Received: 5 April 2018
DOI: 10.1002/mma.5880

RESEARCH ARTICLE

WILEY

(ω, c)-asymptotically periodic functions, first-order Cauchy problem, and Lasota-Wazewska model with unbounded oscillating production of red cells

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Communicated by:T. Roubíček

Funding information
Departamento Administrativo de Ciencia, Tecnología e Innovación, Grant/Award Number: 121556933876; Diubb, Grant/Award Number: 164408 3/R; Fondo Nacional de Desarrollo Científico y Tecnológico, Grant/Award Number: 1170466

In this paper, we study a new class of functions, which we call (ω, c)-asymptotically periodic functions. This collection includes asymptotically periodic, asymptotically antiperiodic, asymptotically Bloch-periodic, and unbounded functions. We prove that the set conformed by these functions is a Banach space with a suitable norm. Furthermore, we show several properties of this class of functions as the convolution invariance. We present some examples and a composition result. As an application, we prove the existence and uniqueness of (ω, c)-asymptotically periodic mild solutions to the first-order abstract Cauchy problem on the real line. Also, we establish some sufficient conditions for the existence of positive (ω, c)-asymptotically periodic solutions to the Lasota-Wazewska equation with unbounded oscillating production of red cells.

KEYWORDS

antiperiodic, completeness, convolution invariance, periodic, $(\omega;c)$-periodic

MSC CLASSIFICATION

34C25; 30D45; 47D06

1 | INTRODUCTION

We say that f is a (ω, c)-periodic function if there is a pair (ω, c), $c \in (\mathbb{C} \setminus \{0\})$, $w > 0$ such that $f(t + \omega) = cf(t)$, for all $t \in \mathbb{R}$ (see Pinto[1]). It represents periodic functions with $c = 1$, antiperiodic functions with $c = -1$, Bloch waves with $c = e^{ik/\omega}$, and unbounded functions for $|c| \neq 1$. Linear systems with periodic coefficients produce, by Floquet's theorem, (ω, c)-periodic solutions. This is the case of the famous Hill and Mathieu equations (see Mathieu[2] and Zounes and Rand[3])

$$\frac{d^2y}{dt^2} + [a - 2q\cos(2t)]y = 0.$$

Mathieu equation is a linearized model of an inverted pendulum, where the pivot point oscillates periodically in the vertical direction (see Nayfeh and Mook[4]). In fluid dynamics, we can find many examples of waves being described by Mathieu equation. The research of Faraday surface waves is very active (see previous studies[5-7]).

Several properties of (ω, c)-periodic functions have been obtained in Alvarez et al.[8] Also, this class of functions appears for example when it used the method of Bloch wave decomposition in order to obtain the homogenization of self-adjoint elliptic operators in arbitrary domains with periodically oscillating coefficients (see Conca and Vanninathan[9]and Orive et al[10] and the references therein).

In this paper, we introduce the space of (ω, c)-asymptotically periodic functions. A continuous function f is said to be (ω, c)-asymptotically periodic function if it can be written as $f = g + h$ where g is a (ω, c)-periodic function and h satisfying $c^{-t/\omega}h(t)$ goes to zero when t goes to infinite. Note that when $c = 1$, we obtain the space of asymptotically periodic functions defined by M. Fréchet[11] (see also other works[12,13] for additional references), when $c = -1$, we obtain the space of asymptotically antiperiodic functions defined in N'Guérékata and Valmorin,[14] and when $c = e^{ik/\omega}$, we obtain the space of asymptotically Bloch-periodic functions studied in previous works.[15,16] Also, it should be noted that the space of (ω, c)-periodic functions are contained in the space of (ω, c)-asymptotically periodic functions. Fréchet developed a remarkable theory in turn to asymptotically almost periodic functions and the solutions of this type in differential equations, see Fink.[17] Then, the new concept of (ω, c)-asymptotically periodic functions is important not only by the unification of several classes of periodicity but also by the projections in the Fréchet theory.

We give several properties of (ω, c)-asymptotically periodic functions including a characterization in terms of the asymptotically periodic functions, uniqueness of the decomposition, algebraic properties, and the fact that the primitive of a (ω, c)-periodic function is, again, (ω, c)-periodic function. Also, we prove a convolution theorem and that the space of (ω, c)-asymptotically periodic functions is a Banach space with the norm $\|\cdot\|_{a\omega c}$ defined below. Furthermore, we prove that the range of these functions is relatively compact with this norm. A composition result is given, and a variety of examples are showed. We point out that the asymptotically periodic, asymptotically antiperiodic and asymptotically Bloch-periodic functions are defined as a subspace of $BC(X)$, while our results includes unbounded functions on $\mathbb{R}$, that is, the cases $|c| < 1$ and $|c| > 1$.

The previous results allow to show the existence and uniqueness of (ω, c)-asymptotically periodic mild solutions for the following class of semilinear abstract differential equations

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A is a closed linear operator defined in a Banach space X which generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$. The results can be extended to delayed systems.

Furthermore, we prove the existence of positive (ω, c)-asymptotically periodic solutions to the Lasota-Wazewska equation with (ω, c)-asymptotically periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0. \tag{1}$$

Wazewska-Czyzewska and Lasota[18] propose this model to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red cells bloods in the time t, $\delta > 0$ is the probability of death of a red blood cell, $a(t)$ is a continuous and positive function which is related with the production of red blood cells by unity of time, τ is the time required to produce a red blood cell, $h(t)$ is a continuous and positive function which describes the generation of red blood cells per unit time.

This paper is organized as follows. In Section 2, we introduce the (ω, c)-asymptotically periodic functions and give some important properties. Also, we show that the space of (ω, c)-asymptotically periodic functions is a Banach space with a suitable norm and the fact that the range of this class of functions is relatively compact with this norm. Convolution and composition theorems will be proved. Several interesting examples are given. In Section 3, we prove the existence and uniqueness of (ω, c)-asymptotically periodic solutions to the first-order abstract Cauchy problem on $\mathbb{R}$. Finally, in Section 4, we prove the existence of positive (ω, c)-asymptotically periodic solutions to the Lasota-Wazewska model with (ω, c)-asymptotically periodic coefficients. Also, we show that the solution is exponentially stable.

2 | (ω,c)-ASYMPTOTICALLY PERIODIC FUNCTIONS

Throughout the paper, $d \in \mathbb{R}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a complex Banach space with norm $\|\cdot\|$, $\Omega \subset X$, and we will denote the space of continuous functions on $[d, \infty)$ by

$$C([d, \infty), X) := \{f : [d, \infty) \to X : f \text{ is continuous}\},$$

the space of asymptotic functions as

$$C_0(X) := \{h \in C([d,\infty),X) : \lim_{t\to\infty} h(t) = 0\},$$

and

$$C_0(\Omega,X) := \left\{h \in C([d,\infty)\times\Omega,X) : \lim_{t\to\infty} h(t,x) = 0 \text{ for all } x \text{ in any compact subset of } \quad \Omega\right\}.$$

Also we will denote, the space of bounded and continuous functions on $\mathbb{R}$ as

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is bounded and continuous}\},$$

the integrable functions in the real line as

$$L^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is integrable}\},$$

and the space of continuous functions on $\mathbb{R}\times X$ by

$$\{f : \mathbb{R}\times X \to X : f \text{ is continuous}\},$$

where $\mathbb{R}\times X$ is a Banach space with the norm $\|(t,x)\| = \max\{|t|,\|x\|\}$.

