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Research paper

Vector Lyapunov-like functions for multi-order fractional systems with multiple time-varying delays

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ABSTRACT

In this paper, a general method to establish the asymptotic behaviour of solutions to multi-order multiple time-varying delays nonlinear systems is proposed. The method, relying on vector Lyapunov-like functions and on comparison arguments, reduces the asymptotic stability problem to verify a Hurwitz property on a suitable matrix. Many results in integer order systems can be easily generalized to multi-order systems since the obtained conditions are order-independent. The latter fact is exploited to obtain robust results when the derivation order is uncertain. To establish the method, robust multi-order multiple time-varying delays linear positive systems are studied generalizing previous results existing in the literature. Two illustrative examples are presented, the main one providing conditions for asymptotic stability of a multi-agent multi-order system with time-varying delay.

1. Introduction

A multi-order (also called multivariable [1], mixed-order [2], or incommensurate) fractional system is a set of fractional differential equations (those involving fractional derivatives) where each equation is allowed to have its own differential order. As fractional systems have been mainly used to get empirical models of complex processes [3], the importance of the qualitative study of multi-order fractional systems becomes evident as they represent the most encompassing structure. Moreover, in applications of fractional operators to classic integer order problems, these systems naturally appear (as we will see in an example).

One of the main topics in the qualitative study of differential equations is stability and asymptotic behavior. Although most of the research has been focused on the study of systems where each equation has the same differentiation order, some results have been obtained for the multi-order case. The linear case has been studied in [4,5]. Convergence for a class of Lipschitz nonlinear systems was studied in [6]. In [2] an integral method is proposed to show boundedness of solutions in the nonlinear case. In [7] a method is developed to study external stability for input-output multi-order initialized fractional systems.

A quite general method to study the stability for the initial value problem of multi-order nonlinear systems, like the Lyapunov method for integer order systems, has not been yet fully developed. A significant contribution in this direction, however, has recently appeared in [1], where the reader can also find related works and their gaps. In that paper, a method

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is proposed based on a comparison argument and on the knowledge of asymptotically stable multi-order fractional systems (this method also appears in [6]). For the latter, the authors invoke Laplace transform and study linear multi-order systems. The obtained conditions are difficult to compute as the number of equations grows and/or the exponents are irrationals. This motivates the present work, whose contributions are summarized below.

- We introduce a general method to establish asymptotic stability of multi-order systems, which reduces to the known method of vector Lyapunov functions [8,9] when particularized to integer order systems. The advantage of this approach is that many results in integer order systems can be easily generalized to multi-order systems when convex Lyapunov functions are used. This is made possible because, in contrast to [1], the obtained conditions are order-independent, which in addition allows establishing a robust result against order perturbation.
- In comparison to [1], a simpler condition to verify asymptotic stability is obtained, robust properties are established and the case of multiple time-varying delays included. In fact, the asymptotic stability problem is reduced to an eigenvalue computation to verify a Hurwitz property on a suited matrix built from the equations of the system, together with a Melzer condition which can be decided by inspection.
- In order to establish the method, robust multi-order multiple time-varying delays positive linear systems are studied, generalizing the results in [5].The properties of positive systems are used to address the analysis of general (not necessarily positive) nonlinear systems with time-varying delays, through vector Lyapunov functions which define positive systems by construction.
- We provide two illustrative examples. The first one shows the control of a multi-order nonlinear system and the second posits conditions for asymptotic stability of a multi-agent multi-order system. Although for integer order systems, scalar Lyapunov functions are enough to characterize the stability (according to classic converse results), they do not seem enough for multi-order systems as all the revised examples of their use deals with single-order systems. The usefulness of the vector Lyapunov functions to address this problem is shown in the first example. The second example aims to combine integer-order with fractional-order agents in multi-agent networks and establishes a procedure which can be applied for consensus or synchronization problems involving multi-orders.

We must mention some limits in our elaboration. First, the differentiation orders must belong to (0, 1]. This restriction appears in the applications, where the use of inequalities for fractional derivatives of order (0, 1] is a key step. Second, we study Caputo systems (or initialized fractional systems when null initial functions are used). For Riemann–Liouville multi-order systems see [10] and for initialized ones see [7], where input/output rather than Lyapunov stability seems more suited. And third, due to the simplicity and to the order-independent delay-independent nature, the main results proposed can become conservative. Notice, however, that in contrast to scalar Lyapunov functions, which are conservative as they use scalar equations which have the same convergence conditions for any order in (0,1], the vector approach allows order-dependent conditions as shown in a side result.

The contributions of this paper are organized as follows. In Section 2, we study the stability of multi-order multiple time-varying delays positive fractional differential systems, which are later used in Section 3. In Section 3, we present the main result, which establishes that if a system has a vector Lyapunov-like function, the asymptotic stability can be asserted. In Section 4, we provide illustrative examples of multi-order nonlinear systems having vector Lyapunov-like functions.

Notation: $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals and nonnegative reals numbers, respectively. For $x \in \mathbb{R}^n$, we use the norm $\|x\|_1 := \sum_{i=1}^n \|x_i\|$ and the Euclidean norm $\|\cdot\|$. $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the set of continuous real-valued functions on $[-\tau, 0]$ endowed with the infinite norm $\|\phi\|_\infty = \sup_{t\in[-\tau,0]} \|\phi(t)\|$.

For $x \in \mathbb{R}^n$, $x \succeq 0$ means $x_i \geq 0$ for any $i \in \{1, \dots, n\}$.

For $A \in \mathbb{R}^{n\times m}$, A' denotes its transpose. $A \in \mathbb{R}^{n\times n}$ is Metzler if the off-diagonal elements are nonnegative. $A \in \mathbb{R}^{n\times n}$ is Hurwitz if all its eigenvalues have negative real part. $A \in \mathbb{R}^{n\times m}$ is nonnegative if all its entries are nonnegative. I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.

A class $\mathcal{K}$ function is an strictly increasing continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(0) = 0$.

For a measurable function $x : [a, b] \to \mathbb{R}$ s.t. $\int_a^b |x(s)|ds < \infty$, the Riemann–Liouville integral operator of order α is defined by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad t \in [a, b] \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function. The Caputo fractional derivative ${}^C D_{0^+}^{\alpha} x(t)$ of a function $x \in AC([a, b], \mathbb{R})$ is defined by $D_{0^+}^{\alpha} x(t) := I_{0^+}^{1-\alpha} \dot{x}(t)$ when $0 < \alpha < 1$. $AC([a, b], \mathbb{R})$ denotes the space of absolutely continuous functions. When clear from context, we will omit the indexes 0^+ and C so that ${}^C D_{0^+}^{\alpha} x(t) \equiv D^{\alpha} x(t)$.

2. Positive systems

In this section, we study positive solutions for the following system

$$D_{0^+}^{\alpha} x(t) = Ax(t) + \sum_{j=1}^{l} A_{d_j} x(t - \tau_j(t)) + Bu(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $D^{\alpha}_{0^+}x = (D^{\alpha_1}_{0^+}x_1, \ldots, D^{\alpha_n}_{0^+}x_n)'$ for $0 < \alpha_i \le 1$ and all $t > 0$. A, A_{d_j}, B are matrices of suited dimensions. This notation includes delayed inputs such as $\bar{u}(t - \tau(t))$ by redefining it as $u(t) = \bar{u}(t - \tau(t))$. The only requirement on the delay functions is that $0 \le \tau_j(t) \le \tau_j$ for all $t \ge -\tau$ where $\tau := \max_j \tau_j < \infty$.

