μ such that $\mu^{(m)}(x) = \mu(x)$ for $|x| \leq m$, $\mu^{(m)}(x) = \mu(m+1)$ for $x \geq m+1$ and $\mu^{(m)}(x) = \mu(-m-1)$ for $x \leq -m-1$. In case the numerical procedure assumes that μ is $\mathcal{C}^k$ (or Lipschitz) we interpolate $\mu^{(m)}$ on $(-m-1, -m) \cup (m, m+1)$ in such a way that $\mu^{(m)}$ is also $\mathcal{C}^k$ (respectively Lipschitz). Similarly, we denote by $\sigma^{(m)}$ a modification of σ.

Given a numerical procedure ϕ, we denote by $(X^{\phi,n})_n = (X^{\phi,n}(x_0, \mu, \sigma, W))_n$ the associated sequence of approximations. This approximations could be discrete or continuous. Here n controls the degree of approximation and typically represents the step size in the procedure. Note that we use the same Brownian motion for every n. When there is no possible confusion we remove the dependence on ϕ in $X^{\phi,n}$.

We assume that for each n the approximation $X^{\phi,n}$ is adapted. This numerical procedure is assumed **local** in the following sense. Assume that $\mu = \tilde{\mu}, \sigma = \tilde{\sigma}$ on the interval $[-m, m]$, where $|x_0| < m$. Then for all n and for $\mathcal{U} = \mathbb{T}^m(X^{\phi,n}(x_0, \mu, \sigma, W))$ it holds

$$(X^{\phi,n}(x_0, \mu, \sigma, W))^{\mathcal{U}} = (X^{\phi,n}(x_0, \tilde{\mu}, \tilde{\sigma}, W))^{\mathcal{U}}$$

almost surely. In particular $\mathbb{T}^m(X^{\phi,n}(x_0, \mu, \sigma, W)) = \mathbb{T}^m(X^{\phi,n}(x_0, \tilde{\mu}, \tilde{\sigma}, W))$ a.s.

This hypothesis is satisfied, for example, by the Euler and Milstein schemes. On the other hand, if (μ, σ) and $(\tilde{\mu}, \tilde{\sigma})$ are regular, the associated solutions satisfy

$$(X(x_0, \mu, \sigma, W))^{\mathbb{T}} = (X(x_0, \tilde{\mu}, \tilde{\sigma}, W))^{\mathbb{T}}$$

almost surely for $\mathbb{T} = \mathbb{T}^m(X(x_0, \mu, \sigma, W))$. Again, we have a.s.

$$\mathbb{T}^m(X(x_0, \mu, \sigma, W)) = \mathbb{T}^m(X(x_0, \tilde{\mu}, \tilde{\sigma}, W)).$$

In what follows a function $f : \mathbb{R} \to \mathbb{R}$ is said to be **locally Lipschitz**, if it is Lipschitz when restricted to any compact set of $\mathbb{R}$. We emphasize that $f(x) = \sqrt{|x|}$ is not locally Lipschitz on $\mathbb{R}$, but it is locally Lipschitz on $(0, \infty)$.

Theorem 1. *Assume the numerical scheme ϕ is local and that $(X^{\phi,n}(x_0, \mu, \sigma, W))_n$ converges in probability uniformly on $[0, T]$, with order $\alpha \geq 0$, to the solution $X(x_0, \mu, \sigma, W)$, that is, for all $C > 0$*

$$\mathbb{P}\left(n^{\alpha} \sup_{0\leq t\leq T} \left|X_t^{\phi,n} - X_t\right| > C\right) \to 0, \quad as\ n \to +\infty, \tag{2}$$

whenever μ, σ are global Lipschitz. Then (2) also holds when global Lipschitz condition is replaced with locally Lipschitz and no finite time explosion condition

$$\lim_{m\to\infty} \mathbb{P}\left(\mathbb{T}^m(X(x_0, \mu, \sigma, W)) \leq T\right) = 0. \tag{3}$$

Proof. We assume that μ, σ are locally Lipschitz. In what follows, to avoid overly burdensome notation, we denote by

$$X = X(x_0, \mu, \sigma, W),\ X^n = X^{\phi,n}(x_0, \mu, \sigma, W),$$

$$Y^{(m)} = X(x_0, \mu^{(m)}, \sigma^{(m)}, W),\ Y^{n,(m)} = X^{\phi,n}(x_0, \mu^{(m)}, \sigma^{(m)}, W).$$

We now fix $0 < C$. Without loss of generality we assume that $C \leq 1$ and we define

$$\mathscr{X}_n = \left\{\omega : n^{\alpha} \sup_{0\leq t\leq T} |X_t^n - X_t| \leq C\right\}, \tag{4}$$

$$\mathscr{Y}_{n,(m)} = \left\{\omega : n^{\alpha} \sup_{0\leq t\leq T} |Y_t^{n,(m)} - Y_t^{(m)}| \leq C\right\}. \tag{5}$$

We also consider $\mathbb{T} = \mathbb{T}^m(X), \mathbb{S} = \mathbb{T}^{m-1}(X)$. It is clear that $X^{\mathbb{S}} = (Y^{(m)})^{\mathbb{S}}$ (actually they are equal up to time $\mathbb{T}$), since they have continuous paths. The numerical procedure is assumed to be local, so for $m > x_0$ we have $\mathcal{U} = \mathbb{T}^m(X^n) = \mathbb{T}^m(Y^{n,(m)})$ and

$$(X^n)^{\mathcal{U}} = (Y^{n,(m)})^{\mathcal{U}}.$$

Consider m large enough such that $|x_0| < m - 1$. Notice that for all $n \geq 1$, we have $C/n^{\alpha} < 1$. For these values of n, m, we show that a.s.

$$\mathscr{Y}_{n,(m)} \bigcap \{\mathbb{S} > T\} \subset \mathscr{X}_n.$$

Indeed, on the set $\{\mathbb{S} > T\}$ the two processes $X, Y^{(m)}$ agree on $[0, T]$ a.s. In particular, we have that $\sup_{0\leq t\leq T} |Y_t^{(m)}| = \sup_{0\leq t\leq T} |X_t| \leq m-1$ a.s. On the other hand on the set $\mathscr{Y}_{n,(m)} \bigcap \{\mathbb{S} > T\}$ we have a.s.

