Soft Computing
https://doi.org/10.1007/s00500-020-04852-5

METHODOLOGIES AND APPLICATION

Covariances with OWA operators and Bonferroni means

Fabio Blanco-Mesa[1] · Ernesto León-Castro[2] · José M. Merigó[3]

Abstract

The covariance is a statistical technique that is widely used to measure the dispersion between two sets of elements. This work develops new covariance measures by using the ordered weighted average (OWA) operator and Bonferroni means. Thus, this work presents the Bonferroni covariance OWA operator. The main advantage of this approach is that the decision maker can underestimate or overestimate the covariance according to his or her attitudes. The article further generalizes this formulation by using generalized and quasi-arithmetic means to obtain a wide range of particular types of covariances, including the quadratic Bonferroni covariance and the cubic Bonferroni covariance. The paper also considers some other extensions by using induced aggregation operators in order to use complex reordering processes in the analysis. The work ends by studying the applicability of these new techniques to real-world problems and presents an illustrative example of a research and development (R&D) investment problem.

Keywords Variance · Covariance · Bonferroni means · OWA operator

1 Introduction

In decision-making problems, it is common to use statistics and probabilistic measures to treat and analyze data and to obtain valid information about the data. These tools allow one to organize and condense the data set and to determine a specific property of a population based on a population sample. The use of these tools ensures that the measurement and accounting data provide objective information. Among the most used procedures and measures are the frequency distribution; the average; measures of the central tendency, dispersion and others related to the probability; and test statistics with greater complexity. Nonetheless, statistics have limitations when capturing and explaining meaning of the information, as they include semantics, linguistic meanings, approximate reasoning, intuition and attitudes. These limitations occur because this sort of data does not support formal patterns and has a broad relationship with human behavior and subjectivity (Blanco-Mesa et al. 2017). Feasibly, measuring these data with sufficient mathematical precision involves a high degree of complexity, which represents a challenge for their mathematical treatment. Hence, measuring complex human reasoning using probability and statistics is difficult since the complexity is negatively related to the precision (Zadeh 1975; Blanco-Mesa et al. 2017).

Based on the above, the idea has arisen to establish formal mathematical methods that address the complexity and uncertainty. Consequently, hybrid models have been proposed that have been applied in various fields such as economics and administrative sciences. In this sense, these methods use fuzzy subsets and the mathematics of uncertainty and have been applied in economics (Kaufmann and Gil Aluja 1986; Kaufmann and Gil-Aluja 1990; Blanco-Mesa et al. 2019a). Likewise, another method has been proposed that is called the ordered weighted averaging

Communicated by V. Loia.

✉ Fabio Blanco-Mesa
fabio.blanco01@uptc.edu.co

Ernesto León-Castro
eleon@delasalle.edu.mx

José M. Merigó
jmerigo@fen.uchile.cl

[1] Facultad de Ciencias Económicas y Administrativas, Escuela de Administración de Empresas, Universidad Pedagógica y Tecnológica de Colombia, Av. Central del Norte, 39-115, Tunja 150001, Colombia

[2] Faculty of Economics and Business Administration, Universidad Católica de la Santísima Concepción, Av. Alonso de Ribera 2850, 4070129 Concepción, Chile

[3] Department of Management Control and Information Systems, School of Economics and Business, University of Chile, Av. Diagonal Paraguay 257, 8330015 Santiago, Chile

Published online: 24 March 2020

aggregation operator that aggregates information while taking into account the attitudes of the decision maker (Yager 1988). Furthermore, using this method, both theoretical mathematical proposals and business and economic decision-making problems applications have been developed. Included in the theoretical mathematical proposals are outlined heavy operators (Yager 2002), induced operators (Yager and Filev 1999; Yager 2003), prioritized operators (Chen and Xu 2014), time series (Yager 2008), Bonferroni means (Yager 2009), moving averages (Merigó and Yager 2013), probabilistic measures (Yager et al. 1995; Merigó 2012), distances (Gil-Lafuente and Merigó 2007; Xu and Chen 2008), linguistics (Herrera and Martinez 2000; Herrera and Herrera-Viedma 2000), Pythagorean theory (Yager 2014; Gao 2018; Wei 2019; Tang and Wei 2019), geometric operators (Chiclana et al. 2002; Xu and Da 2002), power (Yager 2001), variance (Yager 1996), covariance (Merigó et al. 2015a) and variance measures (Verma and Merigó 2019). Furthermore, notable business and economic decision-making problems applications include sales forecasting (Merigó et al. 2015b), forecasting (León-Castro et al. 2016, 2018, 2019a), entrepreneurship (Blanco-Mesa et al. 2016, 2018c), portfolio selection (Laengle et al. 2017), stakeholder management (Blanco-Mesa et al. 2018a, b), enterprise risk management (Blanco-Mesa et al. 2018d, 2019c), transparency (Avilés-Ochoa et al. 2018), competitiveness (Blanco-Mesa and Gil-Lafuente 2017), social networks (Wu et al. 2017; Zhang et al. 2018) and e-services (Carrasco et al. 2017; Dong et al. 2018).

However, it should be noted that in several fields of research it is extremely important to measure information through the interpretation of test statistics. Especially in social sciences such as economics and administration, where the role and influence (attitude, subjectivity, emotions, reasoning, among others) that people have in establishing the instruments and methods of data collection for further processing is inevitable, which entails establishing measures that offer levels of confidence to accept the hypotheses formulated. In this sense, the efforts of this research focused on three methods: the OWA operator, Bonferroni means and covariance, which by their own characteristics can provide a soft test statistic allowing the measurement of subjective aspects such as the attitude of the decision maker.

First, OWA operator allows to represent the maximum and the minimum operators by adding a reordering step (Yager 1988). On the basis of this operator, OWA variance and Bonferroni OWA have been proposed. OWA variance operator is highlighted by the better option which is with the highest expected and lowest variance Yager (1996). Bonferroni OWA operator allows comparing the interrelated information simultaneously (Yager 2009). Thus, in both works OWA operator features have been combined with from methods of probity and statistics. Second, Yager (1996) introduced the concept of variance into problems of decision making in uncertainty, thus opening a new way for the creation of soft statistical measures applied in uncertain environments. Following this proposal, Merigó et al. (2015a) have suggested the ordered weighted mean in combination with variance and covariance including in the new type of Pearson coefficient. Recently, Blanco-Mesa et al. (2019a, b, c) have proposed a new type of variance that combines the characteristics of the OWA operator and the Bonferroni mean, which allows underestimating or overestimating the variance according to the attitudinal character of the decision maker. Third, from the works of Bonferroni and Yager there are multiple proposals that have been made to solve the problems of decision making (Beliakov et al. 2010; Gou et al. 2017; Liu and Liu 2017). Likewise, new extensions have been introduced using linguistic (Merigó et al. 2014) and induced variables (Blanco-Mesa et al. 2018d, 2019b), distance measurements (Blanco-Mesa et al. 2016; Blanco-Mesa and Merigó 2017; Merigó et al. 2017; Blanco-Mesa and Merigó 2020), algorithms (Alfaro-García et al. 2018) among others, which has allowed the creation of new families of aggregation operators.

Hence, this paper proposes a new operator that combines the characteristics of these three methods in a single formulation that allows one to obtain a *soft-measure* to validate subjective information. First, the OWA covariance operator analyzes the strength of the correlation among two or more sets of random variables in an optimistic or pessimistic scenario (Yager 1996; Merigó et al. 2015a). It is worth mentioning that the covariance correlates two or more sets of random variables, which reflects the degree of joint variation with respect to their means. Second, the Bonferroni OWA operator allows for the simultaneous comparison and interrelation of information (Yager 2009). It is also worth recalling that Bonferroni means are used to make multiple comparisons when the null hypothesis is rejected, which allows one to guarantee the significance level α between each of the comparisons that are considered for the data set of the experiment. Thus, this new proposition combines all these characteristics and is called the Bonferroni OWA covariance. This novel methodological proposal provides two new measurement methods: the Bonferroni OWA covariance and the Bonferroni means covariance. Additionally, from this proposition, other extensions, such as the geometric Bonferroni covariance, the quadratic Bonferroni covariance, the cubic Bonferroni covariance and the harmonic Bonferroni covariance, are proposed

Finally, to illustrate this method, it is applied to R&D investment decision making where a decision maker analyzes correlation data. For that, the sales and R&D

investment data from 10 random companies from the National Association of Securities Dealers Automated Quotation (NASDAQ) 100 from 2015 to 2018 are used. These new approaches provide the decision maker with the possibility to obtain a better representation of the potential real scenarios when it is under- or overestimating the covariance using the attitudes of the decision maker. Likewise, this soft covariance can be useful when the decision maker has soft data and needs a valid and reliable measurement. In addition, the use of OWA operators and other related methods are useful when the importance and meaning of the data are not easily known.

