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Differential and integral proportional calculus: how to find a primitive for $f(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2}$

William Campillay-Llanos[a], Felipe Guevara[b], Manuel Pinto[c] and Ricardo Torres[d]

[a]Departamento de Matemáticas, Física y Estadística, Universidad Católica del Maule, Talca, Chile; [b]Departamento de Matemática, Universidad de Atacama, Copiapó , Chile; [c]Departamento de Matemática, Universidad de Chile, Santiago, Chile; [d]Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile

ABSTRACT

We present a type of arithmetic called Proportional Arithmetic. The main properties and objects that emerge with this way of operating quantities are exposed. Finally, the antiderivative and the indefinite integral are defined in order to calculate the primitive of $f(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2}$ in the Proportional context.

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1. Introduction

The problem of finding the primitive $g(x)$ of

$$\int \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2}\,\mathrm{d}x, \tag{1}$$

i.e. a function $g(x)$ such that $g'(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2}$ has been an interesting research topic from mathematicians like *Joseph Liouville (1809–1882)*. He created the theory necessary to study this problem in a series of papers published between 1833 and 1841 (see Liouville, 1833, 1834, 1837, 1838, 1840). Actually, this discipline is called *integration in finite elementary terms*. He proved that it is impossible to find a primitive as an *elementary function*, meaning that it cannot be represented as a combination of polynomial, trigonomic or exponential functions. In that case, we call to $f(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2}$, a *non-elemental function*. This example is a particular case of the following fact: if $p(x)$ is a polynomial of degree ≥ 2, then the integral

$$\int e^{p(x)}\,\mathrm{d}x$$

is non-elemental. For example, we can consider $p(x) = -(x^2/2)$. This type of integrals occurs often in probability and statistics (Kasper, 1980; Ritt, 1948; Rosenlicht, 1972). Some

CONTACT William Campillay-Llanos williamcampillay@gmail.com

other examples of non-elemental integrals are

$$\int e^{e^x}\,\mathrm{d}x, \quad \int \log(\log(x))\,\mathrm{d}x, \quad \int e^x \log(x)\,\mathrm{d}x, \quad \int \frac{\sin(x)}{x}\,\mathrm{d}x.$$

In order to find a primitive for (1), we will define a type of arithmetic called *Proportional Arithmetic*. We consider the multiplication operation as a basic way to add and the quotient as the natural way of comparing quantities. That is, if $a, b \in \mathbb{R}^+_{>0}$ (Set of real numbers strictly greater than zero), we will say that

$$a = b \Leftrightarrow \frac{a}{b} = 1.$$

From this point of view, we present a type of Non-Newtonian Calculus, called *Proportional Calculus* (Campillay-Llanos & Pinto, 2013; Córdova-Lepe & Pinto, 2009). This work presents the proportional algebra constructed considering the notion of bijection and the traditional arithmetic.

Given a set X, the set of real numbers $\mathbb{R}$ and a bijection on $f : X \to Y \subseteq R$, we say that f defines an arithmetic if the following four operations are defined as follows:

$$x \oplus y = f^{-1}(f(x) + f(y)), \tag{2}$$

$$x \ominus y = f^{-1}(f(x) - f(y)), \tag{3}$$

$$x \odot y = f^{-1}(f(x)f(y)), \tag{4}$$

$$x \oslash y = f^{-1}(f(x)/f(y)). \tag{5}$$

If f is considered as the identity function and X as the set of real numbers, Equations (2)–(5) form the four operations studied at school, i.e. the traditional arithmetic. These ideas can contribute to consolidate concepts and techniques of the traditional differential and integral calculus courses.