$$\sup_{0\leq t\leq T} |Y_t^{n,(m)}| \leq m - 1 + \frac{C}{n^\alpha} < m\,.$$

That is $\mathbb{T}^m(Y^{n,(m)}) > T$ a.s. Since ϕ is local, we deduce that on $[0, T]$ the processes X^n and $Y^{n,(m)}$ agree a.s. and therefore on $\mathscr{Y}_{n,(m)} \bigcap \{\mathbb{S} > T\}$

$$\sup_{0\leq t\leq T} |X^n - X_t| = \sup_{0\leq t\leq T} |Y^{n,(m)} - Y_t^{(m)}| \leq \frac{C}{n^\alpha}$$

holds also a.s., proving the desired inclusion. This shows the inequality

$$\mathbb{P}\left(n^\alpha \sup_{0\leq t\leq T} |X_t^n - X_t| > C\right) \leq \mathbb{P}\left(n^\alpha \sup_{0\leq t\leq T} |Y_t^{n,(m)} - Y_t^{(m)}| > C\right) + \mathbb{P}(\mathbb{T}^{m-1}(X) \leq T)\,.$$

Now, given $\epsilon > 0$ the non explosion condition ensures that for large m we have

$$\mathbb{P}(\mathbb{T}^{m-1}(X) \leq T) \leq \epsilon/2.$$

Fix a large m, then according to the hypothesis of the Theorem, there exists $n_0 = n_0(m, \epsilon)$, such that for all $n \geq n_0$

$$\mathbb{P}\left(n^\alpha \sup_{0\leq t\leq T} |Y_t^{n,(m)} - Y_t^{(m)}| > C\right) \leq \frac{\epsilon}{2}\,,$$

giving the result. □

Remark 1. Recall that the hypothesis of no explosion can be checked through the Feller test for explosions, in terms of the behavior of σ and μ for large values of x. This is a one dimensional property, and in multiple dimensions one would have to resort to a Khasminskii type test, which provides sufficient conditions for no explosion (see for example [13,19,23]). Theorem 1, and its proof, can be extended directly to the multidimensional case.

Taking $\alpha = 0$ in Theorem 1, we immediately have if a numerical scheme converges in probability uniformly on compact time interval for SDE with globally Lipschitz coefficients, then the same convergence also holds with locally Lipschitz coefficients and no finite time explosion.

We illustrate the application of Theorem 1 through two most widely used numerical schemes, the Euler scheme and the Milstein scheme. For a discretization of time interval with size as $\frac{T}{n}$, let $n(t) = [\frac{nt}{T}]\frac{T}{n}$, the nearest left time grid point for t. For a function $g : [0, T] \to \mathbb{R}$, define $g_t^{(n)} = g_t - g_{n(t)}$. Then the continuous Euler scheme is defined by

$$X_t^{E,n} = X_{n(t)}^{E,n} + \sigma(X_{n(t)}^{E,n})W_t^{(n)} + \mu(X_{n(t)}^{E,n})t^{(n)}, \;\; X_0^{E,n} = x_0, \tag{6}$$

and the continuous Milstein scheme is defined by

$$X_t^{M,n} = X_{n(t)}^{M,n} + \sigma(X_{n(t)}^n)W_t^{(n)} + \mu(X_{n(t)}^n)t^{(n)} + \frac{1}{2}\sigma(X_{n(t)}^{M,n})\sigma'(X_{n(t)}^{M,n})[(W_t^{(n)})^2 - t^{(n)}] + \frac{1}{2}\mu(X_{n(t)}^{M,n})\mu'(X_{n(t)}^{M,n})(t^{(n)})^2, \quad X_0^{M,n} = x_0.$$

Under the globally Lipschitz condition, it is well known that the continuous Euler and Milstein schemes converge in probability uniformly on compact time interval with any order between $[0, \frac{1}{2})$ and $[0, 1)$, respectively. Interested readers can refer to [24] and [28] for details. Then, applying Theorem 1 leads to the following corollary.

Corollary 1. *If μ, σ are locally Lipschitz, and the solution X of the SDE (1) has no finite time explosion, then the continuous Euler scheme $X^{E,n}$ and continuous Milstein scheme $X^{M,n}$, converge in probability to the unique solution X. Moreover, for all $\gamma \in (0, \frac{1}{2}]$*

$$P\left(n^{\frac{1}{2}-\gamma} \sup_{0\le t\le T} \left|X^{E,n} - X_t\right| > C\right) \to 0, \quad \text{as } n \to +\infty,$$
$$P\left(n^{1-\gamma} \sup_{0\le t\le T} \left|X^{M,n} - X_t\right| > C\right) \to 0, \quad \text{as } n \to +\infty.$$

3. Asymptotic error distribution for the Euler scheme

In this section, we are interested in studying the normalized error processes from the Euler scheme under a locally Lipschitz condition combined with an assumption of no finite time explosion. In the previous section, localization techniques as used in the proof of Theorem 1 transfer the problem of convergence in probability under the locally Lipschitz case into the globally Lipschitz case. The localization technique will be used in this section as well, and we present Lemma 1 to make future proofs concise when applying this technique.

