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Lines in bipartite graphs and in 2-metric spaces

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Abstract

The *line generated* by two distinct points, x and y, in a finite metric space $M = (V, d)$, is the set of points given by

$$\{z \in V : d(x, y) = |d(x, z) + d(z, y)| \,\text{or}\, d(x, y) = |d(x, z) - d(z, y)|\}.$$

It is denoted by $\overline{xy}^M$. A 2-set $\{x, y\}$ such that $\overline{xy}^M = V$ is called a *universal pair* and its generated line a *universal line*. Chen and Chvátal conjectured that in any finite metric space either there is a universal line, or there are at least $|V|$ different (nonuniversal) lines. Chvátal proved that this is indeed the case when the metric space has distances in the set $\{0, 1, 2\}$. Aboulker et al proposed the following strengthenings for Chen and Chvátal conjecture in the context of metric spaces induced by finite graphs: First, the number of lines plus the number of bridges of the graph is at least the number of points. Second, the number of lines plus the number of universal pairs is at least the number of points of the space. In this study, we prove that the first conjecture is true for bipartite graphs different from C_4 or $K_{2,3}$, and that the second conjecture is true for metric spaces with distances in the set $\{0, 1, 2\}$.

KEYWORDS
Chen-Chvátal conjecture, graph metric

1 | INTRODUCTION

In a metric space $M = (V, d)$ a *line* defined by two distinct points $x, y \in V$ is the subset of V defined by

$$\overline{xy}^M = \{z \in V : d(x, y) = |d(x, z) + d(z, y)| \text{ or } d(x, y) = |d(x, z) - d(z, y)|\} (\text{see [7]}).$$

A line $\overline{xy}^M$ is *universal* if $\overline{xy}^M = V$; in this case $\{x, y\}$ is a *universal pair*. The number of distinct lines in M is denoted by $\ell(M)$.

In [5], Chen and Chvátal proposed the following conjecture.

Conjecture 1. *Any finite metric space $M = (V, d)$ with at least two points and $\ell(M) < |V|$ has a universal line.*

Conjecture 1 is a generalization of a classical result in Euclidean geometry asserting that every set of n noncollinear points in the Euclidean plane determines at least n distinct lines (see [9]).

The current best lower bound for the number of lines in a metric space with no universal line is $\ell(M) = \Omega(\sqrt{n})$ ([1]).

Although in general the distance function ranges over the nonnegative reals, to prove Conjecture 1, it was observed in [2] that it is enough to consider nonnegative integers. This motivates the definition of k-metric space, with k a positive integer, to be a metric space in which all distances are integral and are at most k. In this context, it was also proved in [1] that if M is a k-metric space, then the previous bound can be improved to $\ell(M) \geq n/(5k)$, for each $k \geq 3$.

One particular metric space with integer distances is the metric space induced by a graph. Here the points are the vertices of the graph and the distance between two vertices is defined by the length of a shortest path between them. To ease the presentation we will refer to the metric space induced by a graph $G = (V, E)$, just as G. Hence, $\overline{xy}^G$ denotes the line defined by two distinct vertices x and y in V.

In [4] and [2] it was proved that Conjecture 1 holds for metric spaces induced by chordal graphs and for distance-hereditary graphs, respectively.

In [3] two new ingredients were considered: *bridges* and *nontrivial modules*. On the one hand, a set of vertices S of a graph G is a *nontrivial module of G* if it has at least two vertices and for every $x \notin S$, $N_G(x) \cap S = \varnothing$, or $S \subseteq N_G(x)$. On the other hand, an edge e of a graph G is a *bridge* if $G - e$ has more connected components than the graph G. Let $\mathrm{BR}(G)$ denote the number of bridges of G. Notice that if ab is a bridge of G, then the line $\overline{ab}^G$ is universal. The main result in [3] is the following, where $|G|$ denotes the number of vertices of G.

Theorem 2 (Theorem 2.1 in Aboulker et al [3]). *Every graph G such that every induced subgraph of G is either a chordal graph, has a cut vertex or a nontrivial module satisfies $\ell(G) + \mathrm{BR}(G) \geq |G|$, unless G is one of the six graphs depicted in Figure 1.*

Given this result, the authors in [3] proposed the following conjecture:

Conjecture 3 (Conjecture 2.2 in Aboulker et al [3]). *There is a finite set of graphs $\mathcal{F}_0$ such that every connected graph $G \notin \mathcal{F}_0$ either has a pendant edge or satisfies $\ell(G) + \mathrm{BR}(G) \geq |G|$.*

In this study, we prove the following.

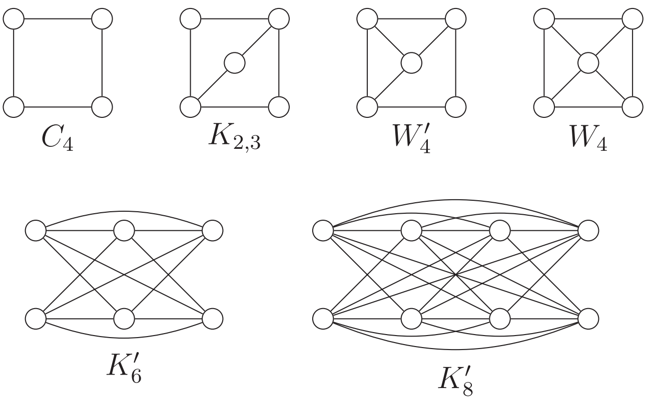

FIGURE 1 Graphs excluded in Theorem 2

Theorem 4. *Let G be a connected bipartite graph with at least three vertices. If* $G \notin \{C_4, K_{2,3}\}$ *then*

$$\ell_2(G) + \mathrm{BR}(G) \geq |G|,$$

where $\ell_2(G)$ *is the number of distinct lines in G, defined by vertices at distance two.*

The proof is based on the study of the lines defined by vertices at distance 2. In this context, we prove two interesting results: first, we prove that given two vertices x and y at distance two in a graph G, the graph induced by $\overline{xy}^G$ either has diameter two or has a cut vertex in $\{x, y\}$. As a consequence, a 2-connected graph G of diameter at least three can not have a universal pair whose vertices are at distance two.

Second, we prove that 2-connected bipartite graphs have more lines than vertices. We do that by counting the lines generated by vertices at distance 2. At first glance, this restriction made the problem harder as it reduces the number of pairs of vertices that can generate lines. However, it also reduces the possibilities for two pairs of vertices to generate the same line. We think that this trade-off can be exploited in other contexts as well, since in general, it is not easy to characterize pairs of vertices that define the same line.

Our result also proves, for bipartite graphs, the following conjecture made by Zwols [11]: if $\ell(G) < |G|$, then either G has a bridge or it contains C_4 as induced subgraph. It also allows us to extend Theorem 2, by adding bipartite graphs as an option for the induced subgraphs.

Notice that graphs of Figure 1 satisfy Conjecture 1 because they have universal lines. Moreover, they have more than one pair of vertices that define universal lines. This is a phenomena that appears in all the examples of graphs with few different lines. Inspired in this observation, the following conjecture was proposed in [3]:

Conjecture 5 (Conjecture 2.3 in Aboulker et al [3]). *Let* $G = (V, E)$ *be a connected graph with at least two vertices. Then,* $\ell(G) + \mathrm{UP}(G) \geq |V|$, *where* $\mathrm{UP}(G)$ *denotes the number of universal pairs in* G.

In this study, we study this conjecture in a more general setting. The main result we get is the following.

Theorem 6. *Every finite 2-metric space with at least three points satisfies* $\ell^*(M) + \mathrm{UP}(M) \geq |V|$, *where* $\ell^*(M)$ *denotes the number of distinct nonuniversal lines in* M *and* $\mathrm{UP}(M)$ *denotes the number of universal pairs in* M.

Notice that when $\mathrm{UP}(M) = 0$ we have that $\ell(M) = \ell^*(M)$. Hence, our result implies that Conjecture 1 holds for 2-metric spaces, a result previously proved in [6,8]. Our proof does not use the result in [6,8], then giving an alternative proof for it.

An important role in this study is played by *pair of twins*. We say that (v, v') is a *pair of twins* of a metric space $M = (V, d)$, if v and v' are two distinct points in V such that $d(v, v') \neq 1$ and for all $u \notin \{v, v'\}$, $d(v, u) = d(u, v')$. In a metric space induced by a connected graph, a pair of twins is usually called a pair of *false* twins.

2 | METRIC SPACES DEFINED BY FINITE GRAPHS

In a metric space induced by a graph G, the distance between two vertices is the length of a shortest path between them. As usual, $N_G(x)$ will denote the neighborhood of the vertex x.

Although our main result is about metric spaces defined by bipartite graphs, we start by proving some preliminaries results which are valid for arbitrary graphs. We shall use them in the proof of our main result.

