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Abstract

We introduce the property of pre-expansivity for cellular automata (CA): it is the property of being expansive on asymptotic pairs of configurations (i.e. configurations that differ in only finitely many positions). Pre-expansivity therefore lies between expansivity and pre-injectivity, two important notions of CA theory.

We show that there exist one-dimensional positively pre-expansive CAs which are not positively expansive and they can be chosen reversible (while positive expansivity is impossible for reversible CAs). We show however that no bi-dimensional CA which is linear over an Abelian group can be pre-expansive. We also consider the finer notion of k-expansivity (expansivity over pairs of configurations with exactly k differences) and show examples of linear CA in dimension 2 and on the free group that are k-expansive depending on the value of k, whereas no (positively) expansive CA exists in this setting.

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1. Introduction

The model of cellular automata is at the crossroads of several domains and is often the source of surprisingly complex objects in several senses (computationally, dynamically, etc).

From the dynamical systems and symbolic dynamics points of view, the theory of cellular automata is very rich [1, 2, 3, 4] and tells us, on the one hand, that

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CA are natural examples of chaotic systems that can perfectly fit the standard notions developed in a general context, and, on the other hand, that they have special properties allowing and justifying the development of a refined and dedicated theory. For instance, the structure of the space of configurations allows to define the notion of an asymptotic pair of configurations: two configurations that differ only on finitely many positions of the lattice. The Garden of Eden theorem, which has a long history [1, 5, 6, 7, 4] and is emblematic of this CA specific theoretical development, then says that surjectivity is equivalent to pre-injectivity (injectivity on asymptotic pairs) if and only if the lattice is given by an amenable group.

Two important lines of questioning have been particularly developed and provide some of the major open problems of the field [8]:

- surjective CA and their dynamics;
- how does CA theory changes when changing the lattice.

In particular, the classical notion of (positive) expansivity has been applied to CA giving both a rich theory in the one-dimensional case [9, 2, 10] and a general inexistence result in essentially any other setting [11, 12]. Even in the one-dimensional case where positive expansivity is equivalent to being conjugated to a one-sided subshift of finite type [13], it is interesting to note that outside the linear and bi-permutative examples, few construction techniques are known to produce positively expansive CA [14]. On the other hand, it is still unknown whether positive expansivity is a decidable property, although it is indeed decidable for some algebraic cellular automata [15, 16].

In this paper, we introduce a new dynamical property called *pre-expansivity* that both generalizes expansivity and refines pre-injectivity: it is the property of being expansive on asymptotic pairs. Our motivation is to better understand surjective CA and expansive-like dynamics, in particular in the higher-dimensional case or in lattices where the classical notion of (positive)expansivity cannot be satisfied by any CA [11, 12]. Pre-expansivity (resp. positive pre-expansivity) is weaker than expansivity (resp. positive expansivity). However, positive pre-expansivity is interesting in that:

1. a reversible CA can be positively pre-expansive (see section 4), while no one can be positively expansive [17];
2. positive pre-expansivity implies sensitivity in all directions (see Proposition 4), while some expansive CA (like the shift map) have equicontinuous directions.

This shows that the notion is useful in the classical setting of one-dimensional cellular automata.

For other settings, the situation is left open: on one hand, we show an impossibility result for Abelian CA in dimension $d \geq 2$ (see Theorem 2), but on the other hand we give several examples of k-expansive CA in this setting and on the free group, where k-expansivity means expansivity over pairs of configurations with exactly k differences.

The paper is organized as follows. In Section 2 we give the main definitions and results we need to work on cellular automata on groups. In Section 3, we introduce pre-expansivity and k-expansivity, and we give some preliminary results which do not depend on the group defining the space. We also consider the particular case of linear cellular automata. In Section 4, we restrict to the group $\mathbb{Z}$ and give examples of cellular automata which are pre-expansive but not positively expansive. In Section 5, we consider the free group and show that k-expansivity is possible for infinitely many values of k although positive expansivity is impossible. Finally, in Section 6, we restrict to the group $\mathbb{Z}^2$ and we study some k-expansive examples for particular values of k but also show that there is no pre-expansive cellular automaton which is linear for a structure of Abelian group on states.

2. Definitions, notations and classical results

We will work on cellular automata defined over a *finitely generated group* $\mathbb{G}$. We will consider Abelian and non-Abelian groups, but since most of our examples are given for Abelian groups, we will prefer the additive notation for $\mathbb{G}$.

Fixing a generator set G, that is closed under inversion, a *norm* can be defined in $\mathbb{G}$: given $z \in \mathbb{G}$, $||z||$ is the length of the shortest sequence $g_1 g_2 ... g_n$ of elements in G such that $z = g_1 + g_2 + ... + g_n$. This norm induces a metric in $\mathbb{G}$ naturally, and given a non-negative integer r we can define also the ball of radius r and center z as the set $B_r(z) = \{x \in \mathbb{G} \mid ||-x+z|| \leq r\}$. Given a point $z \in \mathbb{G}$ and two sets X, $Y \subseteq \mathbb{G}$, we accept the following notation.

$$z + S = \{z + x \mid x \in S\}, \quad \text{and} \quad X + Y = \{x + y \mid x \in X, y \in Y\}$$

Cellular automata are functions defined on the *symbolic space* $Q^{\mathbb{G}} = \{c : \mathbb{G} \to Q \mid c \text{ is a function}\}$. An element c, called *configuration*, assigns a symbol of Q to each element of the group $\mathbb{G}$. We will use both $c(z)$ and c_z to denote the value of c at the cell z. A $\mathbb{G}$-action is defined on $Q^{\mathbb{G}}$: given $z \in \mathbb{G}$, the *shift* function: $\sigma_z : Q^{\mathbb{G}} \to Q^{\mathbb{G}}$ is defined by $\sigma_z(c)(x) = c(z + x)$ for every $x \in \mathbb{G}$. The *Cantor distance* in $Q^{\mathbb{G}}$ is defined for any two configurations c, d as follows.

$$\Delta(c, d) = \begin{cases} 2^{-\min\{||z|| : c(z) \neq d(z)\}} & \text{if } c \neq d \\ 0 & \text{if } c = d \end{cases}$$

Definition 1. *Two configurations c, d are* asymptotic*, denoted $c \overset{\infty}{=} d$, if they differ only in finitely many positions: $\{z \in \mathbb{G} : c(z) \neq d(z)\}$ is finite.*

A cellular automaton (CA) is an endomorphism of $Q^{\mathbb{G}}$, compatible with the shift $\mathbb{G}$-action and continuous for the Cantor metrics. From Curtis-Hedlund theorem [1, 4], every cellular automaton F is characterized by a *local function* $f : Q^V \to Q$, where $V \subset \mathbb{G}$ is finite and called *neighborhood of F*, as follows.

$$\forall c \in Q^{\mathbb{G}}, \forall z \in \mathbb{G}, F(c)(z) = f(\sigma_z(c)|_V)$$

Every function defined in this way is a cellular automaton.

Basic properties of F as surjectivity and injectivity have been considered. The weaker notion of *pre-injectivity* says that, for every pair of different asymptotic configurations c and d, their image by F are different:

$$c \overset{\infty}{=} d \text{ and } c \neq d \Rightarrow F(c) \neq F(d).$$

The so-called Garden-of-Eden theorem establishes that surjectivity is equivalent to pre-injectivity, which in particular implies that injective CAs are also bijective (equivalently reversible by Curtis-Hedlund Theorem, i.e. having an inverse which is also a CA). It was first proved in particular cases [1, 5, 6] and later it was shown that it holds exactly when the group $\mathbb{G}$ is amenable, *i.e.* when it admits a finitely additive measure which is invariant under its action [4].

The pair $(Q^{\mathbb{G}}, F)$ is a dynamical system and can be studied from the point of view of topological dynamics. The present work propose a new particular kind of sensitivity to initial conditions. Weaker and stronger notions in this area are the following.

A CA F is *sensitive* if there exists a number $\delta > 0$, called *sensitivity constant*, such that for every c and every ϵ there exists an instant $t \in \mathbb{T}$ and a configuration d such that $\Delta(F^t(c), F^t(d)) \geq \delta$.

A stronger notion is expansivity, which can be of two kinds, and depends on whether the CA is reversible or not. Given $\mathbb{T}$ be equal to either $\mathbb{N}$ or $\mathbb{Z}$, a CA F is called $\mathbb{T}$-*expansive* if there exists a number $\delta > 0$, called *expansivity constant*, such that for every $c \neq d$ there exists an instant $t \in \mathbb{T}$ such that $\Delta(F^t(c), F^t(d)) \geq \delta$. In this work we will frequently omit the prefix $\mathbb{T}$, assuming by default that $\mathbb{T} = \mathbb{N}$ (positive expansivity).

In $\mathbb{G} = \mathbb{Z}$, given a one-dimensional CA with local rule $f : Q^{[-l,r]} \to Q$ with $l, r > 0$, we say that it is *LR-permutive* when for any $q_{-l}, \ldots, q_r \in Q$ the two following maps are bijective.

$$a \mapsto f(a, q_{-l+1}, \ldots, q_r)$$
$$a \mapsto f(q_{-l}, \ldots, q_{r-1}, a)$$

LR-permutive are always $\mathbb{N}$-expansive.

Definition 2. *Let $(Q, \oplus)$ be a finite group and denote by $\overline{\oplus}$ the component-wise extension of $\oplus$ to $Q^{\mathbb{G}}$ and by $\overline{0}$ the configuration identically equal to 0. A CA F over $Q^{\mathbb{G}}$ is linear if*

$$\forall c, d \in Q^{\mathbb{G}} : F(c \,\overline{\oplus}\, d) = F(c) \,\overline{\oplus}\, F(d).$$

The following lemma shows that linear CA over Abelian groups can be decomposed according to the structure of the group. It is a folklore knowledge that appears often in the particular case of cyclic groups [18, 19], and also in the more general Abelian case [20].

Recall that the product $F \times G$ of two CA F and G is the CA defined on the product alphabet and applying F and G on each component independently.

Lemma 1. *Let $Q = Q_p \times Q'$ be an Abelian group (law $+$ and neutral element $(0,0)$) where Q_p is a p-group (the order of every element is a power of p) for some prime p and the order of Q' is relatively prime with p.*

Then, any linear CA F over Q is isomorphic to $F_p \times F'$ where F_p is a linear CA over Q_p and F' is a linear CA over Q'.

PROOF. By linearity of F, if c satisfies that $n \cdot c = \overbrace{c + \cdots + c}^{n} = \overline{(0,0)}$, then $F(c)$ must satisfy the same: $n \cdot F(c) = \overline{(0,0)}$. We deduce that the subset of states $Q_1 = Q_p \times \{0_{Q'}\}$ induces a subautomaton F_1 of F because any configuration $c \in Q_1^{\mathbb{G}}$ is such that $p^k \cdot c = \overline{(0,0)}$ for some k and no configuration in $\left(Q \setminus Q_1\right)^{\mathbb{G}}$ has this property. Moreover if $n \cdot c = \overline{(0,0)}$ for n relatively prime with p, it implies that $c \in Q_2^{\mathbb{G}} = \left(\{0_{Q_p}\} \times Q'\right)^{\mathbb{G}}$ (because the order of an element must divide the order of the group it belongs to). Therefore Q_2 induces a subautomaton F_2 of F.

Now, any $c \in Q^{\mathbb{G}}$ can be written $c = c_1 + c_2$ where $c_1 \in Q_1^{\mathbb{G}}$ and $c_2 \in Q_2^{\mathbb{G}}$ through cellwise and componentwise decomposition, and $F(c) = F_1(c_1) + F_2(c_2)$. F_1 is isomorphic to a linear CA F_p over Q_p and F_2 to a linear CA F' over Q', and then F is isomorphic to $F_p \times F'$. $\square$

3. Pre-expansivity

Pre-expansivity is the property of expansivity restricted to asymptotic pairs of configurations.

Definition 3. *Let $\mathbb{T}$ be either $\mathbb{N}$ or $\mathbb{Z}$ and let F be a cellular automaton over $Q^{\mathbb{G}}$, supposed reversible in the case $\mathbb{T} = \mathbb{Z}$. F is* $\mathbb{T}$-pre-expansive *if:*

$$\exists \delta > 0 : \forall c, d \in Q^{\mathbb{G}}, c \neq d \text{ and } c \overset{\infty}{=} d \Rightarrow \exists t \in \mathbb{T}, \Delta(F^t(c), F^t(d)) > \delta.$$

The value δ is the pre-expansivity constant.

In the sequel, when $\mathbb{T}$ is not explicitly mentioned, we will always refer to $\mathbb{T} = \mathbb{N}$. In particular this choice does not require the hypothesis of reversibility on the considered cellular automaton.

The notion of pre-expansivity can be further refined by considering only pairs of configurations with a fixed finite number of differences. Given $c, d \in Q^{\mathbb{G}}$, we denote $c \neq_k d$ if $\#\left\{z \in \mathbb{G} : c(z) \neq d(z)\right\} = k$, *i.e.* if c and d differ in exactly k positions.

Definition 4. *Let $\mathbb{T}$ be either $\mathbb{N}$ or $\mathbb{Z}$, let F be a cellular automaton over $Q^{\mathbb{G}}$ (supposed reversible in the case $\mathbb{T} = \mathbb{Z}$) and let $k > 0$. F is* $\mathbb{T}$-k-expansive *if:*

$$\exists \delta > 0 : \forall c, d \in Q^{\mathbb{G}}, c \neq_k d \Rightarrow \exists t \in \mathbb{T}, \Delta(F^t(c), F^t(d)) > \delta.$$

Denote by $T_m : Q^{\mathbb{G}} \to (Q^{B_m(0)})^{\mathbb{N}}$ the *trace function* which to any configuration associates its orbit restricted to $B_m(0)$:

$$T_m(c) = \big(t \mapsto \big(F^t(c)\big)|_{B_m}\big) .$$

Proposition 1. *Let F be any CA over $Q^{\mathbb{G}}$, it holds:*

1. *F is pre-expansive $\Rightarrow \forall k > 0$ F is k-expansive,*
2. *F is k-expansive $\Rightarrow$ F is sensitive to initial configurations,*
3. *F is pre-expansive $\Leftrightarrow T_m$ is pre-injective for some m.*
4. *If L is a CA over $\tilde{Q}^{\mathbb{G}}$, then $F \times L$ is k'-expansive for every $k' \leq k$ if and only if L and F are k'-expansive for every $k' \leq k$.*
5. *If $\mathbb{G}$ is amenable then:*
F expansive $\Rightarrow$ F pre-expansive $\Rightarrow$ F pre-injective $\Rightarrow$ F surjective.

Proof.