The remainder of this paper is structured as follows: Sect. 2 describes the preliminary concepts and formulations related to the covariance, Bonferroni means, OWA and IOWA operators and their existing combinations. Section 3 presents new propositions for the covariance using the Bonferroni means and OWA operators. Section 4 presents the generalized ordered weighted average covariance and a numerical example. Section 5 develops the application and presents the results for the R&D investment decision-making problems. Finally, Sect. 6 provides the main implications and conclusions of the paper.

2 Preliminaries

In this section, the concepts and formulas of the OWA and IOWA operators, the Bonferroni means and their extensions are presented. Similarly, the formula for the covariance measure and its integration with the OWA operator is given.

2.1 Covariance

Covariance is a technique that measures how two random variables will change together and is constantly used in order to calculate the correlation between two arguments, sets or variables. The formulation is as follows.

Definition 1 The covariance is calculated for two data sets, X and Y, as

$$\mathrm{Cov}(x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^{n} v_i (x_i - \mu)(y_i - v), \tag{1}$$

where x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$ and μ and v are the averages of sets X and Y, respectively. Each argument $(x_i - \mu)(y_i - v)$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$.

2.2 Bonferroni mean

An interesting aggregation operator that can assess the relations of the argument variables is the Bonferroni means (Bonferroni 1950). Since its development, extensions include using distance techniques (Blanco-Mesa et al. 2016), heavy and induced aggregation operators (Blanco-Mesa et al. 2018d, 2019b), intuitionistic fuzzy interactions (He et al. 2015), multicriteria decision making (Gou et al. 2017) and so on.

Definition 2 The formulation of the Bonferroni mean is as follows.

$$B^{r,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n} \sum_{k=1}^{n} a_i^r \left(\frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q \right) \right)^{\frac{1}{r+q}}, \tag{2}$$

Some properties of the Bonferroni mean are the following (Zhu and Xu 2013):

(a) Commutativity: Let $(a_1', a_2', \ldots, a_n')$ be any permutation of $(a_1, a_2, \ldots, a_n)$, then: $B^{r,q}(a_1', a_2', \ldots, a_n') = B^{p,q}(a_1, a_2, \ldots, a_n)$.

(b) Idempotency: Let $a_j = a, j = 1, 2, \ldots, n$, then $B^{r,q}(a, a, \ldots, a) = a$.

(c) Monotonicity: Let $a_i (i = 1, 2, \ldots, n)$ and $b_i (i = 1, 2, \ldots, n)$ be two collections of crisp data. $If a_i \geq b_i for\ all\ i, then\ B^{r,q}(a_1, a_2, \ldots, a_n) \geq B^{r,q}(b_1, b_2, \ldots, b_n)$.

(d) Boundedness: The $B^{r,q}$ operator lies between the max and min operators : $\min(a_1, a_2, \ldots, a_n) \leq B^{r,q}(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n)$.

Some specials case of the BM operator is shown as follows:If $r = 1$ and $q = 1$, then Eq. (2) reduces to the following:

$$B^{1,1}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n} \sum_{k=1}^{n} a_i \left(\frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j \right) \right)^{\frac{1}{2}}, \tag{3}$$

If $q = 0$, then Eq. (2) reduces to the following:

$$B^{r,0}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n}\sum_{k=1}^{n}a_i^r\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}a_j^0\right)\right)^{\frac{1}{r+0}}=\left(\frac{1}{n}\sum_{k=1}^{n}a_i^r\right)^{\frac{1}{r}} \tag{4}$$

If $r=2$ and $q=0$, then Eq. (2) reduces to the square mean:

$$B^{2,0}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n}\sum_{k=1}^{n}a_i^2\right)^{\frac{1}{2}} \tag{5}$$

If $r=1$ and $q=0$, then Eq. (2) reduces to the usual average:

$$B^{1,0}(a_1,a_2,\ldots,a_n)=\frac{1}{n}\sum_{k=1}^{n}a_i \tag{6}$$

If $r\to+\infty$ and $q=0$, then Eq. (2) reduces to the max operator:

$$\lim_{r\to\infty}B^{r,0}(a_1,a_2,\ldots,a_n)=\max\{a_i\} \tag{7}$$

If $r\to 0$ and $q=0$, then Eq. (2) reduces to the geometric mean operator:

$$\lim_{r\to 0}B^{r,0}(a_1,a_2,\ldots,a_n)=\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}} \tag{8}$$

2.3 OWA and IOWA operator

The ordered weighted average (OWA) operator (Yager 1988) has as its main advantage the representation of the maximum and the minimum operators by adding a reordering step. Additionally, many applications and frameworks have been developed (Kacprzyk and Zadrożny 2009; Belles-Sampera et al. 2013; Pérez-Arellano et al. 2019). The definition is as follows.

Definition 3 An OWA operator of dimension n is a mapping $\mathrm{OWA}: R^n\to R$ that has an associated weighing vector W of dimension n with $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$ and is represented as follows:

$$\mathrm{OWA}(a_1,a_2,\ldots,a_n)=\sum_{j=1}^{n}w_jb_j, \tag{9}$$

where b_i is the jth largest a_i.

Among the criteria that can be used for the reordering step is the following.

An extension of the OWA operator that further analyzes the reordering characteristic is the induced OWA (IOWA) operator (Yager and Filev 1999; Yager 2003). In this operator, the reordering step is based on the decision maker (or decision makers) using an induced vector that is associated with the arguments. The formulation is as follows.

Definition 4 An IOWA operator of dimension n is an application $\mathrm{IOWA}: R^n\times R^n\to R$ that has an associated weighting vector W of dimension n, where the sum of the weights is 1 and $w_j\in[0,1]$, and an induced set of ordering variables is included (u_i). The formula is as follows:

$$\mathrm{IOWA}(u_1,a_1,u_2,a_2,\ldots,u_n,a_n)=\sum_{j=1}^{n}w_jb_j, \tag{10}$$

where b_j is the a_i value of the OWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i. u_i is the order-inducing variable and a_i is the argument variable.

2.4 Bonferroni IOWA

Authors have developed extensions to the Bonferroni means by using different aggregation operators. Among them are the ones using the OWA and IOWA operators (Blanco-Mesa et al. 2019b). These operators are useful for integrating the relationship between the arguments, obtaining the minimum and maximum operators or using an induced reordering step. Taking into account these characteristics, the operators that can be devolved are the following.

Definition 5 The Bonferroni IOWA (BON-IOWA) is a mean type continuous aggregation operator that can be defined as follows:

$$\mathrm{BON\text{-}IOWA}(u_1,a_1,\ldots,u_n,a_n)=\left(\frac{1}{n}\sum_{i}b_i^r\mathrm{IOWA}_W\left(V^i\right)\right)^{\frac{1}{r+q}}, \tag{11}$$

where b_i is the a_i value of the BON-IOWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i, $\mathrm{IOWA}_W(V^i)=\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}b_j^q\right)$ with (V^i) is the vector of all b_j s except b_i and w is an $n-1$ vector W_i that is associated with α_i whose components w_{ij} are the OWA weights. Let W be an OWA weighing vector of dimension $n-1$ with the components $w_i\in[0,1]$ when $\sum_i w_i=1$, where the weights are associated according to the largest value of u_i, and u_i is the order-inducing variable.

Definition 6 The Bon-OWA operator can be defined as follows (Yager 2009).

$$\text{BON-OWA}(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_i a_i^r \text{OWA}_W(V^i)\right)^{\frac{1}{r+q}}, \tag{12}$$

where $\text{OWA}_W(V^i) = \left(\frac{1}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q\right)$ with (V^i) is the vector of all a_j s except a_i and w is an $n-1$ vector W_i that is associated with α_i whose components w_{ij} are the OWA weights. Let W be an OWA weighing vector of dimension $n-1$ with the components $w_i \in [0,1]$ when $\sum_i w_i = 1$. Then, we can define this aggregation as $\text{OWA}_W(V^i) = \left(\sum_{j=1}^{n-1} w_i a_{\pi_k(j)}\right)$, where $a_{\pi_k(j)}$ is the largest element in the tuple V^i and $w_i = \frac{1}{n-1}$ for all is.