Definition 2.1 (Alvarez et al[8]). A function $g \in C([d,\infty),X)$ is said to be (ω,c)-periodic if $g(t+\omega) = cg(t)$ for all $t \in [d,\infty)$. ω is called the c-period of g. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}([d,\infty),X)$. When $c = 1$ (ω-periodic case) we write $P_{\omega}([d,\infty),X)$ in spite of $P_{\omega 1}[d,\infty),X)$. Using the principal branch of the complex Logarithm (i.e. the argument in $(-\pi,\pi]$) we define $c^{t/\omega} := \exp((t/\omega) \quad \mathrm{Log} \quad (c))$. Also, we will use the notation $c^\wedge(t) := c^{t/\omega}$ and $|c|^\wedge(t) := |c^\wedge(t)| = |c|^{t/\omega}$.

The following proposition gives a characterization of the (ω,c)-periodic functions.

Proposition 2.2 (Alvarez et al[8]). *Let $f \in C([d,\infty),X)$. Then f is a (ω,c)-periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) = c^{t/\omega}, \tag{2}$$

where $u(t)$ is a ω−periodic X-valued function.

In view of (2), for any $f \in P_{\omega c}([d,\infty),X)$ we say that $c^\wedge(t)u(t)$ is the c-factorization of f.

Remark 2.3. From Proposition 2.2, we can write all $f \in P_{\omega c}([d,\infty),X)$ as

$$f(t) = c^\wedge(t)u(t),$$

where $u(t)$ is ω-periodic on $[d,\infty)$. We will call $u(t)$ *the periodic part of f*. With this convention, an antiperiodic function f can be written as $f(t) = (-1)^{t/\omega}u(t)$, where u is ω-periodic. For example, $f(t) = \sin t$ can be considered as an antiperiodic function, with $\omega = \pi$. As $\mathrm{Log}(-1) = i\pi$, f has the decomposition $f(t) = c^\wedge(t)u(t)$ where

$$c^\wedge(t) = (-1)^{t/\pi} = e^{ti} = [\cos t + i\sin t],$$

and

$$u(t) = \sin t(\cos t - i\sin t),$$

which is periodic with period π.

Let $c = e^{2\pi i/k}$ for some natural number $k \geq 2$ and let f be a (ω,c)-periodic function. Then f is a periodic function with period $k\omega$ but, in general can be written as $f(t) = e^{2\pi ti/k\omega}u(t)$, where u is a complex periodic function with period ω. In particular if $k = 4$, a $(\omega, e^{\pi i/2})$-periodic function f can be at the same time a Bloch wave: $f(t+\omega) = e^{\pi i/2}f(t)$, an antiperiodic function with antiperiod 2ω: $f(t+2\omega) = -f(t)$ and a 4ω-periodic function: $f(t+4\omega) = f(t)$.

Definition 2.4. A function $h \in C([d,\infty),X)$ is said to be c-asymptotic if $c^\wedge(-t)h(t) \in C_0(X)$, that is,

$$\lim_{t\to\infty} c^\wedge(-t)h(t) = 0.$$

The collection of those functions will be denoted by $C_{0,c}(X)$. Analogously, a function $h \in C([d,\infty)\times\Omega,X)$ is said to be c-asymptotic if $c^\wedge(-t)h(t,x) \in C_0(\Omega,X)$, that is,

$$\lim_{t\to\infty} c^\wedge(-t)h(t,x) = 0$$

for all x in any compact subset of Ω. The collection of those functions will be denoted by $C_{0,c}(\Omega,X)$.

Definition 2.5. A function $f \in C([d,\infty),X)$ is said to be (ω,c)-asymptotically periodic if $f = g+h$ where $g \in P_{\omega c}([d,\infty),X)$ and $h \in C_{0,c}(X)$. The collection of those functions (with the same c-period ω for the first component) will be denoted by $AP_{\omega c}(X)$.

Remark 2.6. The preceding collection includes the asymptotically periodic functions $AP_{\omega 1}(X) := \{f \in C([d,\infty),X) : f = g+h,\ g \in P_{\omega 1}([d,\infty),X),\ h \in C_0(X)\}$, the asymptotically antiperiodic functions $AP_{\omega\,(-1)}(X) := \{f \in C([d,\infty),X) : f = g+h,\ g \in P_{\omega\,(-1)}([d,\infty),X),\ h \in C_0(X)\}$ and asymptotically Bloch-periodic functions $AP_{\omega\, e^{ik\omega}}(X) := \{f \in C([d,\infty),X) :\ f = g+h,\ g \in P_{\omega\, e^{ik\omega}}([d,\infty),X),\ h \in C_0(X)\}$.

The following proposition gives a characterization of the (ω,c)-asymptotically periodic functions.

Proposition 2.7. *Let $f \in C([d,\infty),X)$. Then f is (ω,c)-asymptotically periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) = c^{t/\omega}, \quad u \in AP_\omega(X). \tag{3}$$

Proof. It is clear that if $f(t)$ satisfies (3) then f is a (ω,c)-asymptotically periodic function. In order to show the inverse statement, let $f \in AP_{\omega c}(X)$. Then there exist $g \in P_{\omega c}([d,\infty),X)$ and $h \in C_{0,c}(X)$ such that $f = g+h$. If we write $u(t) := c^\wedge(-t)f(t) = c^{-t/\omega}f(t)$, then

$$u(t) = c^\wedge(-t)g(t) + c^\wedge(-t)h(t) =: F_1(t) + F_2(t).$$

It follows from Alvarez et al[8], Proposition 2.5 that $F_1 \in P_\omega([d,\infty),X)$ and by definition of $C_{0,c}(X)$ we have that $F_2 \in C_0(X)$. Hence $u \in AP_\omega(X)$. □

Remark 2.8. The decomposition in Definition 2.5 is unique, that is, there exist a unique $g \in P_{\omega c}([d,\infty),X)$ and a unique $h \in C_{0,c}(X)$ such that $f = g+h$. Indeed, suppose that

$$f(t) = g_1(t)+h_1(t) = g_2(t)+h_2(t), \quad g_1,g_2 \in P_{\omega c}([d,\infty),X),\ h_1,h_2 \in C_{0,c}(X),\ t \geq d.$$

Then,

$$u(t) := c^\wedge(-t)f(t) = c^\wedge(-t)g_1(t) + c^\wedge(-t)h_1(t) = c^\wedge(-t)g_2(t) + c^\wedge(-t)h_2(t)$$

belongs to $AP_\omega(X)$ by Proposition 2.2. By the unique representation of the functions in this space, we have that $c^\wedge(-t)g_1(t) = c^\wedge(-t)g_2(t)$ and $c^\wedge(-t)h_1(t) = c^\wedge(-t)h_2(t)$ and consequently $g_1(t) = g_2(t)$ and $h_1(t) = h_2(t)$ for all $t \geq d$.

