Uniform $W^{1,p}$ estimate for elliptic operator with Robin boundary condition in $\mathcal{C}^1$ domain

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Abstract

We study $W^{1,p}$-estimates of inhomogeneous second order elliptic operator of divergence form with Robin boundary condition in $\mathcal{C}^1$ domain. For any $p > 2$, we prove that a weak reverse Hölder inequality holds which in turn provides the $W^{1,p}$-estimates for solutions with Robin boundary condition, independent of α. As a result, we are able to show that uniform $W^{1,p}$-estimate holds for all $p \in (1, \infty)$. Moreover, this shows precisely that the solution of Robin problem converges strongly to the solution of Dirichlet problem in corresponding spaces when the parameter α tends to ∞.

1 Introduction and statement of main result

This paper is concerned with the second order elliptic problem of divergence form with Robin boundary condition

$$\begin{cases} \operatorname{div}(A(x)\nabla)u = \operatorname{div}\boldsymbol{f} + F & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = \boldsymbol{f} \cdot \boldsymbol{n} + g & \text{on } \Gamma \end{cases} \tag{1.1}$$

in a bounded domain (open, connected set) Ω in $\mathbb{R}^n$ with $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$. Here $\boldsymbol{n}$ is the outward unit normal vector on the boundary, $A(x) = (a_{kl}(x))$ denotes an $n \times n$ matrix with real-valued, measurable, bounded entries with uniform ellipticity condition:

$$\mu|\xi|^2 \le a_{kl}(x)\xi_k\xi_l \le \frac{1}{\mu}|\xi|^2 \quad \text{for all} \quad \mu, \xi \in \mathbb{R}^n \quad \text{and some } \mu > 0.$$

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The conormal derivative $\frac{\partial u}{\partial \boldsymbol{n}}$ is defined as

$$\frac{\partial u}{\partial \boldsymbol{n}} = a_{kl}(x)\frac{\partial u}{\partial x_l} n_k.$$

We want to study the well-posedness of the problem (1.1), precisely, the existence, uniqueness of weak solution of (1.1) in $\boldsymbol{W}^{1,p}(\Omega)$ for any $p \in (1,\infty)$ and the bound on the solution, uniform in α. Proving the existence of a unique solution is not that difficult, assuming $\alpha \geq 0$ a constant or a smooth function, using the standard Neumann regularity results. Also $\alpha \leq 0$ corresponds to the Steklov eigen value problem (for a recent survey on this topic, see [5] and the references therein). So our main interest is to obtain some precise estimate on the solution, in particular some estimate uniform in α.

Note that, formally, $\alpha = \infty$ corresponds to the Dirichlet boundary condition whereas $\alpha = 0$ gives the Neumann boundary condition. In both Dirichlet and Neumann cases, we have the estimate of the solution. And so for the Robin problem as follows:

$$\|u\|_{W^{1,p}(\Omega)} \leq C(\alpha)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|F\|_{L^{r(p)}(\Omega)} + \|h\|_{W^{1-\frac{1}{p},p}(\Gamma)}\right).$$

But the continuity constant depends on α whereas the constant in Dirichlet estimate has no α. So it is natural to expect at least for large α, we may obtain α independent bound of the solution of problem (1.1). That is, if we let α tend to ∞, we precisely get back the solution corresponding to Dirichlet problem. The case when α goes to 0 is relatively easier to handle (though not trivial) assuming the compatibility condition of Neumann problem.

The article [1] discusses the Robin boundary value problems for arbitrary domains which gives some generalized result on well-posedness. Among the vast literature on Robin boundary value problem and various related questions to study, we did not find any reference concerning the question of precise dependence of the solution on the parameter α in the existing literature so far, even for Laplacian. So here is the main result of our article.

Throughout this work, the following assumption on α will be considered which we do not mention each time:

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{and} \quad \alpha \geq \alpha_* > 0 \quad \text{on } \Gamma \tag{1.2}$$

where $t(p)$ defined by

$$\begin{cases} t(p) = 2 & \text{if} \quad p = 2 \\ t(p) = 2 + \varepsilon & \text{if} \quad \frac{3}{2} \leq p \leq 3, p \neq 2 \\ t(p) = \frac{2}{3}\max\{p, p'\} + \varepsilon & \text{otherwise} \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is arbitrary, satisfies $t(p) = t(p')$.

Theorem 1.1. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \in (1,\infty)$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$ with*

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if} \quad p > \frac{3}{2} \\ \text{any arbitrary real number} > 1 & \text{if} \quad p = \frac{3}{2} \\ 1 & \text{if} \quad p < \frac{3}{2}. \end{cases}$$

Then the solution $u \in W^{1,p}(\Omega)$ of (1.1) satisfies the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right) \tag{1.4}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Note that, with above estimate result, it is trivial to show that the solution of the Robin boundary problem (1.1) converges strongly to the solution of Dirichlet boundary problem in the corresponding spaces as α goes to ∞. To prove the above theorem, we first obtain the result for $F = 0$, $g = 0$ for $p > 2$ and then for $p < 2$ using duality argument; And finally for $F \neq 0$, $g \neq 0$. The main tool in the proof for $p > 2$ is a weak reverse Hölder inequality (wRHI) satisfied by the solution of the homogeneous problem, which we show in Lemma 2.4. Note that for Lipschitz domain, the weak reverse Hölder inequality is only true for certain values of p, even for Dirichlet boundary condition. It was first proved by Giaquinta [4, Proposition 1.1, Chapter V] in the case of Dirichlet condition, on smooth domain. wRHI in the case of $B(x, r) \subset \Omega$ follows from the classical interior estimate for harmonic functions. But in the case when $x \in \Gamma$, some suitable boundary Hölder estimate is required. In the case of Neumann problem and for general second order elliptic operator, the proof of wRHI has been done in [3, section 4] in Lipschitz domain; Whereas the sketch of the proof for Neumann problem in smooth domain has been given in [7, pp 914].

We obtain the similar result for H^s-bound (on Lipschitz domain) for $s \in (0, \frac{1}{2})$ in Theorem 2.10 and $W^{2,p}$-estimate (on $\mathcal{C}^{1,1}$ domain) in Theorem 3.1.

2 Related results and Proof of Theorem 1.1

To prove Theorem 1.1, we start with studying the existence result. Note that, we consider here only the case $n = 3$ for the sake of clarity but all the results are true for $n = 2$ as well and the exact same proofs follow with the necessary modifications.