A crucial point in our development is that we only count lines defined by vertices at distance two. The following lemma shows part of the structure of these lines:

Lemma 7. *Let* x, y *be vertices of* G *at distance* 2. *If two vertices* a *and* b *are such that* $d(b, x) = d(b, y) + d(y, x)$ *and* $d(a, y) = d(a, x) + d(x, y)$, *then any path* P *between* a *and* b *contained in* $\overline{xy}^G$ *contains the set* $\{x, y\}$.

Proof. For each $v \in G$, we define the function $\Delta(v) := d(y, v) - d(x, v)$. Since $d(x, y) = 2$, the function Δ takes only values in $\{-2, -1, 0, 1, 2\}$; moreover, for every $u \in \overline{xy}^G$, $\Delta(u) \in \{-2, 0, 2\}$.

Since $b \in \overline{xy}^G$ and $d(b, x) = d(b, y) + d(y, x)$, $\Delta(b) = -2$. Equivalently, we deduce $\Delta(a) = 2$. Notice that for two adjacent vertices u and v we have $|\Delta(u) - \Delta(v)| \leq 2$; hence, for u and v adjacent and both in $\overline{xy}^G$, we know that $|\Delta(u) - \Delta(v)| \in \{0, 2\}$. We deduce that there exists a vertex c' in P such that $\Delta(c') = 0$. Let us assume that c' is the first vertex in P from b to a such that $\Delta(c') = 0$. Since $c' \in \overline{xy}^G$, $c' \in N_G(x) \cap N_G(y)$ and the neighbor w of c' in P closer to b satisfies $\Delta(w) = -2$ and $d(x, w) \leq 2$; it follows that $d(w, y) = 0$, which implies that $w = y \in P$. With a similar argument applied to a we can prove that $x \in P$. □

Corollary 8. *Let* x, y *be two vertices of* G *at distance* 2. *Let* $z \in \overline{xy}^G$ *with* $z \notin N_G(x) \cap N_G(y)$ *and let* P *be a path between* z *and* x *such that* $P \subseteq \overline{xy}^G$ *and* $y \notin P$. *Then* $d(z, y) = d(z, x) + d(x, y)$.

Proof. Since $z \notin N_G(x) \cap N_G(y)$, $d(x, y) \neq d(x, z) + d(z, y)$. Suppose that $d(z, x) = d(z, y) + d(x, y)$. Since $P \subseteq \overline{xy}^G$, Lemma 7 implies $y \in P$, which is a contradiction. □

Corollary 9. *Let $G = (V, E)$ be a 2-connected graph and let x, y be two vertices of G at distance 2. If $\overline{xy}^G$ is a universal line, then (x, y) is a pair of twins and $V = \{x, y\} \cup N_G(x)$.*

Proof. Suppose that x and y are not twins. Without loss of generality we can assume that there exists a vertex z which is neighbor of x but not of y; $z \in \overline{xy}^G$ because (x, y) is a universal pair. By Corollary 8 we have $d(z, y) = d(z, x) + d(x, y)$. Moreover, every path between z and y contains x, by Lemma 7. This implies that x is a cut vertex; a contradiction because G is a 2-connected graph. Therefore, x and y are twins. When x and y are twins the line $\overline{xy}^G = \{x, y\} \cup N_G(x)$. As it is a universal line, we obtain the second statement. □

Corollary 9 implies that lines defined by vertices at distance 2 are nonuniversal in 2-connected graphs with diameter at least three. This motivates us to count the number of distinct lines defined by vertices at distance two. The set of lines defined by vertices at distance two is denoted by $\mathcal{L}_2^G$ and its cardinality by $\ell_2(G)$. For a subset U of vertices of G we shall denote $\mathcal{L}_2^G(U)$ the set of lines defined in G by pairs of vertices in U at distance two.

The next lemma is a refinement of part (2) in the proof of Theorem 2.1 in [3]. Here, instead of considering arbitrary lines, we only consider lines defined by vertices at distance two. The proof is the same, but we present it here for the sake of completeness.

Lemma 10. *Let G be a bridgeless graph such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$ and $E(G) = E(G_1) \cup E(G_2)$. Then,*

$$\ell_2(G) \geq \ell_2(G_1) + \ell_2(G_2) - 1 + |N_{G_1}(v)||N_{G_2}(v)|.$$

Proof. Let $V_i = V(G_i)$, for $i = 1, 2$. It is easy to see that for each pair $u, v \in V_i$ the following holds:

$$\overline{uv}^G \in \{\overline{uv}^{G_i}, \overline{uv}^{G_i} \cup V_{3-i}\}, \quad \text{for } i = 1, 2. \tag{1}$$

Therefore, only a universal line can belong to the intersection $\mathcal{L}_2^G(V_1) \cap \mathcal{L}_2^G(V_2)$; hence, there are at least $\ell_2(G_1) + \ell_2(G_2) - 1$ lines in $\mathcal{L}_2^G(V_1) \cup \mathcal{L}_2^G(V_2)$.

Now we prove that there are at least $|N_{G_1}(v)||N_{G_2}(v)|$ lines of G not in $\mathcal{L}_2^G(V_1) \cup \mathcal{L}_2^G(V_2)$. Let u_i be a neighbor of v in G_i, for each $i = 1, 2$. We know that $\overline{u_1u_2}^G \in \mathcal{L}_2^G$ and it contains exactly one neighbor u_i of v in G_i, for each $i = 1, 2$. Since v has degree at least two in G_i, for each $i = 1, 2$, as otherwise G has a bridge, at least one neighbor of v in G_i does not belong to $\overline{u_1u_2}^G$; it follows from (1) that $\overline{u_1u_2}^G \notin \mathcal{L}_2^G(V_1) \cup \mathcal{L}_2^G(V_2)$.

Let u_i, v_i be neighbors of v in G_i, for each $i = 1, 2$. We have that $\{u_1, u_2\} \neq \{v_1, v_2\}$ implies $\overline{u_1u_2}^G \neq \overline{v_1v_2}^G$; then, there are at least $|N_{G_1}(v)||N_{G_2}(v)|$ lines in $\mathcal{L}_2^G \backslash (\mathcal{L}_2^G(V_1) \cup \mathcal{L}_2^G(V_2))$. □

2.1 | Bipartite graphs

In this section, we consider metric spaces defined by bipartite graphs.

Our starting point is the following simple observation: given a vertex v in a bipartite graph and two vertices u and w in $N_G(v)$, we have that $N_G(v) \cap \overline{uw}^G = \{u, w\}$; it follows that for each vertex v in a bipartite graph G, $\ell_2(G) \geq \binom{d(v)}{2}$. Hence, *locally*, a vertex v in a bipartite graph is associate with at least $\binom{d(v)}{2}$ different lines.

Two problems appear when one tries to move this idea from local to global. On the one hand, two or more vertices can have the the same neighborhoods (pairs of twins or modules); on the other hand, the same line can be generated by different pairs in several neighborhoods.

Both problems appear in C_4, where $\ell_2(C_4) = \ell(C_4) = 1$. This graph has two pairs of twins and every pair of vertices at distance two generates a universal line.

The first situation also appears in $K_{2,3}$, where $\ell_2(K_{2,3}) = \ell(K_{2,3}) = 4$. In this case, all the vertices in the bigger independent set have the same neighborhood. In the following figure we show two cases where the second problem appears (Figure 2):

However, the following lemma shows that the existence of many lines locally is enough to satisfy Conjecture 3 for complete bipartite graphs.

Lemma 11. *If $G = K_{p,q}$ with $2 \leq p \leq q$, then $\ell_2(G) = \binom{p}{2} + \binom{q}{2}$ unless $p = q = 2$. In particular, if $p + q \geq 6$, then $\ell_2(G) \geq p + q = |G|$.*

Proof. Let X and Y be the maximal independent sets of $K_{p,q}$. Given two vertices a and b of G at distance two we have $\overline{ab}^G = X \cup \{a, b\}$, if $a, b \in Y$ and $\overline{ab}^G = Y \cup \{a, b\}$, if $a, b \in X$. Hence, when $p + q \geq 5$, each pair of vertices in the same independent set defines a distinct line in $\mathcal{L}_2^G$. $\square$

To control the second problem, we need to characterize the pairs of vertices that define the same line. We define the *width* of a line $\ell \in \mathcal{L}_2^G$ as the number of pairs of vertices $\{x, y\}$ with $d(x, y) = 2$ and $\ell = \overline{xy}^G$. We now prove that the existence of lines of width at least two forces some structure of the graph. We use this structure to prove, in the next section, our main result.