2.5 Covariance with OWA and IOWA operators

2.5.1 General case

As seen in Definition 1, the covariance, as a measure of variability, includes a weighting vector $W = 1/n$ and two averages in its formulations: one for the X sets and one for the Y sets. In this sense, it is possible to combine the OWA operator with the covariance, such as has been done by (Merigó et al. 2015a). The formulation is as follows.

Definition 7 In the case of the OWACov, the definition is as follows:

$$\text{OWACov}(X, Y) = \sum_{j=1}^{n} w_j K_j, \tag{13}$$

where K_j is the jth largest $(x_i - \mu)(y_i - v)$; x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$; y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$; μ and v are the averages (or the OWA operator) of the sets X and Y, respectively; $w_j[0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Finally, is also possible to formulate the IOWACov if an induced vector is used to associate the weights.

Definition 8 The IOWACov operator formula is as follows (Merigó et al. 2015a).

$$\text{IOWACov}(U, X, Y) = \sum_{j=1}^{n} w_j K_j, \tag{14}$$

where K_j is the $(x_i - \mu)(y_i - v)$ value of the IOWACov triplet u_i, x_i, y_i with the jth smallest u_i, and u_i is the order-inducing variable of the set of elements $U = \{u_1, \ldots, u_n\}$.

It is important to note that the main properties of the OWA operator are applied to the IOWACov and OWACov operators. These are the following:

(a) It is monotonic because if $a_i \geq d_i$, for all i, then $\text{IOWACov}(a_1, \ldots, a_n) \geq \text{IOWACov}(d_1, \ldots, d_n)$;
(b) It is commutative because any permutation of the argument has the same evaluation; and
(c) It is bounded because the weighting vector is equal to 1.

2.5.2 Special cases

Some special cases that are not presented by (Merigó et al. 2015a) include that where the OWA operator can be applied to μ and v. The main idea is to aggregate the information in the averages to account for this case, and the averages become the μ_{OWA} and the v_{OWA}. The formulas are as follows.

Definition 9 The μ_{OWA} is defined as

$$\mu_{\text{OWA}}(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} w_j m_j, \tag{15}$$

where m_i is the jth largest x_i.

Definition 10 The v_{OWA} is defined as

$$v_{\text{OWA}}(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{n} w_j n_j, \tag{16}$$

where n_i is the jth largest a_i.

Using this idea, it is possible to generate the total *CovOWA*, and the formula is as follows.

Definition 11 The total OWACov can be defined as

$$\text{total OWACov}(X, Y) = \sum_{j=1}^{n} w_j O_j \tag{17}$$

where O_j is the largest $(x_i - \mu_{\text{OWAcov}})(y_i - v_{\text{OWAcov}})$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, and y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$.

Finally, is important to note that, based on Definition 11, it is possible to generate different cases of the formula based on whether the OWA operator is used in all the elements of the formula or not. These formulations are presented as follows (Table 1).

The cases presented in Table 1 represent the degree of information that want to be added to the OWACov formulation. The covariance is compounded by two different averages μ and v that is based on the average of set X and set Y, respectively. Because of that it is possible to remain the averages without any change and only use the weighting vector and reordering step in the results obtained by $(x_i - \mu)(y_i - v)$, such is case 3, or use the OWA operator or any other extension of the same instead of the traditional average in μ and v. By doing this, it is possible to add more information of the decision maker obtained through the weighting vector and reordering step or also, only applied it to one determined set (X or Y) if the case needs it. Finally, one of the main reasons to use the OWA operator instead of the usual average in μ and v will be based on the complexity of the problem and if the average of the data is not the best option because some arguments are more important than others, or if the sets have outliers that want to be weighted less or more (depending on the attitude of the decision maker). As seen, these ideas are important because, depending on the complexity of the problem that we want to analyze, it is possible to generate both the maximum or the minimum in the general formulation, as has been presented by Merigó et al. (2015a), and include different aggregations in the sets of X and Y; this approach would increase the potential to generate new scenarios that usually cannot be assessed using the traditional covariance.

3 Covariance with Bonferroni means and OWA operators

To understand better the correlation between two variables, the covariance has become a common technique. In addition, as was explained previously, it is possible to aggregate information using methodologies such as the OWA operator and the Bonferroni means. The reason that these methods have been selected is because they integrate the relationships among the arguments (Bonferroni means) and include the expertise of the decision maker (OWA operator). This new formulation is called the Bonferroni ordered weighted average covariance (BONOWACov) operator, which is defined as follows:

Definition 12 For the Bonferroni ordered weighted average covariance (BONOWACov) operator, the formulation is as follows.

$$\mathrm{BONOWACov}(X,Y)=\left(\frac{1}{n}\sum_i a_i^r \mathrm{OWACov}_W\left(E^i\right)\right)^{\frac{1}{r+q}}, \tag{18}$$

where $\mathrm{OWACov}_W(E^i)=\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)$ with (E^i) being the vector of all $(x_i-\mu_{\mathrm{OWACov}})(y_i-v_{\mathrm{OWACov}})$ except $\left(x_j-\mu_{\mathrm{OWACov}}\right)\left(y_j-v_{\mathrm{OWACov}}\right)$, x_i is the argument variable of the first set of elements $X=\{x_1,\ldots,x_n\}$, y_i is the argument variable of the second set of elements $Y=\{y_1,\ldots,y_n\}$ and w is an $n-1$ vector W_i that is associated with α_i whose components w_{ij} are the OWACov weights. Let W be an OWACov weighing vector of dimension $n-1$ with components $w_i\in[0,1]$ when $\sum_i w_i=1$. Then, we can define this aggregation as $\mathrm{OWACov}_W(L^i)=\left(\sum_{j=1}^{n-1} w_i a_{\pi_k(j)}\right)$, where $a_{\pi_k(j)}$ is the largest element in L^i and $w_i=\frac{1}{n-1}$ for all i. This definition also has the total, case 1, case 2 and case 3 operators as extensions, as presented in Table 1.

Additionally, there is the case of $W=1/n$, in which the BONOWACov becomes the covariance Bonferroni mean (BMCov), and it is defined as follows.

Definition 13 The BMCov operator can be defined as

$$\mathrm{BMCov}(X,Y)=\left(\frac{1}{n}\sum_i a_i^r\left(E^i\right)\right)^{\frac{1}{r+q}}, \tag{19}$$

where $(E^i)=\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)$ with (E^i) being the vector of all $(x_i-\mu)(y_i-v)$ except $\left(x_j-\mu\right)\left(y_j-v\right)$, x_i is the argument variable of the first set of elements $X=\{x_1,\ldots,x_n\}$, and y_i is the argument variable of the second set of elements $Y=\{y_1,\ldots,y_n\}$.

Finally, if a reordering step based on induced values is added (such as the IOWA operator), it is possible to generate the induced Bonferroni ordered weighted average covariance (BONIOWACov) operator, and it is defined as follows.

Definition 14 The definition of the BONIOWACov operator is as follows.

$$\mathrm{BONIOWACov}(U,X,Y)=\left(\frac{1}{n}\sum_i b_i^r \mathrm{IOWACov}_W\left(E^i\right)\right)^{\frac{1}{r+q}}, \tag{20}$$

where b_i is the a_i value of the BONIOWACov pair $\langle u_i,a_i\rangle$ having the jth largest u_i, $\mathrm{IOWACov}_W(E^i)=\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} b_j^q\right)$ with (E^i) is the vector of all

Table 1 OWACov operator and its cases

Operator	Total	Case 1	Case 2	Case 3
OWACov	Uses: μ_{OWA} and v_{OWA}	Uses: μ_{OWA} and v	Uses: μ and v_{OWA}	Uses: μ and v

$(x_i - \mu_{\text{IOWAcov}})(y_i - \nu_{\text{IOWAcov}})$ except $\left(x_j - \mu_{\text{IOWAcov}}\right)\left(y_j - \nu_{\text{IOWAcov}}\right)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and w is an $n-1$ vector W_i that is associated with α_i whose components w_{ij} are the IOWACov weights. Let W be an IOWACov weighing vector of dimension $n-1$ with the components $w_i \in [0,1]$ when $\sum_i w_i = 1$, where the weights are associated according to the largest value of u_i, and u_i is the order-inducing variable. This definition also has the total, case 1, case 2 and case 3 operators as extensions, as presented in Table 1. Table 2 summarizes these cases and those of Definitions 12 and 13 as follows:

It is important to note that all operators in this section have the same main properties as the OWA operator (the explanation was provided with the BONIOWACov operator, but the same idea follows for Definitions 12, 13), which are as follows:

(a) *Commutativity* Assume f is the BONIOWACov operator; then, $f(u_i, a_i, \ldots, u_n, a_n) = f(u_i, b_i, \ldots, u_n, b_n)$.