The solutions of (2) are well-defined on $[0, \infty)$ provided that a continuous initial function ϕ for x is specified on $[-\tau, 0]$ and that u is a continuous function (see for instance [1] or the proof of Proposition 1 below). Moreover the function $\xi = x$ for $t > 0$ and $\xi = \phi$ for $t \in [-\tau, 0]$ will be continuous. However, $\|\dot{x}\|$ goes to ∞ as $t \to 0^+$ whenever $D^{\alpha}_{0^+}x(0) \ne 0$, as can be seen from the definition of the Caputo fractional derivative (see also [2]).

System (2) is called *positive* (*negative*, respectively) if $\phi(s) \succeq 0$, $u(t) \succeq 0$ ($\phi(s) \preceq 0$, $u(t) \preceq 0$) for all $s \in [-\tau, 0]$, $t \ge 0$, implies $x(t) \succeq 0$ ($x(t) \preceq 0$) for all $t \ge 0$ [5]. Noting that a positive system is also negative (by replacing $x \longmapsto -x$ and $u \longmapsto -u$ in (2)), the following results hold also *mutatis mutandis* for negative systems.

For positive systems, the next assumption will be shown fundamental.

Assumption 1. For system (2), A is Metzler and A_{d_j}, B are nonnegative for $j = 1, \ldots, l$.

The following results are mostly generalizations (in several ways) of the single-delay statements in [5]. Their proofs can be found in the Appendix. We underline that the results obtained in this section are order-independent and delay-independent once the basic constrains above are fulfilled.

The result in Proposition 1 states a condition to have positive systems which can be verified by simple inspection. The proof is not completely based on [5] since a minor gap was detected there (related to the fact that the infimum in an open set is not necessarily in the set).

Proposition 1. *System (2) is positive for all $\alpha_i \in (0, 1]$, $i = 1, \ldots, n$ if and only if Assumption 1 holds.*

To find conditions for stability a comparison procedure is followed. The next result states a simple criteria for delayed systems.

Proposition 2. *Consider Assumption 1. Let $x(t;\ \phi,\ u)$ be the solution of (2) at time t with initial function ϕ and input u. If $\phi_1 \preceq \phi_2$ and $u_1 \preceq u_2$, then $x(t;\ \phi_1,\ u_1) \preceq x(t;\ \phi_2,\ u_2)$.*

Also for comparison purposes, we will need properties of the solutions for fixed delay and constant initial function as given in the next statement.

Proposition 3. *Let Assumption 1 be satisfied for system (2) with $u \equiv 0$, $\tau_j(t) \equiv \tau_j$. Suppose that there exists a vector $c \in \mathbb{R}^n$ such that $c \succ 0$ and $(A + \sum_{j=1}^{l} A_{d_j})c \prec 0$. Let $\phi \equiv rc$ for arbitrary $r > 0$. Then,*

(i) $0 \preceq x \preceq rc$
(ii) $x(t_1) \succeq x(t_2)$ *for any* $0 \le t_1 < t_2$
(iii) $\lim_{t \to \infty} x(t) = 0$.

Relying on comparison, we are able to show the following basic results. The first, an invariant property, is not included in [5] for the single-delay case.

Proposition 4. *Consider (2) holding Assumption 1 and $u \equiv 0$. The set $\{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) : 0 \preceq \phi \preceq rc\}$, where c is as in Proposition 3 and $r > 0$, is positively invariant in the sense that for any initial function in this set, the solution holds that $0 \preceq x(t) \preceq rc$ for all $t \ge 0$.*

Proposition 5. *Consider (2) holding Assumption 1 and $\phi \equiv 0$. Then*

(i) *If $u \succeq 0$, then $x(t_1) \preceq x(t_2)$ for any $0 < t_1 < t_2$.*
(ii) *$0 \preceq \underline{x} \preceq x \preceq \bar{x}$, where $\underline{x}$ and $\bar{x}$ are the solutions to (2) with null initial function ($\phi \equiv 0$) and constant delays τ_j and 0, respectively.*

3. Main results

The aim of this section is to establish a vector Lyapunov-like method to study the asymptotic behavior of systems with multiple derivations orders and time-varying delays. Based on the results of positive systems, we use a vector of non-negative functions playing the role of Lyapunov functions, although in general, they will not be function of the real state of the system but rather of the (so-called) pseudo-state. This is why the approach is called Lyapunov-like or pseudo-Lyapunov, instead of Lyapunov one.

Consider the following multi-order system with multi-delays

$$D^{\beta}_{0^+}x(t) = f(x, x(t - \tau_1(t)), \ldots, x(t - \tau_l(t)), t) \tag{3}$$

where $x(t) \in \mathbb{R}^n$ for all $t \ge 0$, $D^{\beta}_{0^+}x = (D^{\beta_1}_{0^+}x_1, \ldots, D^{\beta_n}_{0^+}x_n)$ for $0 < \beta_i \le 1$ and $i = 1, \ldots, n$. f is a smooth enough function guaranteeing continuous solutions for any $t \in [0, \infty)$ and such that $f(0, 0, \ldots, 0, t) = 0$ for any $t \in \mathbb{R}$. The delayed functions satisfy

$0 \le \tau_j(t) \le \tau_j$ for all $t \ge 0$, where $\tau := \max \tau_j < \infty$. The continuous initial function ϕ is specified on $[-\tau, 0]$. For a given continuous initial function ϕ, the corresponding solution of (3) is denoted by $x(t;\ \phi)$, or when clear form context, simply $x(t)$ for any $t \ge -\tau$, where $x(t) = \phi(t)$ for any $t \in [-\tau, 0]$. We stress that ϕ is now not necessarily positive.

We recall the stability notion for system (3) (see [11]). This notion relies on in the trivial solution of system (3), i.e. the function $x \equiv 0$ when $\phi \equiv 0$. This trivial solution can be seen as the equilibrium point when $\tau = 0$ or as a trivial limit cycle.

Definition 1. The trivial solution $x \equiv 0$ of (3) is said to be *stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $\|\phi\|_\infty < \delta$ implies $\|x(t;\ \phi)\| < \epsilon$ for any $t > 0$; and *asymptotically stable* if, in addition, $\lim_{t\to 0} \|x(t;\phi)\| = 0$ for any ϕ such that $\|\phi\|_\infty < \bar{\delta}$ for some $\bar{\delta} > 0$.

The vector Lyapunov-like approach is based on the precedent results obtained for positive systems, by associating a vector of non-negative functions to system (3) (the latter is not necessarily positive or linear). Specifically, we will need the following result

Theorem 1. *Consider system (2) such that Assumption 1 is satisfied, $\phi \succeq 0$ and $A + \sum_{i=1}^{l} A_{d_i}$ is Hurwitz.*

(i) *If $u \equiv 0$, then the trivial solution is asymptotically stable.*
(ii) *If u is bounded, x is also bounded and if u is bounded and converges to zero, then x also converges to zero.*

Proof. (i) Let $x_1(t;\ \phi)$ and $x_2(t;\ \phi)$ be the solutions of (2) with initial function ϕ, for arbitrary and fixed delay τ_j, respectively.

Due to Assumption 1, $A + \sum_{j=1}^{l} A_{d_j}$ is Metzler. Since $A + \sum_{j=1}^{l} A_{d_j}$ is in addition Hurwitz, there exists $c \succ 0$ such that $(A + \sum_{j=1}^{l} A_{d_j})c \prec 0$ (see [5, Lemma 2]). Clearly, for each ϕ there exists $r > 0$ s.t. $rc \succ \phi$. Let $e(t) := x_2(t; rc) - x_1(t; rc)$. Then $D^\alpha e(t) = Ae(t) + \sum_{j=1}^{l} A_{d_j} e(t - \tau_j(t)) + \sum_{j=1}^{l} A_{d_j}(x_2(t - \tau_j) - x_2(t - \tau_j(t))$. From Proposition 3(ii), $x_2(t - \tau_j) - x_2(t - \tau_j(t)) \ge 0$. Since $e \equiv 0$ on $[-\tau, 0]$, $e \succeq 0$ according to Proposition 1.