2. Proportional algebra

In the case that f is defined as the natural logarithm function and X be the set of positive real numbers, the defined arithmetic is called *proportional arithmetic* (Campillay-Llanos, 2007; Córdova-Lepe, 2006), and the following operations hold:

$$\left.\begin{aligned} x \oplus y &= xy, \\ x \ominus y &= \frac{x}{y}, \\ x \odot y &= x^{\ln(y)}, \\ x \oslash y &= x^{1/\ln(y)}, \quad y \neq 1. \end{aligned}\right\}$$

With this new arithmetic, we can check the following properties of the operation $\odot$ just using its definition:

Proposition 2.1: *Let $a, b, c \in \mathbb{R}^+_{>0}$*

- *Commutativity:*

$$\begin{aligned} a \odot b = a^{\ln(b)} = a^{\ln(b)/\ln(e)} = a^{1/(\ln(b)/\ln(e))} = a^{1/\log_b e}, \\ &= b^{\log_b(a)/\log_b(e)}, \\ &= b^{\ln(a)}, \\ &= b \odot a. \end{aligned}$$

- *Associativity:*

$$\begin{aligned} a \odot (b \odot c) &= a \odot (b^{\ln(c)}), \\ &= a^{\ln(b)\ln(c)}, \\ &= (a^{\ln(b)})^{\ln(c)}, \\ &= (a \odot b) \odot c. \end{aligned}$$

- *Neutral element for $\odot$: There exists a positive real number e such that for all positive real number a we have $a \odot e = a$. this can be seen as follows:*

$$\begin{aligned} a \odot x &= a, \\ a^{\ln(x)} &= a, \quad \textit{Injectivity of function } \ln(x) \\ \ln(x) &= 1, \quad \textit{Definition of } \ln \\ x &= e. \end{aligned}$$

- *Inverse element: For all positive real number a, there is a positive real number x such that $a \odot x = e$*

$$\begin{aligned} a \odot x &= e, \quad \textit{Definition of exponentiation} \\ x \odot a &= e, \quad \textit{Commutativity} \\ x^{\ln(a)} &= e, \quad \textit{Definition of exponentiation} \\ x &= e^{1/\ln(a)}. \quad \textit{Applying } \ln \textit{ to both members} \end{aligned}$$

  Hence, we define the inverse of exponentiation of a as

$$a^{\{-1\}} = e^{1/\ln(a)}.$$

  As a consequence, it is important to note that $a = 1$ has no inverse associated with the $\odot$ operation. In other words, a = 1 is the traditional 'zero' for the proportional calculus.

We can also obtain the following properties of the proportional product $\odot$:

Proposition 2.2: *Let $a, b \in \mathbb{R}^+_{>0}$. Then:*

(1) $a \odot b^{\{-1\}} = a \oslash b$.
(2) $(a^{\{-1\}})^{\{-1\}} = a$.

(3) $\ln(a \odot b) = \ln(a)\ln(b)$.
(4) $(a \odot b)^{\{-1\}} = a^{\{-1\}} \odot e^{1/\ln(b)} = a^{\{-1\}} \odot b^{\{-1\}}$.

Proof: (1) $a \odot b^{\{-1\}} = a \odot e^{1/\ln(b)} = a^{1/\ln(b)} = a \oslash b$.
(2) $(a^{\{-1\}})^{\{-1\}} = (e^{1/\ln(a)})^{-1} = e^{1/(\ln(e^{1/\ln(a)}))} = e^{\ln(a)} = a$.
(4) $\ln(a \odot b) = \ln(a^{\ln(b)}) = \ln(b)\ln(a) = \ln(a)\ln(b)$.
Recall that the logarithm transforms a product into a sum. In the proportional context, the logarithm transforms a product into a multiplication.
(5) $(a \odot b)^{\{-1\}} = e^{1/\ln(a \odot b)} = e^{1/\ln(a)\ln(b)} = (e^{1/\ln(a)})^{1/\ln(b)} = a^{\{-1\}} \odot e^{1/\ln(b)} = a^{\{-1\}} \odot b^{\{-1\}}$.

■

Based on the properties shown above, it is possible to prove the following result:

Theorem 2.3: $(\mathbb{R}^+_{>0}, \oplus, \odot)$ *is a field.*

Proof: The proof is left as an exercise and it is strongly based on the properties showed before. ■

Example 2.4: As an example, we will solve the following proportional lineal equation $x \odot a = c$:

$$\begin{aligned}
& x \odot a = c, \\
& (x \odot a) \odot a^{\{-1\}} = c \odot a^{\{-1\}}, && \text{Applying inverse} \\
& x \odot (a \odot a^{\{-1\}}) = c \odot a^{\{-1\}}, && \text{Associating} \\
& x \odot e = c \odot a^{\{-1\}} && \text{Inverse and Neutral} \\
& x = c \odot a^{\{-1\}}, \\
& x = c^{1/\ln(a)}.
\end{aligned}$$