1. It follows directly from definitions.
2. It is enough to note that for any configuration c, any $\delta > 0$ and any $k \geq 1$ there always exist a configuration c' with $c \neq_k c'$ and $\Delta(c, c') \leq \delta$.
3. For the fourth item, it is sufficient to note that the existence of some time t such that $\Delta\big(F^t(c), F^t(c')\big) > \delta$ is equivalent to $T_m(c) \neq T_m(c')$ for a suitable choice of m.
4. Finally, if $F \times L$ is k'-expansive, it is enough to take two configurations with their differences in only one of their components, since both automata act independently, k' expansivity of $F \times L$, imply that the perturbations will arrive to the center at the same component, proving the k'-expansivity of the corresponding automaton.
If now L and F are k'-expansive for every $k' \leq k$, we take two configurations with k' differences. They may lay in one or both of their components, in any way there will be $0 < k'' \leq k'$ differences in one of the components of $F \times L$. By the k''-expansivity of the corresponding automaton, we show the expansivity of $F \times L$.
5. It is clear that expansivity implies pre-expansivity (restriction of the universal quantification). Then pre-expansivity implies pre-injectivity because if there is a pair of configurations c, c' with $c \overset{\infty}{=} c'$ and $F(c) = F(c')$ then, eventually applying a translation, we can also suppose them such that $\Delta(c, c')$ is arbitrarily small. Finally, since $\mathbb{G}$ is amenable we have that pre-injectivity implies surjectivity by Garden of Eden Theorem [4].

$\square$

Note however that k-expansivity does not generally imply pre-injectivity or surjectivity as shown by the following example.

Proposition 2. *For any $k \geq 1$ there exists a CA which is not surjective but k'-expansive for any $k' \leq k$.*

PROOF. Consider any pre-expansive one-dimensional CA F of radius 1 over state set $Q = \{0,1\}$ (for instance a bi-permutative CA), and define a CA Ψ over state set Q^{k+1} as follows. It has $k+1$ "layers" and to any configuration c we associate its projection $\pi_i(c)$ on the ith layer. Intuitively it behaves on the k first layers as k independent copies of F, except that the $(k+1)$th layer induce a state flip in the image in the following way: if it has a 1 at position z then, in the image, layer i is flipped at position $z+3i$. Moreover, $(k+1)$th layer is uniformly reset to 0 after one step. Formally, Ψ is defined by:

$$\Psi(c)_z = \big(F(\pi_1(c))_z + \pi_{k+1}(c)_{z-3} \bmod 2, \ldots, F(\pi_k(c))_z + \pi_{k+1}(c)_{z-3k} \bmod 2, 0\big)$$

First it is clear from the definition that it is not surjective since the image of any configuration is always 0 on layer $k+1$. Note also that, when reduced to state set $\{0,1\}^k \times \{0\}$, Ψ is isomorphic to F^k which is pre-expansive. Therefore, to show that Ψ is k'-expansive for any $1 \leq k' \leq k$, it is sufficient to show that for any pair of configurations c and d with $c \neq_{k'} d$ we have $\Psi(c) \neq \Psi(d)$.

So consider such a pair (c,d). Ψ was defined such that, if c and d differ on the $(k+1)$th layer at position z, then, on the ith layer, $F(c)$ and $F(d)$ will differ at position $z+3i$ as soon as c and d are the same on the ith layer at positions $z+3i-1$, $z+3i$ and $z+3i+1$. Therefore, supposing that c and d indeed differ on the $(k+1)$th layer at position z, it implies that $F(c)$ and $F(d)$ differ because c and d having only $k' \leq k$ differences, they can not differ at z and at one of the positions $z+3i-1$, $z+3i$ or $z+3i+1$ for each $1 \leq i \leq k$.

Finally, suppose that c and d are equal on the $(k+1)$th layer. Then they must differ on some layer i with $1 \leq i \leq k$. Therefore, we must have $F(\pi_i(c)) \neq F(\pi_i(d))$. We deduce that $\Psi(c) \neq \Psi(d)$ because their respective ith layers are $F(\pi_i(c))$ and $F(\pi_i(d))$ up to some modification by the $(k+1)$th layer which are identical in c and d. □

The next lemma talks about *linear* CA. When F is supposed to be linear (for law $\oplus$), then T_m is also linear, *i.e.* $T_m(c \,\overline{\oplus}\, d) = T_m(c) \,\overline{\oplus}\, T_m(d)$ where $\overline{\oplus}$ denotes the component-wise application of $\oplus$ either on $Q^{\mathbb{G}}$ or on $\left(Q^{B_m}\right)^{\mathbb{N}}$.

Proposition 3. *Let F be a linear CA for law $\oplus$ and neutral element 0. Let I be the set*

$$I = \{k \in \mathbb{N} : F \textit{ is not } k\textit{-expansive}\}$$

- *if $k_1, k_2 \in I$ then $k_1 + k_2 \in I$,*
- *F is pre-expansive if and only if for some $m > 0$ there is no finite non-empty set $\{c_1, \ldots, c_n\} \subseteq X$ such that*

$$T_m(c_1) \,\overline{\oplus} \cdots \overline{\oplus}\, T_m(c_n) = \overline{0}$$

where $X = \{c : c \neq_1 \overline{0}\}$ is the set of configurations with a single non-0 cell.

PROOF. First, by linearity of the trace functions T_m, we have that $T_m(c) = T_m(c')$ if and only if $T_m\big(c' \,\overline{\oplus}\, (-c)\big) = T_m(\overline{0})$ where $-c$ is the configuration such that

$c\,\overline{\oplus}\,(-c) = \overline{0}$. Moreover we also have $c \neq_k c'$ if and only if $c'\,\overline{\oplus}\,(-c) \neq_k \overline{0}$. Hence F is pre-expansive (resp. k-expansive) if and only if there is m such that no $c \neq \overline{0}$ with $c \overset{\infty}{=} \overline{0}$ (resp. $c \neq_k \overline{0}$) can verify $T_m(c) = T_m(\overline{0})$.

From this we deduce the second item of the proposition.

For the first item, consider k_1 and k_2 in I. From what we said above, for any m_1 there is c_1 such that $c_1 \neq_{k_1} \overline{0}$ and $T_{m_1}(c_1) = T_{m_1}(\overline{0})$. Now choose m_2 large enough so that any non-zero state of c_1 appears at distance at most m_2 from the center. Let us remark that the differences between c_1 and $\overline{0}$ are outside B_{m_1}, otherwise $T_{m_1}(c_1) \neq T_{m_1}(\overline{0})$, thus $m_2 > m_1$. Since $k_2 \in I$ we deduce from what we said earlier that there is c_2 such that $c_2 \neq_{k_2} \overline{0}$ and $T_{m_2}(c_2) = T_{m_2}(\overline{0})$. By our choice of m_2, this implies that $T_{m_1}(c_1\,\overline{\oplus}\,c_2) = \mathbb{T}_{m_1}(\overline{0})$. Moreover $c_1\,\overline{\oplus}\,c_2 \neq_{k_1+k_2} \overline{0}$. Since m_1 was arbitrary, we deduce that F is not $(k_1 + k_2)$-expansive. □

4. 1-dimensional Cellular Automata

The main goal of this section is to show that the notion of k-expansivity, pre-expansivity and expansivity all differ. More precisely we will show the following existential result.

Theorem 1. *For each item in the following list, there exists a CA having the given properties:*

- $\mathbb{N}$-*pre-expansive and reversible and not* $\mathbb{N}$-*expansive,*
- $\mathbb{N}$-*pre-expansive and irreversible and not* $\mathbb{N}$-*expansive,*
- 1-*pre-expansive and reversible and not* $\mathbb{N}$-*pre-expansive, and*
- 1-*pre-expansive and irreversible and not* $\mathbb{N}$-*pre-expansive.*

For this purpose it will be sufficient to focus on linear cellular automata. Nevertheless, before the study of the (linear) examples involved in Theorem 1, we give some additional results which hold in dimension 1 for (non) linear CA. First, the pre-expansivity constant can be fixed canonically as we will prove in Lemma 3. The next lemma is direct and it expresses the locality of CAs.

Lemma 2. *Let F be a CA in $\mathbb{Z}$ with neighborhood $[-l, r]$ the next assertions hold.*

- *If $c_{]-\infty,n]} = d_{]-\infty,n]}$ and there exists an iteration t such that $F^t(c)_{]-\infty,n]} \neq F^t(d)_{]-\infty,n]}$, then there is an iteration $t' \leq t$ such that $F^{t'}(c)_{[n-r,n]} \neq F^{t'}(d)_{[n-r,n]}$.*
- *If $c_{[n,\infty[} = d_{[n,\infty[}$ and there exists an iteration t such that $F^t(c)_{[n,\infty[} \neq F^t(d)_{[n,\infty[}$, then there is an iteration $t' \leq t$ such that $F^{t'}(c)_{[n,n+l]} \neq F^{t'}(d)_{[n,n+l]}$.*

Proof. We will only prove the first assertion, the second one is completely analogous. Let t' be the first time such that $F^{t'}(c)_{]-\infty,n]} \neq F^{t'}(d)_{]-\infty,n]}$, and let $i \in]-\infty, n]$ be a position such that $F^{t'}(c)_i \neq F^{t'}(d)_i$. Since $F^{t'-1}(c)_{]-\infty,n]} = F^{t'-1}(d)_{]-\infty,n]}$, and only the cells in $]n-r, n]$ depend on cells in $]n, \infty[$, $i \geq n-r$. $\square$

This lemma shows a particularity of dimension 1: expansivity properties can be understood through left/right propagation of information. Let us precise this notion.

Definition 5. *Given two configurations $c \neq d$, and a CA F, we define the left and right propagation sequences as follows.*

$$l_t^d(c) = \min\{z \in \mathbb{Z} : \big(F^t(c)\big)(z) \neq \big(F^t(d)\big)(z)\}$$
$$r_t^d(c) = \max\{z \in \mathbb{Z} : \big(F^t(c)\big)(z) \neq \big(F^t(d)\big)(z)\}$$

Lemma 3. *Given a CA F of neighborhood $[-l, r]$ and $k \in \mathbb{N}$, the next assertion hold.*

1. *If F is k-expansive, then $\forall c \neq_k d, (l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded.*
2. *If $\forall k' \leq k, \forall c \neq_{k'} d, (l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded, then F is k-expansive with pre-expansivity constant $2^{-max\{l,r\}}$.*

Proof.
1. Let us suppose that F is k-expansive with pre-expansivity constant m. Let $c \neq_k d$ be two configurations, and let us assume that $l_t^d(c) > n$ for some $n \in \mathbb{Z}$; this means that $F^t(c)_{]-\infty,n]} = F^t(d)_{]-\infty,n]}$ for every $t \in \mathbb{N}$. Thus $T_m(\sigma_{n-m}(c)) = T_m(\sigma_{n-m}(d))$, which is a contradiction. The analogous happens if $(r_t^d(c))_{t\in\mathbb{N}}$ is upper bounded.
2. We need to prove that F is k-expansive with pre-expansivity constant $2^{-m} = 2^{-max\{l,r\}}$. Let $c \neq_k d$ be two configurations, and let us define the next two additional configurations.

$$c^l(i) = \begin{cases} c(i) & \text{if } i < 0 \\ d(i) & \text{if } i \geq 0 \end{cases}$$
$$c^r(i) = \begin{cases} d(i) & \text{if } i < 0 \\ c(i) & \text{if } i \geq 0 \end{cases}$$

If $c_{[-m,m]} \neq d_{[-m,m]}$, $T_m(c) \neq T_m(d)$ and we are done, so let us suppose that $c_{[-m,m]} = d_{[-m,m]}$. Let k' and k'' be such that $c^r \neq_{k'} d$, $c^l \neq_{k''} d$ and $k' + k'' = k$.

If $k' \neq 0$, $(l_t^d(c^r))_{t\in\mathbb{N}}$ is not lower bounded, thus by Lemma 2 and the fact that c^r is equal to d below position m, there is a minimal iteration t_r such that $F^{t_r}(c^r)_{[m-r,m]} \neq F^{t_r}(d)_{[m-r,m]}$. Analogously, if $k'' \neq 0$ there is a minimal iteration t_l such that $F^{t_l}(c^l)_{[-m,-m+l]} \neq F^{t_l}(d)_{[-m,-m+l]}$.

Let us take $\bar{t} = min\{t_r, t_l\}$, by the choice of m we have that $F^{\bar{t}}(c)_{[0,m]} = F^{\bar{t}}(c^r)_{[0,m]}$ and $F^{\bar{t}}(c)_{[-m,0]} = F^{\bar{t}}(c^l)_{[-m,0]}$, and at least one of them is different from $F^{\bar{t}}(d)$ between $[-m,m]$, thus $T_m(c) \neq T_m(d)$.

□

The last lemma establishes that the pre-expansivity constant is uniform in dimension one (it does not depends on k), thus we can conclude the next corollary.

Corollary 1. *If F is k-expansive for every $k \in \mathbb{N}$ then it is pre-expansive.*

Lemma 4. *F is pre-expansive if and only if for all $c \overset{\infty}{\cong} d$, $(l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded.*

PROOF. ($\Rightarrow$) Let $c \neq_k d$ be two asymptotics configurations. Since F is pre-expansive, it is also k-expansive, thus by Lemma 3, $(l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded.

($\Leftarrow$) If for every $k \in \mathbb{N}$ and every pair $c \neq_k d$ we have that $(l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded, then by Lemma 3, F is k-expansive, and by Corollary 1, we conclude that F is pre-expansive.

□

Left and right propagation determines also expansivity. The next lemma can be proven by using the techniques from the last lemmas.

Lemma 5. *F is expansive if and only if for any pair of different configurations c, d, if $c_{]-\infty,0]} = d_{]-\infty,0]}$ then $(l_t^d(c))_{t\in\mathbb{N}}$ is not lower bounded, and if $c_{[0,\infty[} = d_{[0,\infty[}$ then $(r_t^d(c))_{t\in\mathbb{N}}$ is not upper bounded.*

The last two lemmas show a similarity in information propagation between $\mathbb{N}$-pre-expanisvity and $\mathbb{N}$-expansivity: the differences between two configurations are spread both to the left and to the right and thus there is a sensitivity to initial conditions in all directions. This is not the case for $\mathbb{Z}$-pre-expansivity or $\mathbb{Z}$-expansivity: the shift map σ is $\mathbb{Z}$-expansive but has a direction of equicontinuity. We can formalize this following the directional dynamics setting of [21] (we could also use the more general viewpoint of [22] but we prefer to keep a lighter setting for clarity of exposition). Let $\alpha \in \mathbb{R}$. A CA F is said to be $\mathbb{T}$-pre-expansive in *direction* α if there is some $\delta > 0$ such that

$$\forall c, d \in Q^{\mathbb{Z}}, c \neq d \text{ and } c \overset{\infty}{\cong} d \Rightarrow \exists t \in \mathbb{T}, \Delta(\sigma_{\lceil \alpha t \rceil} \circ F^t(c), \sigma_{\lceil \alpha t \rceil} \circ F^t(d)) > \delta.$$

In particular, pre-expansivity in direction 0 is the same as pre-expansivity. Similarly F is said to be sensitive to initial conditions in direction α if there is some $\delta > 0$ such that

$$\forall c \in Q^{\mathbb{Z}}, \forall \epsilon > 0, \exists d \in Q^{\mathbb{Z}} \exists t \in \mathbb{N} : \Delta(c, d) < \epsilon \wedge \Delta(\sigma_{\lceil \alpha t \rceil} \circ F^t(c), \sigma_{\lceil \alpha t \rceil} \circ F^t(d)) > \delta.$$

Remark 1. *Lemma 4 generalizes to the directional dynamics setting: F is pre-expansive in direction β if and only if for all $c \overset{\infty}{=} d$, $(l_t^d(c) - \lceil \beta t \rceil)_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^d(c) - \lceil \beta t \rceil)_{t\in\mathbb{N}}$ is not upper bounded.*

Proposition 4. *Let F be any CA which is $\mathbb{N}$-pre-expansive in some direction α. Then the following holds:*

1. *F is sensitive to initial conditions in any direction;*
2. *if F is also $\mathbb{N}$-pre-expansive in direction $\alpha' > \alpha$ then it is $\mathbb{N}$-pre-expansive in any direction β with $\alpha \leq \beta \leq \alpha'$.*

Proof. We first deduce that if $\alpha \leq \beta \leq \alpha'$ and α' is a direction of pre-expansivity, then for any $c \neq d$ with $c \overset{\infty}{=} d$:

$$l_t^d(c) - \lceil \beta t \rceil \leq l_t^d(c) - \lceil \alpha t \rceil$$

is not lower bounded and

$$r_t^d(c) - \lceil \beta t \rceil \geq r_t^d(c) - \lceil \alpha' t \rceil$$

is not upper bounded so β is also a direction of pre-expansivity.