(b) *Monotonicity* Assume f is the BONIOWACov operator; if $|u_i, a_i| \geq |u_i, b_i|$ for all i, then $f(u_i, a_i, \ldots, u_n, a_n) \geq f(u_i, b_i, \ldots, u_n, b_n)$.

(c) *Bounded* Assume f is the BONIOWACov operator; then, $\min\{a_i\} \leq f(u_i, a_i, \ldots, u_n, a_n) \leq \max\{a_i\}$.

(d) *Idempotency* Assume f is the BONIOWACov operator; if $|u_i, a_i| = a$ for all i, then $f(u_i, a_i, \ldots, u_n, a_n) = a$.

4 The generalized ordered weighted average covariance

4.1 Generalized covariance operators

To generate new cases that can be used in different cases, such as ones where the problem is complex or there is a need for different ways to aggregate the operator in order to generate new scenarios, generalized or quasi-arithmetic means can be used (Yager 2004; Merigó and Gil-Lafuente 2009). Most of the study presents the formulation of the quasi-arithmetic means as a special generalized case (Merigó et al. 2018). The quasi-arithmetic formulations for the new definitions that are proposed in this paper are as follows.

Definition 15 The quasi-BMCov operator can be defined as

$$\text{Quasi-BMCov}(X, Y) = g^{-1}\left[\left(\frac{1}{n}\sum_i a_i^r g\left(E^i\right)\right)^{\frac{1}{r+q}}\right], \tag{21}$$

where $a_i^r g(E^i) = \left(\frac{1}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q\right)$ with (E^i) being the vector of all $(x_i - \mu)(y_i - \nu)$ except $\left(x_j - \mu\right)\left(y_j - \nu\right)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and $g(E^i)$ is a strictly continuous monotonic function. This definition also has the total, case 1, case 2 and case 3 operators as extensions, as presented in Table 1.

Definition 16 For the quasi-BONOWACov operator, the formulation is as follows.

$$\text{Quasi-BonOWACov}(X, Y) = g^{-1}\left[\left(\frac{1}{n}\sum_i a_i^r \text{OWACov}_W g\left(E^i\right)\right)^{\frac{1}{r+q}}\right], \tag{22}$$

where $\text{OWACov}_W g(E^i) = \left(\frac{1}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q\right)$ with (E^i) being the vector of all $(x_i - \mu_{\text{OWAcov}})(y_i - \nu_{\text{OWAcov}})$ except $\left(x_j - \mu_{\text{OWAcov}}\right)\left(y_j - \nu_{\text{OWAcov}}\right)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and w is an $n-1$ vector W_i that is associated with α_i whose components w_{ij} are the OWA weights. Let W be an OWACov weighing vector of dimension $n-1$ with the components $w_i \in [0,1]$ when $\sum_i w_i = 1$. Then, we can define this aggregation as $\text{OWACov}_W g(L^i) = \left(\sum_{\text{j}=1}^{n-1} w_i a_{\pi_k(j)}\right)$, where $a_{\pi_k(j)}$ is the largest element in L^i, $w_i = \frac{1}{n-1}$ for all i and $g(E^i)$ is a strictly continuous monotonic function. This definition also has the total, case 1, case 2 and case 3 operators as extensions and is presented in Table 1.

Definition 17 The definition of the quasi-BonIOWACov operator is as follows.

$$\text{Quasi-BonIOWACov}(U, X, Y) = g^{-1}\left[\left(\frac{1}{n}\sum_i b_i^r \text{IOWACov}_W g\left(E^i\right)\right)^{\frac{1}{r+q}}\right], \tag{23}$$

where b_i is the a_i value of the BONIOWACov pair $\langle u_i, a_i \rangle$ having the jth largest u_i, $\text{IOWACov}_{Wg}(E^i) = \left(\frac{1}{\text{n}-1}\sum_{\substack{j=1 \\ j \neq i}}^{\text{n}} b_j^q\right)$ with (E^i) is the vector of all $(x_i - \mu_{\text{IOWAcov}})(y_i - \nu_{\text{IOWAcov}})$ except $\left(x_j - \mu_{\text{IOWAcov}}\right)\left(y_j - \nu_{\text{IOWAcov}}\right)$, x_i is the argument variable

Table 2 BONOWACov, BMCov and BONIOWACov operators and their cases

Operator	Total	Case 1	Case 2	Case 3
BONOWACov	μ_{OWA} and ν_{OWA}	μ_{OWA} and ν	μ and ν_{OWA}	μ and ν
BMCov	μ and ν	μ and ν	μ and ν	μ and ν
BONIOWACov	μ_{OWA} and ν_{IOWA}	μ_{OWA} and ν	μ and ν_{IOWA}	μ and ν

Table 3 Families of the generalized VarBONIOWA and CovBONIOWA operators

Particular case	Quasi-CovBONIOWA
$u_i = \frac{1}{n}$, for all i	Quasi-arithmetic covariance Bonferroni ordered weighted average (Quasi-CovBONOWA)
$g(b) = b^{\lambda}$	Generalized CovBONIOWA
$g(b) = b$	CovBONIOWA
$g(b) = b^2$	Covariance Bonferroni ordered weighted quadratic average (CovBONOWQA)
$g(b) \to b^{\lambda}$, for $\lambda \to 0$	Covariance Bonferroni ordered weighted geometric average (CovBONOWGA)
$g(b) = b^{-1}$	Covariance Bonferroni ordered weighted harmonic average (CovBONOWHA)
$g(b) = b^3$	Covariance Bonferroni ordered weighted cubic average (CovBONOWCA)
$g(b) \to b^{\lambda}$, for $\lambda \to \infty$	Maximum operator
$g(b) \to b^{\lambda}$, for $\lambda \to \infty$	Minimum operator

of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and w is an $n-1$ vector W_i associated with α_i whose components w_{ij} are the OWACov weights. Let W be an OWACov weighing vector of dimension $n-1$ with the components $w_i \in [0, 1]$ when $\sum_i w_i = 1$, where the weights are associated according to the largest value of u_i; u_i is the order-inducing variable; and $g(E^i)$ is a strictly continuous monotonic function. This definition also has the total, case 1, case 2 and case 3 operators as extensions, presented in Table 1.

4.2 Particular cases

In this section, the main particular cases for each formulation are presented (see Table 3). It is important to note that the most complex formula that is presented is the quasi-BonIOWACov operator, but the idea can be applied to Definitions 15 and 16.

To understand these new formulas, a numerical example is provided that is based on the sales and numbers of clients in the first quarter of 2017 for a Mexican enterprise (see Table 4).

Now, we apply the traditional covariance, OWACov, IOWACov, BMCov BONOWACov and BONIOWACov operators. The results are as follows.