Note that, in the traditional arithmetic, the equivalent equation is $xa = c$, where the solution can be expressed in the following terms

$$x = \frac{c}{a} = ca^{-1}.$$

Example 2.5: We will solve the proportional equation $(a \odot x)b = c$ in order to show the algebraic techniques underlying this way of operating:

$$\begin{array}{ll}
(x \odot a)b = c, & \text{Commutativity of } \odot \\
(x \odot a) = b^{-1}, & \text{Applying multiplicative inverse of } b \\
x \odot (a \odot a^{\{-1\}}) = & \text{Applying the proportional product} \\
\quad (cb^{-1}) \odot a^{\{-1\}}, & \text{with } a^{\{-1\}} \text{ and associating} \\
x \odot e = (cb^{-1}) \odot a^{\{-1\}}, & \text{Neutral proportional product and property 1} \\
x = (cb^{-1}) \oslash a. &
\end{array}$$

Now, we are going to define the *proportional powers of a positive number*:

Definition 2.6: Let $n \in \mathbb{N}$ and $a > 0$. The expression

$$\underbrace{a \odot a \odot \ldots \odot a}_{\text{n times}}$$

is defined as the nth power of a and it will be denoted as $a^{\{n\}}$.

Next, the following properties are stated:

- $a^{\{1\}} = a$
- $a \odot a^{\{-1\}} = e$
- $a^{\{1\}} \odot a^{\{-1\}} = e = a^{\{1-1\}} = a^{\{0\}}$.

As an application of the properties presented before, we give the proportional square binomial:

$$\begin{aligned}
(ab)^{\{2\}} &= (ab) \odot (ab), \\
&= (ab)^{\ln(ab)}, \\
&= (ab)^{\ln(a)+\ln(b)}, \\
&= (ab)^{\ln(a)}(ab^{\ln(b)}), \\
&= a^{\ln(a)} b^{\ln(a)} a^{\ln(b)} b^{\ln(b)}, \\
&= a^{\{2\}}(b \odot a)(a \odot b) b^{\{2\}}, \\
&= a^{\{2\}}(a \odot b)^2 b^{\{2\}}.
\end{aligned}$$

Thus the proportional square binomial corresponds to the product of the proportional square of a, the square of the proportional product of a and b and the proportional square of b.

We can express the correspondence between the usual and the proportional arithmetic as follows:

$$\begin{tikzcd}
(ab)^{\{2\}}: & a^{\{2\}} \arrow[r, "\oplus"] & (a \odot b)^2 \arrow[r, "\oplus"] & b^{\{2\}} \\
(a+b)^2: & a^2 \arrow[u, "\odot"] \arrow[r, "+"] & 2ab \arrow[u, "\odot"] \arrow[r, "+"] & b^2 \arrow[u, "\odot"]
\end{tikzcd}$$

Next, we give the following notable proportional product:

Proposition 2.7: *Let* $a, b \in \mathbb{R}^+_{>0}$. *Then:*

(1) $a \odot b^{-1} = a^{-1} \odot b = (a \odot b)^{-1}$.
(2) $c \odot (ab) = (c \odot a)(c \odot b)$.
(3) $(a/b)^{\{2\}} = a^{\{2\}}(a \odot b)^{-2} b^2$.
(4) $(xa) \odot (xb) = x^{\{2\}}((ab) \odot x)(a \odot b)$.

(5) $(abc)^{\{2\}} = a^{\{2\}}b^{\{2\}}c^{\{2\}}((a \odot b)(a \odot c)(b \odot c))^2$.
(6) $(ab) \odot (ab^{-1}) = (a \odot a)(a \odot b^{-1})(b \odot a)(b \odot b^{-1}) = a^{\{2\}}(b^{\{2\}})^{-1}$.