Theorem 2.1 (Existence result in $\boldsymbol{W^{1,p}(\Omega), p \geq 2}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $p \geq 2$. Then for any $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$, there exists a unique solution $u \in W^{1,p}(\Omega)$ of Problem (1.1).*

Remark 2.2. Note that for $p = 2$, Ω Lipschitz is sufficient to show the existence of solution $u \in H^1(\Omega)$.

Proof. It is trivial to see that $u \in W^{1,p}(\Omega)$ is a solution of (1.1) iff $u \in W^{1,p}(\Omega)$ satisfies the following variational formulation:

$$\forall \varphi \in W^{1,p'}(\Omega), \quad \int_\Omega A(x)\nabla u \cdot \nabla\varphi + \int_\Gamma \alpha u\, \varphi = \int_\Omega \boldsymbol{f} \cdot \nabla\varphi - \int_\Omega F\varphi + \langle g, \varphi\rangle_\Gamma \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality between $W^{-\frac{1}{p},p}(\Gamma)$ and $W^{\frac{1}{p},p'}(\Gamma)$. Note that the boundary integral $\int_\Gamma \alpha u\, \varphi$ is well defined. For $p = 2$, the bilinear form

$$\forall\ u, \varphi \in H^1(\Omega), \quad a(u, \varphi) = \int_\Omega A(x)\nabla u \cdot \nabla\varphi + \int_\Gamma \alpha u\, \varphi$$

is clearly continuous. Also, due to the ellipticity hypothesis on $A(x)$ and by Friedrich's inequality and the assumption $\alpha \ge \alpha_* > 0$ on Γ, we may have

$$a(u,u) = \int_\Omega A(x)\nabla u \cdot \nabla u + \int_\Gamma \alpha |u|^2 \ge C(\alpha_*) \, \|u\|^2_{H^1(\Omega)}$$

which shows that the bilinear form is coercive on $H^1(\Omega)$. And the right hand side of (2.1) defines an element in the dual of $H^1(\Omega)$. Thus, by Lax-Milgram lemma, there exists a unique $u \in H^1(\Omega)$ satisfying (2.1). So we obtain the existence of a unique solution of (1.1) in $H^1(\Omega)$.

Now for $p > 2$, since $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega), L^{r(p)}(\Omega) \hookrightarrow L^{6/5}(\Omega), W^{-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$ and $L^{t(p)}(\Gamma) \hookrightarrow L^2(\Gamma)$, there exists a unique $u \in H^1(\Omega)$ solving (1.1). It remains to show that $u \in W^{1,p}(\Omega)$.

(i) $2 < p \le 3$**.** Since $u \in H^1(\Omega) \hookrightarrow L^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha u \in L^{q_1}(\Gamma)$ where $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But using the Sobolev embedding $L^{q_1}(\Gamma) \hookrightarrow W^{-\frac{1}{p_1},p_1}(\Gamma)$ with $p_1 = \frac{3}{2}q_1$ (since $q_1 > \frac{4}{3}$)

$$\text{i.e.} \quad \frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right),$$

Neumann regularity result (cf. [8]) implies $u \in W^{1,p_1}(\Omega)$ since Ω is $\mathcal{C}^1$. If $p_1 \ge p$, we are done. Otherwise, $u \in W^{1,p_1}(\Omega)$. Hence, $u \in L^{s_1}(\Gamma)$ where

$$\frac{1}{s_1} = \frac{1}{p_1} - \frac{1-\frac{1}{p_1}}{2} = \frac{3}{2p_1} - \frac{1}{2}$$

as $p_1 < p \le 3$. Then $\alpha u \in L^{q_2}(\Gamma)$ where $\frac{1}{q_2} = \frac{1}{s_1} + \frac{1}{2+\varepsilon}$. But, $L^{q_2}(\Gamma) \hookrightarrow W^{-\frac{1}{p_2},p_2}(\Gamma)$ with $p_2 = \frac{3}{2}q_2$ *i.e.*

$$\frac{1}{p_2} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon} - \frac{1}{2} + \frac{1}{2+\varepsilon}\right) = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right).$$

If $p_2 \ge p$, then as before, we have $u \in W^{1,p}(\Omega)$. Otherwise, $u \in W^{1,p_2}(\Omega)$. Proceeding similarly, we get $u \in W^{1,p_{k+1}}(\Omega)$ with

$$\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right).$$

(where in each step, we assumed that $p_k < 3$). Now choosing $k = \lfloor \frac{1}{\varepsilon} - \frac{1}{2} \rfloor + 1$ such that $p_{k+1} \ge 3 \ge p$ (where $\lfloor a \rfloor$ stands for the greatest integer less than or equal to a), we obtain $u \in W^{1,p}(\Omega)$.

(ii) $p > 3$**.** From the previous case, we obtain $u \in W^{1,3}(\Omega)$ which gives $u \in L^q(\Gamma)$ for all $1 < q < \infty$. But $\alpha \in L^{\frac{2}{3}p+\varepsilon}(\Gamma)$ implies $\alpha u \in L^{\frac{2}{3}p}(\Gamma) \hookrightarrow W^{-\frac{1}{p},p}(\Gamma)$. Therefore, using same reasoning as before, from the Neumann regularity result, we get $u \in W^{1,p}(\Omega)$. ■

Next we discuss the estimate of the solution of problem (1.1) for $p > 2$ with $F = 0$ and $g = 0$, independent of α.

Theorem 2.3 (**$W^{1,p}(\Omega)$ estimate, $p \geq 2$ with RHS $\boldsymbol{f}$**). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \geq 2$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Then the solution $u \in W^{1,p}(\Omega)$ of (1.1) with $F = 0$ and $g = 0$, satisfies the following estimate:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{2.2}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

The proof of the above theorem is very much similar to that of Neumann problem [3], once we have the wRHI. Since Ω is $\mathcal{C}^1$, there exists some $r_0 > 0$ such that for any $x_0 \in \Gamma$, there exists a coordinate system (x', x_3) which is isometric to the usual coordinate system and a $\mathcal{C}^1$ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ so that,

$$B(x_0, r_0) \cap \Omega = \left\{(x', x_3) \in B(x_0, r_0) : x_3 > \psi(x')\right\}$$

and

$$B(x_0, r_0) \cap \Gamma = \left\{(x', x_3) \in B(x_0, r_0) : x_3 = \psi(x')\right\}.$$

In some places, we may write B instead of $B(x, r)$ where there is no ambiguity and $aB := B(x, ar)$ for $a > 0$. Before proving the above theorem, we need the following lemma.