Remark 2.9. Note that if $|c| \geq 1$ then $C_0(X) \subset C_{0,c}(X)$, and consequently $P_{\omega c}([d,\infty),X) + C_0(X) \subset AP_{\omega c}(X)$.

As a consequence of Proposition 2.7, we have the following basic properties.

Lemma 2.10. *Let $\alpha \in \mathbb{C}$. Then*

(a) $(f+g) \in AP_{\omega c}(X)$ and $\alpha h \in AP_{\omega c}(X)$ whenever $f,g,h \in AP_{\omega c}(X)$.
(b) If $\tau \geq 0$ is constant, then $f_\tau(t) = f(t+\tau) \in AP_{\omega c}(X)$ whenever $f \in AP_{\omega c}(X)$.
(c) If $f_1 \in AP_{\omega c_1}(X)$ and $f_2 \in AP_{\omega c_2}(X)$, then $f_1 \cdot f_2 \in AP_{\omega\, c_1 c_2}(X)$.

(d) *Let* $g \in P_{\omega c}(X)$ *and* $h \in C_{0,c}(X)$ *such that* $g, h \in C^1([d,\infty), X)$. *Then the derivative of* $f = g + h \in AP_{\omega c}(X)$ *belongs to* $AP_{\omega c}(X)$.

Proof. The proofs of (a) and (b) are consequence of the definition. We see (c) and (d). For (c) let u_1 and u_2 in $AP_\omega(X)$ such that

$$f_1(t) = c_1^\wedge(t)u_1(t), \qquad f_2(t) = c_2^\wedge(t)u_2(t).$$

Since $u_1 \cdot u_2 \in AP_\omega(X)$, we have that

$$f_1(t) \cdot f_2(t) = c_1^\wedge(t)c_2^\wedge(t)\, u_1(t) \cdot u_2(t) = (c_1c_2)^\wedge(t)\, u_1(t) \cdot u_2(t).$$

Hence $f_1 \cdot f_2 \in AP_{\omega\, c_1c_2}(X)$. Now, we show (d). For a fixed $t \in \mathbb{R}$ and every $\beta \neq 0$ with $|\beta|$ small enough, we have

$$g'(t+\omega) = \lim_{\beta\to 0} \frac{g(t+\omega+\beta) - g(t+\omega)}{\beta} = \lim_{\beta\to 0} \frac{cg(t+\beta) - cg(t)}{\beta} = cg'(t),$$

hence $g' \in P_{\omega c}(X)$. On the other hand, let $p \in C_0(X)$ such that $h(t) = c^\wedge(t)p(t)$. Then $h'(t) = (c^\wedge(t))'p(t) + c^\wedge(t)p'(t)$. Since p and p' goes to zero as $t \to \infty$ and $\frac{(c^\wedge(t))'}{c^\wedge(t)}$ is constant then $c^\wedge(-t)h'(t) \to 0$ as $t \to \infty$. □

Example 2.11. Let $X = \mathbb{C}$, $|b| < 2$ and h be a bounded function. Consider

$$f(t) = 2^t \sin t + b^t h(t), \quad t \geq d.$$

Then f is a $(\pi, -2^\pi)$-asymptotically periodic function. Since $c^\wedge(t) = \exp\left(\frac{t}{\pi}\mathrm{Log}(-2^\pi)\right) = 2^t e^{it}$, then by Proposition 2.2 we have that

$$g(t) = 2^t e^{it} u_1(t)$$

where

$$u_1(t) = \sin t(\cos t - i \sin t)$$

is periodic with period $\omega = \pi$. Analogously,

$$b^t h(t) = 2^t e^{it} u_2(t),$$

where

$$u_2(t) = \left(\frac{b}{2}\right)^t h(t)(\cos t - i \sin t)$$

belongs to $C_0([d,\infty), X)$. Hence, f has the decomposition

$$f(t) = 2^t \sin t + b^t h(t) = 2^t(\cos t + i \sin t)\left[\sin t(\cos t - i \sin t) + \left(\frac{b}{2}\right)^t h(t)(\cos t - i \sin t)\right].$$

Remark 2.12. If we put $h = 1$ in Example 2.11, we have that $f(t) = 2^t \sin t + b^t$ is a $(\pi, -2^\pi)$-asymptotically periodic function; however, $f \notin P_{\omega c}([d,\infty), X) + C_0(X)$. This implies that $AP_{\omega c}(X)$ is more general than $P_{\omega c}([d,\infty), X) + C_0(X)$.

Remark 2.13. The sum $P_{\omega c}(X) + C_0(X)$ is not direct. Indeed, let $f(t) = 2^{-t} \sin t$, $t \in \mathbb{R}$. Then f is a nonzero $(2\pi, 2^{-2\pi})$-periodic function and belongs to $C_0(\mathbb{R})$. Furthermore, note that

$$g(t) = 2^{-t} \sin t + h(t), \text{ with } \lim_{t\to\infty} 2^t h(t) = 0$$

belongs to $P_{\omega c}(\mathbb{R}) + C_{0,c}(\mathbb{R})$ where $c = 2^{-2\pi}$.

Example 2.14. Let $u : [d,\infty) \to X$ be a X-valued periodic function with period ω and $v : [d,\infty) \to X$ in $C_0(X)$. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a function with the semigroup property, that is, $\phi(t+s) = \phi(s)\phi(t)$ for all $t, s \in \mathbb{R}$ and such that $\phi(\omega) \neq 0$. Then

$$z(t) = \phi(t)u(t) + \phi(t)v(t), \quad t \geq d,$$

is a $(\omega, \phi(\omega))$-asymptotically periodic function if the function $\varphi(t) := [\phi(\omega)]^{\wedge}(-t)\phi(t)$ is bounded. As a particular case, we take $\phi(t) = e^{ikt}$ and obtain the asymptotically Bloch functions.

Remark 2.15. In general, if u is a (ω, c)-asymptotically periodic function, and ϕ is a function with the semigroup property such that $\phi(\omega) \neq 0$, then $z(t) := \phi(t)u(t)$ is a $(\omega, c\phi(\omega))$-asymptotically periodic if $\varphi(t) := [\phi(\omega)]^{\wedge}(-t)\phi(t)$ is bounded. Moreover, let $(u_k)_{k\in\mathbb{N}}$ be a sequence of (ω, c)-asymptotically periodic functions and $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of functions with the semigroup property and such that $\phi_k(\omega) = p \neq 0$ for all $k \in \mathbb{N}$. Assume that

$$\sum_{k=1}^{\infty} \phi_k(t)u_k(t)$$

is a uniformly convergent series on $\mathbb{R}$. Then,

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t)u_k(t)$$

is a (ω, cp)-asymptotically periodic function if $\varphi_k(t) := p^{\wedge}(-t)\phi_k(t)$ is bounded for $k \in \mathbb{N}$.