Proof: The proof is left to the reader. ■

2.1. Formalizing the way of measuring with proportional arithmetic

In the following, we present the function that represents the absolute value in the proportional sense:

Definition 2.8: Let $x \in \mathbb{R}^+$ the relative value of x, denoted by $[[x]]$, it is defined as

$$[[x]] = \begin{cases} x & \text{if } x \geq 1, \\ \dfrac{1}{x} & \text{if } x < 1. \end{cases}$$

Let x, y, z positive real numbers. The following properties are simple to prove:

- $[[x/y]] \geq 1$.
- $[[x/y]] = 1$ if only if $x = y$.
- $[[x/y]] = [[y/x]]$.
- $[[x/z]] \leq [[x/y]] \odot [[y/z]]$.

In order to show the operability of the proportional absolute value, the following examples are presented:

Example 2.9:

- Let $x \in \mathbb{R}^+$. Find the x such that, $[[x]] \leq 3$.
  We have two cases:
  - If $x \geq 1$, then $1 \leq x \leq 3$.
  - If $x < 1$, we have $1/x \leq 3$. Then $\frac{1}{3} \leq x < 1$

  Hence, we conclude that $\frac{1}{3} \leq x \leq 3$.
- Let $x \in \mathbb{R}^+$. Find the x such that, $[[x]] \geq 2$.
  Again, we have two cases:
  - If $x \geq 1$, we have to $x \geq 2$.
  - If $0 < x < 1$, then $0 < x \leq \frac{1}{2}$.

  Thus, we conclude that $x \geq 2$ or $0 < x \leq \frac{1}{2}$.[1]

3. Proportional geometry

3.1. Proportional lines

Also, in this arithmetic we can define the concepts of *proportional line* and *slope* as follows:

Definition 3.1: Consider two different points on a proportional line: $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. We can form the following system of equations:

$$y_1 = bx_1^r = bx_1 \odot e^r = bx_1 \odot m,$$
$$y_2 = bx_2^r = bx_2 \odot e^r = bx_2 \odot m.$$

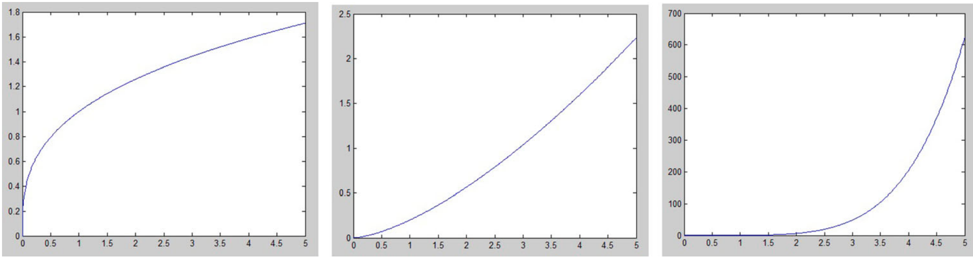

Figure 1. Proportional lines with $\frac{1}{4}$, $\frac{3}{2}$, 5 from left to right.

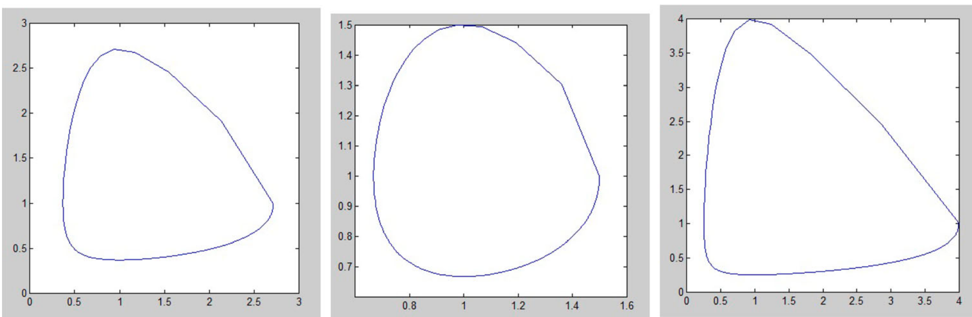

Figure 2. Proportional Circumferences of radius *e*, 1.5 and 4 from left to right.