Lemma 1. *Assume that μ and σ are locally Lipschitz and $X = X(x_0, \mu, \sigma, W)$, the solution to the SDE (1) with coefficients μ, σ, has no finite explosion time. Consider $Y^{(m)} = X(x_0, \mu^{(m)}, \sigma^{(m)}, W)$ the solution to the SDE (1) with coefficients $\mu^{(m)}, \sigma^{(m)}$. If $(X^n)_n$, $(Y^{n,(m)})_n$ are their respective approximations from the continuous Euler scheme, then for all $T > 0$*

$$\lim_{m\to\infty} \limsup_n \mathbb{P}\Big(X \neq Y^{(m)} \text{ or } X^n \neq Y^{n,(m)}, \text{ on } [0, T]\Big) = 0.$$

Proof. Consider as before $\mathbb{T}^m = \mathbb{T}^m(X)$, $\mathcal{U}^{n,m} = \mathbb{T}^m(X^n)$. We assume that m is large enough that $|x_0| < m$. Since SDE (1) has a unique solution and the Euler scheme is local we have

$$\left\{\mathbb{T}^m > T \text{ and } \mathcal{U}^{n,m} > T\right\} \subset \left\{X = Y^{(m)} \text{ and } X^n = Y^{n,(m)}, \text{ on } [0, T]\right\}.$$

Therefore,

$$\mathbb{P}\Big(X \neq Y^{(m)} \text{ or } X^n \neq Y^{n,(m)}, \text{ on } [0, T]\Big) \le \mathbb{P}\Big(\mathbb{T}^m \le T \text{ or } \mathcal{U}^{n,m} \le T\Big). \tag{7}$$

Since X has no finite time explosion, for all $\epsilon > 0$ there exists $m > 0$ such that

$$\mathbb{P}(\mathbb{T}^m \le T) < \frac{1}{3}\epsilon,$$

and a fortiori $\mathbb{P}(\mathbb{T}^{m-1} \le T) < \frac{1}{3}\epsilon$. By the uniform convergence in probability of $(X^n)_n$ on $[0, T]$ (see Corollary 1) there exists $n' = n'(\epsilon)$ such that for all $n > n'$, we have

$$\mathbb{P}\left(\sup_{0\le s\le T} \left|X^n_s - X_s\right| \ge 1\right) < \frac{1}{3}\epsilon.$$

Again we have $\{\mathbb{T}^{m-1} > T\} \cap \left\{ \sup_{0\le s\le T} \left|X^n_s - X_s\right| < 1 \right\} \subset \{\mathcal{U}^{n,m} > T\}$ and therefore

$$\mathbb{P}(\mathcal{U}^{n,m} \le T) \le \mathbb{P}(\mathbb{T}^{m-1} \le T) + \mathbb{P}\left(\sup_{0\le s\le T} \left|X^n_s - X_s\right| \ge 1 \right) < \frac{2}{3}\epsilon,$$

from which the result follows. □

Lemma 2. *Consider SDE (1) and assume that μ, σ are both Lipschitz and bounded. Let $(X^n)_n$ be the sequence of numerical solutions of (1), on $[0, T]$, from the continuous Euler scheme with step size $\frac{T}{n}$. Then, the sequence of normalized error processes $U_n = \sqrt{n}(X^n - X)$ is relatively compact for weak convergence under the uniform topology on compact time sets.*

Proof. Define Z^n as follows

$$Z^{n11}_t = \int_0^t \sqrt{n} s^{(n)} ds, \;\; Z^{n12}_t = \int_0^t \sqrt{n} s^{(n)} dW_s,$$
$$Z^{n21}_t = \int_0^t \sqrt{n} W^{(n)}_s ds, \;\; Z^{n22}_t = \int_0^t \sqrt{n} W^{(n)}_s dW_s.$$

It has been proved that $\sqrt{n}Z^n$ are good sequences (see [17] for definition of good sequences, see also [10] for a result on the weak functional convergence of stochastic integrals. Note that the relationship between the results of [17] and [10] is explained in [16]), and $Z^n \Rightarrow Z$, which means weak convergence under the uniform topology on compact time sets. Z is independent of W and $Z^{1,1} = Z^{1,2} = Z^{2,1} = 0$, $Z^{2,2}$ is a mean zero Brownian motion with $\mathbb{E}(Z^{2,2}(t)) = \frac{t}{2}$. By Corollary 1, we also have $(X^n, Z^n) \Rightarrow (X, Z)$. By the definition of continuous Euler scheme, X^n can also be represented as

$$X^n_t = \int_0^t \mu(X^n_{n(s)}) ds + \int_0^t \sigma(X^n_{n(s)}) dW_s.$$

Then

$$\begin{aligned} U^n_t &= \sqrt{n}(X^n_t - X_t) \\ &= \int_0^t \sqrt{n}\{\mu(X^n_{n(s)}) - \mu(X_s)\} ds + \int_0^t \sqrt{n}\{\sigma(X^n_{n(s)}) - \sigma(X_s)\} dW_s. \end{aligned}$$

For $x \neq y$, define functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ as

$$g(x, y) = \frac{\mu(x) - \mu(y)}{x - y}, \quad h(x, y) = \frac{\sigma(x) - \sigma(y)}{x - y}.$$

Since μ, σ are Lipschitz, $g(x, y), h(x, y)$ are bounded. Now we separate the error process into two terms $U^n = U^{1,n} + U^{2,n}$, where

$$\begin{aligned} U^{1,n}_t &= \int_0^t \sqrt{n}\{\mu(X^n_{n(s)}) - \mu(X_s)\} ds \\ &= \int_0^t \sqrt{n}\{\mu(X^n_s) - \mu(X_s)\} ds - \int_0^t \sqrt{n}\{\mu(X^n_s) - \mu(X^n_{n(s)})\} ds \\ &= \int_0^t g(X^n_s, X_s) U^n_t ds - \int_0^t \frac{\mu(X^n_s) - \mu(X^n_{n(s)})}{X_s - X^n_{n(s)}} (X^n_s - X^n_{n(s)}) \sqrt{n} ds. \end{aligned}$$

Notice that $X^n_s - X^n_{n(s)} = \mu(X^n_{n(s)})s^{(n)} + \sigma(X^n_{n(s)})W^{(n)}_s$, then

$$U^{1,n}_t = \int_0^t g(X^n_{n(s)}, X_s)U^n_t - g(X^n_s, X^n_{n(s)})\big\{\mu(X^n_{n(s)})\sqrt{n}s^{(n)} + \sigma(X^n_{n(s)})\sqrt{n}W^{(n)}_s\big\}ds.$$

Similarly,

$$U^{2,n}_t = \int_0^t h(X^n_{n(s)}, X_s)U^n_t - h(X^n_s, X^n_{n(s)})\big\{\mu(X^n_{n(s)})\sqrt{n}s^{(n)} + \sigma(X^n_{n(s)})\sqrt{n}W^{(n)}_s\big\}dW_s.$$

