Linear Algebra and its Applications 595 (2020) 182–191

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Inverse M-matrix, a new characterization

Claude Dellacherie[a], Servet Martínez[b], Jaime San Martín[b,*]

[a] *Laboratoire Raphaël Salem, UMR 6085, Université de Rouen, Site du Madrillet, 76801 Saint Étienne du Rouvray Cedex, France*
[b] *CMM-DIM, Universidad de Chile, UMI-CNRS 2807, Casilla 170-3 Correo 3, Santiago, Chile*

A R T I C L E I N F O

Article history:
Received 18 June 2019
Accepted 20 February 2020
Available online 22 February 2020
Submitted by P. Semrl

MSC:
15B35
15B51
60J10

Keywords:
M-matrix
Inverse M-matrix
Potentials
Complete Maximum Principle
Markov chains

A B S T R A C T

In this article we present a new characterization of inverse M-matrices, inverse row diagonally dominant M-matrices and inverse row and column diagonally dominant M-matrices, based on the positivity of certain inner products.

Contents

* Corresponding author.
E-mail addresses: Claude.Dellacherie@univ-rouen.fr (C. Dellacherie), smartine@dim.uchile.cl (S. Martínez), jsanmart@dim.uchile.cl (J. San Martín).

https://doi.org/10.1016/j.laa.2020.02.024
0024-3795/

1. Introduction and main results

In this short note, we give a new characterization of inverses M-matrices, inverses of row diagonally dominant M-matrices and inverses of row and column diagonally dominant M-matrices. This is done in terms of a certain inner product to be nonnegative (see (1.5), (1.6) and (1.4), respectively). These characterizations are stable under limits, that is, if an operator $\mathscr{U}$ can be approximated by a sequence of matrices $(U_k)_k$ in such a way that also the corresponding inner products converge (for example in L^2) then the limit operator will satisfy the same type of inequality. This is critical to show, for example, that $\mathscr{U}$ is the 0-potential of a Markov resolvent or a Markov semigroup because as we will see this inequality implies a strong principle called Complete Maximum Principle in Potential Theory (see for example [22], chapter 4). In the matrix case, this corresponds to the inverse of a row diagonally dominant M-matrix (see Theorem 2.1 below).

We continue with the formal definition of a potential matrix.

Definition 1.1. A nonnegative, nonsingular matrix U is called a *potential* if its inverse $M = U^{-1}$ is a row diagonally dominant M-matrix, that is,

$$\forall i \neq j \quad M_{ij} \leq 0; \tag{1.1}$$

$$\forall i \quad M_{ii} > 0; \tag{1.2}$$

$$\forall i \quad |M_{ii}| \geq \sum_{j \neq i} |M_{ij}|. \tag{1.3}$$

Also, U is called a *double potential* if U and U^t are potentials.

We point out that conditions (1.1) and (1.3) imply condition (1.2). Indeed, notice that these two conditions imply that $|M_{ii}| \geq 0$, but if $M_{ii} = 0$, then for all j we would have $M_{ij} = 0$, which is not possible because we assume that M is nonsingular. Finally, M_{ii} cannot be negative, otherwise $1 = \sum_j M_{ij} U_{ji} \leq 0$, which is not possible. We also notice that if U is a symmetric potential, then clearly it is a double potential.

In what follows for a vector x, we denote by x^+ its positive part, which is given by $(x^+)_i = (x_i)^+$. Similarly, x^- denotes the negative part of x, so $x = x^+ + x^-$. Also, we denote by $\langle \, , \, \rangle$, the standard Euclidean inner product, and $\mathbb{1}$ is the vector whose entries are all ones. We are in a position to state our main result.

Theorem 1.1. *Assume that U is a nonsingular nonnegative matrix of size n.*

(i) If U satisfies the following inequality: for all $x \in \mathbb{R}^n$

$$\langle (Ux - \mathbb{1})^+, x\rangle \geq 0 \tag{1.4}$$

then U is a potential.

(ii) Reciprocally, if U is a double potential then it satisfies (1.4).

In particular, if U is a nonnegative nonsingular symmetric matrix, then U is a potential iff it satisfies (1.4).