The following result shows that, under some conditions, the primitive of a (ω, c)-periodic function is also a (ω, c)-periodic function.

Proposition 2.16. *Assume that f is a (ω, c)-periodic function and $b \in [-\infty, \infty)$. Then $F(t) = \int_b^t f(s)ds$ is a (ω, c)-periodic function if and only if $F(b + \omega) = 0$.*

Proof. We have that

$$\begin{aligned} F(t+\omega) - F(b+\omega) &= \int_b^{t+\omega} f(s)ds - \int_b^{b+\omega} f(s)ds \\ &= \int_{b+\omega}^{t+\omega} f(s)ds \\ &= \int_b^t f(s+\omega)ds \\ &= c\int_b^t f(s)ds = cF(t). \end{aligned}$$

From here, we conclude that $F(t + \omega) = cF(t)$ if and only if $F(b + \omega) = 0$. □

Remark 2.17. It follows from Proposition (2.16) that if $F(t) = \int_{-\infty}^t f(s)ds$ is well defined then F is (ω, c)-periodic whenever f is a (ω, c)-periodic function. Analogously, $F(t) = \int_t^{\infty} f(s)ds$ is (ω, c)-periodic whenever f is a (ω, c)-periodic function.

Remark 2.18. The most elementary equation, $y' = f$ can be studied for $f \in P_{\omega c}(X)$. For $|c| \neq 1$ the primitive $\int_{-\infty}^t f(s)ds$ (or $-\int_t^{\infty} f(s)ds$) is well defined and it is a (ω, c)-periodic function.

We recall the following convolution result.

Theorem 2.19. *(Alvarez et al[8], Theorem 2.7) Let $f \in P_{\omega c}(\mathbb{R}, X)$ with $f(t) = c^{\wedge}(t)p(t)$, $p \in P_{\omega}(\mathbb{R}, X)$. If $k^{\sim}(t) := c^{\wedge}(-t)k(t) \in L^1(\mathbb{R})$, then $(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)ds \in P_{\omega c}(\mathbb{R}, X)$.*

We are ready to present the convolution theorem for (ω, c)-asymptotically periodic functions.

Theorem 2.20. *Assume that $d \in [-\infty, \infty)$. Let $f \in AP_{\omega c}(X)$ with $f(t) = c^{\wedge}(t)p(t)$, $p \in AP_{\omega}(X)$. If for some $k(t)$ we have that $k^{\sim}(t) := c^{\wedge}(-t)k(t) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_d^{\infty} k(t-s)f(s)\,ds = c^{\wedge}(t)(k^{\sim} * p)(t).$$

In particular, $k * f \in AP_{\omega c}(X)$.

Proof. Since $p \in AP_{\omega}(X)$ then there exist $p_1 \in P_{\omega}([d,\infty),X)$ and $p_2 \in C_0(X)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$ where $f_1(t) = c^\wedge(t)p_1(t) \in P_{\omega c}([d,\infty),X)$ and $f_2(t) = c^\wedge(t)p_2(t) \in C_{0,c}(X)$. We have

$$\begin{aligned}(k * f)(t) &= \int_d^{\infty} k(t-s)f(s)\,ds \\ &= \int_d^{\infty} k(t-s)f_1(s)\,ds + \int_d^{\infty} k(t-s)f_2(s)\,ds \\ &= (k * f_1)(t) + (k * f_2)(t) =: I_1(t) + I_2(t).\end{aligned}$$

From Theorem 2.19, we have that $I_1 \in P_{\omega c}([d,\infty),X)$. Next, we prove that $I_2 \in C_{0,c}(X)$, that is, $c^\wedge(-t)I_2(t)$ belongs to $C_0(X)$. Indeed, we have that

$$\begin{aligned}\|c^\wedge(-t)I_2(t)\| &\le |c|^\wedge(-t)\int_d^{\infty} |k(t-s)|\|f_2(s)\|ds \\ &= |c|^\wedge(-t)\int_d^{\infty} |k^\sim(t-s)|\,|c|^\wedge(t-s)|c|^\wedge(s)\,\|p_2(s)\|ds \\ &= \int_d^{\infty} |k^\sim(t-s)|\,\|p_2(s)\|ds \to 0, \quad (t \to \infty),\end{aligned}$$

where we have used that $(k * h) \in C_0(X)$ provided $h \in C_0(X)$ (see, eg, Lizama and N'Guérékata[12], Theorem 3.3). Now, from definition of f, we have that $(k * f)(t) = c^\wedge(t)(k^\sim * p)(t)$. Hence, $(k * f) \in AP_{\omega c}(X)$. □

Example 2.21. Consider the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

Let $u(x,t)$ be a regular solution with $u(x,0) = f(x)$. Then, it is known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)ds, \quad t > 0,\ x \in \mathbb{R}.$$

Fix $t_0 > 0$ and assume that $f(x)$ is (ω,c)-asymptotically periodic. Then, by Theorem 2.20, we have that $u(x,t_0)$ is (ω,c)-asymptotically periodic with respect to x.

We recall (see Alvarez et al[8]) that the norm in the space $P_{\omega c}([d,\infty),X)$ is given by

$$\|f\|_{\omega c} := \sup_{t\in[0,\omega]} \||c|^\wedge(-t)f(t)\|.$$

Theorem 2.22. *$AP_{\omega c}(X)$ is a Banach space with the norm*

$$\|f\|_{a\omega c} := \sup_{t\ge d} \left\||c|^\wedge(-t)f(t)\right\|,$$

Proof. Let (f_n) be a Cauchy sequence in $AP_{\omega c}(X)$. Then, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m,n \ge N$ we have

$$\|f_n - f_m\|_{a\omega c} < \epsilon.$$

Since $f_m, f_n \in AP_{\omega c}(X)$, Proposition 2.7 implies that there exist $u_m, u_n \in AP_{\omega}(X)$ such that $f_m(t) = c^\wedge(t)u_m(t)$ and $f_n(t) = c^\wedge(t)u_n(t)$. Now, note that for $m,n \ge N$

$$\begin{aligned}\|u_m - u_n\|_{a\omega} &= \sup_{t\ge d}\|u_m(t) - u_n(t)\| \\ &= \sup_{t\ge d}\|c^\wedge(-t)f_m(t) - c^\wedge(-t)f_n(t)\| \\ &= \sup_{t\ge d}\||c|^\wedge(-t)[f_m(t) - f_n(t)]\| \\ &= \|f_n - f_m\|_{a\omega c} < \epsilon.\end{aligned}$$