For notational convenience, define $\tilde{f}^n$ as

$$\tilde{f}^n = \big[g(X^n_s, X_s), g(X^n_s, X^n_{n(s)}), h(X^n_s, X_s), h(X^n_s, X^n_{n(s)})\big].$$

If μ, σ are also assumed to be continuously differentiable as in Protter and Kurtz [18], then $\tilde{f}^n$ converges weakly uniformly to $[\mu'(X), \mu'(X), \sigma'(X), \sigma'(X)]$ on $[0, T]$. By results on weak convergence of stochastic integrals in Protter and Kurtz [17], U^n converges weakly uniformly on $[0, T]$ as well. However, here σ, μ are only assumed to be Lipschitz and bounded, hence their derivatives might be with poor smoothness or not even exist. This would cause $\tilde{f}^n$ fail to converge weakly. Fortunately, by the boundedness of $\tilde{f}^n$, applying weak convergence technique in [17] would give relative compactness of U^n under uniform topology, which is shown in the following steps. By Prokhorov's Theorem which states that tightness is equivalent to relative compactness in our case, $\tilde{f}^n$ is also relatively compact. Then for every subsequence of $\tilde{f}^n$, there exists a further subsubsequence n' such that $\tilde{f}^{n'}$ converges weakly uniformly on $[0, T]$. It is also known that $(X^{n'}_{n'(.)}, X^{n'}, \sqrt{n'}Z^{n'}) \Rightarrow (X, X, Z)$, and the sequence is a good sequence, see [18] for details. Then we can assume on $[0, T]$,

$$\big[\tilde{f}^{n'}, X^{n'}_{n'(.)}, X^{n'}, \sqrt{n'}Z^{n',1}, \sqrt{n'}Z^{n',2}, \sqrt{n'}Z^{n',3}, \sqrt{n'}Z^{n',4}\big]$$
$$\Rightarrow [G, \tilde{G}, H, \tilde{H}, X, X, 0, 0, 0, \frac{\sqrt{2}}{2}B].$$

Since Z^n is a good sequence and μ, σ bounded then by the proof of Theorem 3.5 in Kurtz and Protter [18], $U^{n'} \Rightarrow R$ on $[0, T]$, where

$$R_t = \int_0^t G_t R_t ds + \int_0^t H_t R_t dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_t)\tilde{H}_t dB_s. \tag{8}$$

Thus every subsequence of $U^n = \sqrt{n}(X^n - X)$ has a subsubsequence that converges weakly uniformly on $[0, T]$, which implies that U^n being relatively compact. □

In what follows, we consider μ, σ to be locally Lipschitz. In particular they are differentiable almost everywhere. We denote by μ', σ' their derivatives and we take $\mu'(x) = 0$ at those points x where μ is not differentiable. Similarly, we take $\sigma'(x) = 0$ when σ is not differentiable at x.

Theorem 2. *Consider the SDE (1), where we assume μ, σ are locally Lipschitz, σ is nonnegative and σ^{-2} is locally integrable. We also assume that the unique solution $X = X(x_0, \mu, \sigma, W)$ has no finite time explosion. Let $(X^n)_n$ be the sequence of approximations from the continuous Euler scheme, and $U_n = \sqrt{n}(X^n - X)$ be the normalized error process.*

Then, the sequence $(U_n)_n$ converges weakly uniformly on $[0, T]$, for all finite T, to the process U, the unique solution of the linear SDE

$$U_t = \int_0^t \mu'(X_s)U_s ds + \int_0^t \sigma'(X_s)U_s dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_s)\sigma'(X_s)dB_s, \ U_0 = 0. \tag{9}$$

Proof. With Lemma 1, applying the localization technique, we can assume without loss of generality that $|\mu|, |\sigma|$ are bounded, globally Lipschitz and σ^{-2} is locally integrable. Notice that we can localize σ in such a way that it is eventually constant and positive.

Define $g(x, y), h(x, y), Z^n$ in the same way as in Lemma 2. It is known that on $[0, T]$

$$(Z^{n11}, Z^{n12}, Z^{n21}, Z^{n22}) \Rightarrow Z = (0, 0, 0, \frac{\sqrt{2}}{2}B). \tag{10}$$

B is a standard Brownian motion and it is independent of W. Lemma 2 shows $(U^n)_n$ is relatively compact, thus for any subsequence n', there exists a subsubsequence n'_k of n' and a process R in $C[0, T]$, such that $U^{n'_k} \Rightarrow R$. It is well known that the SDE (9) has unique a weak solution.

To prove $(U^n)_n$ converges to U, it is sufficient to prove that R is a weak solution to SDE (9). Because $(U^{n'_k}, X, W, Z^{n'_k}) \Rightarrow (R, X, W, Z)$, by the almost sure representation theorem, which is Theorem 1.10.4 on page 59 of van der Vaart and Wellner [4], there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a sequence of processes $(\tilde{Y}_k)_k$ and a process Y that satisfy $\mathcal{L}(Y^k) = \mathcal{L}(U^{n'_k}, X, W, Z^{n'_k})$, for all k, $\mathcal{L}(Y) = \mathcal{L}(R, X, W, Z)$, and $Y^k \overset{a.s.}{\to} Y$ uniformly on $[0, T]$. If we could prove that the first element of Y is a weak solution to SDE (9), we immediately have R is also a weak solution to SDE (9). Thus, without loss of generality, we assume $(U^n, X, W, Z^n)(\omega) \to (R, X, W, Z)(\omega)$, as $n \to \infty$, uniformly on $[0, T]$, except on a set $\mathscr{A}$ of $\mathbb{P}$ measure 0. Now, we try to prove R is a weak solution to (9).