Finally, consider a direction $\beta \geq \alpha$ (the case $\beta \leq \alpha$ is symmetric), some configuration c and some $\epsilon > 0$. Take any $d \neq c$ with $c \overset{\infty}{=} d$ and $l_0^d(c)$ large enough so that $\Delta(c, d) \leq \epsilon$. Since $(l_t^d(c) - \lceil \alpha t \rceil)_{t\in\mathbb{N}}$ is not lower bounded and $\beta \geq \alpha$ then there is some t with $l_t^d(c) - \lceil \beta t \rceil \leq 0$. Let t_0 be the smallest such t. Since $|l_{t+1}^d(c) - l_t^d(c)|$ is bounded by the radius of F we deduce that there exists a constant M, depending only on β and the radius of F, such that $M \leq l_{t_0}^d(c) - \lceil \beta t \rceil \leq 0$. Said differently $\Delta(\sigma_{\lceil \beta t_0 \rceil} \circ F^{t_0}(c), \sigma_{\lceil \beta t_0 \rceil} \circ F^{t_0}(d)) \geq 2^M$. We have thus shown that F is sensitive in direction β. $\square$

The following proposition shows that the simplest form of linearity is not sufficient to achieve the separation between expansivity and pre-expansivity. Let us first introduce some notation. Given $a \in Q$, we denote by c^a the configuration that is equal to 0 everywhere except at cell 0 where its value is a.

Proposition 5. *Let $\mathbb{Z}_n$ be the the group of integers modulo n with addition, and let F be a one-dimensional linear CA over $\mathbb{Z}_n$. Then F is 1-expansive if and only if it is expansive.*

Proof. First by Proposition 1 if F is expansive it is in particular 1-expansive.

For the other direction, it is sufficient to consider the case $n = p^k$ with p a prime number by Lemma 1 because if some $F_1 \times F_2$ is 1-expansive then both F_1 and F_2 must be 1-expansive.

By commutation with shifts we have $l_t^{\overline{0}}(\sigma_n(c^a)) = -n + l_t^{\overline{0}}(c^a)$ and the analogous for r_t. Moreover $c^a = a \cdot c^1$ because we are on a cyclic group.

Let us define $l_t^U(c^a) = min\{i \in \mathbb{Z} \mid \gcd(F^t(c^a)_i, p) = 1\}$. The next properties hold.

1. *If* $\gcd(a,p) \neq 1$*, then* $l_t^U(c^a) = \infty$ *(respectively* $r_t^U(c^a) = -\infty$*).* In fact, in this case the entire evolution of F over c^a is composed by multiples of a, which are multiples of p too.
2. *If* $\gcd(a,p) = 1$*, then* $l_t^U(c^a) = l_t^U(c^1)$ *(respectively* $r_t^U(c^a) = r_t^U(c^1)$*).* In fact, in this case $F^t(c^a)_i = aF^t(c^1)_i$ is coprime with p if and only if $F^t(c^1)_i$ does.
3. $l_t^{\overline{0}}(c^a) \leq l_t^U(c^1)$ *(respectively* $r_t^{\overline{0}}(c^a) \geq r_t^U(c^1)$*).* In fact, if $F^t(c^1)_i$ is coprime with p, then $F^t(c^a)_i = aF^t(c^1)_i$ is not null.
4. $l_t^{\overline{0}}(c^{p^{k-1}}) = l_t^U(c^1)$ *(respectively* $r_t^{\overline{0}}(c^{p^{k-1}}) = r_t^U(c^1)$*).* In fact, $F^t(c^{p^{k-1}})_i = p^{k-1}F^t(c^1) = 0 \Leftrightarrow \gcd(F^t(c^1)_i, p) \neq 1$.

From the last assertion and Lemma 3 (first item) we have that $l_t^U(c^1)$ is not lower bounded and $r_t^U(c^1)$ is not upper bounded.

Now let us take a configuration $v \in (\mathbb{Z}_{p^k})^{\mathbb{Z}}$, such that $v_i = 0$ for every $i < 0$ and $v_0 \neq 0$. Let us define $j = max\{i \in \{0,...,k\} \mid \forall x \in \mathbb{Z}, p^i | v(x)\}$, and let us consider $u = v/p^j$. In this way, there is $y \geq 0$ with $u(y) \neq 0$ and $\gcd(p, u(y)) = 1$. Let y be the smallest integer with this property.

$$F^t(u)_{l_t^U(c^1)+y} = \sum_{x=0}^{l_t^U(c^1)+y+rt} F^t(c^{u_x})_{l_t^U(c^1)+y-x}$$

But $u(x)$ is a multiple of p when $x < y$, thus $F^t(c^{u_x})_i = 0 \bmod p$ for any i. For $x > y$, $F^t(c^{u_x})_{l_t^U(c^1)+y-x}$ is also a multiple of p because the smallest index for which $F^t(c^{u_x})$ is coprime with p is $l_t^U(c^{u_x})$ which is greater than or equal to $l_t^U(c^1)$. In this way we conclude that

$$F^t(u)_{l_t^U(c^1)+y} = F^t(c^{u_y})_{l_t^U(c^1)} \bmod p$$

is coprime with p and is non null. Therefore $F^t(v)_{l_t^U(c^1)+y}$ is non null as well. This implies that $l_t^{\overline{0}}(v)$ is not lower bounded. Symmetrically, $r_t^{\overline{0}}(w)$ is not upper bounded when w is any configuration equal to zero on positive coordinates. Lemma 5 concludes that F is (positively) expansive. □

To establish a separation between expansivity and pre-expansivity, we will focus on linear CA obtained by what is often called "second order method" in the literature [23]. The idea is to turn any CA into a reversible one by memorizing one step of history and combining, in a reversible way, the memorized past step into the produced future step. The interest of this construction for our purpose is that expansivity is excluded from the beginning because no CA can be $\mathbb{N}$-expansive and reversible at the same time [17].

Let $Q = \{0, \ldots, n-1\}$ be equipped with some group law $\oplus$ and consider some CA F over state set Q. The second-order CA associated to F and $\oplus$, denoted $\mathcal{SO}(F, \oplus)$ is the CA over state set $Q \times Q$, which is conjugated through the natural bijection $Q^{\mathbb{Z}} \times Q^{\mathbb{Z}} \to (Q \times Q)^{\mathbb{Z}}$ to the map:

$$(c, d) \mapsto \big(d, F(d) \overline{\oplus} c\big)$$

The following proposition shows that second order construction is useful to separate expansivity from 1-expansivity. Some of the results in the next proposition can be deduced from more general results in [16, 24, 25], but we develop a new specific proof here.

Proposition 6. *Let $\oplus$ be a group law over Q with neutral element 0 and F be a CA over Q which is linear for $\oplus$. It holds:*

1. *$\mathcal{SO}(F, \oplus)$ is bijective and linear for the law $\oplus \times \oplus$;*
2. *if F is LR-permutive then $\mathcal{SO}(F, \oplus)$ is $\mathbb{Z}$-expansive and 1-expansive;*
3. *if F is LR-permutive then for any $m > 0$ the subshift of traces $T_m\big((Q \times Q)^{\mathbb{Z}}\big)$ is an SFT.*

Proof. 1. It is sufficient to check that the CA over the state set $Q \times Q$ is conjugated to the following map:

$$(c, d) \mapsto \big(\bar{\iota}(F(c)) \,\overline{\oplus}\, d, c\big)$$

is the inverse of $\mathcal{SO}(F, \oplus)$, where ι denotes the inverse function for the group law $\oplus$. Moreover $\mathcal{SO}(F, \oplus)$ is linear for $\oplus \times \oplus$ because it is component-wise linear for $\oplus$.

2. Let us suppose that F is LR-permutive with neighborhood $\{-l, ..., r\}$ and denote $\Psi = \mathcal{SO}(F, \oplus)$. To prove $\mathbb{Z}$-expansivity of Ψ it is sufficient to notice that Ψ propagates to left and right when the second Q-component is non-null and Ψ^{-1} propagates to left and right when the first Q-component is non-null. In fact, let us consider a configuration $c \in (Q \times Q)^{\mathbb{Z}}$ equal to $(0, 0)$ on negative coordinates but such that $c(0) \neq (0, 0)$. If the second Q-component of $c(0)$ is non-null then, and since $F(0, ..., 0) = 0$, the leftmost non-null cell of $\Psi(c)$ is at position $-r$ and it is its second Q-component which is non-null, i. e., if the leftmost difference from (0,0) is in the second Q-component, this will be always like this and the difference will propagate to the left. The same holds symmetrically for propagation to the right. In the same way, and given the form of Ψ^{-1}, differences in the first Q-component will propagate to the left and right through Ψ^{-1}, thus by lemma 5, Ψ is $\mathbb{Z}$-expansive.

3. Now let us take $c^{(a,b)}$ as a configuration equal to (a, b) at 0 and $(0, 0)$ everywhere else. By the previous arguments, if $b \neq 0$, $(l_t^{\overline{(0,0)}}(c^{(a,b)}))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^{\overline{(0,0)}}(c^{(a,b)}))_{t\in\mathbb{N}}$ is not upper bounded. But if $b = 0$ and $a \neq 0$, then $(\Psi(c))(0) = (0, a)$ and null everywhere else, then again $(l_t^{\overline{(0,0)}}(c^{(a,b)}))_{t\in\mathbb{N}}$ is not lower bounded and $(r_t^{\overline{(0,0)}}(c^{(a,b)}))_{t\in\mathbb{N}}$ is not upper bounded. Therefore, Ψ (and Ψ^{-1}) is 1-expansive.

   The proof of the third item will be performed in two steps. To simplify notations, for any pair of words $u, v \in Q^*$ of the same length, we will denote by $\binom{u}{v}$ the word over alphabet $Q \times Q$ whose projection on the first (resp. second) component is u (resp. v).

Assertion 1L: *For every word $u \in Q^r$ there exists a configuration c such that $\Psi^t(c)|_{[0,r-1]} = \binom{u}{0^r}$ and $\Psi^k(c)_i = (0,0)$ for every $0 \le k \le t$ and $0 \le i < (t-k)r$.*

Proof of Assertion 1L. By induction on t. If $t = 0$ it is obvious, we just take c equal to $\binom{u}{0^r}$ at $[0, r-1]$ and $(0,0)$ everywhere else. Now, since F is LR-permutive, given a word $w \in Q^{l+r}$, let us define the permutation $\tau_w(a) = f(wa)$ for every $a \in Q$. Given a word $u \in Q^r$, we inductively define another word $v \in Q^r$ as follows: $v_0 = \tau^{-1}_{0^{l+r}}(u_0)$, $v_{i+1} = \tau^{-1}_{0^{l+r-i}v_{[0,i]}}(u_{i+1})$. In this way, $f(0^{l+r}v) = u$. By induction hypothesis, there exists a configuration c such that $\Psi^t(c)|_{[0,r-1]} = \binom{v}{0^r}$ and $\Psi^k(c)_i = (0,0)$ for every $0 \le k \le t$ and $i < (t-k)r$. We take $d = \sigma_{-r}(c)$, then $\Psi^t(d)|_{[r,2r-1]} = \binom{v}{0^r}$, and $(0,0)$ to the left of r; and $\Psi^{t+1}(d)|_{[0,r-1]} = \binom{u}{0^r}$, and $(0,0)$ to the left of 0. Moreover, $\Psi^k(d)_i = (0,0)$ for every $0 \le k \le t+1$ and $0 \le i < (t+1-k)r$.

Assertion 1R: *For every word $u \in Q^l$ there exists a configuration c such that $\Psi^t(c)|_{[-l+1,0]} = \binom{u}{0^l}$ and $\Psi^k(c)_i = (0,0)$ for every $0 \le k \le t$ and $i > (k-t)l$.*

The proof of Assertion 1R is analogous to the proof of Assertion 1L.

Assertion 2: *A sequence $\big(w_t\big)_{t=0}^n$, with $w_t \in (Q \times Q)^{2m+1}$, is a finite subsequence of a trace in $T_m\big((Q \times Q)^{\mathbb{Z}}\big)$ if and only if for any t, there are extensions $w_R \in (Q \times Q)^r$ and $w_L \in (Q \times Q)^l$ verifying*

$$\psi(w_L \cdot w_t \cdot w_R) = w_{t+1}$$

where ψ denotes the action of Ψ over finite words.

Proof of Assertion 2. In one direction, it is clear, so let $\big(w_t\big)_{t=0}^n$ be a sequence such that for any t, there are extensions $w_R \in (Q \times Q)^r$ and $w_L \in (Q \times Q)^l$ verifying $\psi(w_L \cdot w_t \cdot w_R) = w_{t+1}$, and let us prove that it is a subsequence of a trace of Ψ. We perform the proof by induction on n. If $n = 0$ there is nothing to prove, of course any sequence of length 1 can be part of a trace. Now let c be a configuration such that $\Psi^k(c)|_{[-m,m]} = w_k$, for every $k \in \{0, ..., n-1\}$. By locality, only the values of c between $-m-nl$ and $m+nr$ are relevant to this hypothesis, and we take $c(i) = (0,0)$ outside these limits. Let w_R and w_L be such that $\psi(w_L \cdot w_{n-1} \cdot w_R) = w_n$. Let us remark that the first Q-component of w_L and w_R can be chosen arbitrarily, given the form of Ψ. We will suppose that $\pi_1(w_L) = \pi_1(\Psi^{n-1}(c)_{[-m-l,-m-1]})$ and $\pi_1(w_R) = \pi_1(\Psi^{n-1}(c)_{[m+1,m+r]})$, thus we can take $\binom{u_L}{0^l} = w_L - \Psi^{n-1}(c)|_{[-m-l,-m-1]}$ and $\binom{u_R}{0^r} = w_R - \Psi^{n-1}(c)|_{[m+1,m+r]}$. We take from Assertion 1L a configuration c^R that produces the word $\binom{u_R}{0^r}$ at time $n-1$ at position $[m+1, m+r]$ and $(0,0)$ to the left of the light cone that starts at $m+nr$ with slope $-1/r$. From Assertion 1R we take a configuration c^L that produces the word $\binom{u_L}{0^r}$ at time $n-1$ at position $[-m-l, -m-1]$ and $(0,0)$

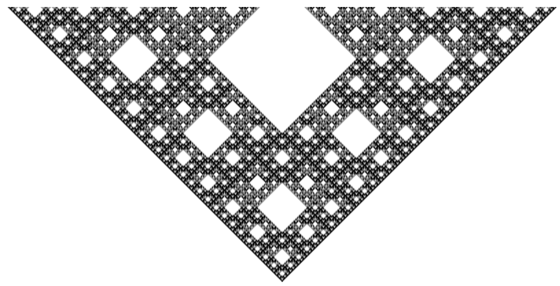

Figure 1: Space-time diagram of Ψ starting from a configuration with a single non-zero cell.

to the right of the light cone that starts at $-m-nl$ with slope $1/l$. By linearity, $\Psi^{n-1}(c^L \overline{\oplus} c^R \overline{\oplus} c)|_{[-m-l,m+r]} = w_L w_{n-1} w_R$, and then $\Psi^n(c_L \overline{\oplus} c_R \overline{\oplus} c)|_{[-m,m]} = w_n$, moreover $\Psi^k(c_L \overline{\oplus} c_R \overline{\oplus} c)|_{[-m,m]} = w_k$, for every $0 \leq k < n$, which completes the proof.