Let $N_G^2(y)$ denotes the set of vertices at distance two from y. Given four vertices y, x, s and t, we say that $[yxst]$ holds if there is a shortest path P between y and t containing x and s, such that x belongs to the subpath of P between y and s. Equivalently, $[yxst]$ holds if and only if

$$\begin{aligned} d(y,t) &= d(y,x) + d(x,t) \\ &= d(y,x) + d(x,s) + d(s,t) \\ &= d(y,s) + d(s,t). \end{aligned}$$

To ease the presentation we denote by xPy the subpath of a path P between two of its vertices x and y.

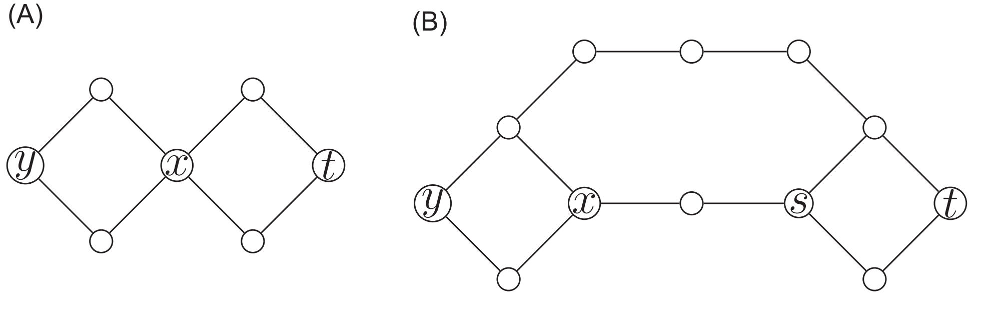

FIGURE 2 A, $\overline{xy} = \overline{xt}$; and B, $\overline{xy} = \overline{st}$

In the next result we need the following definition. A vertex x *dominates* a vertex y (or y is *dominated* by x) if $N_G(y) \subseteq N_G(x) \cup \{x\}$.

Proposition 12. *Let G be a bipartite graph, x, y, s, and t vertices of G such that $d(x, y) = d(s, t) = 2$ and $\overline{xy}^G = \overline{st}^G$. If $[yxst]$ holds, then either y is a cut vertex, or it is dominated by x. The same statement holds for t and x, respectively.*

Moreover, when G is 2-connected we have the following:

(*i*) *For each $z \in N_G^2(y)$ and each $w \in N_G^2(t)$, $d(z, t) = d(x, t) = d(y, s) = d(w, y)$.*
(*ii*) *For each $z \in N_G^2(y)$, $z \neq x$ and for each $v, v \neq t$, for which $d(z, t) = d(z, v) + d(v, t)$ holds, $d(v, s) = d(v, t)$. Similarly, for each $w \in N_G^2(t)$, $w \neq s$ and for each $u, u \neq y$, for which $d(y, w) = d(y, u) + d(u, w)$ holds, $d(u, y) = d(u, x)$.*
(*iii*) *The vertices x and s belongs to an induced cycle of length $2(d(x, s) + 2)$.*

Proof. To prove the first part, suppose the contrary: y is neither a cut vertex nor dominated by x.

Since y is not dominated by x, there exists $b \in N_G(y) \backslash N_G(x)$. From the definition of $\overline{xy}^G$ and the fact that G is bipartite, we have that $b \in \overline{xy}^G$ and $d(b, x) = d(b, y) + d(y, x)$.

Let P be a path from b to t not containing y. It exists as y is not a cut vertex. Since $d(b, x) = d(b, y) + d(y, x)$, $d(t, y) = d(t, x) + d(x, y)$ and y does not belong to P, from Lemma 7, we deduce that P is not completely contained in $\overline{xy}^G$.

Let $w' \in P$ be the closest vertex to b which is not in $\overline{xy}^G$, and let w be its neighbor in P closer to b, which will belong to $\overline{xy}^G$.

If $w \in N_G(x)$, then the path $bPwx$ would be completely contained in $\overline{xy}^G$; but this contradicts Lemma 7 because $d(b, x) = d(b, y) + d(y, x)$ and $d(x, y) = d(x, y) + d(x, x)$. So we may suppose $w \notin N_G(x)$.

Since w' is the first vertex not in $\overline{xy}^G$ of P, the path $ybPw \subseteq \overline{xy}^G$. From Lemma 7 we get that this path does not contain x; otherwise, there is a path contained in $\overline{xv}^G$ between b and x not containing y. It follows from Corollary 8 that

$$d(w, x) = d(w, y) + d(y, x) = d(w, y) + 2. \tag{2}$$

Since the graph G is bipartite and $w' \notin \overline{xy}^G$, we see that $d(x, w') = d(y, w')$. As $|d(y, w) - d(y, w')| \leq 1$ and $|d(x, w) - d(x, w')| \leq 1$, it follows that the only way to satisfy Equation (2) is when

$$d(y, w') = d(y, w) + 1. \tag{3}$$

Since $[yxst]$ holds, there exist a y-s-path Q contained in $\overline{st}^G$ that does not contain t. Hence, the path $wPbQ$ is a w-s-path contained in $\overline{st}^G$ and Corollary 8 implies $d(w, t) = d(w, s) + d(s, t)$. As before, we conclude that

$$d(s, w') = d(s, w) + 1. \tag{4}$$

Since $d(w,t) = d(w,s) + d(s,t)$, a shortest w-s-path must be contained in $\overline{st}^G = \overline{xy}^G$, which implies it contains the vertices y and x (Lemma 7); in particular, we obtain that

$$d(w,s) = d(w,y) + d(y,s). \tag{5}$$

The following chain of equalities holds:

$$\begin{aligned} d(s,w') &= d(s,w) + 1 && \text{(by (4))} \\ &= d(s,y) + d(y,w) + 1 && \text{(by (5))} \\ &= d(s,y) + d(y,w') && \text{(by (3))} \\ &= d(s,x) + d(x,y) + d(y,w') && \\ &= d(s,x) + d(x,y) + d(x,w') && (w' \notin \overline{xy}^G) \\ &\geq d(s,w') + d(x,y) && \end{aligned}$$

which implies $d(x,y) = 0$, a contradiction. Hence, there is no such vertex b and the vertex y is a cut vertex or it is dominated by x.

Now we assume that G is a 2-connected graph.

(i) We first prove that for each $z \in N_G^2(y)$, $d(z,t) = d(x,t)$. It is obvious for $z = x$. For $z \neq x$ we know that $d(x,t) + 2 = d(y,t) \leq d(z,t) + 2$, by the triangle inequality. Hence, $d(x,t) \leq d(z,t)$.

The other inequality comes from the fact that x dominates y; which implies that $d(z,x) = d(z,y) = 2$; then, $z \notin \overline{xy}^G = \overline{st}^G$, then $d(z,t) = d(z,s)$. Since $x \in \overline{st}^G$, $d(z,s) \leq d(x,s) + 2 = d(x,t)$.

By a symmetric argument, for each $w \in N_G^2(t)$ we have that $d(w,y) = d(s,y)$. As $d(y,t) = d(y,s) + 2 = d(x,t) + 2$ we get the result.

(ii) Let v be such that $d(z,t) = d(z,v) + d(v,t)$ holds. On the one hand, since $z \notin \overline{st}^G$, we obtain that

$$d(z,v) + d(v,t) = d(z,t) = d(z,s) \leq d(z,v) + d(v,s)$$

which implies that $d(v,t) \leq d(v,s)$; on the other hand, since s dominates t, we have that $d(v,s) \leq d(v,t)$. Hence, $d(v,s) = d(v,t)$. A symmetric analysis proves the statement for each u satisfying $d(y,w) = d(y,u) + d(u,w)$.

(iii) Let $z \in N_G^2(y)$, $z \neq x$, and let P be a shortest path between z and t. We denote by $\{w\} = P \cap N_G^2(t)$ and by $\{u\} = P \cap N_G(t)$. Notice that by (ii), no vertex in zPw belongs to $\overline{st}^G$.

Let Q be a shortest path between x and s. Then, Q and P are vertex disjoint, because Q is contained in $\overline{st}^G$, which implies that $C = vPuQ$ is a cycle containing x and s, where $v \in N_G(y) \cap N_G(z) \cap N_G(x)$.

Now we prove that the cycle is induced. Assume that there is a chord ab in C. If $a = v$, then $b \in N^2(y)$, contradicting the fact that $d(y,t) = d(x,s) + 4$, when $b \in Q$, or $d(y,t) = d(z,w) + 4$, when $b \in P$. A similar analysis shows that u can not be a vertex of the chord. Hence, we can assume that $a \in P$ and $b \in Q$. From triangular inequality we get that

$$d(z,t)=d(z,s)\le d(z,a)+1+d(b,s)\quad\text{and}\quad d(w,y)=d(w,x)$$
$$\le d(w,a)+1+d(b,x);$$

but, we know from (ii) that $d(z,t)=d(x,s)+2=d(w,y)$. Replacing in the previous inequalities and summing them we obtain

$$2d(x,s)+4\le d(z,a)+d(a,w)+d(x,b)+d(b,s)+2.$$

Since P and Q are shortest paths, it follows that $d(z,w)=d(z,a)+d(a,w)$ and $d(x,s)=d(x,b)+d(b,s)$, which imply

$$d(x,s)+2\le d(z,w)$$

a contradiction because $d(x,s)=d(z,w)$ by (ii).