It follows that (u_n) is a Cauchy sequence in $AP_\omega(X)$. Since $AP_\omega(X)$ is complete, then there exists $u \in AP_\omega(X)$ such that $\|u_n - u\|_{a\omega} \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t)u(t)$. We claim that $\|f_n - f\|_{a\omega c} \to 0$ as $n \to \infty$. Indeed,

$$\begin{aligned}\|f_n - f\|_{a\omega c} &= \sup_{t\geq d} \||c|^\wedge(-t)[f_n(t) - f(t)]\| \\ &= \sup_{t\geq d} \||c|^\wedge(-t)c^\wedge(t)u_n(t) - |c|^\wedge(-t)c^\wedge(t)u(t)\| \\ &= \sup_{t\geq d} \|u_n(t) - u(t)\| \to 0 \quad (n \to \infty).\end{aligned}$$

Hence, $AP_{\omega c}(X)$ is a Banach space with the norm $\|\cdot\|_{a\omega c}$. □

Remark 2.23. Let $f \in P_{\omega c}([d,\infty), X)$. Then the set $\{c^\wedge(-t)f(t) : t \geq d\}$ is relatively compact in X, that is, given $\epsilon > 0$ there exist $\{x_i\}_{i=1}^k$ in X such that $\|c^\wedge(-t)f(t) - x_i\| < \epsilon$ for some $i = 1, \dots, k$ and for all $t \geq d$.

Next, we have the following composition result.

Theorem 2.23. *Let $f(t,x) = g(t,x) + h(t,x)$ where $g(t+\omega, cx) = cg(t,x)$ and $h \in C_{0,c}(X,X)$. Assume the following conditions.*

(a) $h_t(z) = c^\wedge(-t)h(t, c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset of X, uniformly for $t \geq d$ and $h_t(z) \to 0$ as $t \to \infty$ uniformly in z.

(b) There exists a nonnegative bounded function $L_f(t)$ such that

$$\|f(t,x) - f(t,y)\| \leq L_f(t)\|x - y\|, \quad t \geq d,\ x, y \in X.$$

If $\varphi \in AP_{\omega c}(X)$, then $f(\cdot, \varphi(\cdot)) \in AP_{\omega c}(X)$.

Proof. Let $\varphi(t) = \alpha(t) + \beta(t)$ with $\alpha \in P_{\omega c}([d,\infty), X)$ and $\beta \in C_{0,c}(X)$. Then, we have

$$f(t, \varphi(t)) = [f(t,\varphi(t)) - f(t, \alpha(t))] + g(t, \alpha(t)) + h(t, \alpha(t)) =: F(t) + G(t) + H(t).$$

Note that

$$\begin{aligned}\|c^\wedge(-t)F(t)\| &= |c|^\wedge(-t)\|f(t,\varphi(t)) - f(t,\alpha(t))\| \\ &\leq |c|^\wedge(-t)L_f(t)\|\varphi(t) - \alpha(t)\| \\ &= L_f(t)\||c|^\wedge(-t)\beta(t)\| \to 0, \quad t \to \infty.\end{aligned}$$

It follows that $F \in C_{0,c}(X)$. On the other hand, by Alvarez et al[8], Theorem 2.11 we have that $G(t) = g(t, \alpha(t))$ belongs to $P_{\omega c}([d,\infty), X)$. Finally, we prove that $H \in C_{0,c}(X)$. From Remark 2.23, we have that $K := \{c^\wedge(-t)\alpha(t) : t \in [d,\infty)\}$ is relatively compact in X. Then for every $\delta > 0$ there exist $x_1, \dots, x_k \in X$ such that

$$c^\wedge(-t)\alpha(t) \in \bigcup_{j=1}^{k} B(x_j, \delta), \quad t \geq d. \tag{4}$$

Consequently, given $t \geq d$ we can choose $j = 1, \dots, k$ such that

$$\|c^\wedge(-t)\alpha(t) - x_j\| < \delta. \tag{5}$$

Let $\epsilon > 0$. Since $h_t(\cdot) = c^\wedge(-t)h(t, c^\wedge(t)\,\cdot)$ is uniformly continuous on K uniformly for $t \geq d$, then taking $\delta = \delta\left(\frac{\epsilon}{2}\right)$ we obtain that

$$\|c^\wedge(-t)[h(t, c^\wedge(t)c^\wedge(-t)\alpha(t)) - h(t, c^\wedge(t)x_j)]\| < \frac{\epsilon}{2}, \tag{6}$$

uniformly for $t \geq d$. On the other hand, since $\lim_{t\to\infty} c^\wedge(-t)h(t, c^\wedge(t)\cdot) = 0$ on compact subsets of X, then $\lim_{t\to\infty} c^\wedge(-t)h(t, c^\wedge(t)x_j) = 0$ for $j = 1, 2, \dots, k$. Thus, there exists $N \in \mathbb{N}$ such that for all $t \geq N > d$ we have

$$\|c^\wedge(-t)h(t, c^\wedge(t)x_j)\| < \frac{\epsilon}{2}. \tag{7}$$

Next, for all $t \geq N \geq d$, we have

$$\|c^\wedge(-t)h(t,\alpha(t))\| \leq |c^\wedge(-t)h(t,\alpha(t)) - c^\wedge(-t)h(t,c^\wedge(t)x_j)\| + \|c^\wedge(-t)h(t,c^\wedge(t)x_j)\| < \epsilon. \tag{8}$$

Hence,

$$\lim_{t\to\infty} c^\wedge(-t)H(t) = 0.$$

Consequently, $f(\cdot,\varphi(\cdot)) \in AP_{\omega c}(X)$. □

Example 2.25. Let $g(t,x) = u(t)v(x)$ for all $t \geq d$ and $x \in \mathbb{R}$ where $v(c) \neq 0$, u is a $\left(\omega, \frac{c}{v(c)}\right)$-periodic function and v is multiplicative. Suppose that $\left[\frac{c}{v(c)}\right]^\wedge(-t)$ is bounded and $h(t,x) \in C_{0,c}(\Omega,\mathbb{R})$, $(\Omega \subset \mathbb{R})$. Thus, if $f(t,x) = g(t,x) + h(t,x)$ is Lipschitz, then f satisfies the conditions in Theorem 2.23.