We first present one known result for the continuous Euler scheme under the condition of μ, σ being Lipschitz, stated here as (11). The proof of (11) can be found in the book of Kloeden and Platen [14] page 343, from the proof of Theorem 10.2.2 in chapter 10.

$$\sup_n \mathbb{E}\left(\sup_{0<s\leq T} \left|U_s^n\right|^2\right) < \infty. \tag{11}$$

Since $U^n \overset{a.s.}{\to} R$ on $[0, T]$, by Fatou's lemma, we also have

$$\mathbb{E}\left(\sup_{0<s\leq T} |R_s|^2\right) < \infty. \tag{12}$$

From the definition of U^n, we have $U^n = \sqrt{n}(X^n - X) = U^{1,n} + U^{2,n}$, where $U^{1,n}, U^{2,n}$ are the same as in proof of Lemma 2. Since the Lipschitz condition implies differentiability almost everywhere, we can find a set $A \subset \mathbb{R}$ with Lebesgue measure 0 and both μ and σ being differentiable on A^c. Define $I_1 = I_{\{s:X_s\in A^c\}}$ and $I_2 = I_{\{s:X_s\in A\}}$, we analyze the following terms, $i = 1, 2$.

$$G_t^{ni1} = \int_0^t I_i \, \{g(X_{n(s)}^n, X_s)U_t^n - \mu'(X_t)R_t\}ds,$$

$$G_t^{ni2} = \int_0^t I_i \, g(X_s^n, X_{n(s)}^n)\mu(X_{n(s)}^n)\sqrt{n}s^{(n)}ds$$

$$G_t^{ni3} = \int_0^t I_i \, g(X_s^n, X_{n(s)}^n)\sigma(X_{n(s)}^n)\sqrt{n}W_s^{(n)}ds$$

$$F_t^{ni1} = \int_0^t I_i \, \{h(X_{n(s)}^n, X_s)U_t^n - \sigma'(X_t)R_t\}dW_s,$$

$$F_t^{ni2} = \int_0^t I_i \, h(X_s^n, X_{n(s)}^n)\mu(X_{n(s)}^n)\sqrt{n}s^{(n)}dW_s$$

$$F_t^{ni3} = \int_0^t I_i\ h(X_s^n, X_{n(s)}^n)\sigma(X_{n(s)}^n)\sqrt{n}W_s^{(n)}dW_s - \frac{\sqrt{2}}{2}\int_0^t I_i\sigma(X_t)\sigma'(X_t)dB_s$$

Notice that

$$\sum_{i=1}^{2}\sum_{j=1}^{3}(G^{nij}+F^{nij}) = U^n - \Big\{\int_0^t \mu'(X_t)R_t ds + \int_0^t \sigma'(X_s)R_s dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_s)\sigma'(X_s)dB_s\Big\}. \tag{13}$$

Our goal is to show each term of G^{nij}, F^{nij} converges to the 0 process on $[0,T]$ in a proper sense. Consider term G^{n11}, because $(X^n, X^n_{n(.)}, U^n)(\omega) \to (X, X, R)(\omega)$, uniformly on $[0,T]$ for all $\omega \notin \mathscr{A}$ and μ differentiable on A^c, we have

$$I_{\{X_t\in A^c\}}g(X^n_{n(t)}, X_t)U^n_t \to I_{\{X_t\in A^c\}}\mu'(X_t)R_t, \quad \text{pointwise in } t.$$

Since U^n is a continuous process, its limit R will be continuous as well on $\mathscr{A}^c$, which leads to $R_T^* = \underset{0\le s\le T}{sup}|R_s|$ being finite on $\mathscr{A}^c$. We also conclude that $\sup_{0<s\le T}\left|U^n_s\right|$ is finite in $\mathscr{A}^c$. By the Lipschitz condition on μ,

$$\left|g(X^n_{n(t)}, X_t)U^n_t - \mu'(X_t)R_t\right| \le K(\sup_{0<s\le T}\left|U^n_s\right| + \sup_{0<s\le T}|R_s|).$$

Applying the dominated convergence theorem, G^{n11} converges to 0 uniformly almost surely on $[0,T]$.

Consider term G^{n21}, which is clearly bounded by

$$\mathbb{E}\Big(\sup_{0\le t\le T}|G^{n21}_t|\Big) \le \mathbb{E}\Big[\int_0^T I_{\{s:X_s\in A\}}\left|g(X^n_{n(s)}, X_s)U^n_s - \mu'(X_s)R_s\right|ds\Big].$$

We shall prove that $\mathbb{E}\Big[\int_0^T I_{\{s:X_s\in A\}}ds\Big] = 0$, which implies $\mathbb{E}\Big(\sup_{0\le t\le T}|G^{n21}_t|\Big) = 0$ and a fortiori $\sup_{0\le t\le T}|G^{n21}_t| = 0$ $\mathbb{P}$-a.s.

By Corollary 3.8 in Chap 7 of Revuz and Yor [22], let $\mathbb{T}^a = \mathbb{T}^a(X)$, $\mathbb{T}^b = \mathbb{T}^b(X)$ be the hitting times of X at $a < x_0 < b$, then

$$\mathbb{E}\Big[\int_0^{\mathbb{T}^a\wedge\mathbb{T}^b} I_{\{s:X_s\in A\}}ds\Big] = \int_a^b G_I(x_0, y)I_{\{y\in A\}}m(dy),$$

where, for $c \in \mathbb{R}$ fixed, we have

$$G_I(x,y) = \frac{(s(x\wedge y)-s(a))(s(b)-s(x\vee y))}{s(b)-s(a)} \qquad a\le x\wedge y\le x\vee y\le b,$$

$$m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx, \text{ and}$$

$$s(x) = \int_c^x \exp\Big(-\int_c^y 2\mu(z)\sigma^{-2}(z)dz\Big)dy.$$

By the boundedness of μ and the local integrability of σ^{-2}, we have $G_I(x_0, y)$ is locally bounded and $\frac{2}{s'(x)\sigma^2(x)}$ is locally integrable. Since A has Lebesgue measure 0,

$$\mathbb{E}\Big[\int_0^{T\wedge\mathbb{T}^a\wedge\mathbb{T}^b} I_{\{s:X_s\in A\}}ds\Big] = 0.$$

Let $a \to -\infty, b \to \infty$, applying Fatou's lemma,

$$\mathbb{E}\Big[\int_0^T I_2\,ds\Big] = \mathbb{E}\Big[\int_0^T I_{\{s:X_s\in A\}}ds\Big] = 0, \tag{14}$$

as we claimed.