Then, using also Proposition 1, $x_2(t;\ rc) \succeq x_1(t;\ rc) \succeq 0$ for all $t \ge 0$. From Proposition 2, $x_2(t;\ rc) \succeq x_1(t;\ \phi) \succeq 0$. From Proposition 3(iii), $x_2(t;\ rc) \to 0$ as $t \to \infty$. Then $x_1(t;\ \phi) \to 0$ as $t \to \infty$.

Using again $x_2(t;\ rc) \succeq x_1(t;\ \phi) \succeq 0$ and the monotony of x_2 due to Proposition 3(ii), the stability claim can be easily obtained.

(ii) First note that the solution of (2) can be expressed as $x(t) = \xi(t;\phi) + \psi(t;u)$ for any $t \ge 0$, where $\xi(t;\ \phi)$ is the solution to (2) with $u \equiv 0$ and $\psi(t;\ u)$ is the solution to (2) with $\phi \equiv 0$. To see this, note that $\epsilon := x - \xi$ satisfies the equation $D^\alpha \epsilon(t) = A\epsilon(t) + \sum_{j=1}^{l} A_j \epsilon(t - \tau_j(t)) + Bu(t)$ with initial condition $\epsilon \equiv 0$. Therefore, $\psi = \epsilon$.

ξ was studied in part (i); thus, we must study ψ. From Proposition 5, to prove that ψ is bounded, it is enough to show that $\bar{x}$ is bounded. That is, we must prove that the solution to $D^\alpha \bar{x} = (A + \sum_{j=1}^{l} A_{d_j})\bar{x} + Bu$ with null initial function is bounded whenever u is bounded. From Proposition 2, it is enough to prove this for any constant nonnegative u. Since $\sum A_{d_j}$ is nonnegative and $(A + \sum_{j=1}^{l} A_{d_j})$ is Hurwitz, this result follows from [5, Lemma 8] by using $\tau = 0$ and the continuity of ψ. Since ξ is continuous and converges to zero, it follows that x is bounded.

Now suppose that u goes to zero as $t \to \infty$. From Proposition 5, to prove that $\psi \to 0$ as $t \to \infty$, it is enough to show that $\bar{x}$ does it. Let h be the inverse Laplace-transform of the matrix $(diag(s^{\alpha_i}) - (A + \sum_{j=1}^{l} A_{d_j}))^{-1}B$. Then $\bar{x}(t) = \int_0^t h(\tau)u(t-\tau)dt$. From the argument above, we have that for any bounded u, $\bar{x}(t)$ is bounded. By choosing $u = \mathbf{e_i}$ where $\mathbf{e_i}$ is an element of the canonical base of $\mathbb{R}^n$ for $i = 1, \ldots, n$, we conclude that every element of h has bounded integral on $(0, \infty)$. On the other hand, $h\mathbf{e_i}$ is the solution to the impulse input $u = \delta\mathbf{e_i}$. Since δ can be approximated by a sequence of non-negative functions, $h\mathbf{e_i} \succeq 0$. Then each element of the matrix h is integrable on $(0, \infty)$ and thus, its convolution with a function converging to zero, converges to zero as $t \to \infty$. Then $\bar{x} \to 0$ and hence $\psi \to 0$ as $t \to \infty$. Since $\xi \to 0$ as $t \to \infty$ from part (i), the claim follows. □

Now we are ready to state the main result. In this result, vector Lyapunov-like functions can be seen as a reduced-order (pseudo-)state of (3) useful to study its stability (note, however, that the reduced order condition, $m \le n$, is not essential in the proof).

Theorem 2. *Consider that for system (3) there exists a vector function $V : \mathbb{R}^n \to \mathbb{R}^m$ satisfying*

(i) *$\gamma_1(\|x\|) \le \|V(x)\|_1 \le \gamma_2(\|x\|)$ for some class-$\mathcal{K}$ functions γ_1 and γ_2.*
(ii) *The function $V(t) := V(x(t))$, for any solution $x(\cdot)$ of (3), is such that $V(t) \succeq 0$ for any $t \ge -\tau$ and there exist a Metzler matrix A, nonnegative matrices A_{d_i} for $i = 1, \ldots, l$ with $A + \sum_{i=1}^{l} A_{d_i}$ a Hurwitz matrix and*

$$D_{0^+}^{\alpha} V(t) \preceq AV(t) + \sum_{j=1}^{l} A_{d_j} V(t - \tau_j(t)) + C(t), \quad \forall t \ge 0 \tag{4}$$

where $C(t) \succeq 0$ and $D^\alpha V$ is the vector of components $D^{\alpha_i} V_i$ with $\alpha_i \in (0, 1]$ for $i = 1, \ldots, m$.

Then the trivial solution of (3) is asymptotically stable when $C \equiv 0$. In addition, the solutions of (3) remain bounded or converge to zero if C does it.

Proof. Let $u = u(t)$ be the vector function such that (4) is turned into an equality, i.e. $\forall t \geq 0$

$$D^{\alpha}_{0^+} V(t) = AV(t) + \sum_{j=1}^{l} A_{d_j} V(t-\tau_j(t)) + u(t) + C(t).$$

Note that $u \preceq 0$. Consider for comparison purposes the following system $\forall t \geq 0$

$$D^{\alpha}_{0^+} \bar{V}(t) = A\bar{V}(t) + \sum_{j=1}^{l} A_{d_j} \bar{V}(t-\tau_j(t)) + C(t)$$

with $\bar{V} \equiv V$ on $[-\tau, 0]$ as the initial function. By calling $e = \bar{V} - V$, we have

$$D^{\alpha}_{0^+} e(t) = Ae(t) + \sum_{j=1}^{l} A_{d_j} e(t-\tau_j(t)) - u(t)$$

with null initial function. From Proposition 1 and conditions (ii), $e \succeq 0$ i.e. $0 \preceq V \preceq \bar{V}$. Therefore, in order to prove that V converges to zero or it is bounded, it is enough to prove that $\bar{V}$ does it.

Applying Theorem 1, it follows that $\bar{V}$ (and hence V) is bounded when C is bounded and converges to zero when C does it. Using $\|V(x(t))\| \geq \gamma(\|x(t)\|)$, we conclude that x is bounded when C is bounded and converges to zero when C does it. Also due to Theorem 1, the stability of the trivial solution can be similarly obtained. □

Remark 1.

(i) In the scalar and single-delay case without perturbation ($C \equiv 0$), (4) takes the form

$$D^{\alpha} V(t) \leq aV(t) + bV(t-\tau(t)) \tag{5}$$

and the condition (ii) from Theorem 2 reduces to $a,\ b > 0$ and $a + b < 0$. The latter condition has been independently obtained in [12].

(ii) Condition (i) from Theorem 2 is weaker than requiring that each V_i or $\sup_i V_i$ be positive definite.

Note that the orders $\alpha_i \in (0,\ 1]$ and the time-varying delay functions –restricted to take values on $[0,\ \tau]$ for some $\tau < \infty$– are otherwise arbitrary. As an immediate consequence, we get the following robust control result.

Corollary 1. *Consider the nominal system*

$$\dot{x}(t) = f(x, x(t-\tau_1), \ldots, x(t-\tau_l), u) \tag{6}$$

where $x(t) \in \mathbb{R}^n$ for all $t \geq 0$ and f is a smooth enough function guaranteeing continuous solutions on $\mathbb{R}_{\geq 0}$. Suppose that there exist a feedback control $u = u(x)$ and differentiable, convex functions $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $i = 1, \ldots, m$ such that the vector $\dot{V}$ of components $\nabla V_i' f(x, x(t-\tau_1), \ldots, x(t-\tau_l), u(x))$, evaluated along the solution of (6) satisfies

$$\dot{V} \preceq AV(t) + \sum_{j=1}^{l} A_{d_j} V(t-\tau_j),$$

where A, A_{d_i} satisfy the requirements in Theorem 2(ii). Then the feedback control stabilizes the trivial solution and is robust under uncertainties in the orders of derivation (translated to the reduced-order model associated to V) and time-varying delayed functions.