□

To apply Proposition 12 we need to understand in which situations two pairs of vertices x,y, and s,t, with $d(x,y)=d(s,t)=2$, and generating the same line, do satisfy $[yxst]$.

Lemma 13. *Let y,x,s,t such that $\{x,y\}\ne\{s,t\}$, $d(x,y)=d(s,t)=2$ and $\overline{xy}^G=\overline{st}^G$. If G is 2-connected, bipartite and has no pairs of twins, then* $\max\{d(x,s),d(x,t),d(y,s),d(y,t)\}$ *is at least four. Moreover, if $d(y,t)=\max\{d(x,s),d(x,t),d(y,s),d(y,t)\}$, then $[yxst]$ holds.*

Proof. Let $\beta=\max\{d(x,s),d(x,t),d(y,s),d(y,t)\}$ and let y and t be such that $\beta=d(y,t)$. Since $\{x,y\}\ne\{s,t\}$, we see that $\beta\ge1$. If $\beta=1$, then $d(x,s)=d(x,t)=d(y,s)=d(y,t)=1$. Moreover, since (x,y) is not a pair of twins, there is z which is adjacent to y and not adjacent to x; then, $z\in\overline{xy}^G$ and, since $\beta=1, z\notin\{s,t\}$. As G is bipartite, $d(z,s)=d(z,t)=2$ which implies the contradiction $z\notin\overline{st}^G$, since $\overline{xy}^G=\overline{st}^G$.

When $\beta=2$ we cannot have $d(y,s)=d(y,t)+2$. As $y\in\overline{st}^G$ we get that $d(y,t)=d(y,s)+2$ which implies that $s=y$. Similarly, as $t\in\overline{xy}^G$, we conclude that $x=t$. Thus, we get the contradiction $\{x,y\}=\{s,t\}$.

As before, when $\beta=3$ we cannot have $d(y,s)=d(y,t)+2$, hence, $d(y,t)=d(y,s)+2$ and then $d(y,s)=1$. Similarly, as $t\in\overline{xy}^G$ we conclude that $d(x,t)=1$.

Let $z\in N_G(x)\cap N_G(y)$ which imply $z\in\overline{xy}^G$. As $d(x,t)=1$ and $d(y,s)=1$, we have that $d(z,s),d(z,t)\le2$; but $d(y,t)=3$ implies $d(z,t)=2$. Since $z\in\overline{st}^G$, we get that $z=s$. In a similar way we can prove that $N_G(s)\cap N_G(t)=\{x\}$.

We show that $N_G(s)=\{x,y\}$. In fact, if $z\notin\{x,y\}$ is a neighbor of s, then $z\notin\overline{xy}^G=\overline{st}^G$. Then, $1<d(z,t)\ne d(z,s)+d(s,t)$ which forces that $d(z,t)=2$. This is not possible since G would have a cycle of odd length. A similar argument shows that $N_G(x)=\{s,t\}$.

By Corollary 9, in a 2-connected graph with no pairs of twins there are no universal pairs. We shall get a contradiction by proving that (x,y) is a universal pair. Let us assume that $z\notin\overline{xy}^G=\overline{st}^G$. Then, $z\notin\{x,y,s,t\}$. As $\beta=3$, $N_G(x)=\{s,t\}$ and $N_G(s)=\{y,x\}$, a shortest path P between z and x either contains y or contains t. In the first situation, $z\in\overline{xy}^G$, so we can assume that t is in P, that is to say, $d(z,x)=d(z,t)+1$. By a symmetric argument we can assume that a shortest path between z and s must contains y. Hence, $d(z,s)=d(z,y)+1$.

Since $z \notin \overline{xy}^G$ and G is bipartite, we know that $d(z, x) = d(z, y) \geq 2$. Therefore,

$$d(z, s) = d(z, y) + 1 = d(z, x) + 1 = d(z, t) + 2$$

which contradicts $z \notin \overline{st}^G$.

Therefore $\beta = d(y, t) \geq 4$. Since $t \in \overline{xy}^G$ and $y \in \overline{st}^G$, we get that $d(y, t) = d(x, t) + d(x, y)$ and $d(y, t) = d(y, s) + d(s, t)$.

For $z \in N_G(x) \cap N_G(y)$ we have that $d(y, t) \leq d(z, t) + 1 \leq d(x, t) + 2 = d(y, t)$ and then

$$d(z, t) = d(x, t) + 1 = d(y, t) - 1 \geq 3.$$

We also have that $d(z, s) \leq d(y, s) + 1 = d(y, t) - 1 = d(z, t)$. Since $z \in \overline{st}^G$, we get that $d(z, t) = d(z, s) + 2$. By using this equality we get that

$$d(x, s) \leq d(z, s) + 1 = d(z, t) - 1 = d(x, t).$$

Since $x \in \overline{st}^G$, we get that $d(x, s) + 2 = d(x, t)$ and then $[yxst]$ holds. □

2.2 | Proof of the main result

In this section, we prove our main result. We start by considering 2-connected graphs without pairs of twins.

2.2.1 | 2-connected bipartite graphs with no pairs of twins

Before proving our result we need some definitions. Let G be a 2-connected graph with no pairs of twins and let x, y be two vertices of G such that $w(\overline{xy}^G) > 1$ and $d(x, y) = 2$. From Lemma 13 and Proposition 12 we know that x dominates y or y dominates x, since none of them is a cut vertex. As G has no pairs of twins only one of these options can hold. We define X as the set of vertices x such that there is a vertex $y \in N_G^2(x)$ with $w(\overline{xy}^G) > 1$ and such that x dominates y.

For each $x \in X$, let Y_x be the set of vertices $y \in N_G^2(x)$ with $w(\overline{xy}^G) > 1$ and set $Y = \cup_{x \in X} Y_x$.

Lemma 14. *For X and Y defined above, $X \cap Y = \varnothing$ when G is a 2-connected bipartite graph without pairs of twins.*

Proof. By contradiction, suppose there exists $y \in X \cap Y$. As $y \in Y$, there is $x \in X$ such that $y \in Y_x$. Let $s, t \in V$ such that $\overline{xy}^G = \overline{st}^G$. Since G is a 2-connected bipartite graph without pairs of twins, by Lemma 13 we know that $d(y, t) = \max\{d(x, s), d(x, t), d(y, s), d(y, t)\} \geq 4$ and $[yxst]$ holds. By part (i) of Proposition 12 we know that x dominates y.

Since $y \in X$, there is $z \in N^2(y)$ such that y dominates z. From Proposition 12 we know that $d(z, t) = d(x, t) = d(y, t) - 2$. But then we get the contradiction: $d(y, t) \leq d(z, t) < d(y, t)$. □

From Corollary 9 we know that a 2-connected graph G without pairs of twins has no universal pairs (x, y), with $d(x, y) = 2$. Hence, to prove our result for these graphs, we have to prove that there are at least $|G|$ distinct nonuniversal lines defined by pairs of vertices at distance two.

To this end, we define a function f from the set of vertices of the graph into the set of lines of G. The function f associates to each vertex v a line generated by v and a vertex in $N_G^2(v)$, denoted by $g(v)$. If f is injective, then the number of distinct nonuniversal lines defined by pairs of vertices at distance two is al least the number of vertices, and we are done.

Function f could not be injective for two reasons. The first reason is that there are distinct vertices u and v such that $\{v, g(v)\} = \{u, g(u)\}$. This is equivalent to $g^2(u) = u$.

If $g^2(w) = w$ for no vertex $w \in V$, then f still could fail to be injective if there are distinct vertices u and v such that $\overline{vg(v)}^G = \overline{ug(u)}^G$. From Proposition 12 we know that in this case $v \in X$ and $g(v) \in Y$ or $v \in Y$ and $g(v) \in X$. Hence, either v dominates $g(v)$ or $g(v)$ dominates v.

Therefore, when defining $g(v)$ it is important to try to choose $g(v)$ such that $g^2(v) \neq v$ and neither v dominates $g(v)$ nor $g(v)$ dominates v.

One way to guarantee these two properties is that v and $g(v)$ belong to an induced cycle of length at least six. In Figure 3 we show the case of a cycle of length six. If the vertices of the cycle are labeled $v_0, v_1, \ldots, v_{2k+1}$, then by defining $g(v_i) = v_{i+2}$, for every $i = 0, \ldots, 2k - 1$, $g(v_{2k}) = v_0$ and $g(v_{2k+1}) = v_1$, we get the desired property. Indeed, in this case we have that $g^2(v_i) \neq v_i$, for each $i = 0, \ldots, 2k + 1$ and since the cycle is induced, there is no vertex in the cycle dominated by another vertex in the cycle.