(a) Covariance (Table 5)
(b) *OWACov* case 3 (Table 6)

Consider the following information in order to use this operator:

1. The weighting vector $W = (0.20, 0.30, 0.50)$, and
2. A maximum criterion has been used.

(c) *IOWACov* case 3 (Table 7)

Consider the following information in order to use this operator:

1. The weighting vector $W = (0.20, 0.30, 0.50)$, and
2. The induced vector $U = (10, 5, 15)$.

(d) *BMCov* case 3 (Table 8)

Consider the following information in order to use this operator:

1. The weighting vector $W = (0.20, 0.30, 0.50)$,
2. A maximum criterion has been used, and
3. r and q are equal to 1.

$$V_1 = (-1.92 \times 0.33) + (61.08 \times 0.33) = 19.52$$
$$V_2 = (61.08 \times 0.33) + (100.71 \times 0.33) = 53.59$$
$$V_3 = (100.71 \times 0.33) + (-1.92 \times 0.33) = 32.60$$
$$\text{BMCov} = \left(\frac{((19.52 \times 0.33) \times 100.71) + ((53.59 \times 0.33) \times -1.92) + ((32.60 \times 0.33) \times 61.08)}{3}\right)^{\frac{1}{1+1}} = 20.59$$

(e) *BONOWACov* case 3 (Table 9)

1. The weighting vector $W = (0.20, 0.30, 0.50)$,
2. A maximum criterion has been used, and
3. p and q are equal to 1.

$$V_1 = (-1.92 \times 0.20) + (61.08 \times 0.30) = 17.94$$
$$V_2 = (61.08 \times 0.30) + (100.71 \times 0.50) = 68.68$$
$$V_3 = (100.71 \times 0.50) + (-1.92 \times 0.20) = 49.97$$

BONOWACov

$$= \left(\frac{((17.94 \times 0.50) \times 100.71) + ((68.68 \times 0.20) \times -1.92) + ((49.97 \times 0.30) \times 61.08)}{3} \right)^{\frac{1}{1+1}}$$
$$= 24.24$$

(f) *BONIOWACov* case 3 (Table 10)

1. Weighting vector $W = (0.20, 0.30, 0.50)$,
2. A maximum criterion has been used, and
3. r and q are equal to 1.

Finally, by analyzing the results using different operators, it can be seen that there is a positive relation between the sales and numbers of clients in all the operators. In this sense, if a company or seller wants to increase its sales by a certain number, they can assume that increasing the number of clients will help to achieve that goal. The main difference between the operators is when the correlation between the variables is to be calculated. Taking into account that the formula for the lineal correlation is $\rho_{xy} = \frac{\text{Cov}_{xy}}{\sigma_x \sigma_y}$, where ρ_{xy} is the lineal correlation and $\sigma_x$and σ_y are the standard deviations for x and y, respectively, it is interesting to see how much the correlation can change if the covariance ranges from 20.59 to 68.29. In this sense, the generation of new covariance scenarios is important because of the utility and importance that this calculation has in other areas, such as econometrics, engineering, finance and other management fields.

To visualize and better explain the lineal correlation, taking into account that $\sigma_x = 142.66$ and $\sigma_y = 56.33$, then the different linear correlation coefficients (LCCs) are presented in Table 11 and represented in Fig. 1.

$$V_1 = (-1.92 \times 0.20) + (61.08 \times 0.50) = 30.15$$
$$V_2 = (61.08 \times 0.50) + (100.71 \times 0.30) = 60.75$$
$$V_3 = (100.71 \times 0.03) + (-1.92 \times 0.20) = 29.83$$

$$\text{BONIOWACov} = \left(\frac{((30.15 \times 0.30) \times 100.71) + ((60.75 \times 0.20) \times -1.92) + ((29.83 \times 0.50) \times 61.08)}{3} \right)^{\frac{1}{1+1}} = 24.48$$

It is noteworthy that there are two segments of results: the first one includes the Cov, OWACov and IOWACov operators (traditional information analysis), and the second one includes the BMCov, BONOWACov and BONIOWACov operators (Bonferroni information analysis). The main difference between each segment is that when the

Table 4 Sales and numbers of clients for the first quarter of 2017

Date	Sales (in thousands)	Number of clients
Jan-17	85.4	145
Feb-17	65.9	138
Mar-17	63.7	130

Table 5 Covariance calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	w	$(x-\mu)*(y-v)*w$
Jan-17	85.40	13.73	145.00	7.33	100.71	0.33	33.57
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	0.33	− 0.64
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	0.33	20.36
μ	71.67	v	137.67			Covariance	53.29

Table 6 OWACov calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	Max ordered w	[Max ordered $(x-\mu)*(y-v)$]*w
Jan-17	85.40	13.73	145.00	7.33	100.71	0.50	50.36
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	0.20	− 0.38
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	0.30	18.32
μ	71.67	v	137.67			OWACov	68.30

Table 7 IOWACov calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	u	Induced ordered w	[Induced ordered $(x-\mu)*(y-v)$]*w
Jan-17	85.40	13.73	145.00	7.33	100.71	10	0.30	30.21
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	5	0.20	− 0.38
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	15	0.50	30.54
μ	71.67	v	137.67				IOWACov	60.37

Table 8 BMCov calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	w
Jan-17	85.40	13.73	145.00	7.33	100.71	0.33
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	0.33
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	0.33
μ	71.67	v	137.67			

Table 9 BONOWACov calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	Max ordered w
Jan-17	85.40	13.73	145.00	7.33	100.71	0.50
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	0.20
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	0.30
μ	71.67	v	137.67			

Table 10 BONIOWACov calculations

Date	X (sales)	$x-\mu$	Y (clients)	$y-v$	$(x-\mu)*(y-v)$	u	w
Jan-17	85.40	13.73	145.00	7.33	100.71	10	0.30
Feb-17	65.90	− 5.77	138.00	0.33	− 1.92	5	0.20
Mar-17	63.70	− 7.97	130.00	− 7.67	61.08	15	0.50
μ	71.67	v	137.67				

Bonferroni mean is used, it smooths the results, which is why the minimum is 0.0026 and the maximum is 0.0030 (a variation of 15%) instead of the differences in the first stage that range from 0.0066 to 0.0085 (a variation of 28%). With this information, it is possible to appreciate how much the inclusion of the interrelation of the arguments through the Bonferroni means helps to provide a whole new range of possible scenarios with a higher degree of information included.

In the previous example, it used as values for the indices r and q the value of 1. Now, it is showed how the results change when these values change. You can observe how and when the r and q values change (see Table 12); on the one hand, if the r values increase, the results of the BONIOWACov operator are of minimum value; on the other hand, if the q values decrease, the results of the BONIOWACov operator are of maximum value. Likewise, taking the data from the example for linear regression (LR), it is observed that they have the same change behavior. With these results, we can observe the sensitivity of the results when changing the values of these indices.

5 Application in R&D Investment

Among the main ideas of innovation is that when enterprises invest more in R&D, it has important impacts on the company that are usually translated into more sales. In this application, 10 random companies from the NASDAQ 100 were used to analyze the covariance and correlation between sales and R&D investment from 2015 to 2018. Additionally, different aggregation operators were used to

Table 11 Linear correlation coefficients (Case 3)

	Covariance	OWACov	IOWACov	BMCov	BONOWACov	BONIOWACov
Results	0.0066	0.0085	0.0075	0.0026	0.0030	0.0030

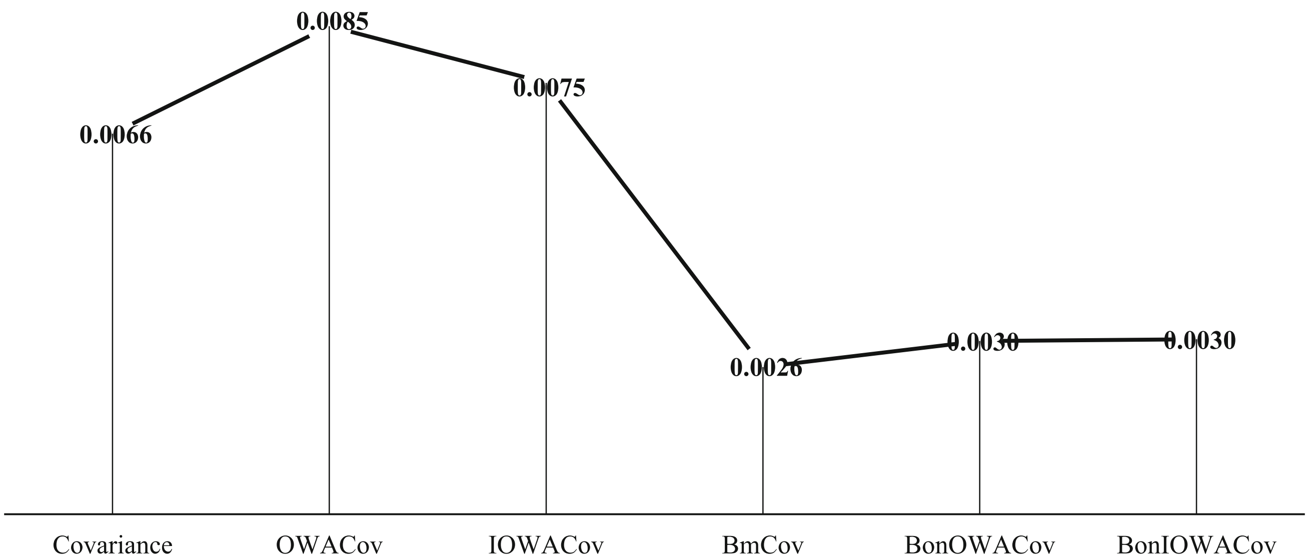

Fig. 1 Linear correlation coefficients (Case 3)