We consider $m = e^r$. Then, operating with last equations we can see that:

$$
\begin{array}{ll}
\dfrac{y_1}{y_2} = \dfrac{bx_1}{bx_2} \odot m & \text{performing the quotient between } y_1 \text{ and } y_2 \\
\dfrac{y_1}{y_2} = \left(\dfrac{x_1}{x_2}\right) \odot m & \text{simplifying and associating} \\
\dfrac{y_1}{y_2} = m \odot \left(\dfrac{x_1}{x_2}\right) & \text{commutativity of } \odot \\
\left(\dfrac{y_1}{y_2}\right) \odot \left(\dfrac{x_1}{x_2}\right)^{\{-1\}} = m & \text{proportional product inverse} \\
\dfrac{y_1}{y_2} \oslash \dfrac{x_1}{x_2} = m & \text{by definition.}
\end{array}
$$

Hence, we have the following result:

Theorem 3.2: *The slope m of the proportional line is the $\oslash$ ratio proportional y_1/y_2, given by the variations of the y-coordinates and x_1/x_2, given by the variations of the x-coordinates.*

Figure 1 shows proportional line graphs for different slope values.

3.2. Proportional conics

According to this notion of distance, a question arises: *is it possible to imagine the algebraic equation of the proportional circumference?* (Campillay-Llanos & Guevara-Morales,

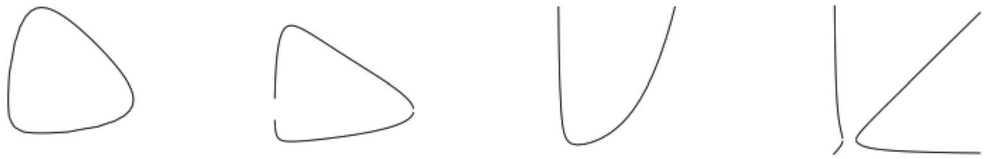

Figure 3. Proportional conics: circle, ellipse, parabola and hyperbola, respectively.

2015, 2018). If we represent the Cartesian circumference centred at the origin with radius 1, i.e.

$$x^2 + y^2 = 1,$$

then it turns out to

$$x^{\{2\}}y^{\{2\}} = (x \odot x)(y \odot y) = x^{\ln(x)}y^{\ln(y)} = e \quad \text{(see Figure 2).}$$

Let a, b positive real numbers not equal to 1. The classic conics are shown in Figure 3. Using a computer program you can verify its graphs, also noting the modifications of the figures by varying the values of a and b:

- Proportional circle: $x^{\{2\}}y^{\{2\}} = a^{\{2\}}$,
- Proportional ellipse: $(x^{\{2\}} \oslash a^{\{2\}})(y^{\{2\}} \oslash b^{\{2\}}) = e$,
- Proportional parabola $y^{\{2\}} = e^4 \odot x \odot a$,
- Proportional hyperbola: $(x^{\{2\}} \oslash a^{\{2\}})/(y^{\{2\}} \oslash b^{\{2\}}) = e$.

4. Proportional calculus

In the following, we give the definitions for the proportional calculus using the proportional arithmetic defined before.

4.1. Basic calculus definitions in the proportional sense

Definition 4.1 (Proportional limit): We say that $\lim_{x\to x_0} f(x) = L$, where $L \in \mathbb{R}^+$, if only if for all $\epsilon > 1$ there exists $\delta > 1$ such that $1 < [[x/x_0]] < \delta$, then $[[f(x)/L]] < \epsilon$.

As usual, the next step is to define the notion of continuity in the proportional sense.

Definition 4.2 (Proportional continuity): We say that f is *Proportionally continuous at* x_0 or $\lim_{x\to x_0} f(x) = f(x_0)$, if and only if for all $\epsilon > 1$ there exists $\delta > 1$ such that if $[[x/x_0]] < \delta$, then $[[f(x)/f(x_0)]] < \epsilon$.

In order to show this definition, we show the following example:

Example 4.3: The function $f(x) = x^{\{2\}}$ is proportionally continuous at a, i.e.