So, in a similar way it is proven that $\sup_{0\le t\le T}|G_t^{n2j}| = 0$ $\mathbb{P}$-a.s., for $j = 2, 3$. To prove that $\sup_{0\le t\le T}|F_t^{n2j}| = 0$ $\mathbb{P}$-a.s., for $j = 1, 2, 3$, we first apply Burkholder–Davis–Gundy inequality and then we use the same argument as above.

Let us now analyze F^{n11}. From the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0<s\le T}\big|F_t^{n11}\big|\Big] &\le C\,\mathbb{E}\Big[\Big(\int_0^T I_1\big(h(X^n_{n(s)},X_s)U_s^n - \sigma'(X_s)R_s\big)^2 ds\Big)^{\frac12}\Big] \\ &\le C\,\mathbb{E}\Big[\Big(\int_0^T I_1\big\{h(X^n_{n(s)},X_s)(U_s^n - R_s)\big\}^2 ds\Big)^{\frac12}\Big] \\ &\quad + C\,\mathbb{E}\Big[\Big(\int_0^T I_1\big\{R_s(h(X^n_{n(s)},X_s) - \sigma'(X_s))\big\}^2 ds\Big)^{\frac12}\Big]\end{aligned} \tag{15}$$

Consider the first term of the right side of (15), since $|h| \le K$, then

$$\mathbb{E}\Big[\Big(\int_0^T I_1\big\{h(X^n_{n(s)},X_s)(U_s^n - R_s)\big\}^2 ds\Big)^{\frac12}\Big] \le K\sqrt{T}\,\mathbb{E}\Big[\sup_{0<s\le T}\big|U_s^n - R_s\big|\Big].$$

The right side of this inequality converges to 0, because the uniform convergence of $(U^n)_n$ to R on $[0, T]$, $\mathbb{P}$-a.s., and (11), (12). Thus the first term of right side of (15) converges to 0.

For the second term, from Hölder's inequality

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_0^T I_1\big\{R_s(h(X^n_{n(s)},X_s) - \sigma'(X_s))\big\}^2 ds\Big)^{\frac12}\Big] \\ &\le \mathbb{E}\Big[\Big(\sup_{0<s\le T}|R_s|^2\int_0^T I_1\big\{h(X^n_{n(s)},X_s) - \sigma'(X_s)\big\}^2 ds\Big)^{\frac12}\Big] \\ &\le \Big(\mathbb{E}\Big[\sup_{0<s\le T}|R_s|^2\Big]\Big)^{\frac12}\Big(\mathbb{E}\Big[\int_0^T I_1\big\{h(X^n_{n(s)},X_s) - \mu'(X_s)\big\}^2 ds\Big]\Big)^{\frac12}.\end{aligned}$$

For each $\omega \in \mathscr{A}$, we have $I_{\{t:X_t\in A^c\}}h(X^n_{n(t)},X_t) \to I_{\{t:X_t\in A^c\}}\sigma'(X_t)$ pointwise in t, and $\big|h(X^n_{n(s)},X_s)\big|, \big|\mu'(X_s)\big|$ are uniformly bounded by K. From the dominated convergence theorem,

$$\lim_{n\to\infty}\mathbb{E}\Big(\int_0^T I_{\{s:X_s\in A^c\}}\big\{h(X^n_{n(s)},X_s) - \sigma'(X_s)\big\}^2 ds\Big) = 0$$

With (12), we have the second term of right side of (15) also converges to 0. Thus

$$\lim_{n\to\infty}\mathbb{E}\sup_{0<s\le t}\big|F_t^{n11}\big| = 0.$$

Consider terms $G^{n12}, G^{n13}, F^{n12}, F^{n22}$, they all converge to the constant process 0 almost surely uniformly on $[0, T]$, because g, h, μ, σ are bounded, and $(Z^{n11}, Z^{n12}, Z^{n21}) \xrightarrow{a.s.} (0, 0, 0)$ uniformly on $[0, T]$.

For dealing with the last term F^{n13}, we first define $\tilde{F}^{n13}$ as

$$\tilde{F}_t^{n13} = \int_0^t I_1\,\big\{h(X_s^n, X^n_{n(s)})\sigma(X^n_{n(s)}) - \sigma'(X_s)\sigma(X_s)\big\}\sqrt{n}W_s^{(n)}dW_s.$$

Define $h_n(s) = h(X^n_s, X^n_{n(s)})$ and $\sigma_n(s) = \sigma(X^n_{n(s)})$. From the Burkholder–Davis–Gundy inequality there exists $C_3 > 0$

$$\mathbb{E}\left(\sup_{0<s\le T}\left|\tilde{F}^{n13}_s\right|\right) \le C_3\ \mathbb{E}\left[\left(\int_0^T I_1\big(h_n(s)\sigma_n(s) - \sigma'(X_s)\sigma(X_s)\big)^2(\sqrt{n}W^{(n)}_s)^2 ds\right)^{\frac{1}{2}}\right].$$

Applying the Cauchy–Schwarz inequality to the right side, there exists $C_4 > 0$, such that

$$\mathbb{E}\left(\sup_{0<s\le T}\left|\tilde{F}^{n13}_s\right|\right) \le \mathbb{E}\Big[\Big(\int_0^T I_1\big(h_n(s)\sigma_n(s) - \sigma'(X_s)\sigma(X_s)\big)^4 ds\Big)^{\frac{1}{4}}\Big(\int_0^T \big(\sqrt{n}W^{(n)}_s\big)^4 ds\Big)^{\frac{1}{4}}\Big]$$
$$\le \Big[\mathbb{E}\Big(\int_0^T \big(\sqrt{n}W^{(n)}_s\big)^4 ds\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(\int_0^T I_1\big(h_n(s)\sigma_n(s) - \sigma'(X_s)\sigma(X_s)\big)^4 ds\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}$$
$$\le C_4\Big[\mathbb{E}\Big(\int_0^T I_1\big(h_n(s)\sigma_n(s) - \sigma'(X_s)\sigma(X_s)\big)^4 ds\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}.$$