Let W be the set of vertices included in some induced cycle of length at least six. If every vertex G is contained in W, then by applying iteratively the idea presented above, we can define $g(v)$ for each vertex v such that v and $g(v)$ are in an induced cycle of length at least six. Then, we will have that $g^2(v) \neq v$ and $\overline{vg(v)}^G \neq \overline{ug(u)}^G$, because neither v dominates $g(v)$ nor $g(v)$ dominates v. Therefore, for 2-connected bipartite graphs we can prove the conjecture of Zwols mentioned in the introduction since in this case every vertex belongs to W.

When a vertex x does not belong to W, then for each $y \in N^2(x)$ there is an induced cycle of length four that contains x and y. In such situation we can still define g such that $g^2(w) \neq w$, for each $w \in V$. But, there are graphs containing a vertex x such that for each $y \in N^2(x)$ the width of line $\overline{xy}^G$ is at least two. In Figure 4 vertex x has this property. We shall prove that when this happens all pair of vertices at distance two defining the line $\overline{xy}^G$ contains x. Hence, to avoid $f(x) = f(z)$ we only need to define $g(z) \neq x$. In the next lemma we prove that this can always be done since when three distinct vertices x, u, and z are such that $d(x, u) = d(x, z) = d(u, z) = 2$, $x \in X$ and $u, z \in Y_x$, then $u, z \in W$.

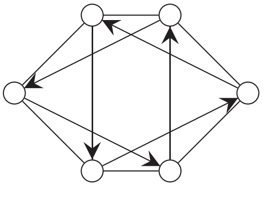

FIGURE 3 Defining function g. An arrow from u to v means that $v = g(u)$

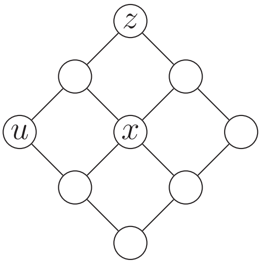

FIGURE 4 The vertex x always generates lines of width 2

Lemma 15. *Let $u, z \in V$ and $x \in X$ be such that $d(u,x) = d(z,u) = d(z,x) = 2$ and $u, z \in Y_x$. Then, $u, z \in W$.*

Proof. From the definition of X and Y_x, there are $s \in X$, $u' \in Y_s$ such that $\overline{su'}^G = \overline{ux}^G$. Let v be a neighbor of z in a shortest path between z and u'. On the one hand, $v \in N_G(z) \subseteq N_G(x)$ since x dominates z, and then $v \in \overline{xu}^G = \overline{su'}^G$; on the other hand, from part (ii) of Proposition 12, $d(v,s) = d(v,u')$; then $d(v,s) = d(v,u') = 1$. This implies that $x = s$ and $v \in N_G(x) \cap N_G(u') \cap N_G(z)$. Since $d(z,u') = 2$, a similar argument shows that there is $w \in N_G(u) \cap N_G(x) \cap N_G(z)$.

Notice that the roles of u and z are symmetric with respect to x. Hence, there is z' such that $\overline{xz}^G = \overline{xz'}^G$ and there are $v' \in N_G(u) \cap N_G(z') \cap N_G(x)$ and $w' \in N_G(u') \cap N_G(x) \cap N_G(z')$. Therefore, the cycle $uwzvu'w'z'v'u$ has length eight and it is an induced cycle because $d(u,u') = d(z,z') = 4$. As u and z belong to this cycle, we get the conclusion. □

Now we can prove our main result for 2-connected bipartite graph without pairs of twins.

Theorem 16. *Let G be a 2-connected bipartite graph without pairs of twins. Then*

$$\ell_2(G) \geq |G|.$$

Proof. Under the assumptions, from Corollary 9 we know that G has no universal pairs at distance two. Moreover, from Lemma 13 we also know that if there are x, y, s, t such that $d(x,y) = d(s,t) = 2$ and $\overline{xy}^G = \overline{st}^G$, then we can assume that $[yxst]$.

We prove that there exists a function $g: V \to V$ satisfying $d(u, g(u)) = 2$ for each $u \in V$, and such that the function $f: V \to \mathcal{L}_2^G$ defined by $f(u) = \overline{ug(u)}^G$ is injective. By Corollary 9 the function f ranges over nonuniversal lines since G has no pairs of twins.

The definition of g is made in several steps:

- We first define g in the set W. Iteratively, we take any induced cycle C of length at least six having vertices where g has not been defined. We define g in all the vertices of the cycle. If for some of them g has been previously defined, we redefine g for these vertices. Let C be a cycle given by $u_0, u_1, \ldots, u_{2k+1}$, with $k \geq 2$. Then

$$g(u_{2k}) = v_0, g(u_{2k+1}) = u_1, \quad \text{and} \quad g(u_i) = u_{i+2}, i \in \{0, 1, \ldots, 2k-1\}.$$

  We have that $w(\overline{u_i g(u_i)}) = 1$, since neither u_i dominates $g(u_i)$, nor u_i is dominated by $g(u_i)$ as C is an induced cycle of size greater than 4. It is clear that $g^2(u_i) \neq u_i$, for each $i = 0, \ldots, 2k+1$.

It is important to point out that by redefining the function g over an induced cycle of length at least six, we keep the property $g^2(u) \neq u$, for every u, for which $g(u)$ is defined, even if u does not belong to the cycle.

To ease the presentation let $Z := \bigcup_{x \in X} N_G^2(x)$ be the set of all the neighbors at distance 2 of vertices in the set X. Notice that $Y \subseteq Z$ and that from Proposition 12 the set $Y^2 := \bigcup_{y \in Y_x} N_G^2(y)$ is included in Z. We also define the set $C := V \backslash (Z \cup X \cup W)$.

- We define g in C. Let $u \in C$:

  If there exists $v \in N_G^2(u)$ with $g(v) \neq u$, then we define $g(u) = v$. Since $u \in C$, we know that $u \notin Z$. Then, $v \notin X$ and $w(f(u)) = 1$.

  If $g(v) = u$ for each $v \in N_G^2(u)$, then we claim that $N_G^2(u)$ has at least two vertices z, z' such that $N_G(u) \cap N_G(z) \cap N_G(z')$ is not empty. In fact, since $u \notin W$ we know that u is contained in a cycle of size four. Let z be the vertex at distance 2 of u in this cycle. Since the other two vertices of the cycle do not form a pair of twins, there exists a vertex z' which is neighbor of only one of them and such that $d(z', u) = 2$. Then, $g(z) = g(z') = u$.

  By using z and z' we define $g(u) = z$ and redefine $g(z) = z'$ (see Figure 5); since $N^2(u) \cap X = \emptyset$, we know that $z, z' \notin X$ and then $w(\overline{zz'}^G) = 1$ and as $u \notin X$, we get $w(\overline{ug(u)}^G) = 1$. Moreover, $g^2(u) = z' \neq u$ and $g^2(z') = z \neq z'$. Notice that with these definitions $g(C) \subseteq W \cup C$.
- Now we define g for $u \in Z \backslash (X \cup W)$ such that there exists $x \in X \cap N_G^2(u)$ with $u \notin Y_x$. In this case, we set $g(u) = x$. We have $w(f(u)) = 1$ because $u \notin Y_x$. If $x \in W$, then $g(x) \neq u$ because $u \notin W$. Otherwise $g(x)$ has not been defined yet, so we will show later that $g(x) \neq u$.
- Now we define g for $u \in Z \backslash (X \cup W)$ such that for all $x \in X \cap N_G^2(u)$ we have that $u \in Y_x$. In this case, we claim that there exists a vertex $z \in N_G^2(u)$ such that $z \notin X$. In effect, if for every $z \in N_G^2(u)$ we have that $z \in X$, then, by the assumption of this case, $u \in Y_z$. By Proposition 12 we obtain that every $z \in N_G^2(u)$ dominates u. This is a contradiction since, when this happens, there is a pair of twins whose common neighborhood is $N_G^2(u) \cup \{u\}$ inside the neighbors of u. Thus, there is $z \in N_G^2(u)$ such that $z \notin X$; so we define $g(u) = z$. Clearly, $w(f(u)) = 1$, by definition.