Proof. Assume that an uncertainty in the orders of derivation α_i and time-varying delayed functions $\tau_j(t)$) occur. Since V_i are differentiable and convex, we have $D^{\alpha_i}_{0^+} V_i \leq \nabla V_i' f(x, x(t-\tau_1), \ldots, x(t-\tau_l), u(x))$, according to [13]. Hence $D^{\alpha}_{0^+} V \preceq AV(t) + \sum_{j=1}^{l} A_{d_j} V(t-\tau_j(t)) + C(t)$ and the application of Theorem 2 concludes the proof. □

We underscore that Theorem 2 has verifiable and order-independent hypotheses (i) and (ii). A more general and order-dependent result than Theorem 2 can be stated if one allows a formal condition. Recall that a function $w : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is quasimonotone nondecreasing [9] if for any $y \in \mathbb{R}^p$, $w_i(x, y) \preceq w_i(\bar{x}, y)$ for all $x_i = \bar{x}_i$, $x \preceq \bar{x}$ and $i = 1, \ldots, n$. Example of quasimonotone nondecreasing function is $f(x) = \Gamma x$, with Γ a constant Hurwitz matrix.

Proposition 6. *Suppose that there exists a function V satisfying hypotheses (i) and (ii) of Theorem 2 and in addition*

$$D^{\alpha}_{0^+} V(t) \preceq w(V(t), V(t-\tau_1(t)), \ldots, V(t-\tau_l(t)), t) \quad \forall t > 0,$$

where w is quasimonotone nondecreasing and $D^{\alpha} V$ is the vector of components $D^{\alpha_i} V_i$ for $\alpha_i \in (0,\ 1]$. Suppose that the trivial solution for system $D^{\alpha} z(t) = w(z(t), z(t-\tau_1(t)), \ldots, z(t-\tau_l(t)), t)$ with initial function $\phi_z \succeq \phi_V$, is asymptotically stable, where $D^{\alpha} z$ is the vector of components $D^{\alpha_i} z_i$ and ϕ_V the initial function of V. Then the trivial solution of (3) is asymptotically stable.

This result, which is proved in the Appendix, has the inconvenient that few multi-order systems are *a priori* known to have asymptotically stable trivial solution.

We close this section with a complement of Theorem 2. If the latter can be seen as an analogue of the scalar Lyapunov condition $\dot{V}(x) \leq -c\|x\|$, the next is the analogue to $\dot{V}(x) \leq 0$.

Proposition 7. *Suppose that there exists a function V satisfying hypotheses (i) and (ii) of Theorem 2 and*

$$D^{\alpha}_{0^+} V \preceq 0 \tag{7}$$

Then $x = 0$ is stable for system (3) without delays.

Proof. Write (7) as $D^{\alpha}V \preceq 0V$, where $0 \in \mathbb{R}^{n\times n}$ is the zero matrix. Obviously, 0 is Metzler. Let $C \succeq 0$ be the vector function such that $D^{\alpha}V = 0V - C$ and consider $D^{\alpha}\bar{V} = 0\bar{V}$ with the initial condition $\bar{V}(0) = V(0)$. Let $e = \bar{V} - V$. Then $D^{\alpha}e = C$ with $e(0) = 0$ and hence, by Proposition 1, $e \succeq 0$ i.e. $V \preceq \bar{V}$. But $\bar{V} \equiv V(0)$. Then $\|x(t)\| \leq \gamma_1^{-1}(\|V(0)\|_1) \leq \gamma_1^{-1}(\gamma_2(x(0)))$ for all $t > 0$ and the stability of $x = 0$ follows according to classic arguments. □

4. Illustrative examples

This section contains two representative examples of the results presented in this paper. But first, a short explanation is introduced, regarding the implementation of the fractional operators used in simulations.

4.1. Implementation of the fractional operators

One of the most common ways of implementing fractional integrals and derivatives in simulations and practical applications is by means of numerical approximations of these operators. The idea is to obtain integer-order transfer functions whose behavior approximates the fractional order Laplace operator:

$$C(s) = ks^{\alpha}. \tag{8}$$

Oustaloup's method [14] is one of the available frequency-domain methods for making this approximation, which uses a recursive distribution of N poles and N zeros of the form:

$$C(s) = k' \prod_{n=1}^{N} \frac{1 + s/\omega_{zn}}{1 + s/\omega_{pn}}. \tag{9}$$

The gain k' is adjusted so that if $k = 1$ then $|C(s)| = 0$ dB at 1 rad/s and ω_{zn}, ω_{pn} represent respectively the zeros and poles of the approximation, which are placed inside a frequency interval $[\omega_l, \omega_h]$ rad/s in which the approximation is valid.

The Oustaloup's method is incorporated in the NID block of the Ninteger Toolbox for Matlab/Simulink [15] specified as the Crone approximation. In this block, if $\alpha < 0$ is set, then the NID simulates a fractional integral, otherwise if $\alpha > 0$ the NID simulates a fractional derivative.

Thus, the implementation of the fractional order operators proposed in this work is made using Matlab/Simulink, along with the Ninteger Toolbox for Matlab/Simulink [15]. Specifically, the Crone approximation was used, with $N = 5$ and $[\omega_l, \omega_h] = [0.01, 100]$ rad/s.

4.2. Examples description, results and analysis

The aim of this section is not only to illustrate the applicability of the results already presented but also the usefulness of the vector Lyapunov-like method to determine the stability of multi-order systems. For the latter reason, we start empathizing that our results include the case $\tau = 0$ with the following example, which at the same time shows an application in control problems.

Example 1. Consider the system

$$\begin{aligned} D^{\alpha_1}x &= xy - \big(5 - \sin(t)\big)x + \cos(t)x^3 \\ D^{\alpha_2}y &= z \\ D^{\alpha_2}z &= y + z + x + u \end{aligned}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ and the neighborhood around the origin $B = \{(x, y, z) \in \mathbb{R}^3 : |x| + |y| + |z| \leq 1\}$. Let $V_1(x) = \frac{1}{2}x^2$ and $V_2(y) = \frac{1}{2}y^2$. Then for any $(x, y, z) \in B$

$$\begin{aligned} D^{\alpha_1}V_1 &\leq -x^2\big(5 + \sin(t)\big) + x^2y + \cos(t)x^4 \\ &\leq -x^2\big(5 + \sin(t)\big) + \frac{1}{2}x^4 + \frac{1}{2}y^2 + \cos(t)x^4 \\ &\leq -(5 + \sin(t) - \frac{1}{2} - |\cos(t)|)x^2 + \frac{1}{2}y^2 \\ &\leq -5V_1 + V_2 \end{aligned}$$

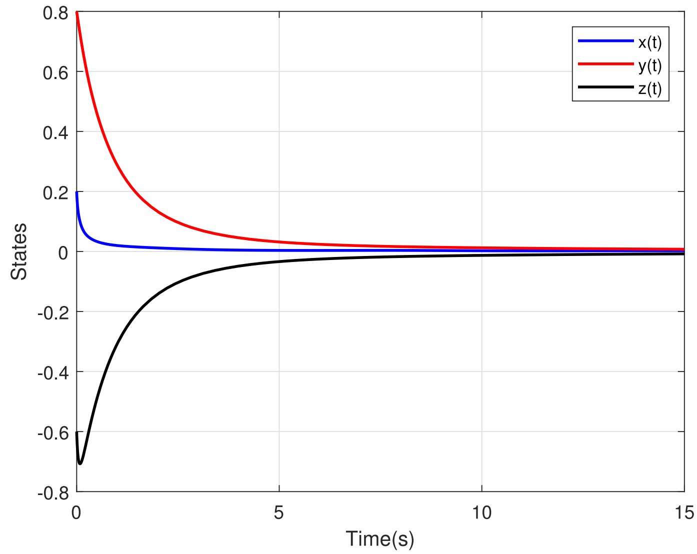

Fig. 1. Evolution of $x(t)$, $y(t)$ and $z(t)$ for Example 1.