3 | EXISTENCE OF A (ω,c)-ASYMPTOTICALLY PERIODIC SOLUTION FOR SEMILINEAR ABSTRACT DIFFERENTIAL EQUATIONS IN BANACH SPACES

In this section, we consider the problem of existence and uniqueness of (ω,c)-asymptotically periodic mild solutions for the following class of semilinear abstract differential equations

$$u'(t) = Au(t) + f(t,u(t)), \quad t \in \mathbb{R}, \tag{9}$$

where A is a closed linear operator defined in a Banach space X which generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ such that there exist constants $M > 0$ and $\alpha > 0$ with

$$\|T(t)x\| \leq Me^{-\alpha t}\|x\|, \quad t \geq 0. \tag{10}$$

A function $u : \mathbb{R} \to X$ is said to be a mild solution of (9) (see N'Guérékata[19]) if the function $s \mapsto T(t-s)f(s,u(s))$ is integrable on (a,t) for each $t \geq a$, $a \in \mathbb{R}$ and

$$u(t) = T(t-a)u(a) + \int_a^t T(t-s)f(s,u(s))ds, \quad t \geq a.$$

The mild solution defined as the first constant variation formula and the solved equation

$$u(t) = \int_{-\infty}^t T(t-s)f(s,u(s))ds \tag{11}$$

are not equivalent. However, they are equivalent when we consider $c^\wedge(t)$-bounded solutions (solutions such that $\sup_{\tau\in\mathbb{R}}\||c|^\wedge(-\tau)u(\tau)\| < \infty$) with $|c| > e^{-\omega\alpha}$ and the semigroup generated by A satisfies (10). Indeed, note that for $u \in AP_{\omega c}(X)$ we have

$$\begin{aligned}\|T(t-a)u(a)\| &\leq Me^{-\alpha(t-a)}|c|^\wedge(a)\||c|^\wedge(-a)u(a)\| \\ &\leq Me^{-\alpha t}e^{\alpha a}|c|^\wedge(a)\sup_{\tau\in\mathbb{R}}\||c|^\wedge(-\tau)u(\tau)\| \\ &\leq Me^{-\alpha t}e^{\left(\alpha+\frac{\log|c|}{\omega}\right)a}\sup_{\tau\in\mathbb{R}}\||c|^\wedge(-\tau)u(\tau)\| \to 0, \quad a \to -\infty.\end{aligned}$$

Now, note that if f is a Lipschitz function, then $\varphi(s) := f(s, u(s))$ is $c^\wedge(s)$-bounded. Consider the integral $\int_r^t T(t-s)\varphi(s)ds$ for each $r < t$. Then,

$$\begin{aligned}\left\|\int_r^t T(t-s)\varphi(s)ds\right\| &\leq M\int_r^t |c|^\wedge(s)e^{-\alpha(t-s)}\||c|^\wedge(-s)\varphi(s)\|ds\\ &\leq M\sup_{\tau\in\mathbb{R}}\||c|^\wedge(-\tau)\varphi(\tau)\|e^{-\alpha t}\int_r^t e^{\left(\alpha+\frac{\log|c|}{\omega}\right)s}ds\\ &\leq \frac{Me^{\frac{\log|c|}{\omega}t}}{\alpha+\frac{\log|c|}{\omega}}\sup_{\tau\in\mathbb{R}}\||c|^\wedge(-\tau)\varphi(\tau)\| < \infty,\end{aligned}$$

which implies that $\int_{-\infty}^t T(t-s)\varphi(s)ds$ is absolutely convergent. These arguments show that

$$\lim_{a\to-\infty}\left[T(t-a)u(a)+\int_a^t T(t-s)f(s,u(s))ds\right] = \int_{-\infty}^t T(t-s)f(s,u(s))]ds.$$

The previous discussion motivates us to consider Equation (11) and the following definition of mild solution.

Definition 3.1. A function $u : \mathbb{R} \to X$ is said to be a mild solution of (9) if

$$u(t) = \int_{-\infty}^t T(t-s)f(s,u(s))\,ds \quad (t\in\mathbb{R}),$$

where $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A satisfying (10).

Note that every mild solution satisfying (11) is solution of the variation of constants equation.
Since the Equation (9) is defined on $\mathbb{R}$, we will take $d = -\infty$ in (2.22), thus

$$\|u\|_{a\omega c} = \sup_{t\in\mathbb{R}}\|c^\wedge(-t)u(t)\|.$$

We recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is uniformly integrable (o strongly integrable) if $\int_0^\infty \|T(t)\|dt < \infty$.
The following result gives us sufficient conditions to obtain a unique mild solution of (9).

Theorem 3.2. *Let $f \in C(\mathbb{R}\times X, X)$. Assume the following conditions.*

(a) Let $f(t,x) = g(t,x) + h(t,x)$ where $h \in C_{0,c}(X,X)$ and $g(t+\omega, cx) = cg(t,x)$ for all $t\in\mathbb{R}$, for all $x\in X$ and for some $(\omega, c) \in \mathbb{R}^+\times(\mathbb{C}\setminus\{0\})$.
(b) $h_t(z) = c^\wedge(-t)h(t, c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset of X, uniformly for $t \geq d$ and $h_t(z) \to 0$ as $t\to\infty$ uniformly in z.
(c) There exists a nonnegative function $L_f(t)$ such that $\|f(t,x)-f(t,y)\| \leq L_f(t)\|x-y\|$ for all $x, y\in X$ and for all $t\in\mathbb{R}$.
(d) A generates a uniformly integrable C_0-semigroup $\{T(t)\}_{t\geq 0}$, $S^\sim$ is integrable and $\sup_{t\in\mathbb{R}}(S^\sim * L_f)(t) < 1$ where $S^\sim(t) := |c|^{-t/\omega}\|T(t)\|$.

Then, Equation (9) has a unique mild solution in $AP_{\omega c}(X)$.

Proof. We define $\mathcal{G} : AP_{\omega c}(X) \to AP_{\omega c}(X)$ by

$$(\mathcal{G}u)(t) = \int_{-\infty}^t T(t-s)f(s,u(s))\,ds,$$

for $u \in AP_{\omega c}(X)$ and $t\in\mathbb{R}$. By Theorem 2.23, we have that $f(\cdot, u(\cdot)) \in AP_{\omega c}(X)$. If $\chi(s)$ is the characteristic function on $(-\infty, t]$, by Theorem 2.20, with $k(t) := \|T(t)\|\cdot\chi(t)$ we have $\mathcal{G}u \in AP_{\omega c}(X)$. Therefore $\mathcal{G}(AP_{\omega c}(X)) \subset AP_{\omega c}(X)$. Now, if

$u, v \in AP_{\omega c}(X)$ we have

$$\begin{aligned}
\|\mathcal{G}(u) - \mathcal{G}(v)\|_{a\omega c} &= \sup_{t\in\mathbb{R}} \left\| |c|^{-t/\omega} \int_{-\infty}^{t} T(t-s)[f(s,u(s)) - f(s,v(s))]\, ds \right\| \\
&\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|T(t-s)|c|^{-(t-s)/\omega}\| \cdot L_f(s) \cdot |c|^{-s/\omega} \|u(s) - v(s)\|\, ds \\
&\leq \|u - v\|_{a\omega c} \sup_{t\in\mathbb{R}} \int_{0}^{\infty} S^{\sim}(s) L_f(t-s)\, ds \\
&= \sup_{t\in\mathbb{R}} (S^{\sim} * L_f)(t) \|u - v\|_{a\omega c}.
\end{aligned}$$

It follows from Banach Fixed Point Theorem that there exists a unique $u \in AP_{\omega c}(X)$ such that $\mathcal{G}u = u$, that is $u(t) = \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds$ for all $t \geq a$. $\square$

Example 3.3. Let $A := -\alpha I$ where $\alpha > 0$ and f as in Theorem 3.2. Then $T(t) = e^{-\alpha t} I$ and we can conclude that for each $f \in AP_{\omega c}(X)$ the equation

$$u'(t) = -\alpha u(t) + f(t, u(t)), \quad t \in \mathbb{R}$$

has a unique mild solution that satisfies

$$u(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)} f(s, u(s)) ds \tag{12}$$

and belongs to $AP_{\omega c}(X)$ whenever $|c| > e^{-\alpha\omega}$ and $\|S^{\sim} * L_f\|_{\infty} < 1$.