Lemma 2.4. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $p \geq 2$. For any $B(x, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x, 2r) \subset \Omega$ or $x \in \Gamma$, the following weak Reverse Hölder inequalities hold:*
(i) if $B(x, 2r) \subset \Omega$,

$$\left(\frac{1}{r^3} \int\limits_{B(x,r)} |\nabla v|^p\right)^{1/p} \leq C \left(\frac{1}{r^3} \int\limits_{B(x,2r)} |\nabla v|^2\right)^{1/2} \tag{2.3}$$

whenever $v \in H^1(B(x, 2r))$ satisfying $\operatorname{div}(A(x)\nabla)v = 0$ in $B(x, 2r)$.
(ii) if $x \in \Gamma$,

$$\left(\frac{1}{r^3} \int\limits_{B(x,r)\cap\Omega} (|v|^2 + |\nabla v|^2)^{p/2}\right)^{1/p} \leq C \left(\frac{1}{r^3} \int\limits_{B(x,2r)\cap\Omega} (|v|^2 + |\nabla v|^2)\right)^{1/2} \tag{2.4}$$

whenever $v \in H^1(B(x, 2r) \cap \Omega)$ satisfying

$$\begin{cases} \operatorname{div}(A\nabla)v &= 0 \quad \text{in} \quad B(x, 2r) \cap \Omega \\ \frac{\partial v}{\partial \boldsymbol{n}} + \alpha v &= 0 \quad \text{on} \quad B(x, 2r) \cap \Gamma. \end{cases} \tag{2.5}$$

The constants $C > 0$ in the above estimates are independent of α.

Proof. The proof of the weak Reverse Hölder inequality for Robin problem follows the similar argument as for the Dirichlet problem, established in [4].

case(i) : $2B \subset \Omega$.
Since v satisfies the equation $\operatorname{div}(A(x)\nabla)v = 0$ in $2B$, we can have the following Caccioppoli inequality,

$$\int_B |\nabla v|^2 \leq \frac{C}{r^2} \int_{2B} |v - \bar{v}|^2, \qquad \bar{v} = \frac{1}{|2B|} \int_{2B} v$$

for some constant $C > 0$ independent of α. Now using the following Sobolev-Poincaré inequality, for any $v \in W^{1,p}(\Omega)$, $p > 1$,

$$\|v - \bar{v}\|_{L^{p^*}(\Omega)} \leq C \|\nabla v\|_{L^p(\Omega)}, \qquad \bar{v} = \frac{1}{|\Omega|} \int_\Omega v$$

where p^* is the Sobolev exponent, we obtain,

$$\int_B |\nabla v|^2 \leq \frac{C}{r^2} \left(\int_{2B} |\nabla v|^{\tilde{q}} \right)^{2/\tilde{q}}$$

with $\tilde{q} = 6/5$ (this value comes from the dimension $n = 3$). Upon normalizing both sides, we can write,

$$\left(\frac{1}{r^3} \int_B |\nabla v|^2 \right)^{1/2} \leq C \left(\frac{1}{r^3} \int_{2B} |\nabla v|^{\tilde{q}} \right)^{1/\tilde{q}}.$$

Here note that in $\mathbb{R}^3$, $|B| = cr^3$. Then setting $g = |\nabla v|^{\tilde{q}}$ and $q = 5/3 = 2/\tilde{q}$, we have,

$$\frac{1}{r^3} \int_B g^q \leq C \left(\frac{1}{r^3} \int_{2B} g \right)^q.$$

Hence, [4, Proposition 1.1] with $f = 0$ and $\theta = 0$ implies

$$\left(\frac{1}{r^3} \int_B |\nabla v|^{2+\varepsilon} \right)^{1/2+\varepsilon} \leq C \left(\frac{1}{r^3} \int_{2B} |\nabla v|^2 \right)^{1/2}.$$

for some $\varepsilon > 0$. Applying [4, Proposition 1.1] a finite number of times, we obtain (2.3).

case(ii) : $x \in \Gamma$.
The proof is very much similar to the above interior estimate. First we want to prove a Caccioppoli type inequality for the problem (2.5) up to the boundary. For that, let $\eta \in C_c^\infty(2B)$ be a cut-off function such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B \quad \text{and} \quad |\nabla \eta| \leq \frac{C}{r}.$$

Now multiplying (2.5) by $\eta^2 v$ and integrating by parts, we get,

$$\int_{2B\cap\Omega} A\nabla v \cdot \nabla(\eta^2 v) + \int_{\partial(2B\cap\Omega)} \alpha \eta^2 v^2 = 0$$

which yields,

$$\mu \int\limits_{2B\cap\Omega} \eta^2|\nabla v|^2 + \int\limits_{2B\cap\Gamma} \alpha\eta^2 v^2 \leq \int\limits_{2B\cap\Omega} \eta^2 A(x)\nabla v\cdot\nabla v + \int\limits_{2B\cap\Gamma} \alpha\eta^2 v^2 = -2\int\limits_{2B\cap\Omega} \eta v\nabla v\cdot\nabla\eta.$$

Using Cauchy's inequality on the right hand side, we obtain,

$$\int\limits_{2B\cap\Omega} |\nabla v|^2\eta^2 + \int\limits_{2B\cap\Gamma} \alpha\eta^2 v^2 \leq 2\left[\frac{1}{4}\int\limits_{2B\cap\Omega} \eta^2|\nabla v|^2 + 4\int\limits_{2B\cap\Omega} v^2|\nabla\eta|^2\right].$$

Simplifying the above estimate gives

$$\int\limits_{2B\cap\Omega} |\nabla v|^2\eta^2 + \int\limits_{2B\cap\Gamma} \alpha\eta^2 v^2 \leq C\int\limits_{2B\cap\Omega} v^2|\nabla\eta|^2,$$

which yields the Caccioppoli-type inequality, up to the boundary,

$$\int\limits_{B\cap\Omega} |\nabla v|^2 + \int\limits_{B\cap\Gamma} \alpha v^2 \leq \int\limits_{2B\cap\Omega} |\nabla v|^2\eta^2 + \int\limits_{2B\cap\Gamma} \alpha\eta^2 v^2 \leq \frac{C}{r^2}\int\limits_{2B\cap\Omega} v^2. \tag{2.6}$$

But we also have,

$$\|v\|^2_{H^1(B\cap\Omega)} \leq C\left(\int\limits_{B\cap\Omega} |\nabla v|^2 + \int\limits_{B\cap\Gamma} v^2\right) \leq C(\alpha_*)\left(\int\limits_{B\cap\Omega} |\nabla v|^2 + \int\limits_{B\cap\Gamma} \alpha v^2\right).$$