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^{\{2\}} = a^{\{2\}} = f(a), \quad \text{where } a > 0.$$

Actually, if $f(x) = x^{\{2\}}$, for the definition of the 2nd-power of a, we have that

$$\ln(f(x)) = (\ln(x))^2.$$

Now, by the continuity of the logarithm, we know that

$$\lim_{x\to a} \ln(f(x)) = \ln\left(\lim_{x\to a} f(x)\right).$$

So, we have

$$\lim_{x\to a} \ln(f(x)) = \lim_{x\to a} \ln^2(x) = \ln^2(a) = \ln\left(a^{\ln(a)}\right) = \ln\left(a^{\{2\}}\right).$$

Hence,

$$\ln\left(\lim_{x\to a} f(x)\right) = \ln\left(a^{\{2\}}\right),$$

so

$$\lim_{x\to a} f(x) = a^{\{2\}}.$$

This is equivalent to

$$\lim_{x\to a} \frac{f(x)}{a^{\{2\}}} = 1.$$

As a corollary, in general we have $x,\ x^{\{2\}}, \ldots, x^{\{n\}}$ are proportionally continuous, and therefore every proportional polynomial is also continuous in this sense. For example, the polynomial of degree 3, $p(x) = (4 \odot x^{\{3\}})(7 \odot x^{\{2\}})(5 \odot x) \odot 9$, is continuous.

4.2. Proportional differentiation

With this arithmetic, the derivative of a function is presented in the following terms:

Definition 4.4 (Proportional derivative): A function f is *Proportionally differentiable* at x_0 if and only if

$$\lim_{x\to x_0} \frac{f(x)}{f(x_0)} \oslash \frac{x}{x_0} = \lim_{x\to x_0} \left(\frac{f(x)}{f(x_0)}\right)^{1/(\ln(x/x_0))},$$

this limit exists. In this case, that limit is designated by $\widetilde{f(x_0)}$ and receives the name Proportional derivative of f at x_0. Also, we say that f is proportionally differentiable, if f is differentiable at x_0 for all x_0 in the domain of f.

Note that if f is derivable at x_0, then it is continuous at x_0, since the following equality holds:

$$\begin{aligned}\lim_{x\to x_0}\frac{f(x)}{f(x_0)} &= \lim_{x\to x_0}\left[\frac{f(x)}{f(x_0)}\oslash\frac{x}{x_0}\right]\odot\left[\frac{x}{x_0}\right],\\ &= \lim_{x\to x_0}\left[\frac{f(x)}{f(x_0)}\oslash\frac{x}{x_0}\right]\odot\lim_{x\to x_0}\left[\frac{x}{x_0}\right],\\ &= \widetilde{f(x_0)}\odot\lim_{x\to x_0}\left[\frac{x}{x_0}\right],\\ &= 1.\end{aligned}$$

Hence, $\lim_{x\to x_0} f(x) = f(x_0)$. To use this definition we present the following example:

Example 4.5: Consider $f:\mathbb{R}^+_{>0}\to\mathbb{R}^+_{>0}$, where $f(x)=x$. Then $\widetilde{f(x)}=e$:

$$\lim_{x\to x_0}\frac{f(x)}{f(x_0)}\oslash\frac{x}{x_0}=\lim_{x\to x_0}\frac{x}{x_0}\oslash\frac{x}{x_0}=\lim_{x\to x_0} e = e.$$

Example 4.6: Consider the function $f:\mathbb{R}^+_{>0}\to\mathbb{R}^+_{>0}$, which represents the power law $f(x)=bx\odot m$. The slope of this line is

$$\begin{aligned}\lim_{x\to x_0}\frac{bx\odot m}{bx_0\odot m}\oslash\frac{x}{x_0} &= \lim_{x\to x_0}\left(\left(\frac{x}{x_0}\right)\odot m\right)\oslash\frac{x}{x_0},\\ &= m\odot\lim_{x\to x_0}\frac{x}{x_0}\oslash\frac{x}{x_0},\\ &= m\odot\lim_{x\to x_0} e,\\ &= m\odot e,\\ &= m.\end{aligned}$$

This coincides with the calculation made in the previous section. Using the proportional derivative again, we can approximate the function $\underset{\smile}{f}(x)=e^x$ around $x=2$ with a proportional line $y=bx\odot m$. It's not hard to get that $\widetilde{e^x}_{|x=2}=e^2$, and b are obtained by solving the equation $e^2=b2\odot e^2$, then $b=1.8473$. This approach is presented in Figure 4. Note that this approximation is better than the linear approximation.