Table 12 Results of the BONIOWACov operator by changing r and q

r	q	BONIOWACov	LR	r	q	BONIOWACov	LR	r	q	BONIOWACov	LR
1	0	599.53	0.0746	1	2	8.43	0.0010	1	4	3.59	0.0004
2	0	24.49	0.0030	2	2	4.95	0.0006	2	4	2.9	0.0004
3	0	8.43	0.0010	3	2	3.59	0.0004	3	4	2.49	0.0003
4	0	4.95	0.0006	4	2	2.9	0.0004	4	4	2.22	0.0003
5	0	3.59	0.0004	5	2	2.49	0.0003	5	4	2.04	0.0003
1	1	24.49	0.0030	1	3	4.95	0.0006	1	5	2.9	0.0004
2	1	8.43	0.0010	2	3	3	0.0004	2	5	2.49	0.0003
3	1	4.95	0.0006	3	3	2.9	0.0004	3	5	2.22	0.0003
4	1	3.59	0.0004	4	3	2.49	0.0003	4	5	2.04	0.0003
5	1	2.9	0.0004	5	3	2.22	0.0003	5	5	1.9	0.0002

analyze the information. The information is presented in Table 13.

Using information obtained from their financial statements, specifically, their income statements,[1] different covariances and correlations will be calculated. To do that, the weighting vector will be assigned for the companies, and the weights will be divided by 4 in order to determine the weight for each data. The same idea will be used for the induced variables. The weights and induced variables that will be used are as follows (see Table 14).

In addition, the values of index p and $q = 1$.

[1] The information was obtained through the webpage www.investing.com.

The results for each special case (explained in Sect. 2.5.2) are presented in Tables 15, 16, 17 and 18.

To visualize better all the results, they are presented in Figs. 2 and 3. As seen in Tables 15, 16, 17 and 18 and in Figs. 2 and 3, there are similar movements between the covariance and the correlation (this is kind of obvious due to the way that the correlation formula works); however, the most important result that we can obtain from this analysis is that, even in the worst scenarios, the correlation is always higher than 0.6000, thus indicating that there is a positive and strong relationship between the sales of the companies and their R&D investment. It is important to note that this conclusion has limitations, as it is primarily

Table 13 Sales and R&D investment in NASDAQ companies

Company	Year	Total revenues	Research and development
Apple	2018	265,595	14,236
	2017	229,234	11,581
	2016	215,639	10,045
	2015	233,715	8067
Microsoft	2018	110,360	14,726
	2017	96,571	12,292
	2016	91,154	11,988
	2015	93,580	12,046
Activision Blizzard	2018	7500	1101
	2017	7017	1069
	2016	6608	958
	2015	4664	646
Alphabet	2018	136,819	21,419
	2017	110,855	16,625
	2016	90,272	13,948
	2015	74,989	12,282
Amazon	2018	72,383	7669
	2017	56,576	7162
	2016	52,886	7247
	2015	51,042	6759
Netflix	2018	15,794	1222
	2017	11,693	1053
	2016	8831	852
	2015	6780	651
Qualcomm	2018	22,732	5619
	2017	22,291	5465
	2016	23,554	5141
	2015	25,281	5476
Pepsi	2018	64,661	680
	2017	63,525	737
	2016	62,799	760
	2015	63,056	754
Tesla	2018	21,461	1460
	2017	11,759	1378
	2016	7000	834
	2015	4046	718
PayPal	2018	15,451	1071
	2017	13,094	953
	2016	10,842	834
	2015	9248	792

Table 14 Induced variables for each company

	Apple	Microsoft	Activision Blizzard	Alphabet	Amazon	Netflix	Qualcomm	Pepsi	Tesla	PayPal
W	0.12	0.12	0.08	0.10	0.10	0.08	0.08	0.08	0.12	0.12
U	5	45	50	35	25	10	15	20	30	40

Table 15 Results for covariance and correlation with the total operator

Operator	Result of covariance	Correlation
Covariance	270,167,400	0.685
OWACov	291,726,392	0.736
IOWACov	298,884,458	0.754
BMCov	243,150,660	0.617
BonOWACov	259,640,483	0.655
BonIOWACov	269,228,156	0.679

Table 16 Results for covariance and correlation with the case 1 operator

Operator	Result of covariance	Correlation
Covariance	270,167,400	0.682
OWACov	291,922,473	0.736
IOWACov	298,673,232	0.753
BMCov	243,150,660	0.613
BonOWACov	259,652,809	0.655
BonIOWACov	269,135,071	0.679

Table 17 Results for covariance and correlation with the case 2 operator

Operator	Result of covariance	Correlation
Covariance	270,167,400	0.682
OWACov	291,771,139	0.736
IOWACov	298,358,394	0.753
BMCov	243,150,660	0.613
BonOWACov	259,547,968	0.655
BonIOWACov	268,639,773	0.678

Table 18 Results for covariance and correlation with the case 3 operator

Operator	Result of covariance	Correlation
Covariance	270,167,400	0.682
OWACov	293,512,859	0.740
IOWACov	299,692,808	0.756
BMCov	243,150,660	0.613
BonOWACov	260,946,424	0.658
BonIOWACov	269,937,145	0.681

applicable to the companies that are used in this paper; these companies are all members of the NASDAQ 100, and it is possible that this correlation could be weaker or stronger for a specific sector or SMEs.

Additionally, with the use of different aggregation operators, it was possible to acquire results other than those obtained by the traditional method, producing values of 270,167,400 for the covariance and 0.6823 for the correlation. With this new information, it is possible to evaluate if the correlation increases or decreases as we examine a specific sector or similar industries. This consideration assumes a difference between exists between the traditional correlation and the correlation obtained via the IOWACov operator; the use of the IOWACov operator revealed a correlation increase of nearly 10%. Finally, this information can provide a new visualization tool for the future of the companies and shows that R&D investment is a must if companies want to increase sales and stay in the market.

6 Conclusions

This work presents the combination of the Bonferroni mean, the OWA operator and the covariance to develop a new aggregated operator within probability and statistics. The objective is to provide better tools for the analysis of the strength of the correlation among two or more sets of random variables. By using the OWA operators, the decision maker can under- or overestimate the covariance according to their attitudes. Likewise, by using the Bonferroni mean, it compensates for the possible error when making several comparisons. This compensation comes from parameters r and q, since they correct the errors when making multiple comparisons guaranteeing a significant adjustment of the analyzed data set (Blanco-Mesa et al. 2019d). This behavior is observed in the example shown in Fig. 1, which shows the significant difference between the methods between the maximum and minimum values in relation to the established correlations. In such way, the results are smoothed showing variations of the 15% (the minimum is 0.0026 and the maximum is 0.0030) and 28% (range from 0.0066 to 0.0085). Likewise, in the results shown in the R&D problem they have the same behavior. Thus, decision makers should consider using attitude and corrections to compare correlated data in the aggregation problem. Hence, decision makers can consider the joint variation of the data according to weights, and the interrelations and results could be higher or lower than when using the classical covariance. This paper presents some simple numerical examples to numerically illustrate the new approach.