Hence, using (2.6), we obtain,

$$\int\limits_{B\cap\Omega} (|v|^2 + |\nabla v|^2) \leq \frac{C(\alpha_*)}{r^2}\int\limits_{2B\cap\Omega} |v|^2 \leq \frac{C(\alpha_*)}{r^2}\left(\int\limits_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2}\right)^{2/\tilde{q}}$$

with $\tilde{q} = 6/5$ so that $(\tilde{q})^* = 2$. Thus,

$$\begin{aligned}\frac{1}{r^3}\int\limits_{B\cap\Omega} (|v|^2 + |\nabla v|^2) &\leq \frac{C(\alpha_*)}{r^5}\left(\int\limits_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2}\right)^{2/\tilde{q}} \\ &= C(\alpha_*)\left(\frac{1}{r^3}\int\limits_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2}\right)^{2/\tilde{q}}.\end{aligned}$$

Now if we set,

$$g(y) = \begin{cases} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2} & \text{if } \ y \in 2B\cap\Omega \\ 0 & \text{if } \ y \in 2B\setminus\Omega \end{cases}$$

and $q = 2/\tilde{q}$, we obtain,

$$\frac{1}{r^3}\int\limits_{B} g^q \leq C(\alpha_*)\left(\frac{1}{r^3}\int\limits_{2B} g\right)^q.$$

Once again [4, Proposition 1.1] with $f = 0$ and $\theta = 0$ implies, for some $\varepsilon > 0$,

$$\left(\frac{1}{r^3}\int_B g^{q+\varepsilon}\right)^{1/q+\varepsilon} \le C\left(\frac{1}{r^3}\int_{2B} g^q\right)^{1/q}$$

i.e.

$$\left(\frac{1}{r^3}\int_{B\cap\Omega} (|v|^2+|\nabla v|^2)^{(q+\varepsilon)\tilde{q}/2}\right)^{1/q+\varepsilon} \le C\left(\frac{1}{r^3}\int_{2B\cap\Omega} (|v|^2+|\nabla v|^2)\right)^{\tilde{q}/2}$$

or equivalently, for some $s > 2$,

$$\left(\frac{1}{r^3}\int_{B\cap\Omega} (|v|^2+|\nabla v|^2)^{s/2}\right)^{1/s} \le C\left(\frac{1}{r^3}\int_{2B\cap\Omega} (|v|^2+|\nabla v|^2)\right)^{1/2}.$$

Now repeating [4, Proposition 1.1] finite times, we get (2.4) which ends the proof. ■

We also need the following lemma which is proved in [3, Theorem 2.2].

Lemma 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $p > 2$. Let $G \in L^2(\Omega)$ and $f \in L^q(\Omega)$ for some $2 < q < p$. Suppose that for each ball B with the property that $|B| \le \beta|\Omega|$ and either $2B \subset \Omega$ or B centers on Γ, there exist two integrable functions G_B and R_B on $2B\cap\Omega$ such that $|G| \le |G_B| + |R_B|$ on $2B\cap\Omega$ and*

$$\begin{aligned}&\left(\frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega} |R_B|^p\right)^{1/p}\\ &\le C_1\left[\left(\frac{1}{|\gamma B\cap\Omega|}\int_{\gamma B\cap\Omega} |G|^2\right)^{1/2} + \sup_{B\subset B'}\left(\frac{1}{|B'\cap\Omega|}\int_{B'\cap\Omega} |f|^2\right)^{1/2}\right]\end{aligned} \tag{2.7}$$

and

$$\left(\frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega} |G_B|^2\right)^{1/2} \le C_2 \sup_{B\subset B'}\left(\frac{1}{|B'\cap\Omega|}\int_{B'\cap\Omega} |f|^2\right)^{1/2} \tag{2.8}$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \gamma$. Then we have,

$$\left(\frac{1}{|\Omega|}\int_\Omega |G|^q\right)^{1/q} \le C\left[\left(\frac{1}{|\Omega|}\int_\Omega |G|^2\right)^{1/2} + \left(\frac{1}{|\Omega|}\int_\Omega |f|^q\right)^{1/q}\right] \tag{2.9}$$

where $C > 0$ depends only on $C_1, C_2, n, p, q, \beta, \gamma$ and Ω.

Proof of Theorem 2.3. Given any ball B with either $2B \subset \Omega$ or B centers on Γ, let $\varphi \in C_c^\infty(8B)$ is a cut-off function such that $0 \le \varphi \le 1$ and

$$\varphi = \begin{cases} 1 & \text{on } 4B \\ 0 & \text{outside } 8B \end{cases}$$

and we decompose $u = v + w$ where v, w satisfy

$$\begin{cases} \operatorname{div}(A(x)\nabla)v = \operatorname{div}(\varphi \boldsymbol{f}) & \text{in } \Omega \\ \dfrac{\partial v}{\partial \boldsymbol{n}} + \alpha v = \varphi \boldsymbol{f} \cdot \boldsymbol{n} & \text{on } \Gamma \end{cases} \tag{2.10}$$

and

$$\begin{cases} \operatorname{div}(A(x)\nabla)w = \operatorname{div}\left((1-\varphi)\boldsymbol{f}\right) & \text{in } \Omega \\ \dfrac{\partial w}{\partial \boldsymbol{n}} + \alpha w = (1-\varphi)\boldsymbol{f} \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{2.11}$$

Multiplying (2.10) by v and integrating by parts, we get,

$$\int_\Omega A(x)\nabla v \cdot \nabla v + \int_\Gamma \alpha |v|^2 = \int_\Omega \varphi \boldsymbol{f} \cdot \nabla v$$

which gives

$$\|\nabla v\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{\mu} \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.12}$$

and since $\alpha \geq \alpha_* > 0$ on Γ,

$$\|v\|^2_{H^1(\Omega)} \leq C(\Omega, \alpha_*) \left(\|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma \alpha |v|^2 \right) \leq C(\Omega, \alpha_*)\ \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla v\|_{\boldsymbol{L}^2(\Omega)}.$$

This yields the complete L^2-estimate

$$\|v\|_{H^1(\Omega)} \leq C(\Omega, \alpha_*)\ \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.13}$$

(i) First we consider the case $4B \subset \Omega$. We want to apply Lemma 2.5 with $G = |\nabla u|, G_B = |\nabla v|$ and $R_B = |\nabla w|$. It is easy to see that

$$|G| \leq |G_B| + |R_B|.$$

Now we verify (2.7) and (2.8). For that, using (2.12) we get,

$$\begin{aligned} \frac{1}{|2B|} \int_{2B} |G_B|^2 = \frac{1}{|2B|} \int_{2B} |\nabla v|^2 \leq \frac{1}{|2B \cap \Omega|} \int_\Omega |\nabla v|^2 &\leq \frac{C(\Omega, \alpha_*)}{|2B \cap \Omega|} \int_\Omega |\varphi \boldsymbol{f}|^2 \\ &\leq \frac{C(\Omega, \alpha_*)}{|8B \cap \Omega|} \int_{8B \cap \Omega} |\boldsymbol{f}|^2 \end{aligned}$$

where in the last inequality, we used that $|8B \cap \Omega| \leq |\Omega|$. This gives the estimate (2.8).