The following properties can be found in Campillay-Llanos (2007), Córdova-Lepe (2006) and Córdova-Lepe and Pinto (2009):

Proposition 4.7: *If f is a constant function, then* $\widetilde{f}(x_0)=1$.

Remark 4.1: The proportional non-variation is associated with 1 (see Proposition 2.1).

In the following property, we have the student's first intuition of the derivative of the product of two functions, that is the product of the derivatives of such functions:

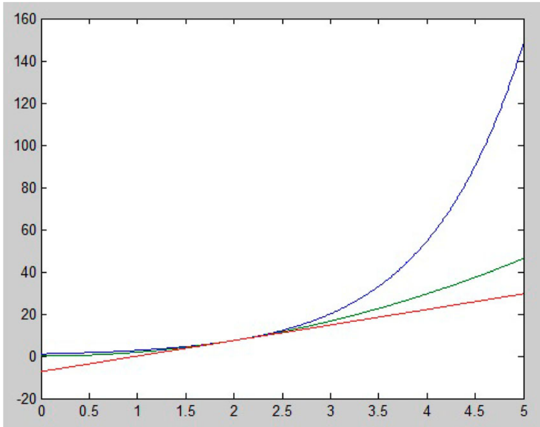

Figure 4. Approximation of the exponential function with a proportional line around $x = 2$. In blue, $f(x) = e^x$. In green, proportional line $y = 1.8473x^2$ and in red, the linear approach, $y = 7.3891x - 7.3891$.

Proposition 4.8: *Let $f, g : \mathbb{R}^+_{>0} \to \mathbb{R}^+_{>0}$, be functions such that the derivatives $\widetilde{f}(x_0)$ and $\widetilde{g}(x_0)$ exist for some $x_0 \in \mathbb{R}^+_{>0}$, then*

$$\widetilde{fg}(x_0) = \widetilde{f}(x_0)\widetilde{g}(x_0).$$

It has greater similarity with the rule of the exposition of functions and the derivative of a usual product of functions:

Proposition 4.9: *Let $f, g : \mathbb{R}^+_{>0} \to \mathbb{R}^+_{>0}$, be functions such that the derivatives $\widetilde{f}(x_0)$ and $\widetilde{g}(x_0)$ exist for some $x_0 \in \mathbb{R}^+_{>0}$, then*

$$\widetilde{f \odot g}(x_0) = ((\widetilde{f}) \odot g)(x_0)((\widetilde{g}) \odot f)(x_0).$$

Proposition 4.10: *If $f'(x_0)$ is the usual derivative, we have to*

$$f'(x_0) = \frac{f(x_0)}{x_0} \log(\widetilde{f}(x_0)).$$

This is the bridge that combines the two ideas of variations.

Finally, we present an important tool, the proportional chain rule:

Proposition 4.11 (Proportional chain rule): *Let $f, g : \mathbb{R}^+_{>0} \to \mathbb{R}^+_{>0}$, be functions such that the derivatives $\widetilde{f}(g(x_0))$ and $\widetilde{g}(x_0)$ exist for some $x_0 \in \mathbb{R}^+_{>0}$, then*

$$\widetilde{f \circ g}(x_0) = (\widetilde{f} \circ g)(x_0) \odot \widetilde{g}(x_0).$$

4.3. Proportional integration

In what follows, we introduce the notion of proportional antiderivative and proportional indefinite integral.

Definition 4.12 (Proportional Antiderivative): The function $F(x)$ is a *Proportional Antiderivative* of the function $f(x)$ on the interval I if $\widetilde{F(x)} = f(x)$ for all $x \in I$.

Definition 4.13 (Proportional indefinite Integral): The Proportional integral of $f(x)$ is the Proportional antiderivative of $f(x)$.

$$\widetilde{\int} f(x) \odot \rho_x = \widetilde{\int} f(x)\rho_x = CF(x),$$

where C is a constant.