Example. Here is an example of a potential matrix U, for which (1.4) does not hold. Consider

$$U = \begin{pmatrix} 2 & 100 \\ 1 & 100 \end{pmatrix},$$

whose inverse is $M = U^{-1}$

$$M = \begin{pmatrix} 1 & -1 \\ -1/100 & 1/50 \end{pmatrix} = \mathbb{I} - \begin{pmatrix} 0 & 1 \\ 1/100 & 49/50 \end{pmatrix}$$

a row diagonally dominant M-matrix. Nevertheless,

$$\langle (Uv - \mathbb{1})^+, v\rangle = (2x + 100y - 1)^+ x + (x + 100y - 1)^+ y = -5.3,$$

for $x = -0.5, y = 0.2$. Notice that U^t is not a potential because its inverse, although it is an M-matrix, it fails to be row diagonally dominant.

To generalize Theorem 1.1 to include all inverse M-matrices we consider the following two diagonal matrices D, E. Here, we assume that U is a nonnegative nonsingular matrix, or more general, it is enough to assume that U is nonnegative and it has at least one positive element per row and column. Let us define D as the diagonal matrix given by, for all i

$$D_{ii} = \left(\sum_j U_{ij}\right)^{-1},$$

as the reciprocal of the i-th row sum. Similarly, consider E the diagonal matrix given by, for all i

$$E_{ii} = \left(\sum_j U_{ji}\right)^{-1},$$

the reciprocal of the i-th column sum. We point out that matrices D, E are computed directly from U.

Theorem 1.2. *Assume that U is a nonsingular nonnegative matrix of size n.*

(i) U is an inverse M-matrix iff DUE is a double potential, which is further equivalent to the following inequality: for all $x \in \mathbb{R}^n$

$$\langle (Ux - D^{-1}\mathbb{1})^+, DE^{-1}x\rangle \geq 0. \tag{1.5}$$

(ii) U is a potential iff UE is a double potential, which is equivalent to the inequality: for all $x \in \mathbb{R}^n$

$$\langle (Ux - \mathbb{1})^+, E^{-1}x\rangle \geq 0. \tag{1.6}$$

Proof. (i) Assume that $M = U^{-1}$ is an M-matrix. Then, $W = DUE$ is a double potential. Indeed, it is clear that $N = W^{-1}$ is an M-matrix. Now, consider $\mu = E^{-1}\mathbb{1}$, then

$$W\mu = DU\mathbb{1} = \mathbb{1},$$

by the definition of D. This means that $N\mathbb{1} = W^{-1}\mathbb{1} = \mu \geq 0$, and so N is a row diagonally dominant matrix. Similarly, if we take $\nu = D^{-1}\mathbb{1}$, we have

$$\nu^t W = \mathbb{1}^t UE = \mathbb{1}^t.$$

This proves that $\mathbb{1}^t N = \nu^t \geq 0$ and therefore, we conclude that N is a column diagonally dominant matrix. In summary, W is a double potential. Conversely, if W is a double potential, in particular it is an inverse M-matrix, which implies that U is an inverse M-matrix.

Let us prove that U being an inverse M-matrix is equivalent to (1.5). We first assume W is a double potential, then from Theorem 1.1, we have for all $x \in \mathbb{R}^n$

$$0 \leq \langle (DUEx - \mathbb{1})^+, x\rangle = \langle (DUy - \mathbb{1})^+, E^{-1}y\rangle = \langle (Uy - D^{-1}\mathbb{1})^+, DE^{-1}y\rangle,$$

which is condition (1.5). Here, we have used the straightforward to prove property that for a diagonal matrix, with positive diagonal elements, it holds $(Dz)^+ = Dz^+$.

Conversely, assume that U satisfies (1.5) then, we obtain that W satisfies (1.4) in Theorem (1.1) and therefore it is an inverse M-matrix. So, U is an inverse M-matrix, proving the desired equivalence.

(ii) This time we take $W = UE$. Since U is a potential, there exists a nonnegative vector μ, such that $U\mu = \mathbb{1}$, then $UEE^{-1}\mu = \mathbb{1}$ and W is a potential. On the other hand, $\mathbb{1}^t UE = \mathbb{1}^t$, and therefore W is a double potential. The rest follows similarly as in the proof of (i). □

The next theorem is a complement to Theorem 1.1. One way to approach this result is by making the change of variables $y = Ux$ in (1.4).