Next, from (2.11), we observe that $\operatorname{div}(A(x)\nabla)w = 0$ in $4B$. Hence, by the estimate (2.3) (using $2B$ instead of B), we have

$$\left(\frac{1}{|2B|} \int_{2B} |\nabla w|^p \right)^{1/p} \leq C \left(\frac{1}{|4B|} \int_{4B} |\nabla w|^2 \right)^{1/2}$$

which implies together with (2.12),

$$\left(\frac{1}{|2B|}\int\limits_{2B}|R_B|^p\right)^{1/p} \le C\left(\frac{1}{|4B|}\int\limits_{4B}|\nabla w|^2\right)^{1/2}$$
$$\le C\left[\left(\frac{1}{|4B|}\int\limits_{4B}|\nabla u|^2\right)^{1/2}+\left(\frac{1}{|4B|}\int\limits_{4B}|\nabla v|^2\right)^{1/2}\right]$$
$$\le C\left(\frac{1}{|4B|}\int\limits_{4B}|G|^2\right)^{1/2}+C(\Omega,\alpha_*)\left(\frac{1}{|8B\cap\Omega|}\int\limits_{8B\cap\Omega}|\boldsymbol{f}|^2\right)^{1/2}.$$

This gives (2.7). So from Lemma 2.5, it follows that

$$\left(\frac{1}{|\Omega|}\int\limits_{\Omega}|\nabla u|^q\right)^{1/q} \le C_p(\Omega)\left[\left(\frac{1}{|\Omega|}\int\limits_{\Omega}|\nabla u|^2\right)^{1/2}+\left(\frac{1}{|\Omega|}\int\limits_{\Omega}|\boldsymbol{f}|^q\right)^{1/q}\right]$$

for any $2 < q < p$ where $C_p(\Omega) > 0$ does not depend on α.

Because of the self-improving property of the weak Reverse Hölder condition (2.3), the above estimate holds for any $q \in (2, \tilde{p})$ for some $\tilde{p} > p$ also and in particular, for $q = p$, which clearly implies (2.2).

(ii) Next consider B centers on Γ. We apply Lemma 2.5 now with $G = |u| + |\nabla u|, G_B = |v| + |\nabla v|$ and $R_B = |w| + |\nabla w|$. Obviously, $|G| \le |G_B| + |R_B|$ and again by (2.13),

$$\frac{1}{|2B\cap\Omega|}\int\limits_{2B\cap\Omega}|G_B|^2 \le \frac{1}{|2B\cap\Omega|}\int\limits_{2B\cap\Omega}(|v|^2+|\nabla v|^2) \le \frac{1}{|2B\cap\Omega|}\ \|v\|^2_{H^1(\Omega)}$$
$$\le \frac{C(\Omega,\alpha_*)}{|2B\cap\Omega|}\int\limits_{\Omega}|\varphi\boldsymbol{f}|^2$$
$$\le \frac{C(\Omega,\alpha_*)}{|8B\cap\Omega|}\int\limits_{8B\cap\Omega}|\boldsymbol{f}|^2$$

which yields (2.8). Also w satisfies the problem

$$\begin{cases} \Delta w = 0 & \text{in } 4B\cap\Omega \\ \dfrac{\partial w}{\partial \boldsymbol{n}} + \alpha w = 0 & \text{on } 4B\cap\Gamma. \end{cases}$$

So by estimate (2.4) and (2.12), we can write,

$$
\begin{aligned}
&\left(\frac{1}{|2B\cap\Omega|}\int\limits_{2B\cap\Omega}|R_B|^p\right)^{1/p}\\
&\le\left(\frac{1}{|2B\cap\Omega|}\int\limits_{2B\cap\Omega}((|w|+|\nabla w|)^2)^{p/2}\right)^{1/p}\\
&\le C\left(\frac{1}{|4B\cap\Omega|}\int\limits_{4B\cap\Omega}(|w|^2+|\nabla w|^2)\right)^{1/2}\\
&\le C\left[\left(\frac{1}{|4B\cap\Omega|}\int\limits_{4B\cap\Omega}(|u|^2+|\nabla u|^2)\right)^{1/2}+\left(\frac{1}{|4B\cap\Omega|}\int\limits_{4B\cap\Omega}(|v|^2+|\nabla v|^2)\right)^{1/2}\right]\\
&\le C\left(\frac{1}{|4B\cap\Omega|}\int\limits_{4B\cap\Omega}|G|^2\right)^{1/2}+C(\Omega,\alpha_*)\left(\frac{1}{|8B\cap\Omega|}\int\limits_{8B\cap\Omega}|\boldsymbol{f}|^2\right)^{1/2}
\end{aligned}
$$

which yields (2.7). Thus we have,

$$
\left(\frac{1}{|\Omega|}\int\limits_{\Omega}(|u|+|\nabla u|)^q\right)^{1/q}\le C_p(\Omega,\alpha_*)\left[\left(\frac{1}{|\Omega|}\int\limits_{\Omega}|u|^2+|\nabla u|^2\right)^{1/2}+\left(\frac{1}{|\Omega|}\int\limits_{\Omega}|\boldsymbol{f}|^q\right)^{1/q}\right]
$$

for any $2<q<p$ where $C_p(\Omega,\alpha_*)>0$ does not depend on α. This completes the proof together with the previous case. ■

The next proposition will be used to study the complete estimate of the Robin problem (1.1). The result is not optimal and will be improved in Proposition 2.8.

Proposition 2.6 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate, $\boldsymbol{p>2}$ with RHS $\boldsymbol{F}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p>2$, and $F\in L^p(\Omega)$. Then the unique solution $u\in W^{1,p}(\Omega)$ of (1.1) with $\boldsymbol{f}=\boldsymbol{0}$ and $g=0$, satisfies the following estimate:*

$$
\|u\|_{W^{1,p}(\Omega)}\le C_p(\Omega,\alpha_*)\ \|F\|_{L^p(\Omega)} \tag{2.14}
$$

where the constant $C_p(\Omega,\alpha_*)>0$ is independent of α.