  If $z \in W$, then $g(z) \in W$. Thus, $g^2(u) \neq u$, since $u \notin W$. By Lemma 15, $z \notin Y_x$, for all $x \in X \cap N^2(u)$. Hence, if $z \in Z \backslash W, g(z)$ was defined in the previous step and satisfies $g(z) \in X$. Hence, $g^2(u) \neq u$, since $u \notin X$.
- The last step is to define g for $u \in X \backslash W$. We pick $y \in Y_u$ arbitrarily and define $g(u) = y$. From the definition of X we conclude that $w(f(u)) > 1$. Notice that $g(y)$ was

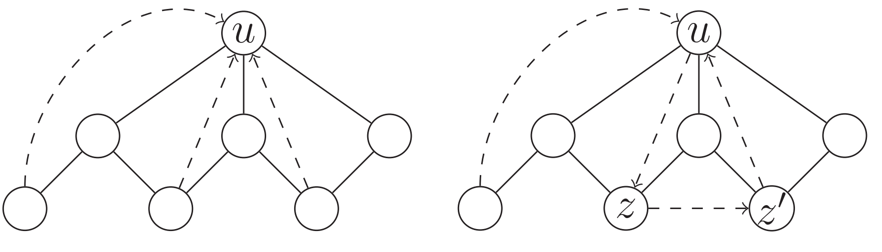

FIGURE 5 Redefining g

already defined in previous steps; moreover, in the previous steps we always have defined $g(v)$ such that $w(f(v)) = 1$; so we have that $g^2(v) \neq v$ for all $v \in Z$ such that $g(v) \in X$.

Finally we prove the injectivity of f. Suppose there exist $u, u' \in X$ such that $f(u) = f(u')$. Since $g^2(u) \neq u$ and $g^2(v) \neq v$, we kwon that $\{u, g(u)\}$ and $\{v, g(v)\}$ generate the same line. Since G is 2-connected and has no pairs of twins, we can assume that $[g(u)uu'g(u')]$ holds. From part (iii) of Proposition 12 we have that u and u' are contained in a cycle of size $2(d(u, u') + 2)$, but $u \notin W$, which implies $d(u, u') = 0$ from where we obtain the injectivity of f. □

2.2.2 | The general case: Proof of Theorem 4

Proof of Theorem 4. We proceed by induction on $n := |G|$. If $n = 3$, then G is a path with three vertices and satisfies $\ell_2(G) = 1$ and $\mathrm{BR}(G) = 2$.

Suppose that G has a pendant edge ab with b a vertex of degree 1. Let $G' := G - b$; if G' is not isomorphic with C_4 or $K_{2,3}$, then by induction hypothesis we obtain that

$$\ell_2(G) + \mathrm{BR}(G) \geq \ell_2(G') + \mathrm{BR}(G') + 1 \geq n.$$

When G' is isomorphic with C_4 or $K_{2,3}$ a case analysis shows that the graph G satisfies $\ell(G) + 1 \geq |G|$. Hence, we can assume in the sequel that G has no pendant edges. If G has a bridge ab, let G_a and G_b be the connected components of $G - ab$ that contain a and b, respectively. As G has no pendant edge, both G_a and G_b have at least two vertices one of them of degree at least two; hence they have at least three vertices. Let G'_a be the subgraph of G induced by $V(G_a) \cup \{b\}$ and G'_b the subgraph of G induced by $V(G_b) \cup \{a\}$. As they have a pendant edge they are neither C_4 nor $K_{2,3}$. For two distinct vertices u and v in G'_a we have that $\overline{uv}^G \in \{\overline{uv}^{G'_a}, \overline{uv}^{G'_a} \cup G_b\}$. The analogous property holds for vertices in G'_b. Then, it follows that:

$$\ell_2(G) \geq \ell_2(G'_a) + \ell_2(G'_b) - 1.$$

On the other hand G'_a and G'_b share a bridge, hence

$$\mathrm{BR}(G) = \mathrm{BR}(G'_a) + \mathrm{BR}(G'_b) - 1.$$

By plugging these two inequalities and using the induction hypothesis we obtain

$$\begin{aligned}\ell_2(G) + \mathrm{BR}(G) &\geq \ell_2(G'_a) + \mathrm{BR}(G'_a) + \ell_2(G'_b) + \mathrm{BR}(G'_b) - 2\\ &\geq |G'_a| + |G'_b| - 2\\ &= n + 2 - 2 = n.\end{aligned}$$

Hence, in what follows we can assume that the graph G is bridgeless. We now consider that G is bridgeless and has a cut vertex v. Let G_1 and G_2 be subgraphs of G

such that $E(G) = E(G_1) \cup E(G_2)$ and $V(G_1) \cap V(G_2) = \{v\}$. Then from Lemma 10 we know that

$$\ell_2(G) \geq \ell_2(G_1) + \ell_2(G_2) + 3.$$

By induction hypothesis this quantity is greater than $|G|$ unless $G_1 = G_2 = C_4$ because $\ell_2(C_4) = 1$ and $\ell_2(K_{2,3}) = 4$. When $G_1 = G_2 = C_4$ we can compute directly the value $\ell_2(G) = 7 = |G|$. Hence, in the rest of the proof we can assume that G is 2-connected.

If G has no pair of twins, then we obtain the conclusion from Theorem 16.

If $G = K_{q,p}$, then from Lemma 11 we get conclusion as $\binom{p}{2} + \binom{q}{2} \geq p + q$ when $p + q \geq 6$. For $p + q \leq 5$, $p + q = 5$ implies that $G = K_{2,3}$ and $p + q = 4$ implies that $G = C_4$.

To end the proof, we assume that G is 2-connected, it has pairs of twins and it is not a complete bipartite graph.

We choose $M = \{v_1, v_2\}$ as a pair of twins, with $G' := G - v_1$ having as few bridges as possible.

The graph $G' \neq C_4$, as otherwise $G = K_{2,3}$, and $G' \neq K_{2,3}$, as otherwise $G = K_{3,3}$ or $G = K_{2,4}$.

From the induction hypothesis, we obtain that $\ell_2(G') + \mathrm{BR}(G') \geq |G'| = |G| - 1$.

Set $\mathcal{L}' = \{\overline{xy}^G : x, y \in V(G'), d(x, y) = 2\}$. Since G' is an isometric subgraph of G (ie, for all $x, y \in V(G')$, the distance between x and y in G' is the same as it is in G), we have, for all $a, b \in V(G')$, $\overline{ab}^G = \overline{ab}^{G'}$ or $\overline{ab}^G = \overline{ab}^{G'} \cup \{v_1\}$. Hence

$$|\mathcal{L}'| = \ell_2(G') \geq |G| - 1 - \mathrm{BR}(G'). \tag{6}$$

Moreover, each line in $\mathcal{L}'$ that contains v_1 must contains v_2.

Since G is not a complete bipartite graph, there is $t \in G - (M \cup N(M))$, with $d(t, v_1) = 2$. It is clear that v_1 is the unique vertex in M which belongs to the line $\overline{v_1 t}^G$. Hence, $\overline{v_1 t}^G \notin \mathcal{L}'$ and thus, if $\mathrm{BR}(G') = 0$, we are done by (6). So we may assume that G' has at least one bridge ab. We will prove that the choice of M guarantees that there is only one bridge in G'. *Claim* 17. For any bridge ab of G', we have that $v_2 \in \{a, b\}$, and there is a connected component in $G' - ab$ whose set of vertices is $\{a, b\} \setminus \{v_2\}$. *Proof.* Set G_a be the connected component that contains a and G_b the one that contains b in the graph $G' - ab$. Without loss of generality we can assume that $v_2 \in G_a$. Since G is bridgeless, v_1 and v_2 must have neighbors in G_a and G_b which implies that $v_2 = a$. Moreover, since G has no cut vertex, it follows that $G_b = \{b\}$. □

Suppose that there exist at least two bridges ab and $a'b'$ in G'. By the claim, we can assume that $a = a' = v_2$ and that $N_G(b) = N_G(b') = \{v_1, v_2\}$. Then, b and b' are twins and the graphs $G - b$ and $G - b'$ are bridgeless, contradicting the choice of M. Hence, G' has only one bridge.

Consider now the line $\overline{v_1 v_2}^G$; since G is bipartite, $N(M)$ is an independent set and thus $\overline{v_1 v_2}^G = M \cup N(M)$. We claim that $\overline{v_1 v_2} \notin \mathcal{L}' \cup \{\overline{v_1 t}^G\}$ which gives the result by (6).

In effect, we first note that $\overline{v_1 v_2}^G \neq \overline{v_1 t}^G$, since $t \notin M \cup N(M)$. So we may assume, for the sake of contradiction, that $\overline{v_1 v_2}^G \in \mathcal{L}'$. Let $x, y \in N(M) \cup M - \{v_1\}$ such that $d(x, y) = 2$ and $\overline{xy}^G = M \cup N(M)$. Then, $x, y \in N(M)$ and $\overline{xy}^G \cap N(M) = \{x, y\}$. Hence,

$N(M) = \{x, y\}$. As G' has a bridge, we can assume that it is v_2x. This leads to a contradiction, since when v_2x is a bridge, y is a cut vertex of G. □

3 | METRIC SPACE WITH FEW DISTANCES

Let $M = (V, d)$ be a metric space. Let $a \in V$ and let $V^{-a} := V \backslash \{a\}$. The set V^{-a} endowed with the restriction of d to V^{-a} is a metric space that, in this study, we shall refer to as $M^{-a} = (V^{-a}, d^{-a})$.