Theorem 1.3. *Assume M is a matrix of size n. Then*

(i) If M satisfies the inequality, for all $x \in \mathbb{R}^n$

$$\langle (x - \mathbb{1})^+, Mx \rangle \geq 0, \tag{1.7}$$

then M satisfies the following structural properties

$$\forall i \neq j \quad M_{ij} \leq 0; \tag{1.8}$$

$$\forall i \qquad M_{ii} \geq 0; \tag{1.9}$$

$$\forall i \qquad M_{ii} \geq \sum_{j \neq i} -M_{ij}. \tag{1.10}$$

That is, M is a Z-matrix, with nonnegative diagonal elements and it is a row diagonally dominant matrix.

(ii) If M is a Z-matrix, with nonnegative diagonal elements and it is a row and column diagonally dominant matrix, then it satisfies (1.7).

There is a vast literature on M-matrices and inverse M-matrices, the interested reader may consult the books by Horn and Johnson [13] and [14], among others. In particular for the inverse M-problem we refer to the pioneer work of [8], [11] and [28]. Some results in the topic can be seen in [1], [2], [5], [6], [9], [10], [12], [15], [16], [17], [18], and [27]. The special relation of this problem to ultrametric matrices in [4], [19], [20], [21], [23], [24], [25], [26]. Finally, for the relation between M-matrices and inverse M-matrices with Potential Theory see our book [7].

2. Proof of Theorem 1.1 and Theorem 1.3

The proof of Theorem 1.1 is based on what is called the *Complete Maximum Principle* (CMP), which we recall for the sake of completeness.

Definition 2.1. A nonnegative matrix U of size n, is said to satisfy the CMP if for all $x \in \mathbb{R}^n$ it holds:

$$\sup_i (Ux)_i \leq \sup_{i : x_i \geq 0} (Ux)_i,$$

where by convention $\sup\limits_{\varnothing} = 0$.

The CMP says that if x has at least one nonnegative coordinate then the maximum value among the coordinates of $y = Ux$ is attained at some coordinate i such that x_i is nonnegative. An alternative equivalent definition, which is the standard in Potential Theory reads as follows, U is a potential if for all x it holds: whenever $(Ux)_i \le 1$ on the coordinates where $x_i \ge 0$, then $Ux \le \mathbb{1}$. The importance of this principle is given by the next result.

Theorem 2.1. *Assume U is a nonnegative matrix.*

(i) If U is nonsingular, then U satisfies the CMP iff U is a potential, that is, $M = U^{-1}$ is a row diagonally dominant M-matrix.
(ii) U satisfies the CMP then for all $a \ge 0$ the matrix $U(a) = a\mathbb{I} + U$ satisfies the CMP and for all $a > 0$ the matrix $U(a)$ is nonsingular.

The proof of (i) in this theorem goes back to Choquet and Deny [3] (Theorem 6, page 89). For a generalization of this result and a more matrix flavor of it, see Theorem 2.9 in [7].

Assume that U is a nonnegative matrix and satisfies the CMP, if the diagonal of U is strictly positive, which happens when U is nonsingular, then there exists an equilibrium potential, that is, a nonnegative vector μ solution of the problem

$$U\mu = \mathbb{1},$$

see for example (v) Lemma 2.7 in [7].

This vector μ plays an important role and it is related to the fact that U^{-1} is row diagonally dominant, when U is nonsingular. In fact, in this case $\mu = U^{-1}\mathbb{1} \ge 0$.

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (i) We shall prove that U satisfies the CMP. For that purpose consider $x \in \mathbb{R}^n$, which has at least one nonnegative coordinate. If $(Ux)_i \le 1$, for those coordinates i such that $x_i \ge 0$, then from condition (1.4) we conclude that

$$0 \le \langle (Ux - \mathbb{1})^+, x\rangle = \langle (Ux - \mathbb{1})^+, x^-\rangle,$$

which implies that $((Ux - \mathbb{1})^+)_i = 0$ if $x_i < 0$, proving that U satisfies the CMP. Hence, from Theorem 2.1 we have that $M = U^{-1}$ is a row diagonally dominant M-matrix.