Proof. The result follows using the same argument as in Theorem 2.3 and hence we do not repeat it. ■

Proposition 2.7 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate with RHS $\boldsymbol{f}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p\in(1,\infty)$ and $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$. Then there exists a unique solution $u\in W^{1,p}(\Omega)$ of (1.1) with $F=0$ and $g=0$, satisfying the following estimate:*

$$
\|u\|_{W^{1,p}(\Omega)}\le C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{2.15}
$$

where the constant $C_p(\Omega,\alpha_*)>0$ is independent of α.

Proof. The existence of a unique solution and the corresponding estimate for $p > 2$ is done in Theorem 2.1 and Theorem 2.3 respectively. Now suppose that $1 < p < 2$. We first discuss the estimate and then the existence of a solution.

(i) Estimate I: Let $\boldsymbol{g} \in C_0^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega)$ be the solution of $\operatorname{div}(A(x)\nabla)v = \operatorname{div} \boldsymbol{g}$ in Ω and $\frac{\partial v}{\partial \boldsymbol{n}} + \alpha v = 0$ on Γ. Since $p' > 2$, from Theorem 2.3, we have

$$\|v\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Also if $u \in W^{1,p}(\Omega)$ is a solution of (1.1) with $F = 0, g = 0$, using the weak formulation of the problems satisfied by u and v, we have

$$\int_\Omega \boldsymbol{f} \cdot \nabla v = \int_\Omega \boldsymbol{g} \cdot \nabla u$$

which gives,

$$\Big|\int_\Omega \boldsymbol{g} \cdot \nabla u\Big| \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|v\|_{W^{1,p'}(\Omega)}$$

and hence,

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{g} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \nabla u \cdot \boldsymbol{g}|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

(ii) Estimate II: Next we prove that

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.16}$$

For that, from Proposition 2.6, we get for any $\varphi \in L^{p'}(\Omega)$, the unique solution $w \in W^{1,p'}(\Omega)$ of the problem

$$\begin{cases} \operatorname{div}(A(x)\nabla)w = \varphi & \text{in } \Omega \\ \dfrac{\partial w}{\partial \boldsymbol{n}} + \alpha w = 0 & \text{on } \Gamma \end{cases}$$

satisfies

$$\|w\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*)\ \|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Therefore using the weak formulation of the problems satisfied by u and w, we obtain,

$$\int_\Omega u\ \varphi = \int_\Omega u \operatorname{div}(A(x)\nabla)w = -\int_\Omega A(x)\nabla u \cdot \nabla w + \int_\Gamma u\frac{\partial w}{\partial \boldsymbol{n}} = -\int_\Omega \boldsymbol{f} \cdot \nabla w$$

which implies

$$\|u\|_{L^p(\Omega)} = \sup_{0 \neq \varphi \in L^{p'}(\Omega)} \frac{|\int_\Omega u\ \varphi|}{\|\varphi\|_{L^{p'}(\Omega)}} \leq C_p(\Omega, \alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

This completes proof of the estimate (2.15).

(iii) Existence and uniqueness: The uniqueness of solution of (1.1) follows from (2.15). For the existence, we will use a limit argument. Let $\{\boldsymbol{f}_k\} \in C_0^\infty(\Omega)$ such that

$$\boldsymbol{f}_k \to \boldsymbol{f} \quad \text{ in } \boldsymbol{L}^p(\Omega)$$

and $u_k \in W^{1,p'}(\Omega)$ be the unique solution of

$$\begin{cases} \operatorname{div}(A(x)\nabla)u_k = \operatorname{div} \boldsymbol{f_k} & \text{in } \Omega \\ \dfrac{\partial u_k}{\partial \boldsymbol{n}} + \alpha u_k = 0 & \text{on } \Gamma \end{cases} \tag{2.17}$$

Note that $u_k \in W^{1,p}(\Omega)$ since $p' > 2$. Also from **(i)** we have,

$$\|u_k\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\boldsymbol{f}_k\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\|u_k - u_l\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\boldsymbol{f}_k - \boldsymbol{f}_l\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus it follows $u_k - u_l \to 0$ in $W^{1,p}(\Omega)$ as $k, l \to \infty$ i.e. $\{u_k\}$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Then as $W^{1,p}(\Omega)$ is a Banach space, there exists $u \in W^{1,p}(\Omega)$ such that

$$u_k \to u \text{ in } W^{1,p}(\Omega)$$

satisfying

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Clearly u also solves the system (1.1). ■

Proposition 2.8 ($W^{1,p}(\Omega)$ estimate with RHS F). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \in (1, \infty)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$. Then the solution $u \in W^{1,p}(\Omega)$ of the problem*

$$\begin{cases} \operatorname{div}(A(x)\nabla)u = F & \text{in } \Omega \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = g & \text{on } \Gamma \end{cases} \tag{2.18}$$

satisfies the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right) \tag{2.19}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Proof. It suffices to prove the estimate since the existence and uniqueness of u follows from the same argument as in Proposition 2.7.

(i) Estimate I: Let $\boldsymbol{f} \in \mathcal{C}_0^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega)$ be the weak solution of $\operatorname{div}(A(x)\nabla)v = \operatorname{div} \boldsymbol{f}$ in Ω and $\frac{\partial v}{\partial \boldsymbol{n}} + \alpha v = 0$ on Γ. By Proposition 2.7, we then have

$$\|v\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*) \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Also, if $u \in W^{1,p}(\Omega)$ is a solution of (2.18), from the weak formulation of the problems satisfied by u and v, we get

$$\int_\Omega \boldsymbol{f} \cdot \nabla u = \int_\Omega A(x)\nabla u \cdot \nabla v + \int_\Gamma \alpha u v = - \int_\Omega F v + \langle g, v \rangle_\Gamma .$$

This implies

$$\Big|\int_\Omega \boldsymbol{f}\cdot\nabla u\Big| \le \|F\|_{L^{r(p)}(\Omega)}\|v\|_{L^{(r(p))'}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\|v\|_{W^{\frac{1}{p},p'}(\Gamma)}$$
$$\le C_p(\Omega)\left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\right)\|v\|_{W^{1,p'}(\Omega)}$$

since $\frac{1}{(p')*} = \frac{1}{p'} - \frac{1}{3} = \frac{1}{(r(p))'}$ for $p > \frac{3}{2}$ and $W^{1,p'}(\Omega) \hookrightarrow L^\infty(\Omega)$ when $p < \frac{3}{2}$. Thus,