Corollary 3.4. *Let $f \in C(\mathbb{R} \times X, X)$. Assume the hypotheses (a) and (b) of Theorem 3.2 and the following conditions.*

(c) There exists a constant $L_f > 0$ such that $\|f(t,x) - f(t,y)\| \leq L_f \|x - y\|$ for all $x, y \in X$ and for all $t \in \mathbb{R}$.
(d) A generates a uniformly integrable C_0-semigroup $\{T(t)\}_{t\geq 0}$, $S^{\sim}(t) := |c|^{-t/\omega} \|T(t)\|$ is integrable and $L_f < \|S^{\sim}\|_1^{-1}$.

Then Equation (9) has a unique mild solution in $AP_{\omega c}(X)$.

Proposition 3.5. *Let $f \in C(\mathbb{R} \times X, X)$. Assume the hypotheses (a) and (b) of Theorem 3.2 and the following conditions.*

(c) There exists a nonnegative function $L_f \in L^1(\mathbb{R})$ such that $\|f(t,x) - f(t,y)\| \leq L_f(t) \|x - y\|$ for all $x, y \in X$ and for all $t \in \mathbb{R}$.
(d) The operator A generates a uniformly integrable C_0-semigroup $\{T(t)\}_{t\geq 0}$ such that $S^{\sim}(t) := |c|^{-t/\omega} \|T(t)\|$ is integrable and $S^{\sim}(t) \leq M$ for all $t \geq 0$ for some $M > 0$.

Then Equation (9) has a unique mild solution in $AP_{\omega c}(X)$.

Proof. We define $\mathcal{G} : AP_{\omega c}(X) \to AP_{\omega c}(X)$ by

$$(\mathcal{G}u)(t) = \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds,$$

for $u \in AP_{\omega c}(X)$ and $t \in \mathbb{R}$. By Theorem 2.23, we have that $f(\cdot, u(\cdot)) \in AP_{\omega c}(X)$. If $\chi(s)$ is the characteristic function on $(-\infty, t]$, by Theorem 2.20, with $k(t) := \|T(t)\| \cdot \chi(t)$ we have $\mathcal{G}u \in AP_{\omega c}(X)$. Therefore $\mathcal{G}(AP_{\omega c}(X)) \subset AP_{\omega c}(X)$. Let

$$(Fu)(t) := c^{\wedge}(-t)(\mathcal{G}u)(t) = c^{-\frac{t}{\omega}} \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Then,

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{a\omega c} = \sup_{t\in\mathbb{R}} \|(Fu)(t) - (Fv)(t)\|.$$

Now, if $u, v \in AP_{\omega c}(X)$ we have

$$\begin{aligned}
\|(Fu)(t)-(Fv)(t)\| &= \left\| |c|^{-t/\omega} \int_{-\infty}^{t} T(t-s)[f(s,u(s))-f(s,v(s))]\,ds \right\| \\
&\leq \int_{-\infty}^{t} \|T(t-s)|c|^{-(t-s)/\omega}\| \cdot L_f(s) \cdot |c|^{-s/\omega}\|u(s)-v(s)\|\,ds \\
&\leq \|u-v\|_{a\omega c} \int_{-\infty}^{t} S^{\sim}(t-s)L_f(s)\,ds \\
&\leq M\|u-v\|_{a\omega c} \int_{-\infty}^{t} L_f(s)\,ds.
\end{aligned}$$

By induction, we can prove that

$$\|(F^n u)(t)-(F^n v)(t)\| \leq \frac{(M\|L_f\|_1)^n}{n!}\|u-v\|_{a\omega c}.$$

Since $\frac{(M\|L_f\|_1)^n}{n!} < 1$ for n sufficiently large, applying the contraction principle we conclude that $\mathcal{G}$ has a unique fixed point in $AP_{\omega c}(X)$. □

Example 3.6. Let $(X; \|\cdot\|) = (L^2(0,\pi), \|\cdot\|_2)$,

$$\begin{aligned}
D(A_D) &= \{u \in L^2(0,\pi) : u'' \in L^2(0,\pi), u(0)=u(\pi)=0\}, \\
A_D u &= \Delta u = u'', \quad \forall u \in D(A_D).
\end{aligned}$$

It is well known that A_D is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(0,\pi)$ with $\|T(t)\| \leq e^{-t}$ for $t \geq 0$ (see Lunardi[20]).

Assume that $|c| > \frac{1}{e^{\omega}}$. Let

$$f(t,x) = a(t)\sin(b(t)x) + \beta^t \cos x =: g(t,x) + h(t,x),$$

with $a \in P_{\omega c}(\mathbb{R})$, $|\beta| < 1$ and $b \in P_{\omega\frac{1}{c}}(\mathbb{R})$ (in the literature, usually β is a power of an exponential e^d). Note that $g(t+\omega, cx) = cg(t,x)$ and $h \in C_{0,c}(X,X)$ satisfies (b) of Theorem 3.2 if $|\beta|^{\omega} < |c|$. Now,

$$\begin{aligned}
\|f(t,x)-f(t,y)\|_2^2 &\leq \int_0^{\pi} |f(t,x(s))-f(t,y(s))|^2 ds \\
&\leq \int_0^{\pi} \left[|a(t)||b(t)| + |\beta|^t\right]^2 |x(s)-y(s)|^2 ds \\
&=: [L_1(t)+L_2(t)]^2\|x-y\|_2^2.
\end{aligned}$$

Note that $L_1(t) := |a(t)||b(t)|$ is ω-periodic and therefore bounded. On the other hand, it is obvious that $L_2(t) := \beta^t$ is bounded. It follows that there exists L_f such that $\|f(t,x)-f(t,y)\|_2 \leq L_f\|x-y\|_2$.

On the other hand, since $|c| > \frac{1}{e^{\omega}}$, we have that $1 + 1/\omega \log|c| > 0$ and therefore

$$\begin{aligned}
\|S^{\sim}\|_1 &= \int_0^{\infty} |c|^{-t/\omega}\|T(t)\|dt \\
&\leq \frac{\omega}{\omega + \log|c|} < \infty.
\end{aligned}$$

Then, by Corollary 3.4, we have that

$$u'(t) = A_D u(t) + f(t,x), \quad t \in \mathbb{R},$$

has a unique (ω, c)-asymptotically periodic mild solution whenever $L_f < \frac{\omega+\log|c|}{\omega}$.