(ii) Assume that U, U^t are potential matrices of size n. Then $M = U^{-1}$ is a column and row diagonally M-matrix, which is equivalent to have $M = k(\mathbb{I} - P)$, for some constant $k > 0$ and a double substochastic matrix P, that is, P is a nonnegative matrix and for all i it holds $\sum_j P_{ij} \le 1, \sum_j P_{ji} \le 1$.

We define $\mu = M\mathbb{1} \geq 0$ and $\xi = U(x-\mu) = Ux - \mathbb{1}$ to get

$$\langle (Ux-\mathbb{1})^+, x\rangle = \langle (Ux - U\mu)^+, x\rangle = \langle \xi^+, M\xi + \mu\rangle = \langle \xi^+, k\xi + \mu\rangle - k\langle \xi^+, P\xi\rangle$$

$$= k\Big(\langle \xi^+, \xi^+\rangle - \langle \xi^+, P\xi\rangle\Big) + \langle \xi^+, \mu\rangle.$$

Since $P \geq 0$, we get

$$\langle \xi^+, P\xi\rangle \leq \langle \xi^+, P\xi^+\rangle = \langle \xi^+, \frac{1}{2}(P+P^t)\xi^+\rangle \leq \langle \xi^+, \xi^+\rangle.$$

The last inequality holds because the nonnegative symmetric matrix $\frac{1}{2}(P+P^t)$ is sub-stochastic and therefore its spectral radius is smaller than 1, which implies that for all $z \in \mathbb{R}^n$ it holds $\langle z, \frac{1}{2}(P+P^t)z\rangle \leq \langle z, z\rangle$. We get the inequality

$$\langle (Ux-\mathbb{1})^+, x\rangle \geq \langle (Ux-\mathbb{1})^+, \mu\rangle \geq 0,$$

which shows the result. □

Proof of Theorem 1.3. The reader may consult [14] Theorem 2.5.3, for some properties about M-matrix that are needed in this proof.

(i). Assume that M is a matrix, of size n, that satisfies (1.7). In order to prove that condition (1.8) holds fix $i \in \{1, \cdots, n\}$ and consider a vector x such that $x_i > 1$ and $x_k = 0, k \neq i$. Then (1.7) implies

$$0 \leq \langle (x-\mathbb{1})^+, Mx\rangle = (x_i - 1)M_{ii}x_i,$$

from where we deduce $M_{ii} \geq 0$, proving that (1.8) holds.

To prove (1.9) consider $i \neq j$ fixed and take a vector x such that $x_i > 1, x_j < 0$ and $x_k = 0, k \neq i, j$. Then

$$0 \leq \langle (x-\mathbb{1})^+, Mx\rangle = (x_i - 1)(M_{ii}x_i + M_{ij}x_j)$$

By taking x_j a large negative number, we conclude that this inequality can hold only if $M_{ij} \leq 0$, proving (1.9).

Now, we prove condition (1.10). For that purpose we consider i fixed and we take $x \in \mathbb{R}^n$ such that $x_i > 1$ and $x_j = 1$ for all $j \neq i$. Then

$$0 \leq \langle (x-\mathbb{1})^+, Mx\rangle = (x_i - 1)\Big(x_i M_{ii} + \sum_{j\neq i} M_{ij}\Big).$$

This implies that $x_i M_{ii} + \sum\limits_{j\neq i} M_{ij} \geq 0$ holds for all $x_i \geq 1$ and therefore $M_{ii} + \sum\limits_{j\neq i} M_{ij} \geq 0$, proving that M satisfies (1.10).