Furthermore, the paper proposes some extensions and generalizations to provide a more robust and general

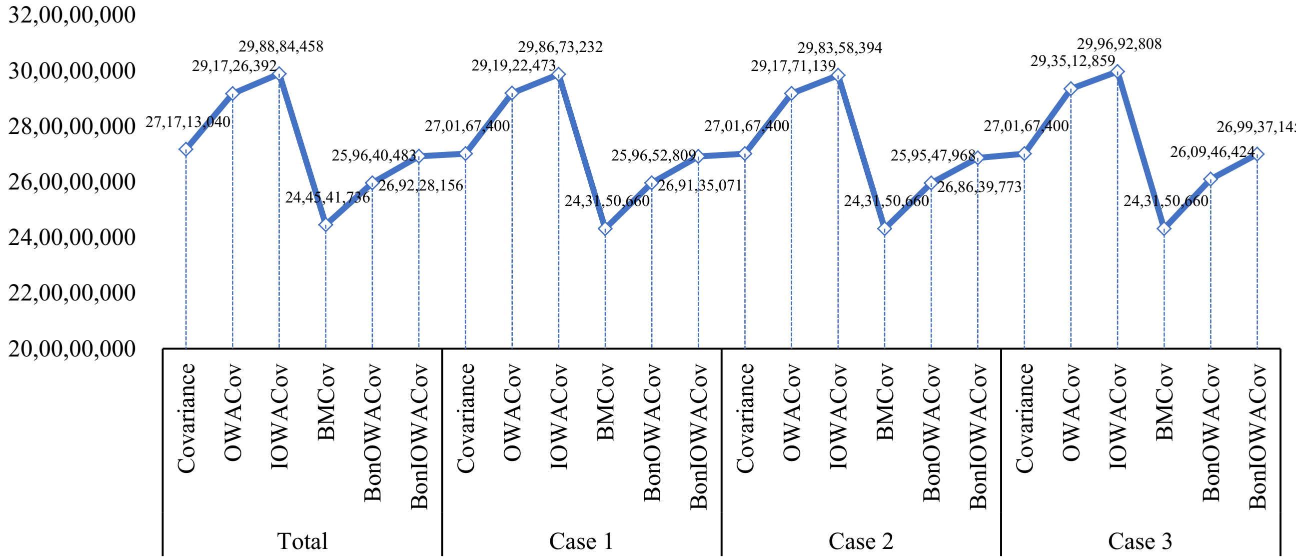

Fig. 2 Covariance results for all operators

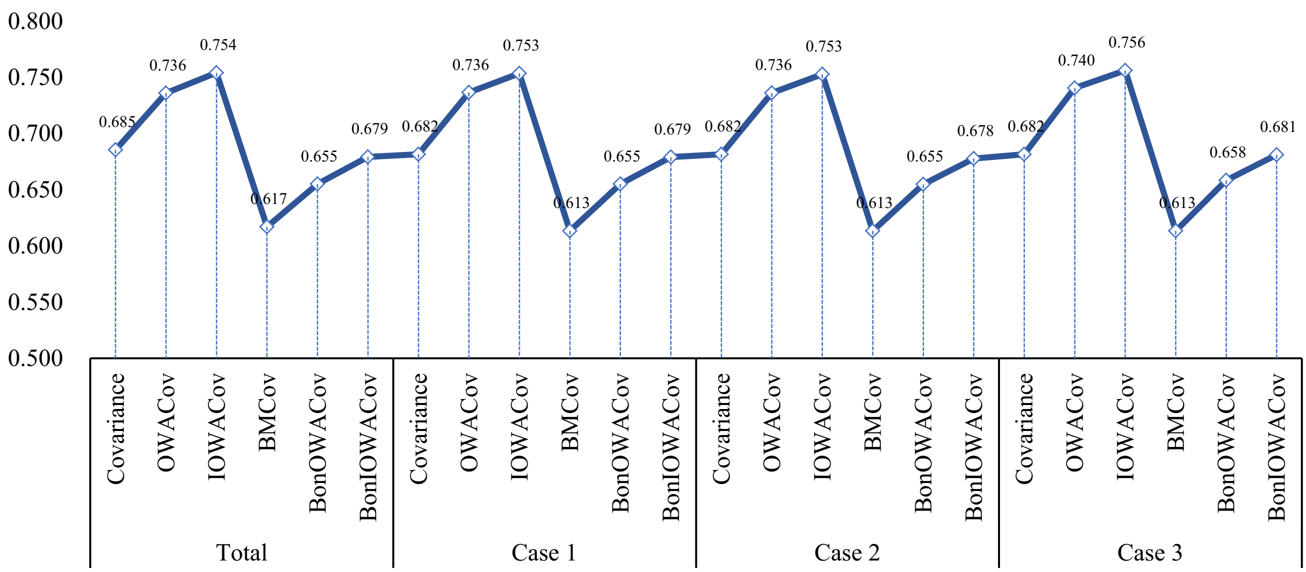

Fig. 3 Correlation for all operators

framework. To this end, induced and generalized aggregation operators are used. With induced aggregation operations, the correlation among two or more sets of random variables can reflect a complex under- or overestimation of the covariance according to the complex factors that affect the attitudes of decision makers. The formula that is proposed is more general for generalized and quasi-arithmetic means, as these aggregations consider a wide range of particular cases, such as the geometric Bonferroni covariance, the quadratic Bonferroni covariance, the cubic Bonferroni covariance and the harmonic Bonferroni covariance.

The work ends with an application of the new approach to R&D investment decision making, in which a decision maker analyzes correlation data. The aim is to analyze the covariance and correlation between sales and R&D investment for 10 random companies from the NASDAQ 100 from 2015 to 2018. These new approaches allow the decision makers the ability to obtain a better representation of the potential real scenarios when it is under- or over-estimating the covariance. Recall that the classic covariance correlates two or more sets of random variables and reflects the degree of joint variation with respect to their means. Thus, the result that is obtained is a representative value of the set of joint variations for which the best result is close to 1 or –1. However, in general terms, the correlation of the whole set could move from positive dependence to negative dependence or no dependence according to the real importance that each data have for the result. In addition, the use of OWA operators and other related methods are useful when the importance and meaning of the data are not obvious. Future work in this direction should consider further covariance extensions using other types of aggregation systems, including orthopair fuzzy numbers (Liu and Wang 2019), heavy, probability and prioritized operators and moving averages (Kacprzyk et al. 2019; León-Castro et al. 2019b). Furthermore, to complement this new proposition, it would also be interesting to develop applications in several fields within which these new covariance measures could be implemented. Thus, based on this perspective, the potential applicability is very broad because a many study in several fields that use covariance measures could be extended to find cases that require the use of this approach.

Acknowledgements This study was funded by Universidad Pedagógica y Tecnológica de Colombia (Grant Number SGI-2640).

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Alfaro-García VG, Merigó JM, Gil-Lafuente AM, Kacprzyk J (2018) Logarithmic aggregation operators and distance measures. Int J Intell Syst 33:1488–1506

Avilés-Ochoa E, León-Castro E, Perez-Arellano LA, Merigó JM (2018) Government transparency measurement through prioritized distance operators. J Intell Fuzzy Syst 34:2783–2794

Beliakov G, James S, Mordelová J, Ruckschlossova T, Yager RR (2010) Generalized Bonferroni mean operators in multi-criteria aggregation. Fuzzy Sets Syst 161:2227–2242

Belles-Sampera J, Merigó JM, Guillén M, Santolino M (2013) The connection between distortion risk measures and ordered weighted averaging operators. Insur Math Econ 52:411–420

Blanco-Mesa F, Gil-Lafuente AM (2017) Towards a competitiveness in the economic activity in Colombia: using Moore’s families and Galois latticies in clustering. Econ Comput Econ Cybern Stud Res 51:231–250

Blanco-Mesa F, Merigó JM (2017) Bonferroni distances with hybrid weighted distance and immedate wieghted distance. Fuzzy Econ Rev 22:29–43

Blanco-Mesa F, Merigó JM (2020) Bonferroni distances and their application in group decision making. Cybern Syst 51:27–58

Blanco-Mesa F, Merigó JM, Kacprzyk J (2016) Bonferroni means with distance measures and the adequacy coefficient in entrepreneurial group theory. Knowl Based Syst 111:217–227

Blanco-Mesa F, Merigó JM, Gil-Lafuente AM (2017) Fuzzy decision making: a bibliometric-based review. J Intell Fuzzy Syst 32:2033–2050

Blanco-Mesa F, Gil-Lafuente AM, Merigo JM (2018a) Dynamics of stakeholder relations with multi-person aggregation. Kybernetes 47:1801–1820

Blanco-Mesa F, Gil-Lafuente AM, Merigó JM (2018b) Subjective stakeholder dynamics relationships treatment: a methodological approach using fuzzy decision-making. Comput Math Organ Theory 24:441–472

Blanco-Mesa F, Gil-Lafuente AM, Merigó JM (2018c) New aggregation operators for decision-making under uncertainty: an applications in selection of entrepreneurial opportunities. Technol Econ Dev Econ 24:335–357

Blanco-Mesa F, León-Castro E, Merigó JM (2018d) Bonferroni induced heavy operators in ERM decision-making: a case on large companies in Colombia. Appl Soft Comput 72:371–391

Blanco-Mesa F, León-Castro E, Merigó JM (2019a) A bibliometric analysis of aggregation operators. Appl Soft Comput 81:1–21

Blanco-Mesa F, León-Castro E, Merigó JM, Xu Z (2019b) Bonferroni means with induced ordered weighted average operators. Int J Intell Syst 34:3–23