□

We will now give an example of pre-expansive CA which is not expansive.

Example 1 (Ψ). *Let $Q = \{0,1,2\}$, $+$ be the addition modulo 3, and F_3 be the CA defined over $Q^{\mathbb{Z}}$ by $F_3 = \sigma \overline{+} \sigma_{-1}$. We define Ψ as the second order construction applied to F_3:*

$$\Psi = \mathcal{SO}\big(F_3, +\big).$$

To establish the pre-expansivity of Ψ we will study its dependency structure, *i.e.* how the value of the cell at position z and time t depends on value of cells at other positions and earlier times. To express these space-time relations we denote by Ψ_z^t the map $\sigma_z \circ \Psi^t$ and by $\oplus$ the component-wise addition modulo 3 over $\{0,1,2\} \times \{0,1,2\}$ naturally extended to configurations of $(Q \times Q)^{\mathbb{Z}}$ and then naturally extended to functions on such configurations.

Lemma 6. *Let c be a configuration in $(Q \times Q)^{\mathbb{Z}}$. Then, for any $k \geq 0$, any $t \geq 0$ and any $z \in \mathbb{Z}$ we have:*

$$\Psi_z^{2\cdot 3^k + t}(c) = \Psi_z^t \oplus \Psi_{z-3^k}^{3^k+t} \oplus \Psi_{z+3^k}^{3^k+t}(c).$$

Proof. First it is straightforward to check that

$$\Psi_0^2(c) = \Psi_{-1}^1 \oplus Id \oplus \Psi_1^1(c).$$

Then, by Lucas' Lemma, we have $(a+b)^{3^k} = a^{3^k} + b^{3^k}$ when doing the arithmetics modulo 3. This identity naturally extends to $\oplus$ and therefore we have:

$$\Psi_0^{2\cdot 3^k}(c) = \Psi_{-3^k}^{3^k} \oplus Id \oplus \Psi_{3^k}^{3^k}(c).$$

Finally, by linearity of both σ and Ψ with respect to $\oplus$ we can compose both sides of the above equality by Ψ_z^t and the lemma follows. □

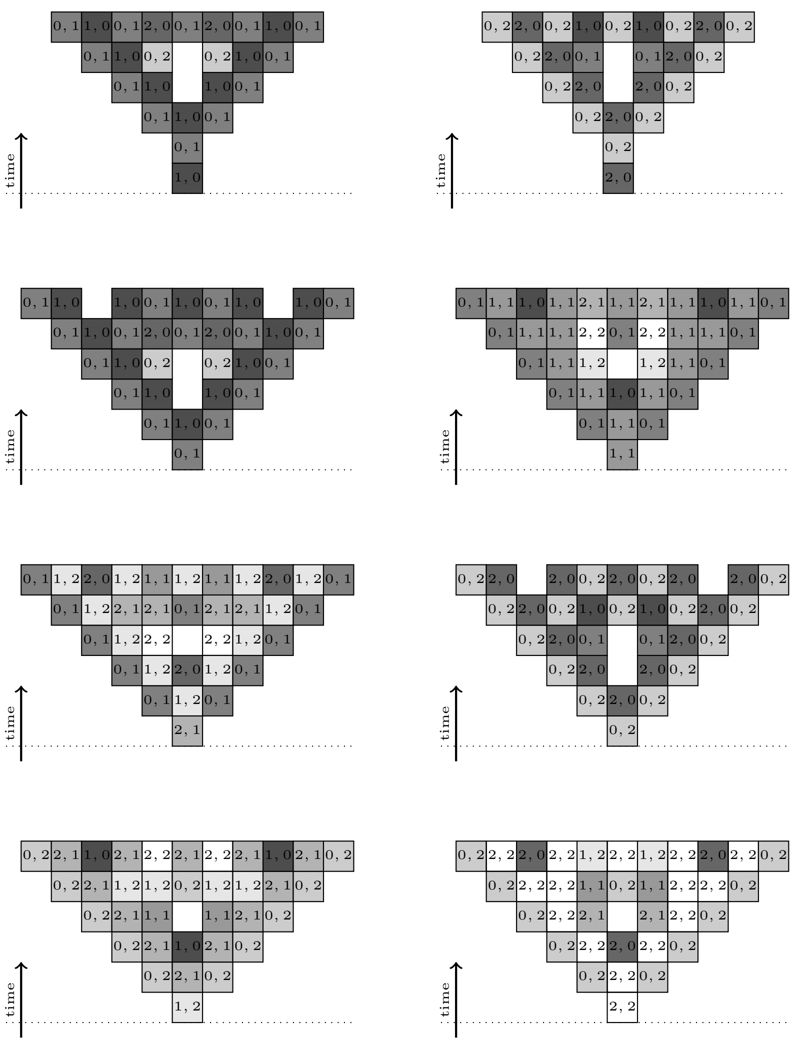

Figure 2: Some space-time diagrams of Ψ (state $(0, 0)$ is represented by empty space)

Using the above lemma, we can show that Ψ has a simple dependency structure at some space-time locations.

Lemma 7. *Let us consider a configuration $c = c^{(a,b)}$ for some pair $(a, b) \in Q \times Q$. For any integers $k \geq 0$ and $M \geq 1$, let $d_{M,k} = \Psi^{M \cdot 3^{k+1}}(c)$. Then we have:*

- $d_{M,k}(-M \cdot 3^{k+1} + 2 \cdot 3^k) = d_{M,k}(M \cdot 3^{k+1} - 2 \cdot 3^k) = \phi(a, b)$
- $d_{M,k}(i) = (0, 0)$ *for* $(M - 1) \cdot 3^{k+1} \leq |i| < (M - 1) \cdot 3^{k+1} + 3^k$

where ϕ is an automorphism of $Q \times Q$ which does not depend on k.

Proof. Let's first show the two items **for** $M = 1$. Denote by c^t_z the state $\big(\Psi^t(c)\big)(z)$. First, by a simple recurrence we can show that $c^n_{-n} = c^n_n = \pi(a, b)$ where π is the projection $\pi(a, b) = (0, b)$. Then, applying Lemma 6 on $\Psi^{2\cdot 3^k + 3^k}_{-3^k}$, we obtain:

$$d_k(-3^k) = c^{2\cdot 3^k}_{-2\cdot 3^k} + c^{3^k}_{-3^k} + c^{2\cdot 3^k}_0.$$

Applying Lemma 6 on $\Psi^{2\cdot 3^k + 0}_0$ we then have:

$$\begin{aligned} d_k(-3^k) &= c^{2\cdot 3^k}_{-2\cdot 3^k} + 2 \cdot c^{3^k}_{-3^k} + c^{3^k}_{3^k} + c^0_0 \\ &= (0, b) + 2(0, b) + (0, b) + (a, b) \\ &= (a, 2b) \end{aligned}$$

where $\phi(a, b) = (a, 2b)$ is an automorphism. The same equality holds for $d_k(3^k)$ by symmetry and the first item of the lemma is shown.

For the second item, first note that $c^t_z = (0, 0)$ whenever $z < -t$ or $z > t$ because Ψ has radius 1. Consider any i with $-3^k < i < 3^k$. Applying Lemma 6 on $\Psi^{2\cdot 3^k + 3^k}_i$, we obtain:

$$d_k(i) = c^{2\cdot 3^k}_{i-3^k} + c^{3^k}_i + c^{2\cdot 3^k}_{i+3^k}.$$

Applying Lemma 6 on $\Psi^{2\cdot 3^k + 0}_{i-3^k}$ and $\Psi^{2\cdot 3^k + 0}_{i+3^k}$ we further get:

$$d_k(i) = c^{3^k}_{i-2\cdot 3^k} + c^0_{i-3^k} + 3 \cdot c^{3^k}_i + c^0_{i+3^k} + c^{3^k}_{i+2\cdot 3^k}.$$

From what we said before and doing the arithmetics modulo 3 we deduce $d_k(i) = (0, 0)$ and the lemma follows.

Now, proceeding by induction on M, suppose we have:

- $d_{M,k}(-M \cdot 3^{k+1} + 2 \cdot 3^k) = d_{M,k}(M \cdot 3^{k+1} - 2 \cdot 3^k) = \phi(a, b)$
- $d_{M,k}(i) = (0, 0)$ for $(M - 1) \cdot 3^{k+1} \leq |i| < M \cdot 3^{k+1} - 2 \cdot 3^k$

Writing $(M+1)\cdot 3^{k+1} = 2\cdot 3^{k+1}+t$ with $t=(M-1)\cdot 3^{k+1}$ and applying Lemma 6 to $\Psi_z^{2\cdot 3^{k+1}+t}$ with $z=(M+1)\cdot 3^{k+1}-2\cdot 3^k$ we get:

$$\begin{aligned} d_{M+1,k}((M+1)3^{k+1}-2\cdot 3^k) &= d_{M,k}(M\cdot 3^{k+1}-2\cdot 3^k)+c^t_{(M+1)3^{k+1}-2\cdot 3^k} \\ &\quad +d_{M,k}((M+2)3^{k+1}-2\cdot 3^k) \\ &= d_{M,k}(M\cdot 3^{k+1}-2\cdot 3^k) \\ &= \phi(a,b) \end{aligned}$$

Applying again Lemma 6 but with $z=(M+1)\cdot 3^{k+1}-2\cdot 3^k-j$ where $-3^k\leq j<0$ we deduce $d_{M+1,k}(i)=(0,0)$ for $M\cdot 3^{k+1}\leq i<(M+1)\cdot 3^{k+1}-2\cdot 3^k$. By symmetry $z\mapsto -z$ we obtain the corresponding equalities and we finally have:

- $d_{M+1,k}(-(M+1)\cdot 3^{k+1}+2\cdot 3^k)=d_{M+1,k}((M+1)\cdot 3^{k+1}-2\cdot 3^k)=\phi(a,b)$
- $d_{M+1,k}(i)=(0,0)$ for $M\cdot 3^{k+1}\leq |i|<(M+1)\cdot 3^{k+1}-2\cdot 3^k$

which completes the induction step. The lemma follows. □

Proposition 7. *Ψ is pre-expansive in direction α for any $\alpha\in]-1,1[$.*

Proof. Let c be any configuration with $c\overset{\infty}{=}\overline{(0,0)}$. Denote $l=l_0^{\overline{(0,0)}}(c)$ and $r=r_0^{\overline{(0,0)}}(c)$ and consider any k such that $3^k>\max(|l|,|r|)$ and M such that $|\alpha|<1-\frac{2}{3M}$. By a finite number of applications of Lemma 7, and by linearity and translation invariance of Ψ, we have:

$$\begin{aligned}\big(\Psi^{M\cdot 3^{k+1}}(c)\big)(r-M\cdot 3^{k+1}+2\cdot 3^k) &= \phi\big(c(r)\big)\\ \big(\Psi^{M\cdot 3^{k+1}}(c)\big)(l+M\cdot 3^{k+1}-2\cdot 3^k) &= \phi\big(c(l)\big).\end{aligned}$$

Since ϕ is a permutation of Q sending $(0,0)$ to itself and since $c(r)$ and $c(l)$ are both different from $(0,0)$ we deduce that $l^{\overline{(0,0)}}_{M\cdot 3^{k+1}}(c)<r-M\cdot 3^{k+1}+2\cdot 3^k$ and $r^{\overline{(0,0)}}_{3^{k+1}}(c)>l+M\cdot 3^{k+1}-2\cdot 3^k$ for any k large enough. This shows that $\big(l_t(c)-\lceil\alpha t\rceil\big)_{t\in\mathbb{N}}$ is not lower-bounded and $\big(r_t(c)-\lceil\alpha t\rceil\big)_{t\in\mathbb{N}}$ is not upper-bounded and concludes the proof by a directional version of Lemma 4. □

We now give an example of CA that is 1-expansive but not pre-expansive.

Example 2 (Υ). *Let $Q=\{0,1\}$, $+$ be the addition modulo 2, and F_2 be the CA defined over $Q^{\mathbb{Z}}$ by $F_2=\sigma\overline{+}\sigma_{-1}$. We define Υ as the second order construction applied to F_2:*

$$\Upsilon=\mathcal{SO}\big(F_2,+\big).$$

Proposition 8. *Υ is not k-expansive when $k\geq 2$ and in particular Υ is not pre-expansive.*

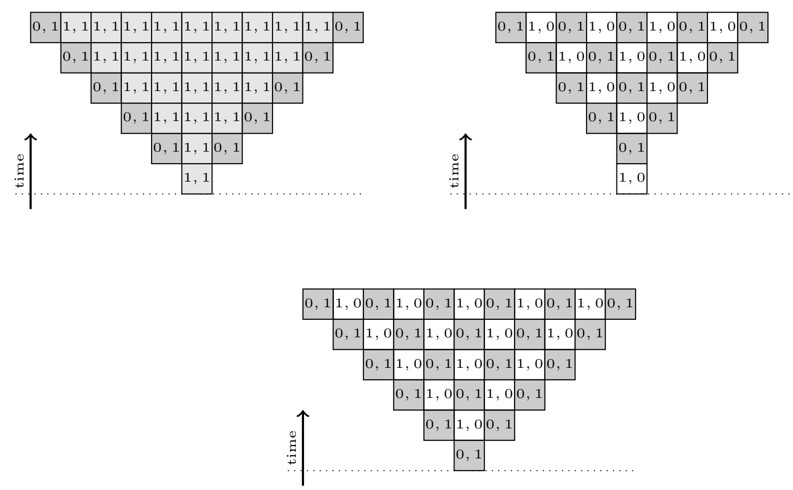

Figure 3: Some space-time diagrams of Υ (state $(0,0)$ is not represented)

Proof. Let $k \geq 2$ be fixed and for each $z \in \mathbb{Z}$ define the configuration c^z by:

$$c^z(z') = \begin{cases} (0,0) & \text{if } z' < z, \\ (0,1) & \text{if } z' = z, \\ (1,1) & \text{if } z < z' \leq z+k-2, \\ (1,0) & \text{if } z' = z+k-1, \\ (0,0) & \text{if } z' \geq z+k. \end{cases}$$

We have $c^z \neq_k \overline{(0,0)}$ and it is straightforward to check that $\Upsilon(c^z) = c^{z-1}$. We conclude that Υ is not k-expansive by Lemma 4. □

We are now ready to prove the main result announced at the beginning of this section.