Part (ii) follows from Theorem 1.1 by considering a perturbation of M. For $\theta > 0$ take $M(\theta) = \theta\mathbb{I} + M$. By hypothesis $M(\theta)$ is a strictly row and column dominant Z-matrix, proving that $M(\theta)$ is an M-matrix. Its inverse, $U(\theta) = (M(\theta))^{-1}$, is a double potential and therefore it satisfies inequality (1.4). Take $y \in \mathbb{R}^n$ and consider $x = x(\theta) = M(\theta)y$ to obtain

$$0 \leq \langle (U(\theta)x - \mathbb{1})^+, x\rangle = \langle (y - \mathbb{1})^+, My\rangle + \theta\langle (y - \mathbb{1})^+, y\rangle.$$

The result follows by taking $\theta \downarrow 0$. □

3. Some complements

In Potential Theory, particularly when dealing with infinite dimensional spaces, most of the time a potential U is singular. According to Theorem 2.1, in case U is nonsingular, our definition of a potential matrix (see Definition 1.1) and the CMP are equivalent. The latter makes sense even for singular matrices and this should be the right definition for a matrix to be a potential. Notice that in (ii) Theorem 2.1 says that a potential in this sense, is the limit of nonsingular potential matrices, which also holds in infinite dimensional spaces.

Theorem 1.1 can be extended to include singular potential matrices as follows.

Theorem 3.1. *Assume U is a nonnegative matrix.*

(i) If U satisfies (1.4) then U satisfies the CMP.
(ii) Reciprocally, if U, U^t satisfy the CMP, then U (and U^t) satisfies (1.4).

That is, for a symmetric matrix U, condition (1.4) and CMP are equivalent.

Proof. The proof of (i) is identical to the one of Theorem 1.1 (i).

(ii). Consider as in Theorem 2.1 a perturbation of U, given by $U(a) = a\mathbb{I} + U$, for $a > 0$. Since $U(a), U^t(a)$ are nonsingular potential matrices we can use Theorem 1.1 to conclude that for all $a > 0$ and $x \in \mathbb{R}^n$ one has

$$0 \leq \langle (U(a)x - \mathbb{1})^+, x\rangle.$$

Now, it is enough to take the limit as $a \downarrow 0$, to conclude the result. □

The question now is: is there a principle like CMP, that characterizes inverses M-matrices? The answer is yes, and it is given by the following principle taken from Potential Theory.

Definition 3.1. A nonnegative matrix U is said to satisfy *the domination principle* (DP) if for any nonnegative vectors x, y it holds $(Ux)_i \leq (Uy)_i$ for those coordinates i such that $x_i > 0$, then $Ux \leq Uy$.

Theorem 2.15 in [7] is exactly this characterization, which we copy here for the sake of completeness.

Theorem 3.2. *Assume that U is a nonnegative nonsingular matrix. Then, U^{-1} is an M-matrix iff U satisfies DP.*

It is interesting to know a relation between CMP and DP, which is given by Lemma 2.13 in [7].

Proposition 3.3. *Assume that U is a nonnegative matrix with positive diagonal elements. Then, the following are equivalent*

(i) *U satisfies the CMP*
(ii) *U satisfies the DP and there exists a nonnegative vector μ solution to $U\mu = \mathbb{1}$.*

Finally let us recall a simple algorithm to check when a nonnegative matrix that satisfies the CMP or DP is nonsingular (see Corollary 2.46 and Corollary 2.47 in [7]).

Proposition 3.4. *Assume that U is a nonnegative matrix, that satisfies either CMP or DP, then the following are equivalent*

(i) *U is nonsingular*
(ii) *not two columns of U are proportional.*

There is a lack of symmetry in this result from columns and rows, because CMP is not stable under transposition. On the other hand DP is stable under transposition, so in this case U is nonsingular iff no two rows are proportional.

Declaration of competing interest

There is no competing interest.

Acknowledgement

Authors S.M. and J.SM. where partially founded by CONICYT, project BASAL AFB170001. The authors thank an anonymous referee for comments and suggestions.

References

[1] J. Brandts, A. Cihangir, Geometric aspects of the symmetric inverse M-matrix problem, Linear Algebra Appl. 506 (2016) 33–81.
[2] A. Carmona, A.M. Encinas, M. Mitjana, On the M-matrix inverse problem for singular and symmetric Jacobi matrices, Linear Algebra Appl. 436 (5) (2012) 1090–1098.
[3] G. Choquet, J. Deny, Modèles finis en théorie du potentiel, J. Anal. Math. 5 (1956) 77–135.