As an exercise we propose to verify the following equalities:
In the following, C and K are constants. Then:

- $\underset{\sim}{\widetilde{\int}} 1\rho_x = K.$
- $\underset{\sim}{\int} e^n \rho_x = Cx^n.$
- $\underset{\sim}{\int} e^{ax} \rho_x = Ce^{ax}.$
- $\underset{\sim}{\int} x^{\{2\}} \rho_x = C(x^{\{3\}})^{13}.$
- $\underset{\sim}{\int} x^{\{3\}} \rho_x = C(x^{\{4\}})^{1/4}.$
- $\underset{\sim}{\int} x^{\{n\}} \rho_x = C(x^{\{n+1\}})^{1/(n+1)}.$
- $\int e \oslash x\rho_x = \ln(x).$

4.3.1. Proportional integration of the Gaussian function: $f(x) = 1/\sqrt{2\pi}e^{-(1/2)x^2}$

Let

$$\int \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2}\, \mathrm{d}x, \tag{6}$$

where the symbol $\int$ represents the integral used in the traditional courses of calculus. From elementary calculus, we know that it is impossible to find an antiderivative g that satisfies (6).

In the next theorem, we will find such function g, in the proportional sense:

Proposition 4.14: *The function* $f(x) = e^{rx^2}$, *where* $r \in \mathbb{R} - \{0\}$, *admits a primitive in the proportional sense. This primitive is given by*

$$\widetilde{\int} e^{rx^2} \rho_x = C(e^{rx^2})^{1/2}.$$

Proof: We considered $h(x) = e^x$ and $s(x) = rx^2$, because $f(x) = h \circ s(x)$. Note that propositions (4.7) and (4.8) turn out that $\widetilde{s}(x) = e^2$. Then, applying the proportional chain rule to $f = h \circ s$, Proposition 4.11, we have

$$\widetilde{f(x)} = \widetilde{h \circ s}(x) = (\widetilde{h} \circ s)(x) \odot \widetilde{s}(x) = e^{rx^2} \odot e^2 = (e^{rx^2})^2.$$

We conclude that

$$\widetilde{\int} e^{rx^2} \rho_x = C(e^{rx^2})^{1/2}.$$

Hence $f(x) = e^{rx^2}$ admits a primitive, in the proportional sense. In particular the Proportional integral of $e^{-(1/2)x^2}$ is $C(e^{-(1/2)x^2})^{1/2}$. ■

Proposition 4.15: *The function $f(x) = K$, where K is constant admits a primitive in the proportional sense. This primitive is given by*

$$\widetilde{\int} K\rho_x = C(K \odot x)$$

Proof: Properties 4.7, 4.8 and 4.9 we get the proportional derivative. Using $\widetilde{C(K \odot x)} = \widetilde{C}\widetilde{(K \odot x)} = 1(K \odot e) = K$. ■

The basic way of adding quantities in the proportional calculation is through multiplication, implies that integral is multiplicative.

Proposition 4.16: *If f and g are positive, integrable functions in the proportional sense, then fg is integrable and*

$$\widetilde{\int} f(x)g(x)\rho_x = \widetilde{\int} f(x)\rho_x \widetilde{\int} g(x)\rho_x.$$

Finally, using propositions 4.14, 4.15 and 4.16, we conclude with the main result of this work:

Theorem 4.17: *The function $f(x) = 1/\sqrt{2\pi}\, e^{-(1/2)x^2}$ admits a primitive in the proportional sense. This primitive is given by*

$$\widetilde{\int} \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} \rho_x = C\left(\frac{1}{\sqrt{2\pi}}\right)^{\ln(x)} (e^{-(1/4)x^2}).$$

Proof: By propositions 4.14–4.16, we can see that

$$\begin{aligned}\widetilde{\int} \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} \rho_x &= \left[\widetilde{\int} \frac{1}{\sqrt{2\pi}} \rho_x\right]\left[\widetilde{\int} e^{-(1/2)x^2} \rho_x\right] \\ &= C\left(\frac{1}{\sqrt{2\pi}} \odot x\right) (e^{-(1/2)x^2})^{1/2} = C\left(\frac{1}{\sqrt{2\pi}}\right)^{\ln(x)} (e^{-(1/4)x^2}).\end{aligned}$$

■

The proportional calculus presented here is not only intended to expose another type of arithmetic and its scope. This way of operating quantities presents new tools to represent natural phenomena. The calculation of the integral of the Gaussian function is an example that could be used in several areas of modern mathematics.

Note

1. Note that in this arithmetic 0 represents $-\infty$. (Think in the homeomorphism $f(x) = \ln(x)$.)

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