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0\neq \boldsymbol{f}\in \boldsymbol{L}^{p'}(\Omega)} \frac{\left|\int_\Omega \nabla u\cdot \boldsymbol{f}\right|}{\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}} \le C_p(\Omega,\alpha_*)\left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\right).$$

(ii) Estimate II: Next we prove the following bound as done in (2.16):

$$\|u\|_{L^p(\Omega)} \le C_p(\Omega,\alpha_*)\left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\right) \tag{2.20}$$

except that we do not need to assume $p < 2$ here as in (2.16). For any $\varphi \in L^{p'}(\Omega)$, there exists a unique $w \in W^{1,p'}(\Omega)$ solving the problem

$$\begin{cases} \operatorname{div}(A(x)\nabla)w = \varphi & \text{in } \Omega \\ \dfrac{\partial w}{\partial \boldsymbol{n}} + \alpha w = 0 & \text{on } \Gamma \end{cases}$$

and satisfying

$$\|w\|_{W^{1,p'}(\Omega)} \le C_p(\Omega,\alpha_*)\|\varphi\|_{L^{p'}(\Omega)}.$$

(For $p < 2$ the above estimate can be proved by the exact same argument as in Proposition (2.7)). Finally we can write,

$$\int_\Omega u\,\varphi = \int_\Omega u\Delta w = \int_\Omega \Delta u\; w - \int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}}w + \int_\Gamma u\frac{\partial w}{\partial \boldsymbol{n}} = \int_\Omega Fw - \langle g, w\rangle_\Gamma$$

which yields as before

$$\|u\|_{L^p(\Omega)} \le C_p(\Omega,\alpha_*)\left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\right)$$

and thus we obtain (2.20). ■

Proof of Theorem 1.1. Let $u_1 \in W^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} \operatorname{div}(A(x)\nabla)u_1 = \operatorname{div}\boldsymbol{f} & \text{in } \Omega \\ \dfrac{\partial u_1}{\partial \boldsymbol{n}} + \alpha u_1 = \boldsymbol{f}\cdot\boldsymbol{n} & \text{on } \Gamma \end{cases}$$

given by Proposition 2.7 and $u_2 \in W^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} \operatorname{div}(A(x)\nabla)u_2 = F & \text{in } \Omega \\ \dfrac{\partial u_2}{\partial \boldsymbol{n}} + \alpha u_2 = g & \text{on } \Gamma \end{cases}$$

given by Proposition 2.8. Then $u = u_1 + u_2$ is the solution of the problem 1.1 which also satisfies the estimate (1.4). ■

Next we prove uniform H^s bound for $s \in (0, \frac{1}{2})$.

Proposition 2.9. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^3$, $g \in L^2(\Gamma)$ and α is a constant. Then the problem*

$$\begin{cases} \operatorname{div}(A(x)\nabla)u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = g & \text{on } \Gamma \end{cases} \tag{2.21}$$

has a solution $u \in H^{\frac{3}{2}}(\Omega)$ which also satisfies the estimate

$$\|u\|_{H^{\frac{3}{2}}(\Omega)} \leq C(\Omega)\|g\|_{L^2(\Gamma)}. \tag{2.22}$$

Proof. A solution $u \in H^1(\Omega)$ of the problem (2.21) satisfies the variational formulation:

$$\forall \varphi \in H^1(\Omega), \qquad \int_\Omega A(x)\nabla u \cdot \nabla \varphi + \int_\Gamma \alpha u \varphi = \int_\Gamma g\varphi.$$

Multiplying the above relation by α and substituting $\varphi = u$, we get

$$\alpha \int_\Omega A(x)\nabla u \cdot \nabla u + \|\alpha u\|^2_{L^2(\Gamma)} = \alpha \int_\Gamma gu \leq \|g\|_{L^2(\Gamma)}\|\alpha u\|_{L^2(\Gamma)}$$

and thus

$$\|\alpha u\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)}.$$

Now from the regularity result for Neumann problem [6, Theorem 2], we obtain

$$\|u\|_{H^{\frac{3}{2}}(\Omega)} \leq C(\Omega)\|g - \alpha u\|_{L^2(\Gamma)} \leq C(\Omega)\|g\|_{L^2(\Gamma)}$$

which gives the required estimate. ■

Theorem 2.10 ($\boldsymbol{H^s(\Omega)}$ estimate). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^3$, $s \in (0, \frac{1}{2})$ and α is a constant. Then for $g \in H^{s-\frac{1}{2}}(\Gamma)$, the problem (2.21) has a solution $u \in H^{1+s}(\Omega)$ which also satisfies the estimate*

$$\|u\|_{H^{1+s}(\Omega)} \leq C(\Omega)\|g\|_{H^{s-\frac{1}{2}}(\Gamma)}.$$

Proof. We obtain the result by interpolation between $H^1(\Omega)$ and $H^{\frac{3}{2}}(\Omega)$ regularity results in Theorem 2.1 and Proposition 2.9 respectively. ■

3 Estimate for strong solution

Theorem 3.1 ($\boldsymbol{W^{2,p}(\Omega)}$ estimate). *Let Ω be a $\mathcal{C}^{1,1}$ bounded domain in $\mathbb{R}^3$, $p \in (1, \infty)$ and α be a constant. Then for $F \in L^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, the solution u of the problem*

$$\begin{cases} \Delta u = F & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = g & \text{on } \Gamma \end{cases} \tag{3.1}$$

belongs to $W^{2,p}(\Omega)$ *and satisfies the following estimate:*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \right) \tag{3.2}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ *is independent of* α.

Remark 3.2. We can in fact show the existence of $u \in W^{2,p}(\Omega)$ for more general α, not necessarily constant; in particular for $\alpha \in W^{1-\frac{1}{q},q}(\Gamma)$ with $q > \frac{3}{2}$ if $p \leq \frac{3}{2}$ and $q = p$ otherwise.

Proof. For the given data, there exists a unique solution u of (3.1) in $W^{1,p}(\Omega)$, by Theorem 1.1. Then it can be shown that in fact u belongs to $W^{2,p}(\Omega)$ by Neumann regularity result using bootstrap argument. But concerning the estimate, we do not obtain a α independent bound on u, using the estimate for Neumann problem. So we consider the following argument.