Notice that for a metric space M defined by a graph, the metric space M^{-a} may not be the same as the metric space defined for the subgraph obtained after removing vertex a.

Recall that $\ell^*(M)$ denotes the number of distinct nonuniversal lines in M. In metric spaces we have the following relation between its lines and the lines of its subspaces.

Lemma 18. *For every metric space $M = (V, d)$, $\ell^*(M) \geq \ell^*(M^{-a})$, for any $a \in V$.*

Proof. Let $V' := V^{-a}, d' := d^{-a}$, and $M' := M^{-a}$. Let $x, x', y, y' \in V'$ such that $l := \overline{xy}^{M'} \neq l' := \overline{x'y'}^{M'}$. Since these lines are different, we can assume there exists a point $z \in V'$ such that $z \in l \backslash l'$. Since the distance between points in V' does not change in V, it follows that $d(x, y) = |d(x, z) \pm d(z, y)|$ and $d(x', y') \neq |d(x', z) \pm d(z, y')|$ which implies that $\overline{xy}^M \neq \overline{x'y'}^M$.

Hence, two different lines in M' extend to two different lines in M. Therefore, $\ell^*(M) \geq \ell^*(M')$. □

Let us recall that (v, v') is a *pair of twins* of a metric space $M = (V, d)$, if v and v' are two distinct points in V such that $d(v, v') \neq 1$ and for all $u \notin \{v, v'\}$, $d(v, u) = d(u, v')$.

The symmetric role of vertices in a pair of twins with respect to the distance function is partially described in the following lemma.

Lemma 19. *Let (v, v') be a pair of twins on $M = (V, d)$ and let x, y two distinct points in $V^{-v'}$. If $v \notin \{x, y\}$, then $v \in \overline{xy}^M$ if and only if $v' \in \overline{xy}^M$.*

Proof. By definition $v \in \overline{xy}^M$ if and only if $d(x, y) = |d(x, v) \pm d(v, y)|$; but as (v, v') is a pair of twins, we can replace v in previous equality by v' and we get the result. □

To ease the presentation we denote by $\mathcal{M}^*$ the set of all metric spaces satisfying

$$\ell^*(M) + \mathrm{UP}(M) \geq |M|.$$

One can check that every metric space with three points belongs to $\mathcal{M}^*$.

Now we prove that a metric space with at least three points which is minimal not in $\mathcal{M}^*$ cannot contain a pair of twins (v, v') such that, for every $u \in V \backslash \{v, v'\}$, $d(u, v) = 1$. For the previous analysis we know that such metric space has at least four points.

Proposition 20. *Let $M = (V, d)$ be a minimal metric space not in $\mathcal{M}^*$ with at least three points. If (v, v') is a pair of twins of M, then there is $u \in V \backslash \{v, v'\}$ such that $d(v, u) \neq 1$.*

Proof. For the sake of contradiction, let M be a minimal metric space not in $\mathcal{M}^*$ and let (v, v') be a pair of twins of V such that for each $u \in V \backslash \{v, v'\}$, $d(v, u) = 1$. Then M has at least four points. Since $d(v, v') \neq 0, 1$, we have that $d(v, v') \geq 2$. As M has at least four points, there is $u \notin \{v, v'\}$. Then, for such u we have $d(v, u) + d(u, v') \leq 2$, which implies that $d(v, v') = 2$. Hence,

$$\overline{vv'}^M = \{v, v'\} \cup \{u \in V: d(u, v) = d(u, v') = 1\} = V.$$

Thus, (v, v') is a universal pair of M.

Let $M' = M^{-v'}$. By the minimality of M, the space M' belongs to $\mathcal{M}^*$. Hence, $\ell^*(M') + \mathrm{UP}(M') \geq |V| - 1$. From Lemma 18, we have that $\ell^*(M) \geq \ell^*(M')$. As $M \notin \mathcal{M}^*$, we see that $|V| - 1 \geq \ell^*(M) + \mathrm{UP}(M)$. Hence, $\mathrm{UP}(M') \geq \mathrm{UP}(M)$.

To get the contradiction we prove that $\mathrm{UP}(M) > \mathrm{UP}(M')$. Let (x, y) be a universal pair in M'. We prove that it is also universal in M. By Lemma 19 this is immediate if $v \notin \{x, y\}$. So, we can assume that $x = v$. As we know that $d(v, y) = 1 = d(v', y)$ and $d(v, v') = 2$, we get that $v' \in \overline{xy}^M$, thus (x, y) is a universal pair in M as well. To prove the strict inequality notice that (v, v') is a universal pair in M but not in M'. □

3.1 | 2-metric spaces

In this section, we prove that 2-metric spaces with at least three points belong to $\mathcal{M}^*$. We first study the case when the metric space has no pairs of twins. To do that, we fix a point v of the metric space and we count the different lines defined by v and the other vertices of the metric space.

The following lemma summarizes the restrictions on a 2-metric space M appearing when there are repeated lines. The first statement appears in [6].

Lemma 21. *Let $M = (V, d)$ be a 2-metric space. Let v, x, y, z points in V.*

(i) If v, x, y are distinct, then $\overline{vx}^M = \overline{vy}^M$ implies $d(v, y) \neq d(v, x)$ or (x, y) is a pair of twins with $d(v, x) = d(v, y) = 1$.
(ii) If $d(v, y) = 2$, then the only point in $\overline{vy}^M$ at distance two from v is y.
(iii) If $d(v, y) = 2$, $d(v, x) = 1$, $\overline{vy}^M = \overline{vx}^M$, $d(v, z) = 2$, and $d(x, z) = 1$, then $z = y$.

Proof. The first statement was proved in [6]. The second statement is direct because if a point u satisfies $d(v, u) = d(v, y) = 2$ and $u \neq y$, then $u \notin \overline{vy}^M$ by definition. For the third statement, it is immediate that $z \in \overline{vx}^M$, since $d(v, z) = 2 = d(v, x) + d(x, z)$. From the second statement we get that $z = y$, since $\overline{vx}^M = \overline{vy}^M$ and $d(v, y) = 2$. □

Let $M = (V, d)$ be a 2-metric space and $v \in V$. We define the sets

$$F = \{x \in V : d(x, v) = 1\} \quad \text{and} \quad S = \{y \in V : d(v, y) = 2\}.$$

Notice that $\{\{v\}, F, S\}$ is a partition of the set V. In the rest of this section, we always will consider this partition, that is to say, v is fixed for the discussion.

We consider the following sets of lines:

- $vF := \{\overline{vx}^M : x \in F, \overline{vx}^M \neq V\}$.
- $vS := \{\overline{vy}^M : y \in S, \overline{vy}^M \neq V\}$.
- $S^* := \{\overline{yw}^M : y, w \in S, y \neq w\}$.
- $FS: =\{\overline{xz}^M : x \in F, z \in S, \quad \text{s.t.} \quad \exists\, y \in S, \overline{vx}^M = \overline{vy}^M, z \notin \overline{xy}^M\}$.

We give example of lines in these sets in the metric space defined by the graph of Figure 6.

- $\overline{va}^M = \{v, a, b, d\} \in vF$.
- $\overline{vd}^M = \{v, a, b, d\} \in vS$.
- $\overline{de}^M = \{d, e, b\} \in S^*$.
- $\overline{ad}^M = \{a, d, v, b, c\}$.
- $\overline{ae}^M = \{a, e, c\} \in FS$.

With previous notation we have the following properties that we shall use further on.

Proposition 22. *Let $M = (V, d)$ be a 2-metric space.*

(*i*) *Let* UP_F *(resp.* UP_S*) be the number of universal pairs (v, u) with $u \in F$ (resp. $u \in S$) and let* UP^{-v} *be the number of universal pairs (u, w) with $v \notin \{u, w\}$. Then,* $\mathrm{UP}(M) = \mathrm{UP}^{-v} + \mathrm{UP}_S + \mathrm{UP}_F$, $|vS| = |S| - \mathrm{UP}_S$, *and when M has no pairs of twins,* $|vF| = |F| - \mathrm{UP}_F$.
(*ii*) $\forall\, \ell \in FS, \exists\, z \in S$ *such that* $\ell \subseteq F \cup \{z\}$. *In particular, if* $FS \neq \emptyset$, *then* $|S| \geq 2$.
(*iii*) $(FS \cup S^*) \cap (vS \cup vF) = \emptyset$.
(*iv*) $|FS| + \mathrm{UP}^{-v} \geq |vF \cap vS|$.
(*v*) $|vF \cup vS \cup FS| + \mathrm{UP}(M) \geq n - 1$, *when M has no pairs of twins.*
(*vi*) $S^* \cap FS = \emptyset$.