Example 3.7. Let $(X; \|\cdot\|) = (L^2(0,\pi), \|\cdot\|_2)$ and $(A_D, D(A_D))$ as in the preceding example. Suppose that $|c| > \frac{1}{e^{\omega}}$ and assume that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant L. As above,$\|S^{\sim}\|_1 < \infty$. Let

$$f(t,x) = e^{-t^2} g(x) = 0 + h(t,x).$$

In this case, the c-periodic part of the function f is zero. On the other hand, $h_t(z) = c^\wedge(-t)e^{-t^2} g(c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset of X and

$$\|h_t(z)\| < Le^{-t^2}[\|z\| + e^{\omega}\|g(0)\|],$$

which implies that $h_t(z) \to 0$ as $t \to \infty$ uniformly in z. Furthermore,

$$\begin{aligned}
\|f(t,x) - f(t,y)\|_2^2 &\le \int_0^{\pi} |f(t,x(s)) - f(t,y(s))|^2 ds \\
&\le \int_0^{\pi} e^{-2t^2} L|x(s) - y(s)|^2 ds \\
&=: L_f(t)^2 \|x - y\|_2^2.
\end{aligned}$$

Note that $L_f(t) = Le^{-t^2}$ belongs to $L^1(\mathbb{R})$. Then, by Proposition 3.5 we have that

$$u'(t) = A_D u(t) + f(t,x), \quad t \in \mathbb{R},$$

has a unique (ω, c)-asymptotically periodic mild solution.

4 | LASOTA-WAZEWSKA MODEL WITH UNBOUNDED OSCILLATING PRODUCTION OF RED CELLS

The theory presented above can be extended to the semilinear abstract problem with delay

$$\begin{cases} y'(t) = Ay(t) + f(t, y(t-\tau)), & t \ge 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0] \end{cases}$$

where $\tau > 0$ and for which a mild solution is a solution of integral equation

$$y(t) = T(t)y(0) + \int_0^t T(t-s) f(s, y(s-\tau)) ds, \quad t \ge 0.$$

Here, we need to know a history φ. Note that $y(t-\tau) = \varphi(t-\tau)$ for $t \in [0,\tau]$ and if y is (ω, c)-asymptotically periodic then $y(t-\tau)$ also is. As an example, we study the important Lasota-Wazewska model with (ω, c)-asymptotically periodic variable coefficients.

The Lasota-Wazewska model is an autonomous differential equation of the form

$$y'(t) = -\delta y(t) + he^{-\gamma y(t-\tau)}, \quad t \ge 0. \tag{13}$$

Wazewska-Czyzewska and Lasota[18] proposed this model to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red cells bloods in the time t, $\delta > 0$ is the probability of death of a red blood cell, h and γ are positive constant related with the production of red blood cells by unity of time and τ is the time required to produce a red blood cell.

In this section, we study the following model:

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \ge 0, \tag{14}$$

where $\tau > 0$, $h(t)$ and $a(t)$ are continuous and positive functions. Equation (14) models several situations in the real life, see, for example, previous studies[21-24] and the references therein. We are looking for positive (ω, c)-asymptotically periodic solutions for certain $\omega > 0,\ c > 0$. Let $f(t, y) = h(t)e^{-a(t)y}$ and assume

(a) $\tau \leq \omega$.
(b) h is (ω, c)-asymptotically periodic.
(c) a is $(\omega, \frac{1}{c})$-asymptotically periodic.
(d) $c > e^{-\delta\omega}$.
(e) $\|ah\|_\infty < \delta$.

By (d) and (e) we have that $f(t, y) = h(t)e^{-a(t)y}$ satisfies the hypotheses of Corollary 3.4 since

$$|f(t, y_1) - f(t, y_2) \leq |a(t)h(t)||y_1 - y_2|, \tag{15}$$

for $y_1, y_2 > 0$ and its (ω, c)-periodic part g satisfies

$$g(t + \omega, cy) = cg(t, y). \tag{16}$$

By the variation of constant formula

$$y(t) = e^{-\delta t}y(0) + \int_0^t e^{-\delta(t-s)}f(s, y(s - \tau))ds, \tag{17}$$

and hence $y(0) > 0$ implies that $y(t) > 0$. Note that the condition (d) is necessary for positive c-periodic solutions y. In fact, (17) and $h(t) > 0$ imply $y(t) > e^{-\delta t}y(0)$ which evaluated at $t = \omega$ implies (d) since $[c - e^{-\delta\omega}]y(0) > 0$.

Moreover, taking $y(0) = \int_{-\infty}^0 e^{\delta s}f(s, y(s - \tau))ds$, which is well defined, we have that y satisfies

$$y(t) = \int_{-\infty}^t e^{-\delta(t-s)}f(s, y(s - \tau))ds. \tag{18}$$

Then by Corollary 3.4, we have that (18) has a unique solution y^* which belongs to $AP_{\omega c}(X)$. Hence, y^* is also solution of type $AP_{\omega c}(X)$ of Equation (14). Moreover, y^* is exponentially stable. Indeed, for any solution y of (14), $z = y - y^*$ satisfies

$$\begin{aligned} z' &= -\delta z + f(t, y) - f(t, y*) \\ &= -\delta z + f(t, y^* + z) - f(t, y*). \end{aligned}$$

Note that

$$|f(t, y^* + z) - f(t, y*)| \leq |a(t)h(t)||z|,$$

Then, taking $\|ah\|_\infty < \delta$, z verifies that

$$|z(t)| \leq e^{-\alpha(t-t_0)} \sup_{t_0-\tau\leq s\leq t_0} |z(s)|$$

for $t \geq t_0 \geq 0$ and $\alpha = \delta - \|ah\|_\infty$.

We have proved the following theorem.

Theorem 4.1. *Assume that the conditions (a) to (e) hold. Then, the Lasota-Wazewska model has a unique (ω, c)-asymptotically periodic solution which is exponentially stable.*

ACKNOWLEDGEMENTS

Many thanks to referees for their very positive comments and observations on the results obtained in this study and their carefully reading of this piece of work will improve it. E. Alvarez is partially supported by Colciencias, Grant Number 121556933876, S. Castillo is partially supported by Diubb164408 3/R, and M. Pinto is partially supported by Fondecyt Grant Number 1170466.

CONFLICT OF INTEREST

The authors declare that they have no competing interests.

AUTHOR CONTRIBUTIONS

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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How to cite this article: Alvarez E, Castillo S, Pinto M. $(\omega$,c)-asymptotically periodic functions, first-order Cauchy problem, and Lasota-Wazewska model with unbounded oscillating production of red cells. *Math Meth Appl Sci*. 2020;43:305–319. https://doi.org/10.1002/mma.5880