As Γ is compact and of class $\mathcal{C}^{1,1}$, there exists an open cover U_i *i.e.* $\Gamma \subset \cup_{i=1}^k U_i$ and bijective maps $H_i : Q \to U_i$ such that

$$H_i \in \mathcal{C}^{1,1}(\overline{Q}),\ J^i := H_i^{-1} \in \mathcal{C}^{1,1}(\overline{U_i}),\ H_i(Q_+) = \Omega \cap U_i \ \text{ and } \ H_i(Q_0) = \Gamma \cap U_i$$

where we denote

$$\begin{aligned} &Q = \{x = (x', x_3); |x'| < 1 \text{ and } |x_3| < 1\} \\ &Q_+ = Q \cap \mathbb{R}^3_+ \\ &Q_0 = \{x = (x', 0); |x'| < 1\}. \end{aligned}$$

Then we consider the partition of unity θ_i corresponding to U_i with supp $\theta_i \subset U_i$. So we can write $u = \sum_{i=0}^k \theta_i u$ where $\theta_0 \in C_c^\infty(\Omega)$. It is easy to see that $v_i = \theta_i u \in W^{2,p}(\Omega \cap U_i)$ and satisfies:

$$\begin{cases} \Delta v_i = \theta_i F + 2\nabla\theta_i \nabla u + u\Delta\theta_i =: f_i & \text{in} \quad \Omega \cap U_i \\ \dfrac{\partial v_i}{\partial \boldsymbol{n}} + \alpha v_i = g + \dfrac{\partial \theta_i}{\partial \boldsymbol{n}} u =: h_i & \text{on} \quad \partial(\Omega \cap U_i). \end{cases}$$

Precisely, we have, for all $\varphi \in W^{1,p'}(\Omega \cap U_i)$,

$$\int\limits_{\Omega\cap U_i} \nabla v_i \cdot \nabla \varphi + \alpha \int\limits_{\Gamma\cap U_i} v_i \varphi = - \int\limits_{\Omega\cap U_i} f_i \varphi + \int\limits_{\Gamma\cap U_i} h_i \varphi \tag{3.3}$$

where $f_i \in L^p(\Omega)$ and $h_i \in W^{1-\frac{1}{p},p}(\Gamma)$. Now to transfer $v_i|_{\Omega\cap U_i}$ to Q_+, set $w_i(y) = v_i(H_i(y))$ for $y \in Q_+$. Then,

$$\frac{\partial v_i}{\partial x_j} = \sum_k \frac{\partial w_i}{\partial y_k} \frac{\partial J^i_k}{\partial x_j}.$$

Also let $\psi \in H^1(Q_+)$ and set $\varphi(x) = \psi(J^i(x))$ for $x \in \Omega \cap U_i$. Then $\varphi \in H^1(\Omega \cap U_i)$ and

$$\frac{\partial \varphi}{\partial x_j} = \sum_l \frac{\partial \psi}{\partial y_l} \frac{\partial J^i_l}{\partial x_j}.$$

Thus, putting these in (3.3), we obtain under this change of variable, for all $\psi \in H^1(Q_+)$,

$$\int\limits_{Q_+} a_{kl}(x) \frac{\partial w_i}{\partial y_k} \frac{\partial \psi}{\partial y_l} + \alpha \int\limits_{Q_0} w_i \psi = - \int\limits_{Q_+} \tilde{f}_i \psi + \int\limits_{Q_0} \tilde{h}_i \psi \tag{3.4}$$

with $a_{kl}(x) = \sum_j \frac{\partial J^i_k}{\partial x_j}\frac{\partial J^i_l}{\partial x_j} |\det Jac\ H_i|$, $\tilde{f}_i = f_i \circ J^i$ and $\tilde{h}_i = h_i \circ J^i$. Here $\det Jac\ H_i$ denotes the determinant of the Jacobian matrix of H_i. Note that $a_{kl} \in \mathcal{C}^{0,1}(\overline{Q_+})$, $\tilde{f}_i \in L^p(Q_+)$ and $\tilde{h}_i \in W^{1/p',p}(Q_0)$. Also (3.4) is a Robin problem of the form (1.1) for w_i on Q_+, since w_i vanishes in a neighbourhood of $\partial Q_+ \setminus Q_0$.

For notational convenience, in this last part, we omit the index i *i.e.* we simply write w instead of w_i. Now denoting $\partial_j = \frac{\partial}{\partial x_j}$, we see that $z_i := \partial_i w, i = 1, 2$ solves the following problem

$$\begin{cases} \operatorname{div}(A(x)\nabla)z_i = \operatorname{div}(\tilde{f}\boldsymbol{e}_i) - \operatorname{div}(\partial_i A(x)\nabla)w & \text{in } Q_+ \\ \dfrac{\partial z_i}{\partial \boldsymbol{n}} + \alpha z_i = \tilde{f}\boldsymbol{e}_i \cdot \boldsymbol{n} - (\partial_i A(x)\nabla)w \cdot \boldsymbol{n} + \partial_i \tilde{h} & \text{on } Q_0 \end{cases} \tag{3.5}$$

where $\boldsymbol{e}_i$ is the unit vector with 1 in i^{th} position Thus, we can apply Theorem 1.1 for the above system and may conclude

$$\|z_i\|_{W^{1,p}(Q_+)} \leq C_p(Q_+) \left(\|\tilde{f}\|_{L^p(Q_+)} + \|\partial_i A(x)\nabla w\|_{\boldsymbol{L}^p(Q_+)} + \|\partial_i \tilde{h}\|_{W^{-\frac{1}{p},p}(Q_0)} \right)$$

which yields, for all $i, j = 1, 2, 3$ except $i = j = 3$,

$$\|\partial^2_{ij} w\|_{L^p(Q_+)} \leq C_p(Q_+) \left(\|\tilde{f}\|_{L^p(Q_+)} + \|w\|_{W^{1,p}(Q_+)} + \|\tilde{h}\|_{W^{\frac{1}{p'},p}(Q_0)} \right). \tag{3.6}$$

Now to show the estimate for $\partial^2_{33} w$, we can write from the equation (3.4) (omitting the index i),

$$\partial^2_{33} w = \frac{1}{a_{33}} \left(\tilde{f} - a_{ij}\ \partial^2_{ij} w - \partial_i a_{ij}\ \partial_j w \right) \quad \text{in} \quad Q_+.$$

But since J is an one-one map, $a_{33} \neq 0$ and thus together with (3.6), we obtain the same estimate (3.6) for $\partial^2_{33} w$. Therefore, we can conclude, for all $i = 1, ..., k$,

$$\|v\|_{W^{2,p}(\Omega \cap U_i)} \leq C_p(\Omega) \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|u\|_{W^{1,p}(\Omega)} \right)$$

and consequently (3.2), using $W^{1,p}$-estimate result. ∎

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