where we have used [13] in the first inequality (so that $D^{\alpha_1}V_1 \leq xD^{\alpha_1}x$), the inequality $2ab \leq a^2 + b^2$ for any a, b in the second one, the fact that $(x, y, z) \in B$ in the third one and the definitions of V_i in the fourth. Let $\xi = y + z$ and $V_3(\xi) = \frac{1}{2}\xi^2$. Then

$$D^{\alpha_2}V_2 \leq -y^2 + \xi y \leq -\frac{1}{2}y^2 + \frac{1}{2}\xi^2 = -V_2 + V_3$$

and

$$\begin{aligned} D^{\alpha_2}V_3 &\leq \xi\left(\xi + (\xi - y) + x + u\right) \\ &\leq 2\xi^2 - y\xi + \frac{1}{2}x^2 + \frac{1}{2}\xi^2 + \xi u \\ &\leq 3\xi^2 + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \xi u = V_1 + V_2 + 6V_3 + \xi u. \end{aligned}$$

By choosing the control $u = -k_0\xi$, we obtain the following system

$$\begin{bmatrix} D^{\alpha_1}V_1 \\ D^{\alpha_2}V_2 \\ D^{\alpha_2}V_3 \end{bmatrix} \preceq \begin{bmatrix} -5 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & (6-2k_0) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} =: \Lambda \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

It follows that the matrix Λ is Metzler and, for $k_0 = 15$, it is also Hurwitz. Applying Theorem 2, the asymptotic stability of $(x, y, z) = (0, 0, 0)$ follows. Note that the stability result and the choice of V implies consistency in the fact that solution remains on B. Applying Corollary 1 and considering $\alpha_1 = \alpha_2 = 1$ as the nominal case, this control is robust against order perturbation α_1, $\alpha_2 \in (0, 1)$.

In Fig. 1, the controlled system was simulated with the following specifications $\alpha_1 = 0.7$, $\alpha_2 = 0.9$. Initial values for x, y and z were used as $x(0) = 0.2$, $y(0) = 0.8$ and $z(0) = -0.6$. As expected, it can be seen from Fig. 1 that the feedback control makes the system states converge to zero.

The main example of this section is presented below and studies a multi-agent system composed by agents defined with different derivation orders and time-varying delays. To the best of our knowledge, fractional nonlinear multi-agent systems have always been studied for agents with a common derivation order (see e.g. the recent work [16]). The present generalization allows combining local agents (i.e. those ruled by integer-order derivative) with non-local or long-memory agents (i.e. those ruled by fractional-order derivative), enhancing the complexity of the network and the range of its applicability.

Example 2. Consider a multiple delayed nonlinear network system with n nodes, each one ruled by

$$D^{\alpha_i}x_i(t) = -c_i x_i + f_i\left(x_i, x_i(t - \tau_1(t)), \dots, x_i(t - \tau_m(t))\right) + \sum_{j=1}^{n} a_{ij}x_j(t) \tag{10}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $a_{ij}, c_i \in \mathbb{R}$, $\alpha_i \in (0, 1]$ and $f_i : \mathbb{R}^{(m+1)n_i} \to \mathbb{R}^{n_i}$ are Lipschitz functions with $f_i(0, 0, \dots, 0) = 0$ and Lipschitz constant l_i, $\tau_l(t) < \tau \in \mathbb{R}$ for $i, j = 1, \dots, n$, $l = 1, \dots, m$ and $t \geq 0$. To enhance readability, we write $f_i(t) := f_i(x_i, x_i(t -$

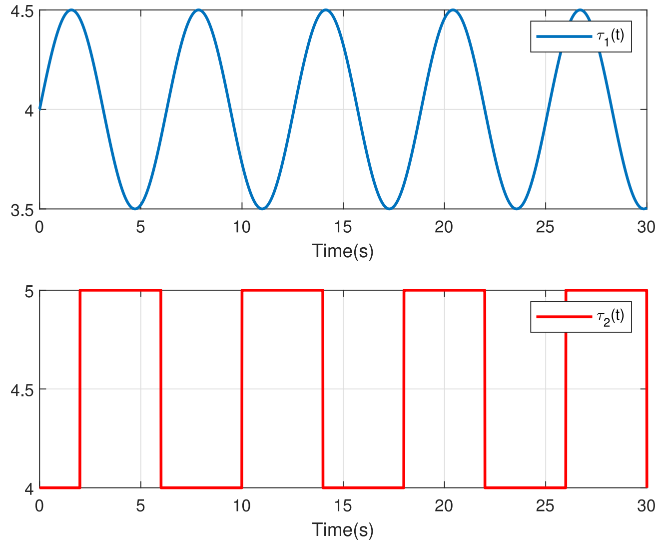

Fig. 2. Time varying delays used in system (12).

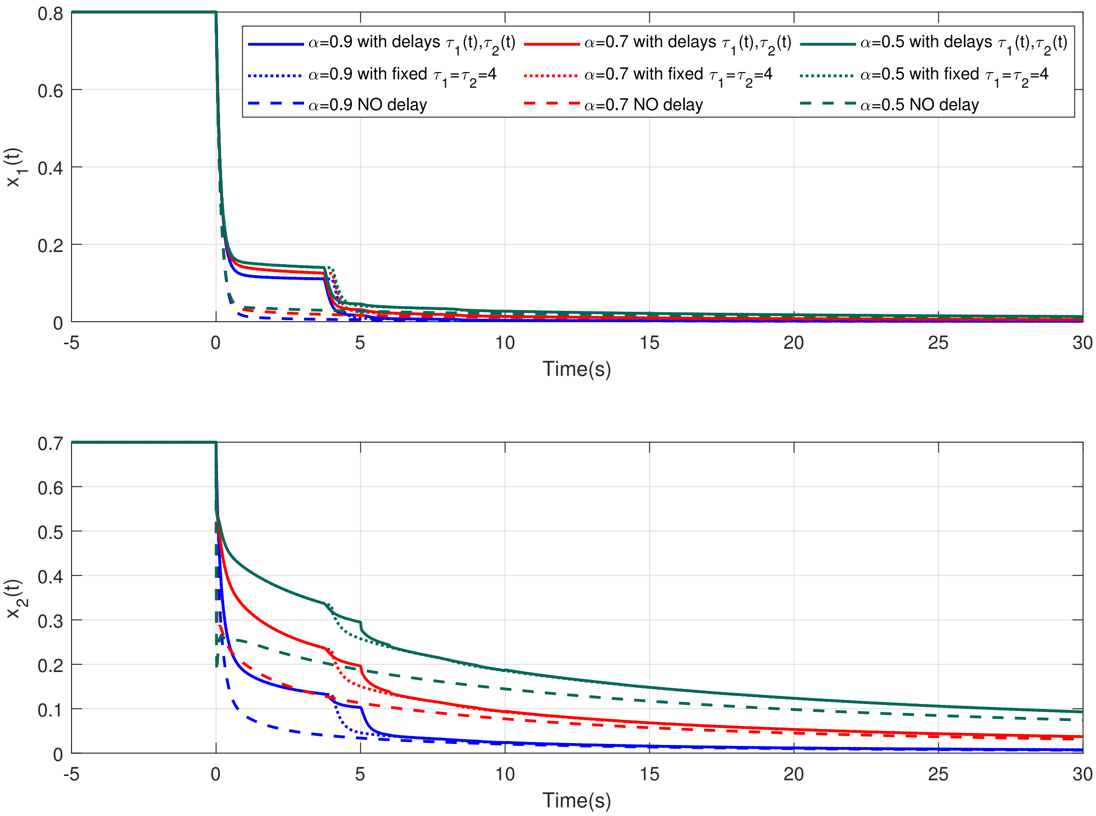

Fig. 3. Evolution of $x_1(t)$ and $x_2(t)$ for different values of α.