Proof.

(i) Direct from Lemma 21.
(ii) Let $\ell \in FS$, then there exist $x \in F, y \in S$, and $z \in S$ such that $\ell = \overline{xz}^M$, $\overline{vx}^M = \overline{vy}^M$ and $z \notin \overline{xy}^M$.
First, notice that since $y \in \overline{vx}^M$, we get that $d(x, y) = d(v, y) - d(v, x) = 1$.
We claim that $d(x, z) = 2$. In effect, if $d(x, z) = 1$, then $d(y, z) = 1$ because $z \notin \overline{xy}^M$; but this would imply that $z \in \overline{vx}^M$ and $z \notin \overline{vy}^M$, which is a contradiction, since these lines are equal.
We have by definition that

$$\overline{xz}^M = \{x, z\} \cup \{u \colon d(u, x) = d(u, z) = 1\},$$

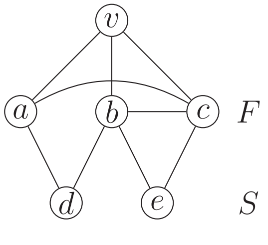

FIGURE 6 The sets F and S defined by a graph

which implies that $v \notin \overline{xz}$.

Now we prove that $\overline{xz}^M \cap S = \{z\}$. By contradiction, suppose there exists a point $u \in \overline{xz}^M \cap S$, with $u \neq z$; it follows that $d(v, u) = 2$ and $d(x, u) = d(u, z) = 1$ which implies that $u \in \overline{vx}^M = \overline{vy}^M$; from part (iii) of Lemma 21 we get that $u = y$, a contradiction, since $d(z, y) = 2$. Hence, $\overline{xz}^M \subseteq F \cup \{z\}$.

Finally, if $FS \neq \emptyset$, there exists a point $z \in S$, which, by definition, is different from y, and then $|S| \geq 2$.

(iii) On the one hand, every line $\ell \in vF \cup vS$ contains the point v by definition; on the other hand, $v \notin \ell'$ when $\ell' \in S^*$ and from part (ii) we get that lines in FS do not contain v.

(iv) Let $y_1, \ldots, y_r \in S$ such that for each $i = 1, \ldots, r$ there exists $x_i \in F$ with $\overline{vx_i}^M = \overline{vy_i}^M$ and $vF \cap vS = \{\overline{vy_i}^M : i = 1, \ldots, r\}$. From part (iii) of Lemma 21 we deduce that if $x_i = x_j$, then $i = j$. Let $I \subseteq \{1, \ldots, r\}$ be the set of all indices such that (x_i, y_i) is a universal pair of M. For each $i \notin I$ there exists a vertex $z_i \notin \overline{x_i y_i}^M$. We claim that $z_i \in S$. In effect, suppose that $z_i \in F$; on the one hand if $d(z_i, y_i) = d(z_i, x_i) = 1$, then $z_i \in \overline{vy_i}^M \backslash \overline{vx_i}$, a contradiction; on the other hand if $d(z_i, y_i) = d(z_i, x_i) = 2$, then $z_i \in \overline{vx_i}^M \backslash \overline{vy_i}$, a contradiction again. Hence $z_i \in S$ and $\overline{x_i z_i}^M \in FS$.

From part (ii) we get that $d(x_i, z_i) = 2$ and $\overline{x_i z_i}^M \subseteq F \cup \{z_i\}$. We shall prove that all the lines defined in this way are different. In effect, suppose there exist i, j such that $\overline{x_j z_j}^M = \overline{x_i z_i}^M$; on the one hand, it follows from (ii) that $z_i = z_j$; on the other hand, from part (ii) of Lemma 21 it follows that $x_i = x_j$, since $d(x_i, z_i) = d(x_j, z_j) = 2$.

Therefore, $|FS| \geq r - |I| \geq |vF \cap vS| - \mathrm{UP}^{-v}$.

(v) Since M has no pairs of twins, from parts (iii) and (i), we get that

$$\begin{aligned}
|vF \cup vS \cup FS| + \mathrm{UP}(M) &= |vF| + |vS| - |vF \cap vS| + |FS| + \mathrm{UP}(M) \\
&= |F| - \mathrm{UP}_F + |S| - \mathrm{UP}_S - |vF \cap vS| + |FS| \\
&\quad + \mathrm{UP}(M) \\
&= |F| + |S| - |vF \cap vS| + |FS| + \mathrm{UP}^{-v},
\end{aligned}$$

since $\mathrm{UP}(M) = \mathrm{UP}^{-v} + \mathrm{UP}_F + \mathrm{UP}_S$.

From part (iv) we get that $|FS| + \mathrm{UP}^{-v} - |vF \cap vS| \geq 0$, which implies the conclusion, since $|F| + |S| = n - 1$.

(vi) Let $\ell \in FS \cap S^*$. On the one hand $|\ell \cap S| \geq 2$ by definition; on the other hand from (ii) we get that $\ell \cap S = \{z\}$ which implies that $FS \cap S^* = \emptyset$. □

Proposition 23. *A 2-metric space with at least three points and no pairs of twins belongs to $\mathcal{M}^*$.*

Proof. If all distance in M are 0 or 1, then every pair of points defines a different line and the result is immediate. Otherwise, there is $v \in V$ such that S is not empty.

From part (v) of Proposition 22 we get that for such v,

$$|vS \cup vF \cup FS| + \mathrm{UP}(M) \geq n - 1.$$

So we only need to find a nonuniversal line not in $vS \cup vF \cup FS$. From parts (iii) and (vi) of Proposition 22, we see that $S^* \cap (FS \cup vF \cup vS) = \emptyset$. Hence, if S^* is not empty we are done.

Let us assume that $S^* = \emptyset$. Hence, S has exactly one element y and the set FS is empty. Since (v, y) is not a pair of twins, there is some $x \in F$ such that $d(x, y) = 2$. In particular, $v \notin \overline{xy}^M$, since $d(v, y) = d(x, y) = 2$. Thus, $\overline{xy}^M \notin vF \cup vS$ and (x, y) is not a universal pair. Therefore, the line $\overline{xy}^M$ belongs to $\ell^*(M)\setminus(vF \cup vS)$ which finishes the proof. □

Now we are in position of proving Theorem 6.

Proof of Theorem 6. For the sake of contradiction, let $M = (V, d)$ a 2-metric space which is minimal not in $\mathcal{M}^*$. Then, M has at least four points. From Proposition 23 we can assume that M has a pair of twins (v, v').

Let $M' = M^{-v'}$ and $V' = V^{-v'}$. Since M is minimal not in $\mathcal{M}^*$ and M' has at least three points we have that

$$\ell^*(M') + \mathrm{UP}(M') \geq n - 1 \geq \ell^*(M) + \mathrm{UP}(M). \tag{7}$$

Let

$$U := \{u \in V' : d(v, u) = 2, \overline{vu}^{M'} = V'\}$$

and

$$W := \{u \in V' : d(v, u) = 2, \overline{vu}^{M'} \neq V'\}.$$

Notice that

$$V = \{v, v'\} \cup \{z : d(v, z) = d(v', z) = 1\} \cup U \cup W.$$

As M is minimal not in $\mathcal{M}^*$, Proposition 20 implies that $|U| + |W| > 0$.

For each point $u \in U \cup W$, the line $\overline{v'u}^M$ contains v' and does not contain v. Part (ii) of Lemma 21 implies that all these lines are distinct and Lemma 19 implies that none of these lines can be generated by two points in V'. Additionally, when $u \in U$, the line $\overline{vu}^{M'}$ is not counted in $\ell^*(M')$ because it is universal in M'. Hence, we get

$$\ell^*(M) \geq \ell^*(M') + 2|U| + |W|.$$

Plugging this inequality with (7) we get

$$\mathrm{UP}(M') - \mathrm{UP}(M) \geq 2|U| + |W|. \tag{8}$$

Let (x, y) be any universal pair of M'. From Lemma 19 we deduce that when $\{x, y\} \cap \{v, v'\} = \emptyset$, (x, y) is also a universal pair in M. This is also true when $x = v$ and $d(v, y) = 1$ since $d(v', v) = 2 = d(v, y) + d(y, v')$. Hence, if (x, y) is not a universal pair in M, we can assume that $x = v$ and $y \in U \cup W$. By definition of W, if $y \in W$, then $\overline{vy}^{M'} \neq V'$. Thus, $y \in U$. Therefore, $|U| \geq \mathrm{UP}(M') - \mathrm{UP}(M)$. By replacing in (8) we obtain

$$0 \geq |U| + |W| > 0$$

which is a contradiction. □

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