Proof (Proof of Theorem 1). Let F be any irreversible and expansive CA. It holds:

- Ψ is $\mathbb{N}$-pre-expansive (by Proposition 7) and it is reversible and not $\mathbb{N}$-expansive (by proposition 6 and [26]);
- therefore $\Psi \times F$ is $\mathbb{N}$-pre-expansive and irreversible and not $\mathbb{N}$-expansive;
- Υ is 1-pre-expansive and reversible (by Proposition 6) but it is not $\mathbb{N}$-pre-expansive (by Proposition 8);
- therefore $\Upsilon \times F$ is 1-pre-expansive and irreversible and not $\mathbb{N}$-pre-expansive.

□

Now we exhibit a non linear family of reversible CA that will provide us with new examples of pre-expansive CA. This family has already been considered with different points of view in the literature [9, 27] and underlined for its links with some Furstenberg problems in ergodic theory [28].

Given two natural numbers k and k', let us consider the cellular automaton $F_{k,k'}$ on the state set $\mathbb{Z}_m$, with $m = kk'$, defined as follows

$$F_{k,k'}(c)_i = kc_i\%m + \lfloor kc_{i+1}/m \rfloor$$

where $i\%j$ denotes $i \bmod j$ with operation precedence as follows: $a + b\%c$ means $a + (b \bmod c)$ and $ab\%c$ means $(ab) \bmod c$. Note that $F_{k,k'}(c)_i$ is always in $\mathbb{Z}_m$ because $kc_i\%m \leq k(k' - 1)$ and $\lfloor kc_{i+1}/m \rfloor < k$. $F_{k,k'}$ can be seen as a multiplication by k in base kk', and the fact that there is no carry propagation ensures that it is a CA.

Proposition 9. *$F_{k,k'}$ is bijective and $F^{-1}_{k,k'} = F_{k',k} \circ \sigma^{-1}$.*

Proof. We will show that $F_{k,k'} \circ F_{k',k} = \sigma$.

$$\begin{aligned}
F_{k,k'} \circ F_{k',k}(c)_i &= (kF_{k',k}(c)_i)\%m + \left\lfloor \frac{kF_{k',k}(c)_{i+1}}{m} \right\rfloor \\
&= 0 + c_{i+1} - c_{i+1}\%k + \left\lfloor \frac{(mc_{i+1})\%(mk) + k\lfloor c_{i+2}/k \rfloor}{m} \right\rfloor \\
&= c_{i+1} - c_{i+1}\%k + \left\lfloor \frac{m(c_{i+1} - k\lfloor c_{i+1}/k \rfloor) + k\lfloor c_{i+2}/k \rfloor}{m} \right\rfloor \\
&= c_{i+1} - c_{i+1}\%k + c_{i+1} - k\left\lfloor \frac{c_{i+1}}{k} \right\rfloor + \left\lfloor \frac{c_{i+2} - c_{i+2}\%k}{m} \right\rfloor \\
&= c_{i+1} + 0 + 0
\end{aligned}$$

from here we obtain that $F_{k,k'}$ is surjective and $F_{k',k}$ is injective, but exchanging the roles of k and k' we obtain that both are bijective and we are done. □

Next lemma establishes some elementary bounds on the way perturbations propagates through $\mathbb{Z}$.

Lemma 8. *Given the function $g : \mathbb{Z}_m^{\mathbb{Z}} \to [0, m]$ defined by*

$$g(c) = \sum_{n=0}^{\infty} c_i m^{-i},$$

and given two configurations $c \overset{\infty}{=} d \in \mathbb{Z}_m^{\mathbb{Z}}$ with $c \neq d$, the next properties hold.

1. $g(F_{k,k'}(c)) = kg(c) - m\lfloor kc_0/m \rfloor$.
2. *If* $c_0 = d_0$, *then* $|g(F_{k,k'}(c)) - g(F_{k,k'}(d))| = k|g(c) - g(d)|$.
3. *If* $i = l_0^d(c) > 0$, *and* $j = r_0^d(c)$, *then* $m^{-j} \leq |g(c) - g(d)| < m^{-i+1}$.
4. $l_t^d(c) < r_0^d(c) + 1 - t\frac{log(k)}{log(m)}$.

5. $l_0^d(c) - 1 - t\frac{log(k)}{log(m)} < r_t^d(c)$.

PROOF. 1. The result is directly obtained from the definition of $F_{k,k'}$ and g.

$$\begin{aligned} g(F_{k,k'}(c)) &= \sum_{i=0}^{\infty}(kc_i\%m + \lfloor c_{i+1}/k' \rfloor)m^{-i} \\ &= \sum_{i=-1}^{\infty}(kc_{i+1}\%m)m^{-(i+1)} + \sum_{i=0}^{\infty}\lfloor c_{i+1}/k' \rfloor m^{-i} \\ &= kc_0\%m + \sum_{i=0}^{\infty}(kc_{i+1}\%m + m\lfloor kc_{i+1}/m \rfloor)m^{-(i+1)} \\ &= kc_0\%m + \sum_{i=0}^{\infty} kc_{i+1}m^{-(i+1)} \\ &= kg(c) + kc_0\%m - kc_0 \\ &= kg(c) - m\lfloor kc_0/m \rfloor \end{aligned}$$

2. It is direct from item 1.
3. $|g(c) - g(d)| = |\sum_{n=i}^{j}(c_n - d_n)m^{-n}|$ from where the result is clear.
4. One can assume, without loss of generality that $l_n^d(c) > 0$, for every $n \in \{0, \ldots, t\}$, and shift c and d if this were not the case. From item 2 and 3, we have that $m^{-l_t^d(c)+1} > |g(F^t(c)) - g(F^t(d))| = k^t|g(c) - g(d)| \geq k^t m^{-r_0^d(c)}$, thus $m^{-l_t^d(c)+1} > k^t m^{-r_0^d(c)}$.
5. Analogously, we have that $m^{-r_t^d(c)} \leq |g(F^t(c)) - g(F^t(d))| = k^t|g(c) - g(d)| < k^t m^{-l_0^d(c)+1}$, thus $m^{-r_t^d(c)} < k^t m^{-l_0^d(c)+1}$, from where we can conclude.

□

Proposition 10. *Let us define q, and p as follows.*

$$\begin{aligned} q &= min\{n \in \{1, ..., m-1\} \mid \exists p \in \mathbb{N}, k' \text{ divides } k^p n\} \\ p &= min\{n \in \mathbb{N} \mid k' \text{ divides } k^n q\} \end{aligned}$$

If $r_0^d(c) < \infty$, the next properties hold.

1. *If $q = 1$, then $r_t^d(c) \leq r_0^d(c) - \frac{t}{p+1}$.*
2. *If $q > 1$ and $k' > k^p$, then $(r_t^d(c))_{t \geq 0}$ is constant after some time t_0.*

PROOF. 1. Let $j = r_0^d(c)$ be the last cell where c and d are different. Since $m = k'k$ divides k^{p+1}, we can see that this difference cannot be preserved for more than $p+1$ steps.

$$\begin{aligned} F(c)_j - F(d)_j &= k(c_j - d_j)\%m \\ F^2(c)_j - F^2(d)_j &= k^2(c_j - d_j)\%m \\ &\cdots \\ F^{p+1}(c)_j - F^{p+1}(d)_j &= k^{p+1}(c_j - d_j)\%m = 0 \end{aligned}$$

2. We remark first that when $q > 1$, the difference at cell $j = r^d_0(c)$ can be preserved for ever, this will be the case if $c_j - d_j$ is not a multiple of q. If $k' > k^p$ we claim that there must be some time t_0 such that $F^{t_0}(c)_{r^d_{t_0}(c)} - F^{t_0}(d)_{r^d_{t_0}(c)}$ is not a multiple of q, proving assertion 2. Let us suppose the contrary, that is: q divides $F^t(c)_{r^d_t(c)} - F^t(d)_{r^d_t(c)}$ for all t. We will prove that under this assumption, $r^d_t(c) \leq r^d_0(c) - t/(p+1)$, as in item 1. Considering any time t and $j = r^d_t(c)$ we have q divides $F^t_{k,k'}(c)_j - F^t_{k,k'}(d)_j$, thus $F^{t+p+1}_{k,k'}(c)_j - F^{t+p+1}_{k,k'}(d)_j = 0$. Therefore, $r^d_{t+p+1}(c) < r^d_t(c)$. Inductively, this implies that $r^d_{(p+1)t}(c) \leq r^d_0(c) - t$, or equivalently $r^d_t(c) \leq r^d_0(c) - t/(p+1)$.
   Now, from item 5 of Lemma 8 we have that
$$l^d_0(c) - 1 - t\frac{\log(k)}{\log(m)} < r^d_t(c) \leq r^d_0(c) - t\frac{1}{p+1}$$
   Which implies that $r^d_0(c) + 1 - l^d_0(c) \geq t\left(\frac{1}{p+1} - \frac{\log(k)}{\log(m)}\right) = t\epsilon$, where $\epsilon > 0$, since $k' > k^p$; but this cannot hold for every t, thus $r^d_t(c)$ is lower bounded.

□

Corollary 2. *Considering the definitions of Proposition 10, if $q > 1$ and $k' > k^p$, then $F_{k,k'}$ is α-pre-expansive for every $\alpha \in]-log(k)/log(m), 0[$.*

Proof. It is direct from Remark 1, Lemma 8 item 4 and Proposition 10. □

The last corollary applies in particular to the case where k and k' are co primes, because in this case $q = k'$ and $p = 0$. An example of this case is the universal pattern generator introduced in [27]. Figure 4 shows the evolution of a finite configuration in a background of 0 under $F_{3,2}$. From the former lemmas we know that finite perturbations over non-zero configurations propagates in a similar way.

In [9], it was prooved that $F_{k,k'}$ is left-expansive if and only if $q = 1$, where *left-expansive* means $\{l^d_t(c)\}_{t\in\mathbb{N}}$ not lower bounded for every $d \neq c$. The proof is performed through an argument using interesting results about the entropy of left-expansive CA. The present work allows us to establish the same result, in fact, if $q = 1$ and $g(c) \neq g(d)$, Lemma 8 item 3 shows the unboundness of $l^d_t(c)$. If, otherwise $g(c) = g(d)$, it can be proved that there exists j such that $d_{[j,\infty[} = \overline{m-1}_{[j,\infty[}$ and $c_{[j,\infty[} = \overline{0}_{[j,\infty[}$ (or viceversa), but in this case Proposition 10 imply that $r^d_t(\overline{m-1})$ and $r^c_t(\overline{0})$ are not lower bounded, thus $l^d_t(c)$ can not be lower bounded. On the other hand, if $q > 1$, taking $c = ...0.10...$ and $d = ...0.0(m-1)...$, we have that $l^d_t(c) = 0$ for every $t \in \mathbb{N}$.

As a last remark, let us notice that if $q = 1$, then $\frac{\log(k)}{\log(m)} \geq \frac{1}{p+1}$, with equality only if $k' = k^p$, in those cases $F_{k,k'}$ is a kind of shift with slope $p+1$ which will be pre-expansive in no direction. Nevertheless, with more sofisticated tools we can exhibit examples where $k' \neq k^p$ and $F_{k,k'}$ is pre-expansive, we conclude that left-expansivity and directional pre-expansivity are independent properties.

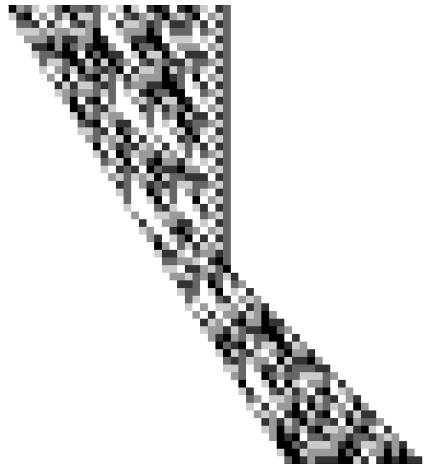

Figure 4: Space-time diagram of $F_{3,2}$ starting from a configuration with a finite number of non-zero cells (time goes from bottom to top, states are represented by shades of gray from white for 0 to black for 5).

5. Cellular automata over the free group

Some of the properties proved in the last section come from the fact that the graph $(\mathbb{Z}, \{(i, i+j) \mid j \in V\})$ can be always disconnected by extracting a finite part from $\mathbb{Z}$. The graph of any free group where the edges are given by any finite neighborhood has this feature, and we would be able to extend some of the previous properties to the case of a cellular automaton over the free group. In particular, the pre-expansivity constant is strictly related with the neighborhood size, as in $\mathbb{Z}$, and it does not depend on k for a k-expansive CA. We denote by $\mathbb{F}_n$ the free group with n generators ($\mathbb{F}_1$ is $\mathbb{Z}$).

Proposition 11. *If F is a cellular automaton with a neighborhood $V \subseteq B_r(0)$ of radius r over the free group $\mathbb{F}_n$ and it is k'-expansive for all $k' \leq k$, then F is k-expansive with pre-expansivity constant equal to 2^{-r}.*

Proof. The proof takes the ideas of Lemma 3. Let $c \neq_k d$ be two configurations in $\mathbb{F}_n$. Let us call S the set of generators of $\mathbb{F}_n$, including their inverses (i. e. $|S| = 2n$), and for each $s \in S$, let us call R_s the branch of $\mathbb{F}_n$ that hangs from s; we mean the set of elements whose shortest description in terms of S start with s. In this way $\mathbb{F}_n = \{0\} \sqcup (\sqcup_{s \in S} R_s)$.

Now let us define $D = \{i \in \mathbb{F}_n \mid c(i) \neq d(i)\}$ and $D_s = D \cap R_s$. We want to prove that $T_r(c) \neq T_r(d)$ so let us suppose the opposite. This imply that $D = \sqcup_{s \in S} D_s$, and we can consider $k_s = |D_s| \leq |D| = k$. As in the case of Lemma 3, we define configurations c^s which are equal to d everywhere except on branch s, as follows.

$$c^s(i) = \begin{cases} c(i) & \text{if } i \in R_s \\ d(i) & \text{otherwise} \end{cases}$$

At the beginning, c^s differs from d only on branch R_s. We will see that this will be always the case. Let us suppose that, for some $t \in \mathbb{N}$, $F^t(c^s)_i = F^t(c)_i$ for all $i \in R_s$ and $F^t(c^s)_j = F^t(d)_j$, for all $j \notin R_s$. We remark that, we assumed that if $j \in B_r(0)$, $F^t(c^s)_j = F^t(d)_j = F^t(c)_j$. Since $\mathbb{F}_n$ is a tree and F is a CA of radius r, if $i \in R_s$, $B_r(i) \subset R_s \cup B_r(0)$, thus $F^{t+1}(c^s)_i = F^{t+1}(c)_i$.If $j \in \{0\} \sqcup \left(\sqcup_{s' \in S \setminus \{s\}} R_{s'}\right)$, $B_r(j) \subset \left(\sqcup_{s' \in S \setminus \{s\}} R_{s'}\right) \cup B_r(0)$, then $F^{t+1}(c^s)_j = F^{t+1}(d)_j$.

We conclude that $T_r(c^s) = T_r(c) = T_r(d)$, for every $s \in S$.