Since h, σ, σ' are bounded, by the dominated convergence theorem,

$$\lim_{n\to\infty}\mathbb{E}\Big(\sup_{0<s\le t}\left|\tilde{F}^{n13}_t\right|\Big) = \lim_{n\to\infty}\mathbb{E}\Big(\int_0^T I_1\big(h_n(s)\sigma_n(s) - \sigma'(X_s)\sigma(X_s)\big)^4 ds\Big)^{\frac{1}{2}} = 0.$$

Thus $\tilde{F}^{n13} \xrightarrow{L^1} 0$ uniformly on $[0, T]$. We define $\bar{F}^{n13}$ as,

$$\bar{F}^{n13}_t = \int_0^T I_1\ \sigma'(X_s)\sigma(X_s)dZ^{n22} - \int_0^T I_1\ \sigma'(X_s)\sigma(X_s)dB_s. \tag{16}$$

Since $Z^{n22} \xrightarrow{a.s.} B_s$ uniformly on $[0, T]$, Z^{n22} is a good sequence, applying results on convergence in probability of stochastic integrals in Kurtz and Protter [17] leads to $\bar{F}^{n13} \xrightarrow{\mathbb{P}} 0$ uniformly on $[0, T]$. Since $F^{n13} = \tilde{F}^{n13} + \bar{F}^{n13}$, we have $F^{n13} \xrightarrow{\mathbb{P}} 0$, uniformly on $[0, T]$.

Combining all the G and F terms, each of them converges to 0 uniformly on $[0, T]$ in a sense that would lead to convergence in probability. Then by (13), we have $U^n \xrightarrow{\mathbb{P}} \tilde{R}$ uniformly on $[0, T]$, where

$$\tilde{R}_t = \int_0^t \mu'(X_s)R_s ds + \int_0^t \sigma'(X_s)R_s dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_s)\sigma'(X_s)dB_s.$$

Since we also have $U^n \xrightarrow{a.s} R$ on $[0, T]$, the two limits must be equal to each other, and R follows

$$R_t = \int_0^t \mu'(X_t)R_t ds + \int_0^t \sigma'(X_s)R_s dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_s)\sigma'(X_s)dB_s. \tag{17}$$

This concludes the proof. □

Remark 2. We have stated and proved Theorem 2 in the one dimensional case. The theorem and the previous proof can be extended to include the multidimensional case, analogous to the way it is done in Jacod and Protter [9]. In particular we would obtain tightness and then mimic our proof to get the convergence $\sqrt{n}(X^n - X) \to U$, where U is the unique solution of the multidimensional version of (9). In order to do so, the key ingredients are that X does not explode and that for all $T > 0$ and all $A \subset \mathbb{R}^n$, Lebesgue measurable sets of measure 0, one has

$$\int_0^T \mathbb{E}(X_t \in A)dt = 0.$$

Here A is the set where σ is not differentiable. In the multidimensional case, these two facts are not so simple to characterize in terms of the coefficients σ, μ, but there are still some tractable sufficient conditions, which we do not go into here (see Remark 1).

4. Study of normalized limit error process

With the weak limit of the normalized error process for the Euler scheme being derived, we are interested in further analyzing its properties. Though in Kurtz and Protter and [18], they derived the form of the normalized error process of Euler scheme under C^1 and a bounded coefficient condition, its properties have barely been studied in the existing literature. In this section, we focus on the mean, variance and martingale nature of the limit error process under a globally Lipschitz condition. The locally Lipschitz case is more complicated, and it is studied through examples in this section as well.

4.1. Globally Lipschitz case

Theorem 3. *When μ and σ are globally Lipschitz, for the normalized error process $U_n = \sqrt{n}(X^n - X)$ from the continuous Euler scheme, there exists $0 < C_t < \infty$ increasing with t such that*

$$\mathbb{E}[U_t^2] \leq \mathbb{E}[U_t^{*2}] \leq C_t,$$

where $U_t^* = \sup_{0\leq s\leq t} |U_s|$. Moreover, the process $V = (e^{-\int_0^t \mu'(X_s)\,ds} U_t)_t$ is a mean 0 square integrable martingale. In particular, we have $\mathbb{E}(U_t) = 0$ if μ' is constant.

Proof (*Suggested by the Referee*)**.** In the globally Lipschitz case the process U satisfies Eq. (9), that is,

$$U_t = \int_0^t \mu'(X_s)U_s ds + \int_0^t \sigma'(X_s)U_s dW_s + \frac{\sqrt{2}}{2}\int_0^t \sigma(X_s)\sigma'(X_s)dB_s, \; U_0 = 0,$$

where B, W are independent B.M. This equation has a unique solution given by

$$U_t = \frac{\sqrt{2}}{2} Y_t \int_0^t \frac{\sigma\sigma'(X_s)}{Y_s}\,dB_s, \text{ with } Y_t = \exp\left(\int_0^t \mu'(X_s)\,ds\right)\mathcal{E}\left(\int_0^\bullet \sigma'(X_s)dW_s\right).$$

Here $\mathcal{E}\left(\int_0^\bullet \sigma'(X_s)dW_s\right)$ and $\int_0^\bullet \frac{\sigma\sigma'(X_s)}{Y_s}\,dB_s$ are orthogonal local martingales and L^p integrable, because σ' is bounded by hypothesis, $\sigma\sigma'$ has linear growth and X_t^* is in L^p for all $p \geq 1$. This finishes the proof of the result. □