Blanco-Mesa F, Rivera-Rubiano J, Patiño-Hernandez X, Martinez-Montaña M (2019c) The importance of enterprise risk management in large companies in Colombia. Technol Econ Dev Econ 25:600–633

Blanco-Mesa F, León-Castro E, Merigó JM, Herrera-Viedma E (2019d) Variances with Bonferroni means and ordered weighted averages. Int J Intell Syst 34:3020–3045

Bonferroni C (1950) Sulle medie multiple di potenze. Boll dell’Unione Mat Ital 5:267–270

Carrasco RA, Sánchez-Fernández J, Muñoz-Leiva F, Blasco MF, Herrera-Viedma E (2017) Evaluation of the hotels e-services quality under the user’s experience. Soft Comput 21:995–1011

Chen L, Xu Z (2014) A prioritized aggregation operator based on the OWA operator and prioritized measure. J Intell Fuzzy Syst 27:1297–1307

Chiclana F, Herrera F, Herrera-Viedma E (2002) The ordered weighted geometric operator: Properties and application in MCDM problems. In: Bouchon-Meunier BB, Gutierrez-Rios J, Magdalena L, Yager RR (eds) Technologies for constructing intelligent systems 2. Studies in fuzziness and soft computing. Physica, Heidelberg, pp 173–183

Dong Y, Zhao S, Zhang H et al (2018) A self-management mechanism for non-cooperative behaviors in large-scale group consensus reaching processes. IEEE Trans Fuzzy Syst 26:3276–3288

Gao H (2018) Pythagorean fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. J Intell Fuzzy Syst 35:2229–2245

Gil-Lafuente AM, Merigó JM (2007) The ordered weighted averaging distance operator. Lect Model Simul 8:84–95

Gou X, Xu Z, Liao H (2017) Multiple criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information. Soft Comput 21:6515–6529

He Y, He Z, Chen H (2015) Intuitionistic fuzzy interaction Bonferroni means and its application to multiple attribute decision making. IEEE Trans Cybern 45:116–128

Herrera F, Herrera-Viedma E (2000) Linguistic decision analysis: steps for solving decision problems under linguistic information. Fuzzy Sets Syst 115:67–82

Herrera F, Martinez L (2000) A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Trans Fuzzy Syst 8:746–752

Kacprzyk J, Zadrożny S (2009) Towards a general and unified characterization of individual and collective choice functions under fuzzy and nonfuzzy preferences and majority via the ordered weighted average operators. Int J Intell Syst 24:4–26

Kacprzyk J, Yager RR, Merigó JM (2019) Towards human centric aggregation via the ordered weighted aggregation operators and linguistic data summaries: a new perspective on Zadeh's inspirations. IEEE Comput Intell Mag 14:16–30

Kaufmann A, Gil Aluja J (1986) Introducción de la teoría de los subconjuntos borrosos a la gestión de las empresas, 2nd edn. Milladoiro, Santiago de Compostela

Kaufmann A, Gil-Aluja J (1990) Las matemáticas del azar y de la incertidumbre: elementos básicos para su aplicación en economía. Centro de Estudios Ramón Areces, Madrid

Laengle S, Loyola G, Merigó JM (2017) Mean-variance portfolio selection with the ordered weighted average. IEEE Trans Fuzzy Syst 25:350–362

León-Castro E, Áviles-Ochoa E, Gil-Lafuente AM (2016) Exchange rate usd/mxn forecast through econometric models, time series and howma operators. Econ Comput Econ Cybern Stud Res 50:135–150

León-Castro E, Avilés-Ochoa E, Merigó JM, Gil-Lafuente AM (2018) Heavy moving averages and their application in econometric forecasting. Cybern Syst 49:26–43

León-Castro E, Blanco-Mesa F, Merigó JM (2019a) Weighted averages in the ordered weighted average inflation. In: Kearfott R, Batyrshin I, Reformat M et al (eds) Fuzzy techniques: theory and applications. IFSA/NAFIPS 2019. Advances in intelligent systems and computing, 1000th edn. Springer, Cham, pp 87–95

León-Castro E, Espinoza-Audelo LF, Aviles-Ochoa E, Merigó JM (2019b) A new measure of volatility using induced heavy moving averages. Technol Econ Dev Econ 25:576–599

Liu Z, Liu P (2017) Intuitionistic uncertain linguistic partitioned Bonferroni means and their application to multiple attribute decision-making. Int J Syst Sci 48:1092–1105

Liu P, Wang P (2019) Multiple-attribute decision-making based on Archimedean Bonferroni operators of q-Rung orthopair fuzzy numbers. IEEE Trans Fuzzy Syst 27:834–848

Merigó JM (2012) The probabilistic weighted average and its application in multiperson decision making. Int J Intell Syst 27:457–476

Merigó JM, Gil-Lafuente AM (2009) The induced generalized OWA operator. Inf Sci 179:729–741

Merigó JM, Yager RR (2013) Generalized moving average, distance measures and OWA operators. Int J Uncertain Fuzziness Knowl Based Syst 21:533–559

Merigó JM, Casanovas M, Palacios-Marqués D (2014) Linguistic group decision making with induced aggregation operators and probabilistic information. Appl Soft Comput 24:669–678

Merigó JM, Guillén M, Sarabia JM (2015a) The ordered weighted average in the variance and the covariance. Int J Intell Syst 30:985–1005

Merigó JM, Palacios-Marqués D, Riberio-Navarrete B (2015b) Aggregation systems for sales forecasting. J Bus Res 68:2299–2304

Merigó JM, Palacios-Marqués D, Soto-Acosta P (2017) Distance measures, weighted averages, OWA operators and Bonferroni means. Appl Soft Comput 50:356–366

Merigó JM, Zhou L, Yu D, Alrajeh N, Alnowibet K (2018) Probabilistic OWA distances applied to asset management. Soft Comput 22:4855–4878

Pérez-Arellano LA, León-Castro E, Avilés-Ochoa E, Merigó JM (2019) Prioritized induced probabilistic operator and its application in group decision making. Int J Mach Learn Cybern 10:451–462

Tang X, Wei G (2019) Multiple attribute decision-making with dual hesitant Pythagorean fuzzy information. Cognit Comput 11:193–211

Verma R, Merigó JM (2019) Variance measures with ordered weighted aggregation operators. Int J Intell Syst. 34:2556–2583

Wei GW (2019) Pythagorean fuzzy Hamacher power aggregation operators in multiple attribute decision making. Fundam Inform 166:57–85

Wu J, Chiclana F, Fujita H, Herrera-Viedma E (2017) A visual interaction consensus model for social network group decision making with trust propagation. Knowl Based Syst 122:39–50

Xu Z, Chen J (2008) Ordered weighted distance measure. J Syst Sci Syst Eng 17:432–445

Xu ZS, Da QL (2002) The ordered weighted geometric averaging operators. Int J Intell Syst 17:709–716

Yager RR (1988) On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans Syst Man Cybern 18:183–190

Yager RR (1996) On the inclusion of variance in decision making under uncertainty. Int J Uncertain Fuzziness Knowl Based Syst 04:401–419

Yager RR (2001) The power average operator. IEEE Trans Syst Man Cybern A Syst Hum 31:724–731

Yager RR (2002) Heavy OWA operators. Fuzzy Optim Decis Mak 1:379–397

Yager RR (2003) Induced aggregation operators. Fuzzy Sets Syst 137:59–69

Yager RR (2004) Generalized OWA aggregation operators. Fuzzy Optim Decis Mak 3:93–107

Yager RR (2008) Time Series smoothing and OWA aggregation. IEEE Trans Fuzzy Syst 16:994–1007

Yager RR (2009) On generalized Bonferroni mean operators for multi-criteria aggregation. Int J Approx Reason 50:1279–1286

Yager RR (2014) Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst 22:958–965

Yager RR, Filev DP (1999) Induced ordered weighted averaging operators. IEEE Trans Syst Man Cybern B Cybern 29:141–150

Yager RR, Engemann KJ, Filev DP (1995) On the concept of immediate probabilities. Int J Intell Syst 10:373–397

Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning III. Inf Sci 9:43–80

Zhang H, Dong Y, Herrera-Viedma E (2018) Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. IEEE Trans Fuzzy Syst 26:884–898

Zhu B, Xu ZS (2013) Hesitant fuzzy Bonferroni means for multi-criteria decision making. J Oper Res Soc 64:1831–1840