$\tau_1(t)), \ldots, x_i(t - \tau_m(t)))$. Let $V_i(x_i) = \frac{1}{2}x_i' x_i$. Then

$$D^{\alpha_i} V_i \leq -2c_i V_i + x_i' f_i(t) + \sum_{j=1}^{n} a_{ij} x_i' x_j$$

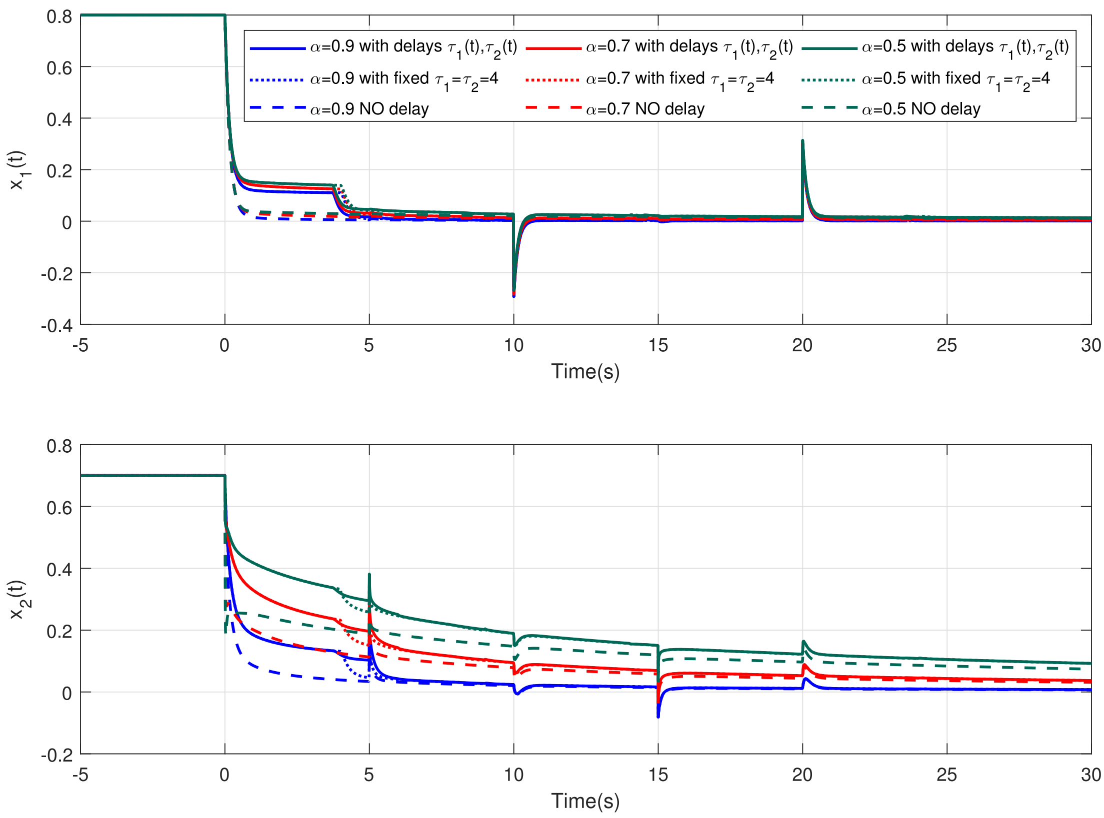

Fig. 4. Evolution of $x_1(t)$ and $x_2(t)$ for different values of α and additive disturbances in the states.

$$
\begin{aligned}
&\leq -2(c_i - a_{ii})V_i + \|x_i\|\|f_i(t)\| + \sum_{j=1, j\neq i}^{n} |a_{ij}|\|x_i\|\|x_j\| \\
&\leq -2(c_i - a_{ii} - l_i)V_i + l_i\|x_i\|(\|x_i(t-\tau_1(t))\| + \ldots + \|x_i(t-\tau_m(t))\|) + \sum_{j=1, j\neq i}^{n} |a_{ij}|\|x_i\|\|x_j\| \\
&\leq -2\left(c_i - a_{ii} - \sum_{j=1, j\neq i}^{n} |a_{ij}| - l_i - ml_i\right)V_i + l_iV_i(t-\tau_1(t)) \\
&+ \ldots + l_iV_i(t-\tau_m(t)) + \sum_{j=1, j\neq i}^{n} |a_{ij}|V_j \\
&=: -\gamma_iV_i + l_iV_i(t-\tau_1(t)) + \ldots + l_iV_i(t-\tau_m(t)) + \sum_{j=1, j\neq i}^{n} |a_{ij}|V_j
\end{aligned}
$$

Let $\Lambda \in \mathbb{R}^{n\times n}$ be the matrix with $\Lambda_{ii} = -\gamma_i$, $\Lambda_{ij} = |a_{ij}|$ for $i, j = 1, \ldots, m$. Clearly Λ is Metzler. Let $A_d = diag(l_1, \ldots, l_n)$, $V = (V_1, \ldots, V_n)$ and $D^{\alpha}V = (D^{\alpha_1}V_1, \ldots, D^{\alpha_n}V_n)$, then the trivial solution of

$$
D^{\alpha}V \preceq \Lambda V + \sum_{j=1}^{m} A_dV(t-\tau_j(t)) \tag{11}
$$

is asymptotically stable if $\Lambda + mA_d$ is Hurwitz, according to Theorem 2. Moreover, if a perturbation ν_i occurs in the $i-$agent's equation, which vanishes after some time, the convergence to zero is preserved. In fact, in the notation of Theorem 2, a term $c_i = x_i'\nu_i$ appears in each component of (11) which vanishes after some time, whereby the convergence is ensured since $C = (c_1, \ldots, c_n)'$ converges to zero. Notice the reduced-order dimension n of (11) in comparison to the original dimension $\sum_{i=1}^{n} n_i$ of system (10).

We indicate two other interesting cases that can be studied with essentially the same procedure as above (of course, one can always combine them). The case of delayed nonlinear communication among agents, e.g. $\sum_{j=1}^{n} f_{ij}(x_j(t-\tau_j(t)))$ in

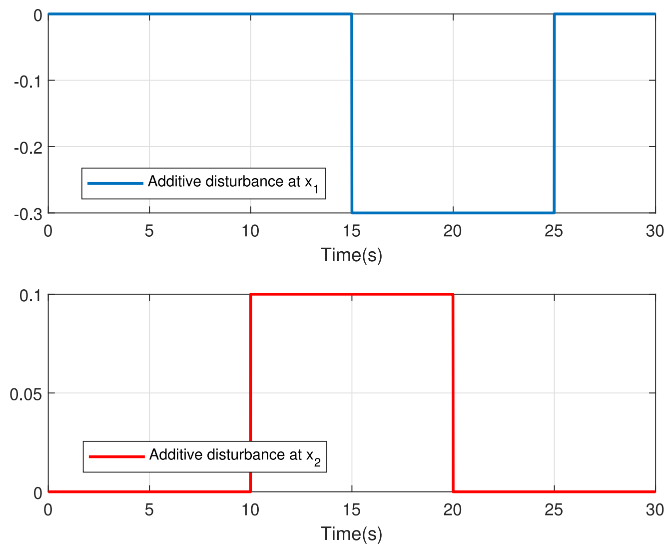

Fig. 5. Additive disturbances acting on the system (12).