4.2. Locally Lipschitz case and examples

When f is Lipschitz locally but not globally, Theorem 3 may not hold even with no finite time explosion condition. One example is the inverse Bessel process which is a solution to the SDE

$$dX_t = X_t^2\,dW_t, \quad X_0 > 0.$$

The coefficient $\sigma(x) = x^2$ is locally Lipschitz and X has no finite explosion. From Theorem 2, we have the error process $U_t^n = \sqrt{n}(X_t^n - X_t)$ converges weakly uniformly to U_t on $[0, T]$. U_t is solution to

$$dU_t = 2X_tU_tdW_t + \sqrt{2}X_t^3 dBt,$$

where B is a Brownian motion independent of W. It has been proved in Hutzenthaler, Jentzen and Kloeden [8] that $\mathbb{E}[(X_t^n)^p]$ diverges to infinity as $n \to \infty$ for all $p \geq 1$. Thus $\mathbb{E}\left|U_t^n\right|^2$ will diverge to infinity as well. The expected quadratic variation is as

$$\mathbb{E}(\langle U, U\rangle_t) = \mathbb{E}\int_0^t \left\{X_s^2 U_s^2 + 2X_s^6\right\}dt \geq 2\mathbb{E}\int_0^t X_s^6 dt.$$

Since the inverse Bessel process can also be represented as inverse of the norm of a three dimensional Brownian motion starting from $X_0(1, 0, 0)$, its explicit distribution can be obtained, for example in [5]. Calculation shows if $X_0 > 0$, then for all $t > 0$, $\mathbb{E}X_t^6 = \infty$. This leads to $\mathbb{E}(\langle U, U\rangle_t) = \infty$. Applying the Burkholder–Davis–Gundy inequality, we have $\mathbb{E}\{U_t^{*2}\} = \infty$. This indicates that under the locally Lipschitz condition, the asymptotic distribution for the normalized error process might have a heavier tail than the globally Lipschitz case. In order to preserve the same properties in Theorem 3, besides locally Lipschitz and no finite explosion assumption, we need extra conditions on μ, σ and moments condition on X_t and X_t^n. This is shown in the following theorem.

Theorem 4. *Consider SDE (1), assume that μ and σ are locally Lipschitz and of at most polynomial growth. We also assume that σ' has at most polynomial growth. Further assume its discrete Euler scheme solution $\bar{X}^n$ and continuous Euler scheme solution X^n satisfy that for all $p \geq 1$ and all T*

$$\mathbb{E}[(X_T^*)^p] < \infty,\ \mathbb{E}[(X_T^{n*})^p] < \infty,\ \mathbb{E}[(\bar{X}_T^{n*})^p] < \infty. \tag{18}$$

Then for the normalized error process $U_n = \sqrt{n}(X^n - X)$, we have $\exists\ C_t > 0$ increasing with t, s.t.

$$\mathbb{E}[U_t^2] \leq \mathbb{E}[U_t^{*2}] \leq C_t. \tag{19}$$

where $U_t^* = \sup_{s\leq t}|U_s|$. Furthermore, when $\mu' = 0$, U is a uniformly integrable martingale.

Proof. With the assumptions in Theorem 4, it is known that there exists $C_t > 0$ increasing with t such that

$$\sup_n \mathbb{E}[(U_t^{n*})^2] < C_t.$$

For details of the proof, check Theorem 4.4 in H. Desmond and X.Mao [6]. Then by the same argument as in Theorem 3, we have

$$\mathbb{E}[U_t^2] \leq \mathbb{E}[U_t^{*2}] \leq C_t.$$

Using the Burkholder–Davis–Gundy inequality with $1 \leq p < 2$, we get

$$\begin{aligned}
\mathbb{E}(U_T^{*p}) &\leq C_p\mathbb{E}(\langle U, U\rangle_T^{p/2}) = C_p\,\mathbb{E}\Big[\Big(\int_0^T \{(\sigma'(X_s))^2U_s^2 + (\sigma'(X_s))^2\sigma^2(X_s)\}ds\Big)^{p/2}\Big] \\
&\leq C_p\,\mathbb{E}\Big[\Big(\int_0^T (\sigma'(X_s))^2U_s^2\,ds\Big)^{p/2} + \Big(\int_0^T (\sigma'(X_s))^2\sigma^2(X_s)ds\Big)^{p/2}\Big] \\
&\leq C_pT^{p/2}\,\mathbb{E}\Big[(\sigma'(X))_T^{*p}\,U_T^{*p} + (\sigma'(X))_T^{*p}\,(\sigma(X))_T^{*p}\Big].
\end{aligned}$$

Now, we use Hölder's inequality with exponents $2/p$ and its conjugated $r = \frac{2}{2-p}$, to deduce that

$$\mathbb{E}[(\sigma'(X))_T^{*p}\,U_T^{*p}] \leq (\mathbb{E}[(\sigma'(X))_T^{*pr}])^{1/r}(\mathbb{E}[U_T^{*2}])^{p/2} < \infty,$$

because the growth condition on σ', hypothesis (18) and the inequality (19). Similarly, we prove that $\mathbb{E}\left[(\sigma'(X))_T^{*p}\,(\sigma(X))_T^{*p}\right] < \infty$, and the result is shown. □

Remark 3. There is a classic example that Theorems 2 and 4 leave undressed. The function $x \mapsto \sqrt{x}$ is not far from being locally Lipschitz, but of course it fails at the point $x = 0$. Additionally, if one were to simulate a sequence of numerical approximations one would inevitably be taking the square roots of negative numbers. To handle situations like this, consider the following SDE

$$dX_t = \sqrt{X_t}\,dB_t + m(X_t)\,dt, \qquad X_0 > 0 \tag{20}$$

where B is a standard Brownian motion and m is locally Lipschitz.

We define $Y_t = \log(X_t)$, using the well known fact that the unique solution X of (20) is strictly positive for all $t > 0$. Using Itô's formula we have

$$dY_t = \exp(-Y_t/2)\,dB_t + (m(\exp(Y_t)) - 1)\exp(-Y_t)\,dt \tag{21}$$

The coefficients of (21) are locally Lipschitz in $\mathbb{R}$, and we can apply our results for Y and then deduce the results for X from that, by transforming back.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We wish to thank Jean Jacod and Denis Talay for helpful discussions during our work on this paper. We also thank an anonymous referee for his/her useful comments that improve the presentation of this work, specially Theorem 3. The results in this paper constitute part of the PhD thesis of Lisha Qiu, in the Statistics Department of Columbia University. The third author is very grateful for the hospitality of the Statistical Department of Columbia University.

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