[4] C. Dellacherie, S. Martínez, J. San Martín, Ultrametric matrices and induced Markov chains, Adv. Appl. Math. 17 (1996) 169–183.
[5] C. Dellacherie, S. Martínez, J. San Martín, Hadamard functions of inverse M-matrices, SIAM J. Matrix Anal. Appl. 31 (2) (2009) 289–315.
[6] C. Dellacherie, S. Martínez, J. San Martín, Hadamard functions that preserve inverse M-matrices, SIAM J. Matrix Anal. Appl. 33 (2) (2012) 501–522.
[7] C. Dellacherie, S. Martínez, J. San Martín, Inverse M-Matrices and Ultrametric Matrices, Lecture Notes in Mathematics, vol. 2118, Springer, 2014.
[8] M. Fiedler, Relations between the diagonal elements of an M-matrix and the inverse matrix, Mat.-Fyz. Casopis Sloven. Akad. Vied. 12 (1962) 123–128 (in Russian).
[9] M. Fiedler, Some characterizations of symmetric inverse M-matrices, in: Proceedings of the Sixth Conference of the International Linear Algebra Society, Chemnitz, 1996, Linear Algebra Appl. 275/276 (1998) 179–187.
[10] M. Fiedler, Special ultrametric matrices and graphs, SIAM J. Matrix Anal. Appl. 22 (2000) 106–113.
[11] M. Fiedler, H. Schneider, Analytic functions of M-matrices and generalizations, Linear Multilinear Algebra 13 (1983) 185–201.
[12] L. Hogben, The symmetric M-matrix and symmetric inverse M-matrix completion problems, Linear Algebra Appl. 353 (2002) 159–168.
[13] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[14] R. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[15] C.R. Johnson, R. Smith, The completion problem for M-matrices and inverse M-matrices (English summary), in: Proceedings of the Fourth Conference of the International Linear Algebra Society, Rotterdam, 1994, Linear Algebra Appl. 241/243 (1996) 655–667.
[16] C.R. Johnson, D.D. Olesky, Rectangular submatrices of inverse M-matrices and the decomposition of a positive matrix as a sum, Linear Algebra Appl. 409 (2005) 87–99.
[17] I. Koltracht, M. Neumann, On the inverse M-matrix problem for real symmetric positive-definite Toeplitz matrices, SIAM J. Matrix Anal. Appl. 12 (2) (1991) 310–320.
[18] M. Lewin, M. Neumann, On the inverse M-matrix problem for (0, 1)-matrices, Linear Algebra Appl. 30 (1980) 41–50.
[19] S. Martínez, G. Michon, J. San Martín, Inverses of ultrametric matrices are of Stieltjes types, SIAM J. Matrix Anal. Appl. 15 (1994) 98–106.
[20] J.J. McDonald, M. Neumann, H. Schneider, M.J. Tsatsomeros, Inverse M-matrix inequalities and generalized ultrametric matrices, in: Proceedings of the Workshop "Nonnegative Matrices, Applications and Generalizations" and the Eighth Haifa Matrix Theory Conference, Haifa, 1993, Linear Algebra Appl. 220 (1995) 321–341.
[21] J.J. McDonald, M. Neumann, H. Schneider, M.J. Tsatsomeros, Inverse tridiagonal Z-matrices, Linear Multilinear Algebra 45 (1998) 75–97.
[22] M. Marcus, J. Rosen, Markov Processes, Gaussian Processes and Local Times, Cambridge University Press, 2006.
[23] R. Nabben, On Green's matrices of trees, SIAM J. Matrix Anal. Appl. 22 (4) (2001) 1014–1026.
[24] R. Nabben, R.S. Varga, A linear algebra proof that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix, SIAM J. Matrix Anal. Appl. 15 (1994) 107–113.
[25] R. Nabben, R.S. Varga, Generalized ultrametric matrices – a class of inverse M-matrices, Linear Algebra Appl. 220 (1995) 365–390.
[26] R. Nabben, R.S. Varga, On classes of inverse Z-matrices, in: Special Issue Honoring Miroslav Fiedler and Vlastimil Pták, Linear Algebra Appl. 223/224 (1998) 521–552.
[27] M. Neumann, N. Sze, On the inverse mean first passage matrix problem and the inverse M-matrix problem, Linear Algebra Appl. 434 (7) (2011) 1620–1630.
[28] R.A. Willoughby, The inverse M-matrix problem, Linear Algebra Appl. 18 (1) (1977) 75–94.