(10) where the nonlinearity is Lipschitz, and the case of non-diagonal dynamic, i.e. A_ix_i instead of c_ix_i, where A_i is a Hurwitz matrix so that there exists $P_i > 0$ such that $A_i'P_i + P_iA_i = Q < 0$. In the latter case, it should be used $V_i = x_i'P_ix_i$.

Notice that according to the expression $c_i - a_{ii} - \sum_{j=1,j\neq i}^{n} |a_{ij}| - l_i - ml_i$, the Hurwitz property of $\Lambda + mA_d$ depends on the coefficients c_i, corresponding to the intuition of the dominance of these quadratic terms to avoid instability. If a feedback control term is allowed in each node, i.e. $u_i = k_ix_i$ is added in (10), the stability could be ensured if bounds on the other terms are available.

For numerical visualization, let us consider the following example, which can be seen not only as a multi-order network but also as an integer system with fractional dynamic

$$\begin{aligned} \dot{x}_1(t) &= -c_1x_1(t) + x_1^2(t - \tau_1(t)) + a_{12}x_2(t) \\ D^{\alpha}x_2(t) &= -c_2x_2(t) + x_2^2(t - \tau_2(t)) + a_{21}x_1(t) \end{aligned} \tag{12}$$

where $\alpha \in (0, 1]$ but otherwise unknown. It follows that the quadratic nonlinear functions are Lipschitz around the origin with Lipschitz constant 2 in the ball of radius 1 around the origin $(x_1, x_2) = (0, 0)$. Therefore, in this case $\Lambda = [-2(c_1 - |a_1| - 2), |a_1|; |a_2|, -2(c_2 - |a_2| - 2)]$ and $A_d = [2, 0; 0, 2]$.

The following parameter values and simulation specifications were used: $a_{12} = 1,\ a_{21} = 2,\ c_1 = 7,\ c_2 = 8$. This guarantees that matrix $\Lambda + A_d$ is Hurwitz, with eigenvalues located at -4.58 and -7.41. Time-varying delays are used as specified in Fig. 2. Initial functions for x_1 and x_2 were chosen as $x_{10} = 0.8,\ \forall t \in [-5, 0]$ and $x_{20} = 0.7,\ \forall t \in [-5, 0]$.

Different values of α were used ($\alpha = 0.9$, $\alpha = 0.7$ and $\alpha = 0.5$) in simulations, and results are shown in Fig. 3. For comparison purposes, particular cases of system (12) with no delay ($\tau_1 = \tau_2 = 0$) and with fixed delay ($\tau_1 = \tau_2 = 4$) are shown as well.

For the sake of visibility in the transient stage, plots are shown only until $t = 30$ s. For this time window, it can be seen from Fig. 3 that the states are converging towards zero for all the cases (delayed and non delayed), with a convergence speed higher as α gets closer to 1.

An additional simulation was carried out, where external additive disturbances are applied to the states x_1, x_2. Fig. 4 shows the response of the system in the presence of the disturbances, which are shown in Fig. 5. As can be seen, the system is also robust, since it is able to correct the effect caused by the disturbances.

5. Conclusion

This paper has introduced a method to study the asymptotic behaviour of solutions to nonlinear multi-order fractional differential equations with or without time-varying delays. In particular, we have obtained a simple condition to assure asymptotic stability which amounts to verify a Hurwitz property on a suited matrix together with a Melzer property which can be decided by simple inspection. Using that this condition is order-independent, robust results under uncertainty in the derivation order are obtained. An example in multi-agent multi-order systems is presented, which can be easily generalized to multi-agent systems with delayed communication and applied to consensus or synchronization problems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Proof of Proposition 1 (Sufficiency)**.** Assume first that $\phi \succ 0$ (a function in $\mathcal{C}([-\tau,0],\mathbb{R}^n)$). Since A is Metzler, there exists $r>0$ such that rI_n+A is a nonnegative matrix. Then (2) is equivalent to

$$D^{\alpha_i}x_i(t) = -rx_i(t) + \sum_{j=1}^{n}(rI+A)_{ij}x_j(t) + \sum_{k=1}^{n}\sum_{j=1}^{l}(A_{d_j})_{ik}x_k(t-\tau_j(t)) + \sum_{j=1}^{m}(B)_{ij}u_j(t) \quad (13)$$

$$=: -rx_i(t) + v_i(t), \quad (14)$$

and the solution to (2) can be expressed by $x_i(t) = E_{\alpha_i}(-rt^{\alpha_i})x(0) + \int_0^t \tau^{\alpha_i-1}E_{\alpha_i,\alpha_i}(-r\tau^{\alpha_i})v(t-\tau)d\tau$, where $E_{\alpha_i}(-rt^{\alpha_i}), E_{\alpha_i,\alpha_i}(-rt^{\alpha_i})$ are the Mittag-Leffler functions, which are nonnegative [17]. Since $\phi \succ 0$, $x_i(0)>0$ for any $i=1,\dots,n$ and $v(0)>0$.

Suppose that there exist $\bar{t}, \bar{\delta} > 0$ such that $x(t) \succ 0$ for any $t \in [0,\bar{t})$ and $x(t) \nsucc 0$ for $t \in [\bar{t}, \bar{t}+\delta)$. From continuity, for some i_0 it must hold that $x_{i_0}(\bar{t}) = 0$. Then, $v \succcurlyeq 0$ for any $t \in [0,\bar{t}]$ since $rI+A, A_{d_j}, B$ are nonnegative. Hence, $x_{i_0}(t) \geq E_{\alpha_{i_0}}(-rt^{\alpha_{i_0}})x(0) > 0$ for any $t \in [0,\bar{t})$ and any $i = 1,\dots,n,$ which implies $x(\bar{t}) \succ 0,$ yielding a contradiction with the definition of $\bar{t}$. Therefore, such $\bar{t}$ does not exist and $x(t) \succ 0$, for any $t>0$.

Now consider $\phi \succeq 0$ and define $\phi_n = \phi + \frac{1}{n}\mathbf{1}$ where $n \in \mathbb{N}$ and $\mathbf{1}$ is a vector of ones with dimension n. Let x_n be the corresponding solution of (2) associated to ϕ_n. From the above argument $x_n \succ 0$. For any $n>m$, the function $e = x_m - x_n$ holds Eq. (2) with initial function $(\frac{1}{m} - \frac{1}{n})\mathbf{1} \succ 0$. Therefore, $x_m \succ x_n \succ 0$ and the sequence of continuous functions $(x_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below. Therefore, it converges to some function $\bar{x}$ uniformly on any closed interval [0, T] and it holds that $\bar{x} \succeq 0,$ since otherwise a contradiction is obtained from the fact that $x_n \succ 0$. Noting that $\bar{x}$ is the solution to (2) for ϕ_n as $n \to \infty$, i.e. to a ϕ, due to the continuity of the solutions w.r.t the initial functions, the claim follows.

(Necessity) Suppose that (2) is positive for any $\alpha \in (0, 1]$. In particular, it is positive for $\alpha = 1$. From [18, Theorem II.2], Assumption 1 follows. □

Proof of Proposition 2. Define $e = x(t;\phi_2,u_2) - x(t;\phi_1,u_1)$ and use Proposition 1. □

Proof of Proposition 3.

(i) From Proposition 1, $x \succeq 0$. To prove $x \preceq rc$, note that $e = rc - x$ holds

$$D^{\alpha}e = Ae(t) + \sum_{j=1}^{m}A_{d_j}e(t-\tau_j(t)) - r\left(A + \sum_{j=1}^{m}A_{d_j}\right)c$$

with initial function $\phi_e \equiv 0$. Using Proposition 1 the claim follows.