But we know, by hypothesis, that F is k'-expansive, let us take its pre-expansivity constant as $\epsilon = 2^{-m}$. Let us consider now the configuration $\sigma_{-ms}(c^s)$, the shift of c^s by $ms \in \mathbb{F}_n$. By construction, $\sigma_{-ms}(c^s)$ and $\sigma_{-ms}(d)$ are equal over $B_m(0)$. By the k'-expansivity of F, there exists a time t and $j \in B_m(0)$ such that $F^t(\sigma_{-ms}(c^s))_j \neq F^t(\sigma_{-ms}(d))_j$. But, as shown before, $F^t(c^s)$ and $F^t(d)$ differ only on branch R_s. Therefore $F^t(\sigma_{-ms}(c^s))$ and $F^t(\sigma_{-ms}(d))$ differ only on branch R_{ms} which is disjoint from $B_m(0)$: this is a contradiction. $\square$

Not all the properties survive from $\mathbb{F}_1 = \mathbb{Z}$ to $\mathbb{F}_n$, when $n > 1$; k-expansivity is possible for infinitely many k's in $\mathbb{F}_n$ even without pre-expansivity, as the next example shows.

Example 3 (Λ_n). *Let $Q = \{0, 1\}$, $+$ be the addition modulo 2, and Λ_n be the CA defined over $Q^{\mathbb{F}_n}$ by*

$$\Lambda_n(c)_i = c(i) + \sum_{j \in S} c(i + j).$$

Lemma 9. *If $||x|| = ||y||$, then for every $t \in \mathbb{N}$ $\Lambda_n^t(c^1)_x = \Lambda_n^t(c^1)_y$, moreover $\Lambda_n^{||x||}(c^1)_x = 1$.*

Proof. We prove by induction on l that for every $t \leq l$ and every $x, y \in B_l(0)$, $\left[||x|| = ||y|| \Rightarrow \Lambda_n^t(c^1)_x = \Lambda_n^t(c^1)_y\right]$ and that $\Lambda_n^l(c^1)_x = 1$ if $||x|| = l$.

For $l = 0$ is clear since in this case $x = 0 = y$ and $\Lambda_n^0(c^1)_0 = 1$. Now let us suppose it true for some l, and let us prove it for $l + 1$.

Case 1, $t \leq l$. By the induction hypothesis, we only need to verify the property for $x, y \in B_{l+1}(0) - B_l(0)$, but $\Lambda_n^t(c^1)_x = 0 = \Lambda_n^t(c^1)_y$ because at time $t \leq l$ no perturbation at 0 has the time to arrive to these cells.

Case 2, $t = l + 1$. We first remark that any cell x in $\mathbb{F}_n$ has exactly $2n - 1$ neighbors farther and exactly one neighbor closer than x to 0; we also remark that the local rule of Λ_n is totalistic, only the quantity of neighbors at a given state counts. If $x, y \in B_l(0)$, all of their neighbors are in $B_{l+1}(0)$, thus by Case 1, their state at time l depends only on their distance to 0, thus $\Lambda_n^{l+1}(c^1)_x = \Lambda_n^{l+1}(c^1)_y$. If $x, y \in B_{l+1}(0) \setminus B_l(0)$, then their neighbors outside $B_l(0)$ and themselves have all state 0 at time l; their unique neighbors in $B_l(0)$ have both state 1, by induction hypothesis. Thus, by the definition of Λ_n, $\Lambda_n^{l+1}(c^1)_x = \Lambda_n^{l+1}(c^1)_y = 1$.

$\square$

Proposition 12. *Λ_n is k-expansive for every k odd and it is not 2-expansive if $n \geq 2$.*

Proof. Let $c \neq_k \overline{0}$. Let $D = \{i \mid c(i) \neq 0\}$ and let $D_l = D \cap (B_l(0) \setminus B_{l-1}(0))$. It is clear that $c = \sum_{i \in D} \sigma_{-i}(c^1)$. Since $|D| = k$ is odd, there exists some l such that $|D_l|$ is odd, let us take $\bar{l}$ as the smallest one. For every $x, y \in D_l$, $T_0(\sigma_{-x}(c^1)) = T_0(\sigma_{-y}(c^1))$, thanks to lemma 9. Therfore, given a cell $y \in D_{\bar{l}}$,

$$\begin{aligned}
\Lambda_n^{\bar{l}}(c)_0 &= \Lambda_n^{\bar{l}}(\sum_{l \in \mathbb{N}} \sum_{x \in D_l} \sigma_{-x}(c^1))_0 \\
&= \Lambda_n^{\bar{l}}(\sum_{l=0}^{\bar{l}} \sum_{x \in D_l} \sigma_{-x}(c^1))_0 \\
&= \sum_{l=0}^{\bar{l}} \Lambda_n^{\bar{l}}(\sum_{x \in D_l} \sigma_{-x}(c^1))_0 \\
&= \Lambda_n^{\bar{l}}(\sum_{x \in D_{\bar{l}}} \sigma_{-x}(c^1))_0 \\
&= \Lambda_n^{\bar{l}}(\sigma_{-y}(c^1))_0.
\end{aligned}$$

By Lemma 9, this last term is equal to 1 which proves the k-expansivity when k is odd.

The second part of the proposition is almost direct from lemma 9. In fact, let $m \in \mathbb{N}$ be any natural number and let us take $z = ms$ for some fixed generator s. Now let s' be another generator, different from s and $-s$ and define $x = z + s'$ and $y = z - s'$. This imply that $||x|| = ||y|| = ||z|| + 1 = m + 1$. Lemma 9 says that $T_0(\sigma_x(c^1)) = T_0(\sigma_y(c^1))$, but also that $T_m(\sigma_x(c^1)) = T_m(\sigma_y(c^1))$, because x and y are equidistant from z, as well as from all the other members of $B_m(0)$. $\square$

6. Cellular Automata on $\mathbb{Z}^n$, with $n \geq 2$

Expansivity is not possible in dimension $n \geq 2$ or more, due to combinatorial reasons: the number of possible n-dimensional patterns grows too quickly to be uniformly conveyed into a 1-dimensional array without loss (see [12] for a general result of inexistence of expansive CA).

This argument does not apply to pre-expansivity because the definition allows loss of information in some cases: only finite differences should be propagated, but nothing is required for non-asymptotic pairs of configurations.

Nevertheless, in linear CA over Abelian groups, the information propagates in a very regular way, and pre-expansivity is impossible as we will show.

6.1. No pre-expansivity for linear CA in dimension 2 or higher

Given a linear CA F, spot configurations (*i.e.* configurations c with $c \neq_1 \overline{0}$) form a basis of the whole set of configurations and we get a complete knowledge on the orbit of any configuration from the orbits of spot configurations. The main argument of this section comes from the fact that the space time diagram of spot configurations in a linear CA is substitutive.

Lemma 10 (Main Theorem of [20]). *Let F be any d-dimensional linear CA on a p-group Q. Then there exists a substitution of factor M describing space-time dependency, that is to say, there exists:*

- $M \geq 2$,
- *a finite set E,*
- $e : \mathbb{Z}^d \times \mathbb{N} \to E$,
- $\Psi : E \to (Q \to Q)$,
- *for any $0 \leq a_1, .., a_d, s < M$, a function $\Phi^s_{\vec{a}} : E \to E$, such that,*
$$e(M\vec{z} + \vec{a}, Mt + s) = \Phi^s_{\vec{a}}\big(e(\vec{z}, t)\big)$$
- $\Gamma^t_{\vec{z}} = \Psi(e(\vec{z}, t))$

where $\Gamma^t_{\vec{z}}$ is the space-time dependency function given by:

$$\Gamma^t_{\vec{z}} : q \mapsto \sigma_{\vec{z}} \circ F^t(c^q).$$

Proof (Proof sketch). We review the proof of [20] written in the one-dimensional setting to show that it works straightforwardly for CA over $\mathbb{Z}^d$.

It is sufficient to consider Q a p-group of the form $Q = \mathbb{Z}^D_{p^l}$ because, firstly, any linear CA on a p-group is a subautomaton of some linear CA on a group of this form (Proposition 1 of [20]), and secondly, if the statement of the Lemma holds for some CA, then it clearly holds for any of its subautomata. Then, a linear CA in that case can be viewed as a $D \times D$ matrix whose coefficients are Laurent polynomials with d variables $u_1, \dots, u_d$ and coefficients in $\mathbb{Z}_{p^l}$.

Formally, we denote by $\mathbb{Z}_{p^l}[u_i, u_i^{-1}]_{1 \leq i \leq d}$ the ring of Laurent polynomials with variables $u_1, \dots, u_d$, *i.e.* the ring of linear combinations of monomials made with positive or negative powers of the variables. A monomial corresponds to a vector of $\mathbb{Z}^d$, hence we use the notation $u^{\vec{i}}$ for any $\vec{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$ to denote the monomial $u_1^{i_1} \cdots u_d^{i_d}$. A linear cellular automaton is identified with some $T \in \mathcal{M}_D\big(\mathbb{Z}_{p^l}[u_i, u_i^{-1}]_{1 \leq i \leq d}\big)$ where the coefficient $a_{\vec{z}} \in \mathbb{Z}_{p^l}$ of the monomial $u^{\vec{z}}$ of the coefficient $T_{i,j}$ of the matrix T means intuitively that, when applying the CA, the layer j of cell $\vec{z_0}$ receives $a_{\vec{z}}$ times the content of the layer i of cell $\vec{z} + \vec{z_0}$, and all these individual contributions are summed-up. This matrix representation works well with the powers of F in the sense that F^n is represented by T^n.

We now mimic step by step the proof of [20]:

- by the Cayley-Hamilton theorem (Laurent polynomials form an Abelian ring), the characteristic polynomial of T gives a relation of the form:

$$T^m = \sum_{j=0}^{m-1} \sum_{\vec{i} \in I} \lambda_{\vec{i},j} u^{\vec{i}} T^j$$

  for some m, some finite $I \subseteq \mathbb{Z}^d$, and where $\lambda_{\vec{i},j} \in \mathbb{Z}_{p^l}$;

- first, by standard techniques (binomial theorem and binomial coefficients modulo p), we have the following identity on any commutative algebra of characteristic p^l:

$$\left(\sum_i X_i\right)^{p^{n+l-1}} = \left(\sum_i X_i^{p^n}\right)^{p^{l-1}}$$

  for any positive n. Then, applying this to the expression of T^m obtained above we get:

$$T^{m \cdot p^{n+2(l-1)}} = \left(\sum_{j=0}^{m-1} \left(\sum_{\vec{i} \in I} \lambda_{\vec{i},j}^{p^n} u^{p^n \cdot \vec{i}} \right)^{p^{l-1}} T^{p^{n+l-1} \cdot j} \right)^{p^{l-1}}$$

- noting that the sequence $(\lambda_{\vec{i},j}^{p^n})_n$ is ultimately periodic, with common period N for all $\vec{i}, j$, denoting $k = p^N$ and expanding the above equality, we have:

$$T^{k^n m'} = \sum_{j=0}^{m'-1} \sum_{\vec{i} \in I'} \mu_{\vec{i},j} u^{k^n \vec{i}} T^{k^n j}$$

  for some m' large enough and $n \geq 1$, some finite $I' \subseteq \mathbb{Z}^d$, and where $\mu_{\vec{i},j} \in \mathbb{Z}_{p^l}$;

- we are interested in the space-time dependency function $T_{\vec{i}}^j$ of T which is the matrix of $\mathcal{M}_D(Q)$ corresponding to the terms in $u^{\vec{i}}$ of the matrix T^j, so that $T^j = \sum_{\vec{i}} T_{\vec{i}}^j u^{\vec{i}}$; we therefore have the following:

$$T_{\vec{x}}^{k^n m' + y} = \sum_{\vec{i} \in I'} \sum_{j=0}^{m'-1} \mu_{\vec{i},j} T_{\vec{x} - k^n \vec{i}}^{y + k^n j}$$

- choosing $n = \lfloor \log_k \frac{y}{m'} \rfloor$, we can rewrite it for all $y > m'$ as:

$$T_{\vec{x}}^y = \sum_{\vec{i},j} \mu_{\vec{i},j} T_{\vec{x} + f_{\vec{i},j}(y)}^{g_{\vec{i},j}(y)} \tag{1}$$

  where $\vec{i}$ and j range over fixed finite sets, and where for any $0 \leq t < k$ we have:

1. $g_{\vec{i},j}(y) < y$,
2. $f_{\vec{i},j}(ky+t) = kf_{\vec{i},j}(y)$,
3. $g_{\vec{i},j}(ky+t) = kg_{\vec{i},j}(y) + t$;

- starting from any point $(\vec{x}, y) \in \mathbb{Z}^d \times \mathbb{N}$ and applying recursively the equation above as much as we can, we get:

$$T_{\vec{x}}^{y} = \sum_{\vec{i},j<m'} \alpha_{\vec{i},j}(\vec{x}, y) T_{\vec{i}}^{j} \tag{2}$$

  we take this as a definition of $\alpha_{\vec{i},j}(x, y)$;

- now, by the properties of the functions $f_{\vec{i},j}$ and $g_{\vec{i},j}$ we have that

$$\alpha_{\vec{i},j}(\vec{x}, y) = \alpha_{\vec{0},j}(\vec{x} - \vec{i}, y),$$

  and in the equation (1) the transformation $(\vec{x}, y) \mapsto (k\vec{x} + \vec{s}, ky + t)$ (for $t < k$ and $\|s\|_\infty < k$) applied to the left term $T_{\vec{x}}^{y}$, translates into the same transformation on each term $T_{\vec{a}}^{b}$ of the right hand side; therefore, denoting $\alpha_j = \alpha_{\vec{0},j}$, the recursive application of (1) in (2) can be reproduced identically starting from $T_{k\vec{x}+\vec{s}}^{ky+t}$ so that we have

$$T_{k\vec{x}+\vec{s}}^{ky+t} = \sum_{\vec{i_1},j<m'} \alpha_j(\vec{x} - \vec{i_1}, y) T_{k\vec{i_1}+\vec{s}}^{kj+t} \tag{3}$$

  that we want to compare to the final decomposition of $T_{k\vec{x}+\vec{s}}^{ky+t}$ which is by definition of the α_j:

$$T_{k\vec{x}+\vec{s}}^{ky+t} = \sum_{\vec{i_2},j<m'} \alpha_j(k\vec{x} + \vec{s} - \vec{i_2}, ky + t) T_{\vec{i_2}}^{j} \tag{4}$$

- since $kj + t$ is bounded by $k(m + 1)$, going from (3) to (4) involves only a bounded number of applications of (1) to each term of the right hand side; therefore, there is some finite set $\Delta \subseteq \mathbb{Z}^d$ such that each term $T_{\vec{a}}^{b}$ on the right will produce only terms in $T_{\vec{a}+\vec{i}}^{j}$ for $j < m'$ and $\vec{i} \in \Delta$.

  Therefore, $\alpha_j(k\vec{x} + \vec{s} - \vec{i_2}, ky + t)$ only depends on the $\alpha_j(\vec{x} + \frac{\vec{s}-\vec{i_2}}{k} - \vec{i'}, y)$ for $\vec{i'} \in \Delta$;

- we choose $X \subseteq \mathbb{Z}^d$ large enough so that:
  1. $\vec{i_2} \in X$ implies $\vec{i'} - \frac{\vec{s}-\vec{i_2}}{k} \in X$ for any $\vec{i'} \in \Delta$;
  2. $T_{\vec{i}}^{j}$ is null for $j < m'$ and $\vec{i} \notin X$;

  with such an X we finally set $M = k$ and

$$e(\vec{x}, y) = \big(\alpha_j(\vec{x} - \vec{i}, y) : j < m', \vec{i} \in X\big).$$

The lemma follows. $\square$

The existence of this substitution has strong consequences on the structure of traces: the trace of a finite configuration is determined by a prefix of linear size in the distance of the farthest non-zero cell. Let us first define some notation.

- For $z \in \mathbb{Z}^d$, $\|z\|_\infty = \max\{|z_i| \ : i \in \{1, .., d\}$.
- The *size of a configuration* $c \overset{\infty}{=} \overline{0}$ is the smallest $n \in \mathbb{N}$ such that $c(z) \neq 0$ imply $\|z\|_\infty \leq n$.

Lemma 11. *Let F be any d-dimensional linear CA on a p-group. Let $m > 0$ and denote by T_m the trace function associated to F and m. There exist a function $\lambda : \mathbb{N} \to \mathbb{N}$ with $\lambda \in O(n)$ and such that for any n and for any pair of configurations c_1, c_2 with:*

- *the size of c_i is less than or equal to n,*
- *$T_m(c_1)(t) = T_m(c_2)(t)$ for any $t \leq \lambda(n)$,*

then $T_m(c_1) = T_m(c_2)$.

Proof. First F fullfils the hypothesis of Lemma 10 so we have the existence of the substitution and adopt the notations of the lemma.

Let's focus on the substitution given by the function e and consider $k \geq 0$, $t \geq M^k$, and $\vec{z} \in \mathbb{Z}^d$ with $\|z\|_\infty \leq M^k$. By $k-1$ applications of the substitution we get the following expression for $e(\vec{z}, t)$:

$$e(\vec{z}, t) = \Phi^{t \bmod M}_{\vec{z} \bmod M} \circ \cdots \circ \Phi^{t/M^{k-1} \bmod M}_{\vec{z}/M^{k-1} \bmod M}\big(e(\rho(\vec{z}), t/M^k)\big)$$

where $\|\rho(\vec{z})\|_\infty \leq M$, and where the division/modulus correspond to the standard Euclidean division on $\mathbb{Z}^d$.

The sequence of superscripts in this expression only depends on $t \bmod M^k$. The sequence of subscripts depends only on $\vec{z}$. Therefore we can write this functional dependency of $e(\vec{z}, t)$ on $e(\rho(\vec{z}), t/M^k)$ in the following way:

$$e(\vec{z}, t) = \chi^{t \bmod M^k}_{\vec{z}}\big(e(\rho(\vec{z}), t/M^k)\big). \tag{5}$$

Now consider a time t_0 sufficiently large to see any possible vector of the form $(e(\vec{z_0}, t))_{\|\vec{z_0}\|_\infty \leq M}$ before t_0, precisely:

$$\forall t, \exists t' \leq t_0, \forall \vec{z_0}, \|\vec{z_0}\|_\infty \leq M : e(\vec{z_0}, t) = e(\vec{z_0}, t').$$

Given an index set I, consider any tuple $(\vec{z}_i)_{i \in I}$, with $\|\vec{z}_i\|_\infty \leq M^k$, and any $\mathcal{P} \subseteq E^I$. For any time t, we can define the property $\mathcal{P}_I(t)$ by:

$$\mathcal{P}_I(t) \Leftrightarrow \big(e(\vec{z}_i, t)\big)_{i \in I} \in \mathcal{P}.$$

Claim: if $\mathcal{P}_I(t)$ holds for every $t \leq (t_0 + 1) \cdot M^k$ then $\mathcal{P}_I(t)$ holds for every $t \in \mathbb{N}$.

Indeed, take some time $t > (t_0 + 1) \cdot M^k$. Then by choice of t_0 there exists $t' \leq t_0$ such that:

$$\forall \vec{z_0}, \|\vec{z_0}\|_\infty \leq M : e(\vec{z_0}, t/M^k) = e(\vec{z_0}, t').$$

Now we can choose $t'' \leq (t_0 + 1) \cdot M^k$ with

$$t''/M^k = t' \text{ and}$$
$$t'' \bmod M^k = t \bmod M^k$$

and equation 5 yields the equalities:

$$e(\vec{z_i}, t) = \chi_{\vec{z}}^{t \bmod M^k}\big(e(\rho(\vec{z_i}), t/M^k)\big) = \chi_{\vec{z}}^{t'' \bmod M^k}\big(e(\rho(\vec{z_i}), t')\big) = e(\vec{z_i}, t'').$$

It shows that $\mathcal{P}_I(t) \Leftrightarrow \mathcal{P}_I(t'')$ and the claim follows.

Since the space-time dependency function is completely determined by the substitution e (Lemma 10), the fact that the trace of a finite configuration at time t is null can be expressed by a property of the form $\mathcal{P}_I(t)$. More precisely, for any configuration c of size M^k, we can define

$$\begin{aligned} D &= \{z \in \mathbb{Z}^d \ : \ c(z) \neq 0\}, \\ I &= \bigcup_{z \in D} B_m(z), \\ \mathcal{P} &= \{f \in E^I \ : \ \forall x \in B_m(0), \sum_{z \in D} \Psi(f_{z+x})(c(z)) = 0\}. \end{aligned}$$

We then have that

$$\begin{aligned} \mathcal{P}_I(t) &\Leftrightarrow \forall x \in B_m(0), \sum_{z \in D} \Psi(e(z+x, t))(c(z)) = 0 \\ &\Leftrightarrow \forall x \in B_m(0), \sum_{z \in D} \Gamma^t_{z+x}(c(z)) = 0 \\ &\Leftrightarrow \sum_{z \in D} T_m(c^{c(z)}) = 0 \\ &\Leftrightarrow T_m(c)_t = 0. \end{aligned}$$

We deduce that if $F^t(c)$ is null on $B_m(0)$ until time $(t_0 + 1) \cdot M^k$ then it is null forever. By linearity of F, equality of two traces is equivalent to nullity of their difference. We have thus shown the lemma for $m \geq 0$ by choosing $\lambda(n) = (t_0 + 1) \cdot M^k$ for $k = \lceil log_M(n) \rceil$. □

Theorem 2. *If a CA of dimension $d \geq 2$ is linear over an Abelian group then it is not pre-expansive.*

Proof. First, if G is the Abelian group of the theorem it can be decomposed in a direct product $G = G_p \times G'$ where G_p is a finite p-group for some prime p and G' is a group whose order m is such that p doesn't divide m (structure theorem for finite Abelian groups, see [29]). Then F is isomorphic to $F_p \times F'$ according to Lemma 1, where F_p is linear over G_p. Moreover if F is pre-expansive, then F_p must also be pre-expansive (by Proposition 1). It is therefore sufficient to show the Theorem for p-groups.

Now consider F of dimension $d \geq 2$ linear over a p-group and some $m \geq 0$. By Lemma 11, we know that the trace T_m of a finite configuration of size n is determined by its prefix of size $\lambda(n)$ where $\lambda \in O(n)$. The number of such finite configurations grows like α^{n^d} for some $\alpha > 0$ and the number of prefixes of T_m of length $\lambda(n)$ grows like $\beta^{\lambda(n)}$ for some $\beta > 0$ which depends only on m, G_p and d. Since $d \geq 2$ and λ is linear we deduce for n large enough that two finite configurations of size n have the same trace T_m. Therefore T_m is not pre-injective and by Proposition 1, F is not pre-expansive. □

Note that this does not avoid a priori the existence of a linear CA which is k-expansive for any $k \in \mathbb{N}$ or for infinitely many k.

6.2. Linear CA with $Q = \mathbb{Z}_p$

In general, in a CA with neighborhood $V \subset \mathbb{G}$, we can remark that the influence of the cell 0 is restricted to the set generated by linear combinations of $-V$. More precisely, at time t, its influence is restricted to the following set:

$$-V_t(0) = \left\{ \sum_{i=1}^{t} v_i \mid (\forall i \in \{1,..,t\})\ v_i \in -V \right\}$$

A perturbation in a cell $u \in \mathbb{G}$ can produce a change in the state of cells in $-V_t(u) = u - V_t(0)$ up to time t.

If $\mathbb{G}$ is commutative, for example $\mathbb{G} = \mathbb{Z}^n$ and $V = \{v_1, .., v_m\}$, this set can be computed as follows.

$$\begin{aligned}
-V_t(0) &= \left\{ \sum_{k=1}^{m} n_k(-v_k) \mid \sum_{k=1}^{m} n_k = t \text{ and for each } k, n_k \in \mathbb{N} \right\} \\
&= \left\{ \sum_{k=1}^{m} \frac{n_k}{t}(-tv_k) \mid \sum_{k=1}^{m} n_k = t \text{ and for each } k, n_k \in \mathbb{N} \right\} \\
&\subseteq \left\{ \sum_{k=1}^{m} \lambda_k(-tv_k) \mid \sum_{k=1}^{m} \lambda_k = 1 \text{ and for each } k, \lambda_k \in [0,1], \right\} \\
&\subseteq co(-tV)
\end{aligned}$$

Where $tV = \{tv \mid v \in V\}$, and $co(\cdot)$ stands for the *convex hull* (in $\mathbb{R}^n$).

In the simpler case where $G_p = \mathbb{Z}_p$, any linear CA F can be expressed as

$$F = \sum_{z \in V} a_z \sigma_z,$$

where $(a_z)_{z \in V}$ is a sequence of elements of $\mathbb{Z}_p$. When p is prime, we have Lucas' Lemma, which gives even stronger properties to linear CAs, more precisely.

$$F^{p^k} = \sum_{z \in V} a_z \sigma_{p^k z} \tag{6}$$

The next lemma establishes that the constant of k-expansivity in a linear CA on $\mathbb{Z}_p$ depends only on the radius of the neighborhood. The radius of a neighborhood V is the smallest integer r such that $V \subseteq B_r(0)$.

Lemma 12 (Amplification). *Let F be a linear CA with neighborhood $V \subset \mathbb{Z}^n$ of radius r, with state set $\mathbb{Z}_p$. If there exists a configuration $c \neq \overline{0}$ such that $T_r(c) = 0$, then for any $m \geq r$ there exists a configuration $c' \neq \overline{0}$ such that $T_m(c') = 0$.*

Proof. Let c be such that $T_r(c) = 0$ and let k be such that $m \leq p^k - 1$. We define c' by $c'_{p^k x} = c_x$ for every $x \in \mathbb{Z}^n$ and 0 elsewhere.

From Equation 6, it is easy to see that $F^{tp^k}(c')_{p^k x} = F^t(c)_x$, and 0 elsewhere. Therefore, for every $t \in \mathbb{N}$ and every $v \in B_r(0)$, $F^{tp^k}(c')_{p^k v} = 0$.

Now, between iterations tp^k and $(t+1)p^k$, we know, from the former remarks, that only cells in $\Omega = \bigcup_{x \notin B_r(0)} (p^k x - V_{p^k}(0))$ can have a state different from 0. Since $-V \subseteq B_r(0)$, we have that $-V_{p^k}(0) \subseteq B_{rp^k}(0)$, and the complement of Ω contains $B_{p^k-1}(0)$, which is what we were looking for, in fact,

$$\begin{aligned}
y \quad &\in \quad \Omega = \left(\bigcup_{x \notin B_r} p^k x - V_{p^k}(0) \right) \\
&\Rightarrow \quad (\exists x \notin B_r(0))(\exists v \in -V_{p^k}(0)) \; y = p^k x + v \\
&\Rightarrow \quad ||y|| \geq ||p^k x|| - ||v|| \geq p^k(r+1) - p^k r = p^k \\
&\Rightarrow \quad y \notin B_{p^k-1}(0).
\end{aligned}$$

□

Corollary 3. *Let F be a linear CA in $\mathbb{Z}_p$. It holds:*

- *F is k-expansive, if and only if F is k-expansive with pre-expansivity constant 2^{-r};*
- *F is k-expansive for all $k \in \mathbb{N}$ if and only if F is pre-expansive.*

With this lemma we can establish k-expansivity just by looking at T_r. We will show a CA in that setting which is 1-expansive, 3-expansive and non 2-expansive, and another which is non 1-expansive (and so non k-expansive for every k).

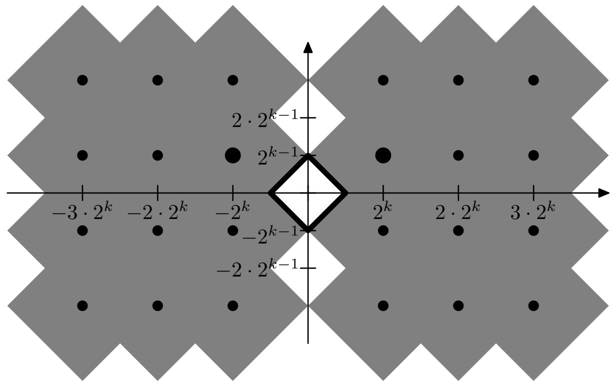

Figure 5: Potentially active cells at iteration $t2^k$ cannot influence $B_{2^{k-1}-1}(0)$ before iteration $(t+1)2^k$. Big dots represent the initially active cells.

6.2.1. The rule $\oplus_2$ with von Neumann neighborhood in $\mathbb{Z}^2$

The rule that simply sums the state of its 5 neighbors in the von Neumann neighborhood: $(0,0),(0,1),(1,0),(0,-1),(-1,0)$ is not 2-expansive. This can be seen through a simple picture: let us suppose that we start with the configuration c that has a '1' in cell $(-2^k,2^{k-1})$ and in cell $(2^k,2^{k-1})$. By symmetry, the vertical line $\{0\}\times\mathbb{Z}$ will be always null. Thus, at iterations $t2^k$ only cells at $2^k(\mathbb{Z}\setminus\{0\})\times\mathbb{Z}$ will be activated. Between iterations $t2^k$ and $(t+1)2^k$ these cells cannot influence the ball $B_{2^{k-1}-1}(0,0)$ (see figure 5) and this ball will have a null trace: $T_{2^{k-1}}(c)=0$.

In order to establish the 3-expansivity of this CA, we will start by proving some lemmas that describe the form of the traces $T_1(\sigma_z(c^1))$ of the evolution of the configuration c^1 at the different points of $\mathbb{Z}^2$. In order to achieve this, we start by computing the partial traces $T_0(\sigma_z(c^1))|_{[0,2^k-1]}$ and $T_0(\sigma_z(c^1))|_{[2^k,2^{k+1}-1]}$. We first give a way for compute them, and afterwards we prove they are effectively the partial traces.