(ii) Define $e = x(t) - x(t+\delta)$ for arbitrary $\delta > 0$. Noting that e satisfies

$$D^{\alpha}e(t) = Ae(t) + \sum_{j=1}^{m}A_{d_j}e(t-\tau_j(t))$$

with initial condition $\phi_e \succeq 0$ due to Proposition 3(i), the claim follows using Proposition 1.

(iii) From part (ii), x is nonincreasing in each coordinate. From part (i), x is nonnegative. Then $x(t)$ converges to some value $\bar{x} \succeq 0$ as $t\to\infty$. Suppose that $\bar{x} \neq 0$. Since A is Metzler and $\sum_{j=1}^{m}A_{d_j}$ is nonnegative, then $(A+\sum_{j=1}^{m}A_{d_j})$ is Metzler. From the hypotheses, there exits $c \succ 0$ such that $(A+\sum_{j=1}^{m}A_{d_j})c \prec 0$. Using [5, Lemma 2], it follows that $(A+\sum_{j=1}^{m}A_{d_j})$ is a Hurwitz matrix, and therefore, invertible. Then $(A+\sum_{j=1}^{m}A_{d_j})\bar{x} \neq 0$. Since $x(t) \to \bar{x}$ as $t\to\infty$ and $\tau_j(t) < \tau < \infty$, we also have $x(t-\tau_j) \to \bar{x}$ as $t\to\infty$. Therefore, Eq. (2) with $u \equiv 0$ can be written as $D^{\alpha}x(t) = (A+\sum_{j=1}^{m}A_{d_j})\bar{x} + \nu(t)$ where $\nu(t)$ goes to zero as $t\to\infty$. Since $(A+\sum_{j=1}^{m}A_{d_j})\bar{x} \neq 0$, $\bar{x} \succeq 0$ and $(A+\sum_{j=1}^{m}A_{d_j})$ is Metzler, there exist numbers i, T, $\lambda > 0$ such that $D^{\alpha_i}x_i(t) > \lambda > 0$ for all $t>T$. By comparison and writing the latter inequality in its integral form, we have $x_i(t) \geq x_i(0) + \int_0^T (t-s)^{\alpha-1}\Psi(s)ds + \int_T^t (t-s)^{\alpha-1}\lambda ds,$ where Ψ_i is the i–component of the right-hand side of Eq. (13), i.e. a continuous function (since x is continuous) and hence it is bounded in [0, T]. Then the second term converges to zero and the third diverges to $+\infty$, implying $x_i \to \infty$ as $t \to \infty$, which is in contradiction with part (i). Therefore, $\bar{x} = 0$ and we have $\lim_{t\to\infty} x(t) = 0$. □

Proof of Proposition 4. Use Propositions 1, 2 and 3(i). □

Proof of Proposition 5.

(i) Let $e = x(t+\delta) - x(t)$ for arbitrary $\delta > 0$. Then

$$D^\alpha e = Ae(t) + \sum_{j=1}^{m} A_{d_j} e(t - \tau_j(t)).$$

Note that the initial function, $e(s)$ for any $s \in [-\tau, 0]$, is nonnegative since $x(s) = 0$ and $x(s+\delta) \geq 0$ for any $s \in [-\tau, 0]$. The claim follows from Proposition 1.

(ii) Let $e = x - \underline{x}$. Then $D^\alpha e(t) = Ae(t) + \sum_{j=1}^m A_{d_j} e(t-\tau_j(t)) + \sum_{j=1}^m A_{d_j} (\underline{x}(t-\tau_j(t)) - \underline{x}(t-\tau_j))$ and $e \equiv 0$ on $[-\tau, 0]$. Using the part (i) and the fact $\tau_j(t) \leq \tau_j$, it follows that $\underline{x}(t-\tau_j(t)) \geq \underline{x}(t-\tau_j)$. Using Proposition 1, it follows that $x \succeq \underline{x}$. A similar reasoning shows that $x \preceq \bar{x}$. □

Proof of Proposition 6. It is enough to show $0 \preceq V \preceq z$. Let $C \preceq 0$ be the function making $D^\alpha V(t) = w(t) + C(t)$. Let $e = z - V$. Then $D^\alpha e = w(z, \ldots, t) - w(z, \ldots, t) - C$ with initial function $\phi_e \succeq 0$. Suppose first that $\phi_e \succ 0$. Due to the fact that w is monotone increasing, the assertion of a time where some component of e changes sign leads to a contradiction similar to the proof of Proposition 1 since by integration $e(t) = e(0) + I^\alpha[w + C](t)$. For the case of $\phi_e \succeq 0$ a similar argument as in the proof of Proposition 1, shows that $e \succeq 0$. Therefore, $V \preceq z$ and from hypothesis, $0 \preceq V$. □

References

[1] Lenka BK. Fractional comparison method and asymptotic stability of multivariable fractional order systems. Commun Nonlinear Sci Numer Simul 2019;69:398–415.
[2] Gallegos JA, Aguila-Camacho N, Duarte-Mermoud MA. Smooth solutions to mixed-order fractional differential systems with applications to stability analysis. J Integral Equ Appl 2019;31:59–84.
[3] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. Fractional calculus: models and numerical methods. World Scientific; 2016.
[4] Diethelm K, Siegmund S, Tuan HT. Asymptotic behavior of solutions of linear multi-order fractional differential equation systems. Fract Calc Appl Anal 2017;20:1165–95.
[5] Shen J, Lam J. Stability and performance analysis for positive fractional-order systems with time-varying delays. IEEE Trans Autom Control 2016;61:2676–81.
[6] Wang Z, Yang D, Zhang H. Stability analysis on a class of nonlinear fractional-order systems. Nonlinear Dyn 2016;86:1023–33.
[7] Gallegos JA, Duarte-Mermoud MA. A dissipative approach to the stability of multi-order fractional systems. IMA J Math Control Inform 2018;00:1–16.
[8] Lakshmikantham V, Matrosov VM, Sivasundaram S. Vector lyapunov functions and stability analysis of nonlinear systems. Kluwer Academic Publishers; 1991.
[9] Xu D, Wang X, Hong Y, Jiang Z, Xu S. Output feedback stabilization and estimation of the region of attraction for nonlinear systems: a vector control Lyapunov function perspective. IEEE Trans Autom Control 2016;61:4034–40.
[10] Gallegos JA, Duarte-Mermoud MA. Asymptotic analysis of riemann-liouville fractional systems. Electron J Qual Theory Differ Equ 2018;73:1–16.
[11] Tuan HT, Trinh H. A linearized stability theorem for nonlinear delay fractional differential equations. IEEE Trans Autom Control 2018;63:3180–6.
[12] Wang H, Yu Y, Wen G, Zhang S, Yu J. Global stability analysis of fractional-order hopfield neural networks with time delay. Neurocomputing 2015;154:15–23.
[13] Tuan HT, Trinh H. Stability of fractional-order nonlinear systems by Lyapunov direct method. IET Control Theory A 2018;12:2417–22.
[14] Oustaloup A. La commande CRONE: commande robuste d'ordre non entier. Hermes; 1991.
[15] Valerio D, Da Costa JS. Ninteger: a non-integer control toolbox for matlab. Fractional Derivatives and Applications; 2004.
[16] Wang F, Yang Y. Quasi-synchronization for fractional-order delayed dynamical networks with heterogeneous nodes. Appl Math Comput 2018;339:1–14.
[17] Diethelm K. The analysis of fractional differential equations. an application-oriented exposition using differential operators of caputo type. Lecture Notes in Mathematics. Springer-Verlag; 2010.
[18] Ngoc P. Stability of positive differential systems with delay. IEEE Trans Autom Control 2013;58:203–9.