Definition 6. *Given $k\geq 0$ and $z\in B_{2^k-1}(0,0)$, we recursively define $u_k(z)$ and $v_k(z)$ as follows. Let us define $S_k=\{(0,0),(0,2^k),(2^k,0),(0,-2^k),(-2^k,0)\}$, the active cells of iteration 2^k.*

$$u_0(z) = v_0(z) = 1;$$

$$u_k(z) = \begin{cases} u_{k-1}(z)v_{k-1}(z) & \text{if } z\in B_{2^{k-1}-1}(0,0) \\ 0^{2^{k-1}}u_{k-1}(z-x) & \text{if } z\in B_{2^{k-1}-1}(x)\setminus B_{2^{k-1}-1}(0,0) \text{ and } x\in S_{k-1} \\ 0^{2^k} & \text{otherwise} \end{cases}$$

$$v_k(z) = \begin{cases} u_{k-1}(z)u_{k-1}(z) & \text{if } z\in B_{2^{k-1}-1}(0,0) \\ u_{k-1}(z-x)u_{k-1}(z-x) & \text{if } z\in B_{2^{k-1}-1}(x)\setminus B_{2^{k-1}-1}(0,0) \text{ and } x\in S_k \\ 0^{2^k} & \text{otherwise} \end{cases}$$

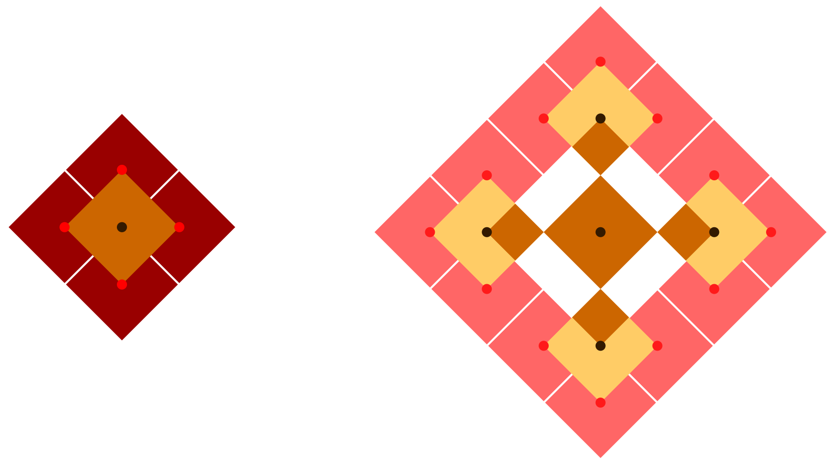

Figure 6: (a) Represents the evolution from iteration 0 to 2^k. Non-null cells at iterations 0 and 2^{k-1} are marked with brown and red dots, respectively (cell (0,0) is active at both instants). The balls of radius 2^{k-1} around these cells are colored with similar colors. (b) Represents the evolution from iteration 2^k to 2^{k+1}. Brown dots represent non-null cells at iterations 2^k and 2^k+2^{k-1}, while red dots represent the cells which are non-null at iteration 2^k+2^{k-1}. The balls of radius 2^{k-1} around these cells are colored with similar colors. Faded colors represent the cells outside the ball of radius 2^k.

Lemma 13. *If $z \in B_{2^k-1}(0,0)$, then $u_k(z)$ and $v_k(z)$ represent the trace of z from 0 to 2^k-1 and from 2^k to $2^{k+1}-1$ respectively.*

Proof. When $k=0$, $B_0(0,0)=\{(0,0)\}$, and the trace of $(0,0)$ is constant and equal to 1.

Let us suppose the lemma true for $k-1 \geq 0$. Let $z \in B_{2^k-1}(0,0)$.

Figure 6(a) depicts the first two cases in the definition of $u_k(z)$, the last one corresponds to cells over the central diagonals.

- Case 1) $z \in B_{2^{k-1}-1}(0,0)$. In this case, the induction hypothesis says that the trace until 2^k-1 is given by $u_{k-1}(z)v_{k-1}(z)$
- Case 2) $z \in B_{2^k-1}(0,0) \setminus B_{2^{k-1}-1}(0,0)$. From 0 to $2^{k-1}-1$, z has not been touched jet, thus its trace until $2^{k-1}-1$ is $0^{2^{k-1}}$. At iteration 2^{k-1}, only the cells in S_{k-1} are in state 1, thus z is influenced by only one of the cells in S_{k-1}, say x, its trace from 2^{k-1} to 2^k-1 is equal to the trace of the cell $z-x$ from 0 to $2^{k-1}-1$, thus by induction again, it is equal to $u_{k-1}(z-x)$.
- Case 3) If z belong to none of these balls, it is over one of the two diagonals that pass by $(0,0)$, and its trace is null.

Figure 6(b) depicts the three cases in the definition of $v_k(z)$.

- Case 1) $z \in B_{2^{k-1}1}(0,0)$. At iteration 2^k, only the cells in S_k are in state 1. Therefore, from 2^k to $2^k+2^{k-1}-1$, z will be influenced only by the cell $(0,0)$, and its trace will be equal to $u_{k-1}(z)$. At iteration 2^k+2^{k-1}, the active cells corresponds to *red* and *brown* cells in Figure 6(b), and again only cell $(0,0)$ reaches z, the process is repeated.

- Case 2) $z \in B_{2^k-1}(0,0) \setminus B_{2^{k-1}-1}(0,0)$. At iteration 2^k, only the cells in S_k are in state 1, thus, before iteration 2^k+2^{k-1}, z is touched only if it is at distance less than 2^{k-1} from one of the cells in S_k, say x, (orange zone in Figure 6(b)), then its trace is equal to $u_{k-1}(z-x)$. At iteration 2^k+2^{k-1}, the active cells are far again, and z is influenced only by x again.

- Case 3) If z belong to none of these balls, its trace is null.

$\square$

This lemma proves that the traces can be obtained through the substitution $u \to uv$ and $v \to uu$. The basic u and v for a given cell z are obtained at iteration 2^{k+1} if $B_{2^k-1}(0,0)$ is the smallest ball containing z. From the next lemma we can conclude the 1-expansiveness of this automaton with pre-expansivity constant equal to 2^{-1}.

Lemma 14. *If $i+j$ is odd and smaller than 2^k, then the trace of the cell $z=(i,j)$, $T_0(\sigma_z(c^1))$ is not null and its first non null index is odd and smaller than 2^k, in particular $u_k(i,j)$ is not null.*

Proof. For $k=1$, the trace of the odd cells inside $B_1(0,0)$ from 0 to 1 is $u=01$, the result holds. Let us assume the result true for $k \geq 1$, and let $z=(i,j)$ be an odd cell in $B_{2^{k+1}-1}(0,0) \setminus B_{2^k-1}(0,0)$. Since the cell is odd, it is not over the diagonals, and it belongs to the ball of radius 2^k of one of the four cells of $S_k \setminus \{(0,0)\}$, say x. Then, by lemma 13, its trace from 2^k to $2^{k+1}-1$ is given by $u_k(z-x)$. Since the cells of S_k are even, $(z-x)$ is also odd, and the conclusion follows by induction. $\square$

Now we will prove several properties that will be useful to prove 3-expansivity.

Lemma 15. *Given $k>0$, if $z \in B_{2^k-1}(0,0)$ then $v_k(z)$ is a square.*

Proof. It is clear from Definition 6 and the fact that $k>0$. $\square$

Lemma 16. *Given $k>0$, if $z \in B_{2^k-1}(0,0) \setminus \{(0,0)\}$ and $u_k(z) \neq 0^{2^k}$, then $u_k(z)$ is not a square.*

Proof. By induction on k. For $k=1$, if z is inside the von Neumann neighborhood of $(0,0)$, thus $u_1(z)=01$ which is not a square. Let us suppose the assertion true for $k-1 \geq 1$. Since $k \geq 1$, from Definition 6, we recognize two cases for $u_k(z)$.

Case 1: $u_k(z) = 0^{2^k} u_{k-1}(z-x)$**, for some** $x \in S_{k-1}$**.** The only way for $u_k(z)$ to be a square is to be equal to 0^{2^k}.

Case 2: $u_k(z) = u_{k-1}(z)v_{k-1}(z)$**.** By the induction hypothesis u is either null or not a square. From Lemma 15 $v_{k-1}(z)$ is a square, then $u_k(z)$ is a square if and only if $u_k(z) = 0^{2^k}$.

□

Lemma 17. *If* $|i| = |j|$ *and* $(i,j) \neq 0$*, then the trace of the cell* (i,j)*,* $T_0(\sigma_{(i,j)}(c^1))$ *is equal to* 0^ω*.*

Proof. Cells in the diagonals systematically falls in the boundaries of the zones given by the substitution, then their traces are systematically assigned equal to 0. □

Lemma 18. *If* $i+j$ *is even, smaller than* 2^k *and the trace of the cell* $z = (i,j)$*,* $T_0(\sigma_z(c^1))$ *is not null, then the first non null index of the trace is even and it is smaller than* 2^k*.*

Proof. For $k = 0$, the trace of the even cell inside $B_0(0,0)$ from 0 to 0 is $u = 1$, the result holds. Let us assume the result true for $k \geq 0$, and let $z = (i,j)$ be an even cell in $B_{2^{k+1}-1}(0,0) \setminus B_{2^k-1}(0,0)$. Since the cell is attained, from Lemma 17, it is not over the diagonals, then it belongs to the ball of radius 2^k of one of the four cells in S_k, say x. Then, its trace from 2^k to $2^{k+1}-1$ is given by $u_k(z-x)$. Since the cells in S_k are even, $z - x$ is also even, and the conclusion follows by induction. □

Now we are ready to prove that this automaton is 3-expansive.

Lemma 19. *If* z_1*,* z_2 *and* z_3 *are three different cells and* $T_0(\sigma_{z_1}(c^1)) + T_0(\sigma_{z_2}(c^1)) + T_0(\sigma_{z_3}(c^1)) = 0^\omega$*, then there exists* $z \in \{z_1, z_2, z_3\}$ *such that* $T_0(\sigma_z(c^1)) = 0^\omega$*.*

Proof. We will prove a stronger assertion:

> If z_1, z_2 and z_3 are three different cells in $B_{2^k-1}(0,0)$ and $u_k(z_1) + u_k(z_2) + u_k(z_3) = 0^{2^k}$, then there exists $z \in \{z_1, z_2, z_3\}$ such that $u_k(z) = 0^{2^k}$.

It is stronger because from Lemmas 14 and 18, if $z \in B_{2^k-1}(0,0)$ and $u_k(z) = 0^{2^k}$, then $T_0(\sigma_z(c^1)) = 0^\omega$.

By contradiction, let z_1, z_2 and z_3 be three different cells in a ball $B_{2^k-1}(0,0)$ such that $u_k(z_1) + u_k(z_2) + u_k(z_3) = 0^{2^k}$ and for all $i \in \{1,2,3\}$, $u_k(z_i) \neq 0^{2^k}$. Let us choose these cells such that k is as small as possible.

Let be t_{z_1}, t_{z_2} and t_{z_3} the indices where the respective traces equals 1 by the first time. It is clear that two of these numbers are equal and smaller than the third. Let us suppose that $t_{z_1} = t_{z_2} < t_{z_3}$.

Case 1: z_1, $z_2 \in B_{2^{k-1}-1}(0,0)$. In this case, the trace of z_3 from 2^{k-1} to $2^k - 1$ is equal to $u_k(z_3 - x)$, for some $x \in S_k$, thus $u_k(z_3) = 0^{2^{k-1}} u_{k-1}(z_3 - x)$. On the other hand, $u_k(z_i) = u_{k-1}(z_i) v_{k-1}(z_i)$ for $i \in \{1, 2\}$. From Lemma 15, $v_{k-1}(z_1)$ and $v_{k-1}(z_2)$ are squares if $k > 0$, which is the case, since $B_0(0,0)$ contains only one cell. Thus $v_{k-1}(z_1) + v_{k-1}(z_2)$ is also a square, then it cannot be equal to $u_{k-1}(z_3 - x)$ which is not a square thanks to Lemma 16.

Case 2: z_1, z_2, $z_3 \notin B_{2^{k-1}-1}(0,0)$. In this case, for each $i \in \{1, 2, 3\}$ there exists some $x_i \in S_{k-1}$ such that $u_k(z_i) = 0^{2^{k-1}} u_{k-1}(z_i - x_i)$. Of course $z_i - x_i \in B_{2^{k-1}-1}(0,0)$ for each $i \in \{1, 2, 3\}$ and $u_{k-1}(z_1 - x_1) + u_{k-1}(z_2 - x_2) + u_{k-1}(z_3 - x_3) = 0^{2^{k-1}}$, which contradicts the minimality of k.

□

With this last lemma we know that if three cells produces a null trace of radius 1, thus one of them has a null trace of radius 0, this means that this cell is even, and its four neighbors are odd. When looking at the neighbors of these cells, their sum is also null, for each if its neighbors. Thus at least one cell must have even neighbors with a null trace, but in this case, two of these cells are odd, and its four even neighbors cannot equal the odd neighbors of the first cell, and then the trace of radius 1 of the sum of the three cells cannot be null.

6.2.2. The rule $\oplus_2$ with triangular neighborhood

The last rule is 1 and 3 expansive, now we present a linear rule which is even not 1-expansive. Thanks to Proposition 3, it implies in particular that is not k-expansive, for every $k \in \mathbb{N}$. It correspond to addition modulo 2 as the last one but with a triangle shaped neighborhood: $N = \{(-1, 1), (1, 1), (0, 0), (0, -1)\}$.

Proposition 13. *$T_2(\sigma_{(0,36)}(c^1))$ is null.*

Proof. We will prove, by induction, that $T_2(\sigma_{(0,36)}(c^1))$ is null from 0 to 2^k. For $k = 0$ to 5 is clear since cell $(0, 36)$ is too far to touch $B_2(0,0)$. At iteration 2^k the only active cells are $(-2^k, -2^k + 36)$, $(2^k, -2^k + 36)$, $(0, 2^k + 36)$ and $(0, 36)$. The first three are too far to touch $B_2(0,0)$ from iterations 2^k to 2^{k+1}. By induction hypothesis $(0, 36)$ does not attain $B_2(0,0)$ in 2^k iterations, thus $B_2(0,0)$ remains null until iteration 2^{k+1}. □

7. Open Problems

We showed in this paper that dynamics like pre-expansivity or k-expansivity can exist without necessarily implying positive expansivity. We also showed that some combinations of the space structure and the local rule structure forbid pre-expansivity (Theorem 2). However, we left many open questions concerning pre-expansivity and k-expansivity:

- is there a pre-expansive cellular automata on $\mathbb{Z}^d$ when $d \geq 2$?

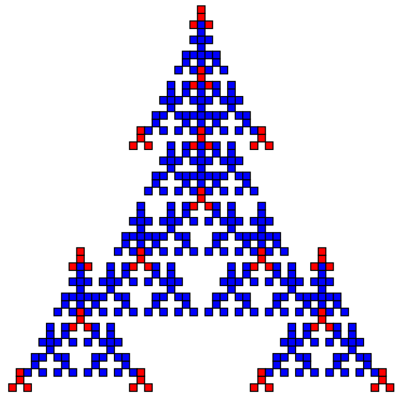

Figure 7: Simulation of $\oplus_2$ with a triangular neighborhood at iteration 25 starting from an initial configuration with a single 1 inside 0s: cells in state 1 are in red, cells that have been in state 1 between iteration 0 and 24 but not at 25 are in blue, and the others cells are not drawn.

- is there a 2-expansive cellular automata on $\mathbb{Z}^2$? on the free group? more generally which is the set of integers k such that a given group admits k-expansive cellular automata?
- are pre-expansive CA always transitive? mixing? open?
- is the property of pre-expansivity decidable?

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